Uncertainty, Pessimism and Economic Fluctuations*

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Abstract. This paper develops a novel theory of uncertainty-driven business cycles that accommodates the notion of non-inflationary aggregate demand shocks out of variations in uncertainty. Instead of thinking uncertainty as risk, we regard uncertainty as ambiguity. We demonstrate that within the real business cycle model, ambiguity shocks, namely shock to the variance of agents' prior belief over possible models, can generate comovements across real quantities without commensurate movements in labor productivity under the condition that agents are ambiguity averse and there exists a certain type of coordination friction among them. In response to a positive ambiguity, agents behave as if they believe aggregate demand is turning bad and becoming more volatile. The former translates into depressed market confidence, which makes all real quantities plummets. While the latter incentivizes agents use more of their private information both when making economic decisions and forecasts, which heightens the cross-sectional dispersions of beliefs. These predictions regarding agents' belief in our theory are consistent with survey data evidence. Finally, the quantitative potential of our theory is illustrated within a dynamic RBC model.

Keywords. Business cycles, ambiguity, ambiguity aversion, smooth model of ambiguity, aggregate demand, confidence, uncertainty.

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1. Introduction

Recessions are times with heightened uncertainty. Measure uncertainty by the cross-sectional dispersion of real GDP forecasts in Survey of Professional Forecasters (SPF). Depicted in Figure 1, over the last thirty years, all three recessions experienced by the US economy feature a large increase in uncertainty.[1]

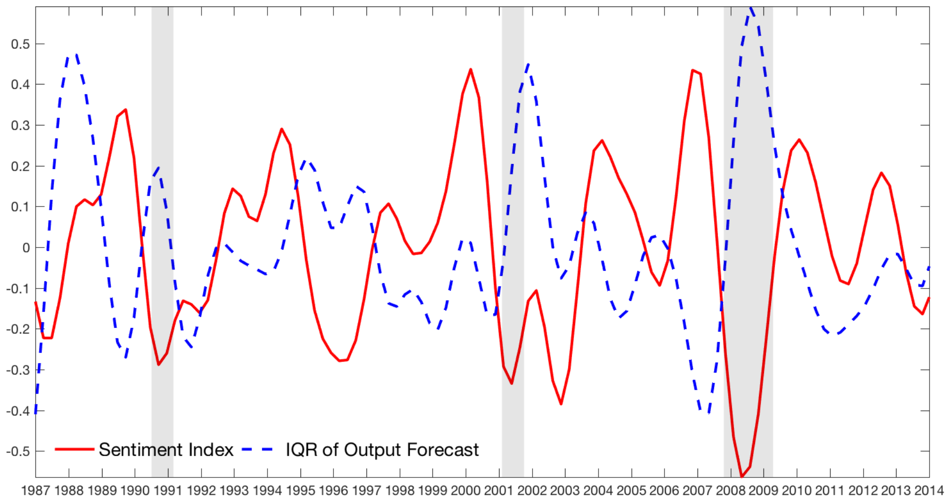

Figure 1. *Market Confidence and Uncertainty.*

Note: The figure plots consumer sentiment index (red line) and cross-sectional dispersion of real GDP forecasts by professional forecasters (dashed blue line) over the period 1987Q1-2014Q4. Correlation is around -0.45 over the entire period. Both of the time series are bandpass-filtered at frequencies 6-32 quarters and re-scaled. Consumer sentiment index is from Michigan Survey of Consumers, and real GDP forecast data is from Survey of Professional Forecasters provided by the Federal Reserve Bank of Philadelphia.

Business cycles tend to be uncertainty-driven. In the theory of uncertainty shock, Bloom [2009] and Bloom et al. [2016] demonstrate that if the economy features non-convex adjustment cost, exogenous variations in risk, namely the volatility of underlying shocks, can generate variations in the cross-sectional misallocation of resources, which eventually map into co-movements across real quantities. However, the byproduct of their transmission mechanism of uncertainty shock is the strong pro-cyclical movements in labor productivity, which is at odds with the data where the correlation between output and labor productivity is near zero and slightly negative and the correlation between labor productivity and hours is significantly negative[2]. The data prefers a theory of non-inflationary aggregate demand shock instead of a

[1]See Bloom [2009] and Bloom et al. [2016] for more measures of uncertainty and their behaviours during crisis.

[2]See Angeletos et al. [2016a] for a recent empirical demonstration that business cycles are disconnected with labor productivity and inflation

theory of measured TFP shock or supply shock. To fill in this gap, in this paper, we develop a novel theory of uncertainty-driven business cycle that differs with Bloom [2009] and Bloom et al. [2016] (1) in the notion of uncertainty and uncertainty shock; (2) in the transmission mechanism of uncertainty shock; and importantly (3) in the empirical properties of the shock.

Instead of thinking uncertainty as risk, we regard uncertainty as ambiguity[3]. When uncertainty refers to risk, uncertainty shock is defined to be exogenous variations in the variance of underlying shocks and agents are risk averse. In our model, when uncertainty refers to ambiguity, uncertainty (or ambiguity) shock is defined to be exogenous variations in the variance of agents' prior beliefs over possible models and agents are ambiguity averse. We formalize these notions of uncertainty and uncertainty shock within the smooth model of ambiguity axiomatized by Klibanoff et al. [2005, 2009]. A positive ambiguity shock that makes agents more uncertain across possible models in their priors, creates an increased degree of pessimism over the outlook of the economy, which we call market confidence. The particular transmission mechanism from uncertainty to confidence is consistent with survey data evidence. As depicted in Figure 1, when we proxy market confidence with Consumer Sentiment Index in Michigan Survey of Consumers, we do find that periods with heightened uncertainty are associated with depressed market confidence. Finally, when the economy features coordination frictions due to the existence of incomplete information, the pessimism is over the others' beliefs about the state of the economy, which, within a general equilibrium setup, translates into a pessimistic belief over aggregate demand. The economy responds, at the aggregate level, as if there presents a negative aggregate demand shock.

Framework and Mechanism. We formalize an otherwise standard real business cycle model that additionally features (a) aggregate demand externalities; (b) ambiguity averse agents (households, workers or firm managers) and finally (c) incomplete information over ambiguous aggregate fundamentals.

In the model, firms differ in productivities, which consists of an aggregate component as well as an idiosyncratic component. The average productivities of all firms are ambiguous in the sense that the cross-sectional mean of idiosyncratic components can be anything on the real-line. Firms have perfect knowledge over own productivity but incomplete information over the ambiguous average productivities of all other firms. The key thing here is that when making output decision, firms need to make expectations over the average productivities of the other firms since it is the sufficient statistics of own demand conditions under aggregate demand externalities. Since firms are op-

[3] Ambiguity refers to subjective uncertainty over probabilities due to lack of ex-ante information to pin down a specific model for the economy in the course of decision making.

erated for the interests of the ambiguity averse households, firms behave as if they are ambiguity averse by themselves. Therefore, output decisions are made as if firms possess a pessimistic belief over models when making expectations over demand conditions. Pessimism at the firm level means that models with a lower demand on average are perceived to be more likely than optimistic models with a higher demand on average. In response to a positive ambiguity shock, firms become more uncertain across different possible models of demand conditions. Such a mean-preserving spread of belief over models amplifies the degree of pessimism of firms resulting in a depressed expectation over own demand conditions. At the aggregate level, there would be an increase in the economy-wide degree of pessimism, which is interpreted as depressed **market confidence**.

Also, firms rely on the observation of own productivity form expectations over demand. This is because individual firm productivity equals to the average productivities of all other firms plus an idiosyncratic shock. In this sense, the observation of own productivity serves as a private information over average productivities of all other firms or own demand conditions. When forming expectation over demand, firms face trade-offs between the use of prior and private information. To note that, we formulate ambiguity shock as the shock to the variance of prior belief over possible models. A positive ambiguity shock results in a less informative prior information over demand conditions from the perspective of firms. Hence it incentivizes more use of private information in forming expectations over demand. Therefore, firm-level output responds more to private information, which further implies that aggregate output responds more to average productivities of all firms. From the perspective of the professional forecasters who are required to submit a personal forecast over aggregate output, this indicates that there are more to be estimated. Also, following the same arguments as in the case of firms, professional forecasters would also use more of their private information to make forecast over the average productivities of the economy. In combine, output forecast of professional forecasters respond more to private information, which heightens **belief divergence** or measured uncertainty.

Finally, firms hire less labor and cut down output in response to depressed expectation over own demand conditions, which eventually leads to a drop in aggregate output. From this perspectives, ambiguity shock in this paper is nothing more than a particular formulation of **aggregate demand shock**.

To note that, incomplete information is crucial in driving all these results in our model. Technically, this is because if the information is complete, all agents including firms and households can perfectly coordinate not only their beliefs but also their actions. Common knowledge of the economy implies no heterogeneity in beliefs hence zero belief divergence. Also, all firms will have not only perfect knowledge of own productivity but also perfect knowledge of own demand conditions. Therefore, there exists no room for ambiguity shock having any impact on market confidence, belief divergence or aggregate economy. The broader insights are that incomplete information helps us to accommodate a situation where ambiguity is mostly over others' productivity rather than own productivity. Alternatively, ambiguity is mostly over the short-run outlooks of the economy rather than long-run perspectives, which is crucial in understanding the aggregate demand shock nature of ambiguity shock.

Results. The paper starts with a simple business cycle model that abstracts out capital accumulation that allows us to deliver a couple of analytical results that clarifies the main mechanism of the paper. We demonstrate that a positive ambiguity shock generates lower market confidence hence lower aggregate output and larger belief divergence if agents inside the economy are ambiguity averse and the information is incomplete.

At the core of these results are the interplay between incomplete information and dual impacts of ambiguity shocks. On the one hand, a positive ambiguity shock makes agents (firms and households) believe that the aggregate fundamental becomes more volatile and on the other hand, aggregate fundamental is turning bad when they are ambiguity averse. With the existence of incomplete information, the former provides increased incentives for the use of private information both in making forecasts and decisions resulting in heightened belief divergence and the latter translates into increased pessimism over aggregate demand resulting in depressed market confidence. Aggregate output plummets since all firms believe their demand is turning bad and in response cut down output.

We also discuss the dual impacts of ambiguity shock within a game theoretic interpretation of the market equilibrium, which resembles the idea in Angeletos and La'O [2009] such that any business cycle models with incomplete information can be transformed into a beauty contest. This game-theoretic interpretation allows us to clarify the role of incomplete information in driving aggregate fluctuations. It turns out that incomplete information acts as an amplification mechanism for ambiguity. In the extreme case when there presents complete information, ambiguity shock has no impact at the aggregate level. Finally, to note that our aggregate demand channel relies on incomplete information but not on any forms of nominal rigidities. Therefore, aggregate fluctuations generated by ambiguity shock are consistent with the fact that most of the aggregate fluctuations observed in the US data are disconnected from productivity or inflation[4]. Our novel theory of uncertainty-driven business cycles features

[4] Angeletos et al. [2016a] identifies a business cycle factor, which is a one-dimensional summary of aggregate movements. The business cycle factor turns out to disconnect from technology or inflation. Further see Gali [1999] and Basu et al. [2006] for the former point. And **?** for a survey of the empirical

non-inflationary demand-driven aggregate fluctuations.

We further conduct a couple of quantitative evaluations of the impacts of ambiguity shock within the dynamic RBC framework. Ambiguity shock is shown to be able to generate aggregate co-movements in output, consumption, hours and investment without commensurate movements in labor productivity similar to confidence shock alias Angeletos et al. [2016b] and Huo and Takayama [2015]. This further allows us to interpret ambiguity shock as aggregate demand shock quantitatively. What drive the co-movements in quantities are fluctuations in the degree of pessimism over the short run outlooks of the economy, i.e., fluctuations in market confidence due to ambiguity shock. The model is capable of generating empirically plausible counter-cyclical labor wedge. In this sense, we can alternatively interpret ambiguity shock as the counter-cyclical tax on labor supply or pro-cyclical subsidy on labor demand. Moreover, quantitatively the model can capture cyclical behaviors in cross-sectional dispersions in output forecasts indicated by the SPF dataset. Finally, the estimated market confidence closely tracks Sentiment Index in Michigan Survey of Consumer manifesting the fact that our theory captures movements in market confidence quantitatively.

Contributions. The contributions of our paper are four folds. First of all, we propose a theory of uncertainty-driven business cycles that is capable of generating non-inflationary demand-driven aggregate fluctuations. We complement the existing literature of uncertainty shock with an alternative formulation and transmission that accommodates a notion of non-inflationary aggregate demand shock out of variations in uncertainty. Our paper further extends the conventional wisdom in understanding the co-movements across confidence, uncertainty, and aggregate economy by arguing that these can be the endogenous outcomes of ambiguity shock not only qualitatively but also quantitatively.

Secondly, we contribute to the confidence shock literature Angeletos and La'O [2013], Angeletos et al. [2016b], Huo and Takayama [2015] and Benhabib et al. [2015] with the empirical linkages between confidence and uncertainty shocks. The confidence shock literature has been critized by the lack of empirical counter-parts of the higher-order beliefs shock in the theory. Our theory reconciles such concern by demonstrating that variations in confidence can originate from variations in uncertainty where we do have a lot of empirical measures in the literature.

Thirdly, we also contribute to the ambiguity literature with a Bayesian formulation of ambiguity shock based on the smooth model of ambiguity. Conceptually, it differs

evidence on NKPC that corresponds to the latter point, namely inflation puzzle. Also, see Beaudry and Portier [2014] for a couple of evidence that US business cycles are mainly non-inflationary demand-driven.

with the existing literature in the sense that it is a shock to the amount of ambiguity rather than a shock to agents' taste over ambiguity (Bhandari et al. [2016]) or a mix of both (Ilut and Schneider [2014]). The unique insight for our Bayesian formulation of ambiguity shock that cannot be shared with the others is that it induces endogenous fluctuations in measured uncertainty, such as dispersion measures. Our paper further contributes to the literature with a particular channel to let second-moment shock, i.e., ambiguity shock, have the first-moment impact. Such a channel relies on the interplay between ambiguity aversion and incomplete information rather than the non-convex adjustment costs as in the theory of uncertainty shock.

Finally, we make one methodological contribution by the provision of a linkage between (a) business cycle models featuring ambiguity aversion and incomplete information and (b) games of incomplete information with ambiguity aversion. The game theoretic interpretation of the business cycle model allows us to build the key economic intuition behind main mechanisms of our paper and to deliver more insights into the interplay between ambiguity, ambiguity aversion, and incomplete information. It turns out incomplete information acts as an amplification mechanism for ambiguity when agents are ambiguity averse.

Layout. The rest of the paper is organized as follows. Section 2 reviews the literature. Section 3 sets up the simple model without capital. Section 4 characterizes the equilibrium by a formal definition as well as a set of optimality conditions. Then in Section 5, impacts of ambiguity shocks are closely studied within the simple model without capital. Section 6 sets up a dynamic RBC model where a couple of quantitative evaluations are conducted. Finally, Section 7 concludes.

2. Related Literature

This paper is related to the literature of expectation-driven business cycles including (a) the news shock literature, Beaudry and Portier [2004, 2006] and Jaimovich and Rebelo [2009]; (b) noise shock literature, Lorenzoni [2009], Barsky and Sims [2012] and Blanchard et al. [2013]; (c) confidence shock literature, Angeletos and La'O [2013], Angeletos et al. [2016b], Huo and Takayama [2015] and Benhabib et al. [2015]; (d) misspecification shock literature, Bhandari et al. [2016]; (e) non-bayesian ambiguity shock literature, Ilut and Schneider [2014] and Ilut and Saijo [2016] and finally (f) uncertainty shock literature, Bloom [2009], Bidder and Smith [2012], Bloom et al. [2016] and Arellano et al. [2018]. We contribute this line of literature with an alternative "exotic" shock, i.e., Bayesian formulation of ambiguity shock, which generates not only aggregate fluctuations but also endogenous co-movements across market confidence and belief divergence within the lens of the business cycles.

Our theory of uncertainty-driven business cycles, on the one hand, directly connects to the theory of confidence shock alias Angeletos and La'O [2013], Angeletos et al. [2016b], Huo and Takayama [2015] and Benhabib et al. [2015] in generating animal-spirit-like aggregate fluctuations, at the core of which is the interplay between ambiguity aversion and incomplete information. In response to a positive ambiguity shock, firms behave as if their beliefs over average productivities of the others are depressed but beliefs over own productivity, on which they have perfect information, are unaffected. The unique insights we provide in this paper is the empirical linkages between confidence and uncertainty shocks.

On the other hand, our paper also relates to the theory of uncertainty shock alias Bloom [2009] and Bloom et al. [2016] in generating time-varying measures of uncertainty, specifically for belief divergence. Under our Bayesian formulation, the nature of ambiguity shock is a second-moment "model uncertainty" shock, a positive realization of which incentivizes more use of private information for agents when making output forecasts hence heightened uncertainty. In the broader context, we share the same research spirit with uncertainty shock literature in identifying a particular mechanism that enables second-moment shocks to have first-moment impacts. Uncertainty shock literature relies on non-convex adjustment costs while we rely on the interplay between ambiguity aversion and incomplete information. However, as noted by Angeletos et al. [2016b], uncertainty shocks as in Bloom [2009] and Bloom et al. [2016] "can only generate realistic aggregate co-movements through inducing strong procyclical movements in aggregate TFP". From this perspective, it is an alternative formulation of aggregate productivity shock. While in our paper, ambiguity shock is by nature aggregate demand shock. Notable exceptions in the uncertainty shock literature Bidder and Smith [2012] and Arellano et al. [2018]. The former demonstrates that interaction between robust preference and stochastic volatility generates animal spirits fluctuations. We differentiate with them in the underlying preference structure and also with the transmission mechanism from animal spirit to aggregate fluctuations. They rely on news shock channel while we rely on the confidence channel. The latter relies on financial frictions in generating aggregate-demand like fluctuations due to risk shocks. We differ with them in the notion of uncertainty and also in terms of the underlying transmission mechanisms where they rely on hedging behaviours of firms and in our paper it is the variations in as if belief channel.

There are some other works that study the implications of ambiguity aversion in the context of business cycle models, but with the different mathematical representation of the preferences. For example, Ilut and Schneider [2014] and Bhandari et al. [2016]. Apart from the above-mentioned difference in the ability to capture fluctuations in belief divergence or measured uncertainty in general, our paper differs with both of the above-mentioned works in a few other aspects. Ilut and Schneider [2014] uses multiple priors preference axiomatized by Gilboa and Schmeidler [1989] to model ambiguity aversion, and ambiguity shock is modeled in a classical statistics fashion. Bhandari et al. [2016] uses robust preference proposed by Hansen and Sargent [2001a,b] and focus on time-varying concerns for model misspecification, which can be understood as time-variations in the degree of ambiguity aversion. While in our paper, ambiguity aversion is modeled by (recursive) smooth model of ambiguity axiomatized by Klibanoff et al. [2005, 2009] and learning over models features smooth rule updating proposed by Hanany and Klibanoff [2009] to ensure dynamic consistency. Our Bayesian formulation of ambiguity shock is a pure shock to the amount of ambiguity rather than a shock to agents' taste over ambiguity (Bhandari et al. [2016]) or a mix of both (Ilut and Schneider [2014]). Our concept of ambiguity shock is consistent with empirical studies on the recent financial crisis that demonstrate that sudden increases in credit spreads observed during the 2007-2008 crisis are mainly due to the increase in the amount of ambiguity rather than the increase in agents' taste over ambiguity[5].

A notable exception in this area is Ilut and Saijo [2016]. In their paper, ambiguity averse preference is represented by the multiple priors preference axiomatized by Epstein and Schneider [2003]. Ambiguity aversion is modeled as an amplification mechanism of business cycles rather than devices for exogenous variations. In their paper, recessions are periods of less learning. With an exogenous entropy constrain, reduced learning translates into a larger range of models. Therefore, there features endogenous counter-cyclical ambiguity. They focus on the aggregate implications of their model and succeed in capturing aggregate fluctuation in the data. However, their model implications for dispersion measures are ambiguous. Less learning in recession implies less use of private information when making forecasts. In most of the cases, this implies lower cross-sectional dispersions either in output forecasts or realized outputs. We differ with them in generating the right co-movement pattern in cross-sectional dispersion measures.

Ambiguity averse preferences, especially recursive multiple prior preferences, have been intensively used in the literature to generate asymmetric responses of aggregate variables to aggregate shocks in recessions and boom. For example Epstein and Schneider [2008], Ilut [2012], Ilut et al. [2016], Baqaee [2017] and Zhang [2017]. These papers assume the existence of ambiguity over precisions of various sources of information to provide agents with state-dependent subjective belief over precisions of related information. Then negative realizations of shocks are associated with a subjective belief of higher precisions hence increased responses to shocks in bad times.

[5]See Boyarchenko [2012] for more details.

In our paper, ambiguity is over the first-moment of shocks but ambiguity shock is of second-moments, which is the unique feature of the smooth model of ambiguity. The interplay between ambiguity aversion and incomplete information in our paper also generates counter-cyclical responses to aggregate (productivity) shock. This is because a positive ambiguity shock incentivizes the use of private information in making expectations hence in making output decision. Eventually, aggregate output responds more to aggregate productivity shock. In sum, as far as we know, we are the only paper in the literature that studies joint implications of ambiguity aversion and incomplete information within the lens of business cycles. Also, ambiguity aversion has been studied in the context of asset pricing extensively, see Ju and Miao [2012] and Collard et al. [2018].

Our paper also relates to the theory of beauty contest pioneered by Morris and Shin [2002] then extended by Angeletos and Pavan [2007]. We further extend the theory by allowing for ambiguity averse preference. Finally, in a broader context, our paper is also related to those studying coordination games with model uncertainty, such as Chen et al. [2016] and Chen and Suen [2016].

3. The Simple Model without Capital

In this section, we construct a static general equilibrium model in the vein of Angeletos and La'O [2009]. The model embeds three key features in an otherwise standard real business cycle environment: (a) aggregate demand externalities, (b) incomplete information over the ambiguous aggregate state of the economy and finally (c) smooth model of ambiguity together with ambiguity shock. We first describe the physical environment along with the uncertainty structure and the evolving of information sets of all agents. We then specify the preferences and interim belief systems. Finally, we close up this section by a couple of remarks and interpretations of the setup.

3.1. Physical Environment, Shocks and Information Structure

Geography, markets and timing. The economy consists of a continuum of islands, indexed by $j \in J = [0,1]$ and a mainland. On each island j, there exists a continuum of firms, indexed by $(i,j) \in I \times J = [0,1]^2$ and a continuum of workers, indexed by $(m,j) \in M \times J = [0,1]^2$. Island firms and workers interact with each other in the locally competitive labor market for the production of differentiated island commodities indexed by j. These commodities are traded in a centralized market operated on the mainland, where a continuum of consumers, indexed by $h \in H = [0,1]$ and a large number of competitive final good producers inhabit. We assume that consumer h and a continuum of workers $\{(h,j)\,;j \in J\}$ constitute a large household indexed by $h \in H$, who owns a continuum of firms $\{(h,j)\,;j \in J\}$. By doing so, we ensure the existence of a representative household at the mainland and a continuum of representative firms and workers on every island. This is because, as it will become evident later, there exist no heterogeneities, either in fundamental or in information, within an island.

We focus on the static setup in this section. There is only one period, say period t, which is decomposed into three stages. At stage zero, period t shocks are realized. At stage 1, island-specific competitive labor markets open up. The representative household sends out workers to each islands. On island j, firms make labor demand decisions and symmetrically, workers make labor supply decisions on the basis of incomplete information over the ambiguous aggregate state of the economy. At stage 2, on the mainland, the centralized commodities market opens up. All uncertainty, either risk or ambiguity, is resolved. Final good producers produce. And the representative household makes consumption decisions upon receiving all the transfers from workers and firms on the basis of perfect information.

Households. The utility of the representative household is given by:

$$\frac{C_t^{1-\gamma}-1}{1-\gamma}-\chi\int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon}dj$$

where γ is the relative risk aversion and ϵ is the inverse Frisch elasticity of labor supply. To note that, in our static setup, γ also controls for the income effects of labor supply. The corresponding budget constraint of household is such that

$$P_tC_t=\int_J W_{j,t}N_{j,t}dj+\int_J\int_I \Pi_{i,j,t}d\left(i,j\right)$$

where P_t denotes the price of final goods, $\int_J W_{j,t}N_{j,t}dj$ denotes the total labor income and finally $\int_J\int_I \Pi_{i,j,t}d\left(i,j\right)$ denotes the total realized firm profits.

Island firms. Island j firms use labor only for the production of island j commodity. The production function of firm (i,j) is given by

$$Y_{j,t}=A_{j,t}N_{i,j,t}^{1-\alpha} \tag{1}$$

where $A_{j,t}$ is the island-specific productivity and the realized profit is given by

$$\Pi_{i,j,t}=P_{j,t}Y_{i,j,t}-W_{j,t}N_{i,j,t}$$

where $W_{j,t}$ denotes the nominal wage on island j in period t and $P_{j,t}$ denotes the market price of island j commodity to be determined at stage 2 when the centralized markets opens up. Since it is assumed that it is the representative household who owns the firm, any realized profits are to be transferred to the consumer for the purchase of final goods for consumption. Therefore, in the absence of any uncertainty concerns, island j firms care about the consumer valuation over its profits given by

$$\frac{u'\left(C_t\right)}{P_t}\Pi_{i,j,t}$$

where P_t is the price of final goods normalized to 1.

Final-good producers. The competitive final-good sector employs a CES production technology given by

$$Y_t = \left(\int_J Y_{j,t}^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}$$

where θ is elasticities of substitution among island commodities that controls the strength of aggregate demand externalties. Demand function for island j commodity is, therefore, given by

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\theta} Y_t$$

where $P_t \equiv \left(\int_J P_{j,t}^{1-\theta} di\right)^{\frac{1}{1-\theta}}$ denotes the price of final goods that is normalized to 1.

Productivity and ambiguity shocks. Aggregate productivity $a_t \equiv \log A_t$ follows a Normal distribution with mean 0 and variance σ_ζ^2

$$a_t \sim N\left(0, \sigma_\zeta^2\right)$$

Island-specific productivity, defined by $a_{j,t} \equiv \log A_{j,t}$, equals to aggregate productivity plus some idiosyncratic productivity shock $\iota_{j,t}$:

$$a_{j,t} = a_t + \iota_{j,t}$$

Idiosyncratic productivity shocks $\iota_{j,t}$ are assumed to be i.i.d normally distributed with mean ω_t and variance σ_ι^2. Objectively, the cross-sectional mean of idiosyncratic productivity shock is zero, i.e. $\omega_t = 0$. However, agents inside the economy cannot fully "understand" it. They possess some ambiguity over it. Specifically, they believe that anything on the real-line can be a potential candidate for ω_t. And they possess a common zero-mean[6] Normal prior belief over $\omega_t \in \mathcal{R}$:

$$\omega_t \sim N\left(0, e^{\psi_t}\right) \quad \text{with}$$

where ψ_t measures the amount of ambiguity perceived by the agents[7]. We assume that ψ_t is chosen by nature such that

$$\psi_t = \overline{\psi} + \tau_t \quad \text{with} \quad \tau_t \sim N\left(0, \sigma_\tau^2\right) \tag{2}$$

where $\overline{\psi}$ denotes the amount of ambiguity perceived by all agents at the ambiguous steady state[8]. And we interpret τ_t as ambiguity shock, which is Normally distributed with mean 0 and variance σ_τ^2.

Information structure. Denote $\mathcal{I}_{t,0}$, $\mathcal{I}_{j,t,1}$ and $\mathcal{I}_{t,2}$ as the information sets that are available to all agents at stage 0 of period t, are only available to island j agents at stage 1 of period t and are available to all agents at stage 2 of period t, respectively. We define these information sets by

$$\mathcal{I}_{t,0} = \{\psi_t\} \qquad \mathcal{I}_{j,t,1} = \mathcal{I}_{t,0} \cup \left\{a_{j,t}\right\} \qquad \mathcal{I}_{t,2} = \cup_j \mathcal{I}_{j,t,1} \cup \{\zeta_t\} \tag{3}$$

There are a couple of implicit assumptions behind this information structure. First of all, information is symmetric within each island but is asymmetric across islands. Secondly, ambiguity shock τ_t happens at the beginning of each period t. Thirdly, at stage 1 of period t, island j productivity $a_{j,t}$ is only accessible for island j agents. In this sense, $a_{j,t}$ serves as the private information of island j agents over the average productivity $\int_J a_{j,t} dj$, which is ambiguous. Therefore, labor supply and demand decisions on each island are made under incomplete information over the ambiguous aggregate state of the economy. Fourthly, $\mathcal{I}_{t,2}$ contains the complete set of local information that is originally dispersed at stage 1. This is justifiable since commodities prices or transfers perfectly reveal the island-specific productivity that is previously dispersed at stage 1. Fifthly, all uncertainty, either risk (state uncertainty over a_t) or ambiguity (model uncertainty over ω_t) is resolved at stage 2 of period t. Hence consumption decisions are made under perfect information.[9]

[6]The objective model ω_t are assumed to be inside the set of possible models of all agents. By assuming this, we rule out any mis-specification concerns and focus on ambiguity. See Peter Hansen and Marinacci [2016] for a detailed discussion of the differences between mis-specification and ambiguity

[7]Maccheroni et al. [2013] proposes to use the variance of the ex-ante expected utility of a particular model to quantify the amount of ambiguity in general information structure, which is shown to be consistent with a quadratic approximation akin to Arrow-Pratt approximation. Our measure of the amount of ambiguity is consistent with theirs ordinally under Normality.

[8]Ambiguous steady state refers to the state the economy converges to in the absence of any shocks but taking into account of the existence of ambiguity.

[9]The assumption that all period t uncertainty resolves at the second stage of period t is in some sense

We close the description of the physical environment, shocks and information structure by the timeline of our model in Figure 2.

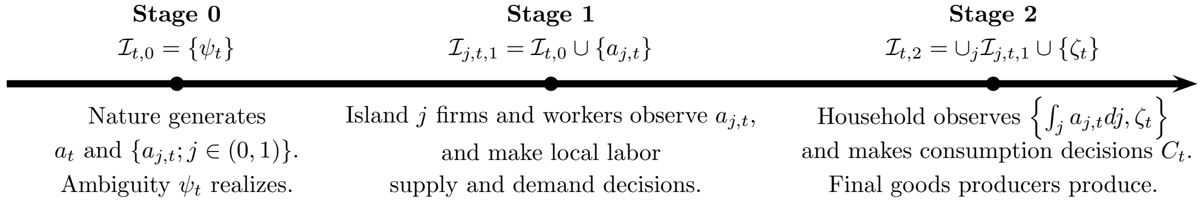

***Figure* 2.** *Timeline for Period t*

3.2. Preferences and Interim Belief Systems

All agents inside the economy are ambiguity averse in the sense that they dislike ambiguity more than risk. Ambiguity averse preferences can be mathematically represented by the smooth model of ambiguity first proposed by Klibanoff et al. [2005], which features a separation between the amount of ambiguity (characteristics of subjective belief) and degree of ambiguity aversion (characteristics of decision makers' tastes). Such a separation is appealing because only with it, we can seriously study the impacts of ambiguity shock upon controlling for decision makers' taste over it. It is widely acknowledged that Bayesian updating is only dynamic consistent under expected utility preferences. To ensure dynamic consistency *across stages* within a period, we employ the smooth rule of updating, which is a re-weighted Bayesian updating rule proposed in Hanany and Klibanoff [2009][10].

At stage 2, when all uncertainty is resolved, the representative household's preferences are represented by a standard utility function. In what follows, we carefully specify the preferences and interim belief systems for the representative household at stage 1 of period t and formulate the stage 1 workers' and firms' problems when there features incomplete information is over the ambiguous fundamentals.

Preference of the representative household at stage 1. At stage 1, preference of the representative household is represented by the smooth model of ambiguity proposed

ad-hoc to ensure tractability. However, most of the key messages delivered in this paper do not rely on this particular assumption on information structure.

[10]See Appendix for a discussion over dynamic consistency

in Klibanoff et al. [2005]. The corresponding island j workers' problem is such that

$$\max_{N_{j,t}} \int_{\mathcal{R}} \phi \left(E^{\omega_t}_{j,t,1} \left[\frac{C_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right) \widetilde{f}^h_{j,t,1} (\omega_t)\, d\omega_t \tag{4}$$

$$\text{s.t.} \quad P_t C_t = \int_J W_{j,t} N_{j,t} dj + \int_J \int_I \Pi_{i,j,t} d\,(i,j) \tag{5}$$

where the island j workers decide how much labor $N_{j,t}$ to supply into island-specific competitive labor market given nominal wage $W_{j,t}$.

Here $\phi(x)$ is some strictly increasing and concave function, whose curvature captures decision makers' taste over ambiguity, i.e., degree of ambiguity aversion and $E^{\omega_t}_{j,t,1}[\cdot]$ denotes the mathematical expectation conditioned on $\mathcal{I}_{t,1}$ under a particular model ω_t for the cross-sectional mean of idiosyncratic productivity shock ω_t. And $\widetilde{f}^h_{j,t,1}(\omega_t)$ stands for the probability density function of interim belief of island j workers over period t ambiguity ω_t. It is nothing more than the posterior belief over possible models ω_t, which follows smooth rule of updating

$$\widetilde{f}^h_{j,t,1}(\omega_t) \propto \underbrace{\frac{\phi' \left(E^{\omega_t}_{j,t,0} \left[\frac{C_t^{1-\gamma}-1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right)}{\phi' \left(E^{\omega_t}_{j,t,1} \left[\frac{C_t^{1-\gamma}-1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right)}}_{\text{Weights}} \underbrace{f\left(a_{j,t}|\omega_t\right) f_t(\omega_t)}_{\text{Bayesian Kernel}} \tag{6}$$

Here $f\left(a_{j,t}|\omega_t\right)$ is the conditional probability density function of $a_{j,t}$ under a particular model ω_t, which is the Normal density with mean ω_t and variance $\sigma^2_\zeta + \sigma^2_\iota$, and $f_t(\omega_t)$ stands for the period t prior belief density over ω_t, which is the Normal density with mean 0 and variance e^{ψ_t}.

Relative to the standard Bayesian updating, the smooth rule puts more weight to the model that provides higher marginal incentive to act ex-ante (at stage 0) when comparing to its ex-post (at stage 2) counterparts:

$$\phi' \left(E^{\omega_t}_{t,0} \left[\frac{C_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right) > \phi' \left(E^{\omega_t}_{t,2} \left[\frac{C_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right)$$

Through re-weighting in interim belief, incentives to act in ex-ante (at stage 0) and in ex-post (at stage 2) can be aligned with each other, which leads to dynamic consistency across stages within a period.

Firm problem at stage 1. Firm problem can be formulated as follows:

$$\max_{N_{i,j,t}} \int_{\mathcal{R}} \phi \left(E^{\omega_t}_{j,t,1} \left[\frac{C_t^{-\gamma}}{P_t} \left(P_{j,t} Y_{i,j,t} - W_{j,t} N_{i,j,t} \right) \right] \right) \widetilde{f}^{f}_{j,t,1} \left(\omega_t \right) d\omega_t \tag{7}$$

subject to the production function of island j firms (1). The firm decides how much labor to hire by taking nominal wage $W_{j,t}$ as given and by making expectation over its terms of trade $P_{j,t}$ to be determined at stage 2 by

$$Y_{j,t} \equiv \int_I Y_{i,j,t} di = \left(\frac{P_{j,t}}{P_t} \right)^{-\theta} Y_t \tag{8}$$

Interim belief systems of island j firms follows an extended smooth rule of updating given by

$$\widetilde{f}^{f}_{j,t,1} \left(\omega_t \right) \propto \underbrace{\frac{\phi' \left(E^{\omega_t}_{j,t,0} \left[\frac{C_t^{1-\gamma}-1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right)}{\phi' \left(E^{\omega_t}_{j,t,1} \left[\frac{C_t^{-\gamma}}{P_t} \left(P_{j,t} Y_{j,t} - W_{j,t} N_{j,t} \right) \right] \right)}}_{\text{Weights}} \underbrace{f \left(a_{j,t} | \omega_t \right) f_t \left(\omega_t \right)}_{\text{Bayesian Kernel}} \tag{9}$$

Unlike the standard smooth rule of updating, firms' incentives to act are not aligned in a pure ex-ante versus ex-post sense. Instead, the proposed extended smooth rule of updating aligns incentives to act of the representative household in ex-ante with that of the firms in ex-post. This ensures *dynamic consistency from the perspective of the representative household*. That is if we allowed the household to make ex-ante contingency plans of production for island firms, the contingency plans are to be respected ex-post by firms when it is their turn to move.

To note that, we formulate the firms' problem in a way that the firms are ambiguity averse by themselves. We can justify the formulation of firms' problem by arguing that firms' are maximizing the shareholder value. Therefore, firms behave as if they are ambiguity averse by themselves and share the same belief with their shareholders when evaluating the marginal benefit of labor demand. The additional concavity introduced by the ϕ function manifests the former point, and the extended smooth rule of updating takes care the latter. Therefore, we can alternatively formulate the firms' problem by

$$\max_{N_{i,j,t}} \int_{\mathcal{R}} E^{\omega_t}_{j,t} \left[SDF_t \left(P_{j,t} Y_{i,j,t} - W_{j,t} N_{i,j,t} \right) \right] f_{j,t,1} \left(\omega_t \right) d\omega_t \tag{10}$$

where the stochastic discount factor SDF_t is given by

$$SDF_t \equiv \phi' \left(E^{\omega_t}_{j,t,0} \left[\frac{C_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right) \frac{C^{-\gamma}}{P_t}$$

Here the stochastic discount factor takes cares not only households' risk attitude $\frac{C^{-\gamma}}{P_t}$ but also ambiguity attitude $\phi' \left(E^{\omega_t}_{j,t,0} \left[\frac{C_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right)$. The two formulations (7) and (10) are isomorphic to each other.

To close up the description of the model, we assume that $\phi(x)$ takes the constant absolute ambiguity aversion (CAAA) form for simplicity and tractability:

Assumption 1. (CAAA) We assume $\phi(x) = -\frac{1}{\lambda} e^{-\lambda x}$ where $\lambda \geq 0$ measures degree of ambiguity aversion of all agents.

3.3. Remarks and Interpretations

We conclude this section by some remarks and interpretations on the three key features of our model that have been listed at the beginning of this section.

1. Ambiguity is introduced into the model in the form of the cross-sectional mean of idiosyncratic productivity shock. But this does not mean there is any ambiguity over local economic conditions. Instead, firms and workers on island j have the perfect understanding of own island productivity, but an incomplete and ambiguous understanding of average productivity of all other islands $\int_J a_{j,t} dj$. This is because from the perspective of island j agents, cross-sectional mean of idiosyncratic productivity shock ω_t can be regarded as some temporary aggregate productivity shocks. Therefore, if local economic decisions are made solely depending on expectations over local economic conditions, output or labor will not respond to ambiguity shock at all[11]. This is the exact reason why we need aggregate demand externalities.

2. In our model, with aggregate demand externalities, incomplete and ambiguous information over the average productivity of all other islands $\int_J a_{j,t} dj$ can be translated into incomplete and ambiguous information over own demand conditions. We show in later sections that under the smooth model of ambiguity and the smooth rule of updating, fluctuations in the amount of ambiguity, i.e., ambiguity shock τ_t, generate fluctuations in island j agents' belief over local demand conditions, which eventually maps into aggregate fluctuations. In this sense, we can formally interpret our ambiguity shock as a particular formulation of aggregate demand shocks. This differentiates

[11]There are still some inter-temporal impact of ambiguity shock on consumption-saving trade-offs. But this would, in general, imply no aggregate co-movements, which is the sentence to death for any business cycle model.

our paper with Ilut and Schneider [2014], where ambiguity shocks are designed to be some form of news shocks about future productivity.

3. Ambiguity shock τ_t in our model is, by design, a second-moment shock. This is the primary reason we can generate fluctuations in belief divergence, i.e., cross-sectional dispersion of ex-ante output forecast. The key question here is whether or not such a second-moment shock can generate the first-moment impact at the aggregate level. The answer is yes if we have the smooth model of ambiguity together with the smooth rule of updating. From this perspective, we share the same spirit with Bloom [2009] and Bloom et al. [2016] in identifying possible mechanisms that enable the second-moment shocks to have first-moment impacts[12].

4. Equilibrium Characterization

In this section, we first define the equilibrium of the model and then derive a set of optimality conditions that jointly describe the equilibrium allocations and beliefs of all agents. Finally, we demonstrate how to characterize the *conditional log-normal equilibrium* associated with these optimality conditions.

4.1. Equilibrium Definition

Definition 1 (**Equilibrium**). *An equilibrium consists of a set of*

- *allocations* $\left\{\left\{N_{i,j,t}, Y_{i,j,t}\right\}_{(i,j)\in J}, \left\{N_{j,t}\right\}_{j\in J}, Y_t\right\}$;
- *factors and commodities prices* $\left\{\left\{W_{j,t}\right\}_{j\in J}, \left\{P_{j,t}\right\}_{j\in J}, P_t\right\}$;
- *information sets* $\left\{\mathcal{I}_{t,0}, \left\{\mathcal{I}_{j,t,1}\right\}_{j\in J}, \mathcal{I}_{t,2}\right\}$;
- *exogenous shocks* $\left\{\tau_t, \zeta_t, \left\{\iota_{j,t}\right\}_{j\in J}\right\}$;
- *and finally interim beliefs over the set of possible models* $\left\{\widetilde{f}^h_{j,t,1}\left(\omega_t\right), \widetilde{f}^f_{j,t,1}\left(\omega_t\right)\right\}_{j\in J}$

such that:

- *Information sets* $\left\{\mathcal{I}_{t,0}, \left\{\mathcal{I}_{j,t,1}\right\}_{j\in J}, \mathcal{I}_{t,2}\right\}$ *are defined in (30)*
- *At stage* 1, *given factors and commodities prices* $\left\{\left\{W_{j,t}\right\}_{j\in J}, \left\{P_{j,t}\right\}_{j\in J}, P_t\right\}$ *and the interim beliefs over the set of possible models* $\left\{\widetilde{f}^h_{j,t,1}\left(\omega_t\right), \widetilde{f}^f_{j,t,1}\left(\omega_t\right)\right\}_{j\in J}$, $\left\{N_{j,t}\right\}$ *solves the workers' problem (4) and* $\left\{N_{i,j,t}, Y_{i,j,t}\right\}_{(i,j)\in J}$ *solves the firms' problem (7)*

[12]In Bloom [2009] and Bloom et al. [2016], the existence of non-convex adjustment costs generates the real value of wait-and-see when "uncertainty" (risk in its nature) goes up, which has first moment impact on firms hiring and investment decisions.

- *Interim beliefs are such that:* $\widetilde{f}^{h}_{j,t,1}(\omega_t)$ *is given by (6) and* $\widetilde{f}^{f}_{i,j,t,1}(\omega_t)$ *is given by (9).*
- *Market clears for island-specific labor markets*

$$\int_I N_{i,j,t} di = N_{j,t}$$

- *Market clears for island commodities*

$$\int_I Y_{i,j,t} dj \equiv Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\theta} Y_t \tag{11}$$

with

$$Y_t = \left(\int_J Y_{j,t}^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}$$

where the price of final goods $P_t \equiv \left(\int_J P_{j,t}^{1-\theta} di\right)^{1/(1-\theta)}$ *is normalized to* 1.

- *Market clears for final good*

$$Y_t = C_t$$

4.2. Optimality Conditions

We can characterize the equilibrium with a set of optimality conditions. Detailed derivations can be found in Appendix.

First of all, within island j labor market, optimal labor supply is governed by the following condition

$$\chi N_{j,t}^{\epsilon} = W_{j,t} \int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[u'(C_t)\right] \widetilde{f}_{j,t,1}(\omega_t)\, d\omega_t \tag{12}$$

Workers on island j equate stage 1 valuation of the marginal benefit of labor (RHS) with marginal disutility of labor. On the other side of the labor market, optimal labor demand condition is given by

$$W_{j,t} \int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[u'(C_t)\right] \widetilde{f}_{j,t,1}(\omega_t)\, d\omega_t = \left(\int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[u'(C_t) P_{j,t}\right] \widetilde{f}_{j,t,1}(\omega_t)\, d\omega_t\right) \left((1-\alpha)\frac{Y_{j,t}}{N_{j,t}}\right) \tag{13}$$

Firms on island j equate stage 1 valuation of the marginal cost of labor (LHS) with the marginal benefit (RHS). Unlike standard expected utility preferences, ambiguity aversion implies that when evaluating marginal effects at stage 1 of period t, firms and workers on island j employs a distorted posterior belief over the set of possible models given by

$$\widetilde{f}_{j,t,1}\left(\omega_t\right) \propto \underbrace{\phi'\left(E^{\omega_t}_{j,t,0}\left[\frac{C_t^{1-\gamma}-1}{1-\gamma}-\chi\int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon}dj\right]\right)}_{\text{Belief Distortion}}\underbrace{f\left(a_{j,t}|\omega_t\right)f_t\left(\omega_t\right)}_{\text{Bayesian Kernel}} \tag{14}$$

It tells that whenever a model ω_t generates lower ex-ante (stage 0) expected utility for the representative household at the margin of ω_t, island j firms and workers tend to regard it as the more likely one in their posteriors. Put it differently, ambiguity aversion implies a pessimistic posterior belief over the set of possible models. To note that, firms on island j have the same distorted posterior belief over the set of possible models as island j workers, which is the by-product of the extended smooth rule we assumed in the interim belief systems of island firms.

Combing (12) and (13), equilibrium allocation of labor can be summarized by the following key equation for labor market:

$$\chi N_{j,t}^{\epsilon} = \left(\underbrace{\int_{\mathcal{R}} E^{\omega_t}_{j,t,1}\left[u'\left(C_t\right)\left(\frac{Y_{j,t}}{Y_t}\right)^{-\frac{1}{\theta}}\right]\widetilde{f}_{j,t,1}\left(\omega_t\right)d\omega_t}_{\text{marginal utility of island } j \text{ commodity}}\right)\left(\underbrace{\left(1-\alpha\right)\frac{Y_{j,t}}{N_{j,t}}}_{\text{marginal productivity}}\right) \tag{15}$$

The LHS of this key equation is the marginal disutility of labor, and the RHS is the multiplication of (a) marginal utility of island j commodity and (b) marginal productivity of labor. The equation simply says, in the labor market equilibrium, stage 1 valuation of private benefit of labor equates with the private cost of labor. Similar condition also appears in Angeletos and La'O [2009] and Angeletos et al. [2016c]. There are two main differences between ours and theirs. First of all, there is another round of integration over models due to the existence of ambiguity. Second, agents use a distorted posterior belief due to ambiguity aversion.

4.3. Joint Approximation of Allocation and Belief

Using island production function $Y_{j,t} = A_{j,t}N_{j,t}^{1-\alpha}$ and the market clearing condition for final goods $Y_t = C_t$, we can transform (15) into a fixed point condition over allocation $\{Y_{j,t}\}_{j\in J}$:

$$\chi Y_{j,t}^{\frac{1+\epsilon}{1-\alpha}-1+\frac{1}{\theta}} = \left(1-\alpha\right)A_{j,t}^{\frac{1+\epsilon}{1-\alpha}}\left(\int_{\mathcal{R}} E^{\omega_t}_{j,t,1}\left[Y_t^{\frac{1}{\theta}-\gamma}\right]\widetilde{f}_{j,t,1}\left(\omega_t\right)d\omega_t\right) \tag{16}$$

where the distorted posterior belief $\widetilde{f}_{j,t,1}(\omega_t)$ is given by

$$\widetilde{f}_{j,t,1}(\omega_t) \propto \underbrace{\phi'\left(E^{\omega_t}_{j,t,0}\left[\frac{Y_t^{1-\gamma}-1}{1-\gamma} - \chi\int_J \frac{\left(Y_{j,t}/A_{j,t}\right)^{(1+\epsilon)/(1-\alpha)}}{1+\epsilon}dj\right]\right)}_{\text{Belief Distortion}} \underbrace{f\left(a_{j,t}|\omega_t\right) f_t(\omega_t)}_{\text{Bayesian Kernel}} \tag{17}$$

To ensure complementarity in productions across islands, we make the following parametric restriction for the simple model without capital[13]:

Assumption 2. (Complementarity) It is assumed that $\frac{1}{\theta} > \gamma$ when there is no capital.

Increase in output of all other islands $k \neq j \in J$, on the one hand, raises the demand for island j commodities due to aggregate demand externalities, but on the other hand generates upward pressure on the wage rate of island j due to income effect of labor supply. Assumption 2 ensures that income effect of labor supply is so weak that the channel of aggregate demand externalities dominates that of the income effects. Later in Section 6, we drop this assumption when there exists capital accumulation and assume $\theta = 1$, i.e., a Cobb-Douglas technology for the aggregation of island output $\{Y_{j,t}\}_{j\in J}$ and $\gamma = 1$, i.e., log utility for consumption. Such a parameterization also ensures complementarity in production given the existence of consumption smoothing incentive, which weakens income effect of labor supply.

The technical complication here is that the distorted posterior belief $\widetilde{f}_{j,t,1}(\omega_t)$ is not orthogonal to equilibrium allocation. Allocations and beliefs have to be solved simultaneously in equilibrium. As a result, the equilibrium of the economy is the solution to a double fixed point conditions: one solves (16) characterizing the equilibrium cross-sectional allocation $\{Y_{j,t}\}_{j\in J}$ conditional on any stage 1 distorted posterior belief over possible models $\widetilde{f}_{j,t,1}(\omega_t)$ and the other one solves (17) characterizing equilibrium stage 1 distorted posterior belief over the set of possible models conditional on any cross cross-sectional allocation $\{Y_{j,t}\}_{j\in J}$ of the economy.

Definition 2 (Conditional Log-Normal Equilibrium). *An allocation $\{Y_{j,t}, Y_t\}_{j\in J}$ constitutes a conditional Log-Normal equilibrium if both $Y_{j,t}|\psi_t$ and $Y_t|\psi_t$ are Log-Normally distributed.*

[13] To see why this is the case, observe that under perfect information, (16) can be simplified into

$$\chi Y_{j,t}^{\frac{1+\epsilon}{1-\alpha}-1+\frac{1}{\theta}} = \left(\frac{\eta-1}{\eta}\right)(1-\alpha) A_{j,t}^{\frac{1+\epsilon}{1-\alpha}} Y_t^{\frac{1}{\theta}-\gamma}$$

It is straight-forward to show $\partial Y_{j,t}/\partial Y_t > 0$ if and only if $\frac{1}{\theta} - \gamma > 0$.

Lemma 1. Up to second order, distorted posterior belief over the set of possible models $\widetilde{f}_{j,t,1}(\omega_t)$ can be Normally approximated if allocation $\{Y_{j,t}, Y_t\}_{j\in J}$ constitutes a conditional Log-Normal equilibrium.

Proof. See Appendix. ∎

Lemma 2. Allocation $\{Y_{j,t}(a_{j,t}, \psi_t)\}_{j\in J}$ constitutes a conditional Log-Normal equilibrium if distorted posterior belief over possible models $\widetilde{f}_{j,t,1}(\omega_t)$ is Normal.

Proof. Directly follows Angeletos and La'O [2009]. ∎

The complication of the joint determination (or approximation) of allocations and beliefs can be greatly simplified once we narrow our analysis down to the focus of conditional log normal equilibrium defined in Definition 2. On the one hand, conditional Log-Normal equilibrium embeds the standard Log-Normal equilibrium or log-linearized equilibrium as a special case when there is no ambiguity shock. While on the other hand, it can be justified, up to an approximation sense, by Lemma 1 and Lemma 2 in a self-fulfilling fashion. The following proposition characterizes the approximated conditional log-normal equilibrium.

Proposition 1 (Equilibrium Characterization)**.** *Under some regularity conditions, there exists a unique* ***approximated*** *symmetric conditional Log-Normal equilibrium where the allocation $\{Y_{j,t}, Y_t\}_{j\in J}$ is such that*

$$y_{j,t} \equiv \ln Y_{j,t} = \left(\underbrace{y^* + \overline{h}_y\left(\overline{\psi}\right)}_{\text{Ambiguous SS}}\right) + \underbrace{\kappa_{ya_j}\left(\psi_t, \lambda\right) \cdot a_{j,t}}_{\text{Use of Private Info.}} + \underbrace{\widehat{h}_y\left(\psi_t, \lambda\right)}_{\text{Impact of Amb. Shock}} \tag{18}$$

and

$$y_t \equiv \ln Y_t = \left(\underbrace{y^* + \overline{h}_y\left(\overline{\psi}\right)}_{\text{Ambiguous SS}}\right) + \underbrace{\kappa_{ya_j}\left(\psi_t, \lambda\right) \cdot \int_J a_{j,t} dj}_{\text{Use of Private Info.}} + \underbrace{\widehat{h}_y\left(\psi_t, \lambda\right)}_{\text{Impact of Amb. Shock}} \tag{19}$$

where $y^* + \overline{h}_y\left(\overline{\psi}\right)$ denotes the ambiguous steady state output. And $\kappa_{ya_j}\left(\psi_t, \lambda\right)$, the slope of output w.r.t. productivity, is called the use of private information, which is a function of the amount of ambiguity ψ_t and degree of ambiguity aversion λ. Finally, $\widehat{h}_y\left(\psi_t; \lambda\right)$ denotes the impact of ambiguity shock on output satisfying

$$\widehat{h}_y\left(\overline{\psi}, \lambda\right) = 0$$

Finally, the distorted posterior belief over the set of possible models is Normal with mean μ_t and variance σ_t^2 such that

$$\mu_t = \left(\frac{e^{\psi_t} + g_\sigma(\psi_t, \lambda)}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)} \right) x_{j,t} + \left(\frac{\sigma_\zeta^2 + \sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)} \right) g_\mu(\psi_t, \lambda)$$

and

$$\sigma_t^2 = \left(\frac{\sigma_\zeta^2 + \sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)} \right) \left(e^{\psi_t} + g_\sigma(\psi_t, \lambda) \right)$$

where the distortion in mean $g_\mu(\psi_t, \lambda)$ and in variance $g_\sigma(\psi_t, \lambda)$ are given by

$$g_\mu(\psi_t, \lambda) = -\lambda \kappa_{J\omega} \left(\frac{1}{1 + \lambda \kappa_{J\omega\omega} e^{\psi_t}} \right) e^{\psi_t} \quad g_\sigma(\psi_t, \lambda) = -\left(\frac{\lambda \kappa_{J\omega\omega} e^{\psi_t}}{1 + \lambda \kappa_{J\omega\omega} e^{\psi_t}} \right) e^{\psi_t} \tag{20}$$

with $\kappa_{J\omega}$ and $\kappa_{J\omega\omega}$ being functions of κ_{ya_j} such that

$$\kappa_{J\omega} = e^{(1-\gamma)y^*} \left(\left(1 + (1-\gamma)\overline{h}_y\right) \kappa_{ya_j} - \left(1 + \left(\frac{1+\epsilon}{1-\alpha}\right)\overline{h}_y\right)\left(\kappa_{ya_j} - 1\right) \right) > 0 \tag{21}$$

$$\kappa_{J\omega\omega} = e^{(1-\gamma)y^*} \left[\left(1 + (1-\gamma)\overline{h}_y\right)(1-\gamma)\kappa_{ya_j,t}^2 - \left(1 + \left(\frac{1+\epsilon}{1-\alpha}\right)\overline{h}_y\right)\left(\frac{1+\epsilon}{1-\alpha}\right)\left(\kappa_{ya_j} - 1\right)^2 \right] > 0 \tag{22}$$

Proof. See Appendix. ∎

The approximation turns out to be quite accurate. In Appendix B, we conduct a numerical check and demonstrate that our approximation is not only accurate but also conservative in the sense it under-estimates the impact of ambiguity shock.

The allocations (18) and (19) are akin to those of the Angeletos and La'O [2009]. Log deviations from the ambiguous SS for island output $\widehat{y}_{j,t} \equiv y_{j,t} - \left(y^* + \overline{h}_y(\overline{\psi})\right)$ and aggregate output $\widehat{y}_t \equiv y_j - \left(y^* + \overline{h}_y(\overline{\psi})\right)$ can be expressed into a linear function of island $a_{j,t}$ and average productivity $\int_j a_{j,t} dj$, respectively. We name the slope of these linear functions $\kappa_{ya_j}(\psi_t, \lambda)$ as the use of private information, which is a function of the amount of ambiguity ψ_t. Furthermore, the intercept term $\widehat{h}_y(\psi_t; \lambda)$ controls the impact of ambiguity shock on aggregate output, given the fact that it is zero when evaluated at the ambiguous steady state. These two terms (a) the use of private information $\kappa_{ya_j}(\psi_t, \lambda)$ and (b) the impact of ambiguity shock on aggregate output $\widehat{h}_y(\psi_t; \lambda)$ are at the core of our analysis. Later in this Section 5, we study the impacts of ambiguity shocks through comparative static analysis over these terms and demonstrate how ambiguity shock can possibly generate co-movements across market confidence, belief divergence and aggregate economy.

5. Impacts of Ambiguity Shock

In this section, we analyze the impacts of ambiguity shock by conducting a couple of comparative static analysis. We start by providing a game theoretic interpretation of the equilibrium allocations of our business cycle model. Such a game theoretic interpretation clarifies the main mechanisms of our paper, which are the dual impacts of ambiguity shock. We then demonstrate how such dual impacts of ambiguity shock are to be mapped into fluctuations in aggregate output, market confidence, and belief divergence. Finally, we close up this section with some discussions about the interplay between incomplete information and ambiguity aversion.

5.1. Game Theoretic Interpretation

To build up the economic intuition behind impacts of ambiguity shock, we present a game theoretic interpretation of the equilibrium of our business cycle model, which resembles the beauty contest in Morris and Shin [2002] and Angeletos and Pavan [2007], but with a distorted information structure that captures the belief distortion or pessimism in belief due to ambiguity aversion.

Proposition 2. *The approximated equilibrium allocations $\{Y_{j,t}, Y_t\}_{j\in J}$ are identical to that of a beauty contest such that*

$$y_{j,t} = \kappa_a a_{j,t} + \kappa_y E_{j,t}[y_t]$$

where the coefficients κ_a and κ_y are such that

$$\kappa_a = \frac{\frac{1+\epsilon}{1-\alpha}}{\frac{1+\epsilon}{1-\alpha} - 1 + \frac{1}{\theta}} \qquad \kappa_y = \frac{\frac{1}{\theta} - \gamma}{\frac{1+\epsilon}{1-\alpha} - 1 + \frac{1}{\theta}} \in (0,1)$$

The information structure is distorted such that

$$\widetilde{a}_{j,t} = \widetilde{a}_t + \widetilde{\iota}_{j,t}, \quad \widetilde{\iota}_{j,t} \sim N\left(0, \sigma_\iota^2\right)$$
$$\widetilde{a}_t \sim N\left(g_\mu(\psi_t, \lambda), \sigma_\zeta^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)\right)$$

where distortions $\{g_\mu(\psi_t, \lambda), g_\sigma(\psi_t, \lambda)\}$ are given by (20) and satisfies the followings

$$g_\mu(\psi_t, \lambda) \leq 0, \quad g_\mu(-\infty, \lambda) = 0, \quad g_\mu(\psi_t, 0) = 0, \quad \frac{\partial g_\mu(\psi_t, \lambda)}{\partial \psi_t} < 0.$$

and

$$g_\sigma(\psi_t, \lambda) \leq 0, \quad g_\sigma(-\infty, \lambda) = 0, \quad g_\sigma(\psi_t, 0) = 0, \quad \frac{\partial\left(e^{\psi_t} + g_\sigma(\psi_t, \lambda)\right)}{\partial \psi_t} > 0$$

Proof. See Appendix. ■

Within the beauty contest interpretation, island output $y_{j,t}$ is the linear combination of island productivity $a_{j,t}$ and island j expectation of aggregate output because the former controls the marginal cost of production on island j and the latter manifests island j's forecast over own demand conditions. However, unlike standard beauty contest, perceived distribution of aggregate productivity is distorted both in mean and variance due to the existence of ambiguity and ambiguity aversion. Here κ_y corresponds to the notion of coordination motive in the beauty contest literature. Its magnitude $\kappa_y \in (0, 1)$ is the product of Assumption 2, which ensures complementarity in action and uniqueness in allocations once we fix a distorted information structure. Such a game theoretic interpretation provides us a natural laboratory to study the impacts of ambiguity shock. In what follows, we utilize this beauty contest interpretation and discuss in more details how ambiguity generates fluctuations in market confidence, belief divergence and aggregate economy once and for all.

5.2. Market Confidence, Belief Divergence and Aggregate Fluctuations

What are the impacts of ambiguity shocks? As evident from Proposition 1, ambiguity shock affects the allocations of the economy by fluctuating decision makers' distorted posterior belief over the set of possible models. Proposition 2 further decomposes the impacts of a positive ambiguity shock into two parts: one generates an increased pessimism over aggregate fundamental and the other one creates higher perceived volatility of aggregate fundamentals. We call them the dual impacts of ambiguity shock.

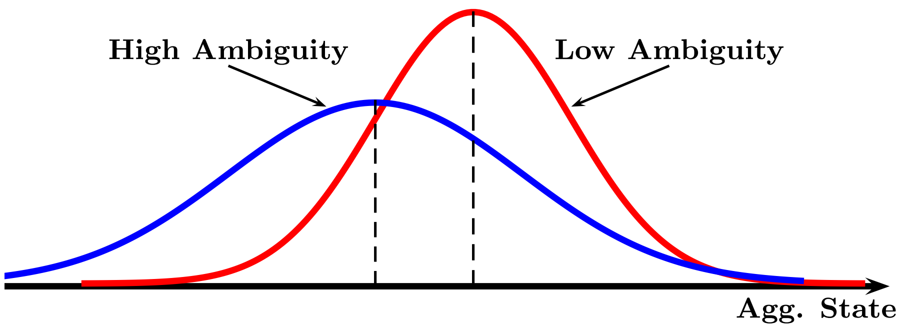

Figure 3. *Impact of Ambiguity Shock: Main Mechanism*

Figure 3 plots the perceived "as if" distribution of aggregate fundamental for the low level of ambiguity, i.e., ψ_t is small, and for the high level of ambiguity, i.e., ψ_t is large. It is named "as if" because these are the subjective beliefs over aggregate fundamentals that would deliver exactly the same allocation as our baseline model when agents have expected utility preferences. Since island j agents have perfect understanding over own productivity, distorted prior belief over aggregate fundamental $\tilde{a}_t$ can be translated into distorted prior belief over aggregate demand of island j. Therefore, a positive ambiguity shock makes decision-makers believe that the aggregate demand becomes worse on average and more volatile in their "as if" subjective prior. The former maps into lower output, either island or aggregate, at the margin. While the latter maps into an increased incentive in the use private information when making expectation over aggregate demand hence when making factors demand and supply decisions. We summarize these results in the following proposition

Proposition 3. *A positive ambiguity shock that increases the amount of ambiguity ψ_t generates lower aggregate output in the sense that*

$$\frac{\partial \widehat{h}_y(\psi_t, \lambda)}{\partial \psi_t} < 0 \tag{23}$$

if agents are ambiguity averse, i.e. $\lambda > 0$. Moreover, equilibrium use of private information $\kappa_{ya_j}(\psi_t, \lambda)$ is an increasing function of amount of ambiguity ψ_t:

$$\frac{\partial \kappa_{ya_j}(\psi_t, \lambda)}{\partial \psi_t} > 0. \tag{24}$$

Proof. See Appendix. ∎

At the core of understanding (23) is the increased degree of pessimism over the set of possible models. In fact, there are two forces at work that deepen agents' degree of pessimism over the set of possible models, one fundamental and one strategic. A positive ambiguity shock, on the one hand, increases the amount of ambiguity faced by all agents. In response, agents behave more pessimistic. This is the fundamental or direct channel. On the other hand, a positive ambiguity shock induces all other agents to use more of their private information when making output decisions. Under aggregate demand externalities, this raises the amount of ambiguity in firms' demand structure, which further increases the degree of pessimism. This is the strategic or indirect channel. Through the fundamental and strategic channels, a positive ambiguity shock raises all agents' degree of pessimism over the outlooks of the economy, which eventually drives down economic activities.

Market confidence. We define market confidence as the "economy-wide average of agents' first-order expectations of aggregate output of the economy". Such a definition is consistent with the practice of most of the survey exercise. For example, in the Michigan Survey of Consumer, consumers are asked about whether they believe output will go up or down in the following year. In a broader context, it is a short-cut for agents forecasts about the outlooks of the economy.

Definition 3 (**Market Confidence**). *Market confidence is mathematically defined to be*

$$Conf.\left(\psi_t,\lambda\right) \equiv \int_J \int_{\mathcal{R}} E^{\omega_t}_{j,t}\left[y_t\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t dj$$

To note that, market confidence is defined under the distorted belief over the set of possible models $\widetilde{f}_{j,t,1}\left(\omega_t\right)$. By doing so, we implicitly assume that ambiguity averse agents would use their as if belief, i.e., the pessimistic belief to make forecasts. Also, we integrate individual agents' first-order belief over j to be consistent with the idea that market confidence is an economy-wide concern. Following Proposition 1, it is directly to have that

$$Conf.\left(\psi_t,\lambda\right) = y^* + \overline{h}_y\left(\overline{\psi}\right) + \kappa_{ya_j}\left(\psi_t,\lambda\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2 + e^{\psi_t} + g_\sigma\left(\psi_t,\lambda\right)}\right) g_\mu\left(\psi_t,\lambda\right) + \widehat{h}_y\left(\psi_t,\lambda\right)$$

Observe that economy-wide average of agents first-order belief over average productivity is given by

$$\int_J \int_{\mathcal{R}} E^{\omega_t}_{j,t}\left[\int_J a_{j,t} dj\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t d = \kappa_{ya_j}\left(\psi_t,\lambda\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2 + e^{\psi_t} + g_\sigma\left(\psi_t,\lambda\right)}\right) g_\mu\left(\psi_t,\lambda\right) \tag{25}$$

Being hit by a positive ambiguity shock, all agents become more pessimistic over the aggregate fundamental, i.e., $g_\mu\left(\psi_t,\lambda\right)$ decreases. On the other hand, all agents perceive the aggregate fundamental become more volatile $e^{\psi_t} + g_\sigma\left(\psi_t,\lambda\right)$ increases. The former increases the economy-wide pessimism. But the latter reduces economy-wide pessimism because agents find it optimal to use more of the private information, the economy-wide average of which is objectively zero. However, in equilibrium, the former always dominates the latter implying that all agents are becoming more pessimistic over average productivity. The point is all agents understand that all the others are more pessimistic. And they also understand that the other agents understand this increase in economy-wide pessimism. Further, they all understand that all the others understand that the others understand this, etc. The consequence of such an higher-order thinking over each others eventually leads to a drop in aggregate output, i.e., $\widehat{h}_y\left(\psi_t,\lambda\right)$. This further depresses market confidence. Finally, all agents also understand that the others all perceive aggregate fundamental being more volatile. Hence they understand that all the others will use more of their private information. Therefore, they know that aggregate output will respond more to aggregate fundamental, i.e., $\kappa_{ya_j}(\psi_t, \lambda)$ increases. This raises output forecasts' reliance on the pessimistic belief over average productivity, which further depresses market confidence. We summarize this result in the Corollary 1.

Corollary 1. (Depressed Market Confidence) A positive ambiguity shock that increases the amount of ambiguity perceived by agents depresses market confidence in the sense that

$$\frac{\partial Conf.(\psi_t, \lambda)}{\partial \psi_t} < 0$$

Proof. Directly following Proposition 2 and 3. ■

To note that in the model, island j agents have perfect information over own productivity. An increase in the amount of ambiguity depresses island j agents' belief over the average productivity of the other islands without changing beliefs over own productivity. These movements in belief are isomorphic to those of a negative confidence shock under heterogenous prior setup alias Angeletos et al. [2016b]. In this sense, our paper provides an alternative micro-foundation for the heterogeneous prior setup by generating endogenous movements in confidence.

Belief Divergence. To discuss the impact of ambiguity shock on belief divergence, we need to define the expectation formation process of professional forecasters formally. Equivalently, we ask the following question: what are professional forecasters' attitudes towards ambiguity?

Figure 4 plots realized real GDP growth rates and its average year-ahead forecasts from Survey of Professional Forecasters. There displays no significant pessimism over time in SPF forecasts. To be consistent with this observation, we assume that professional forecasters are ambiguity neutral. If not, ambiguity aversion would imply a systematic pessimism in average forecasts due to distorted belief over the set of possible models.

Assumption 3. Professional forecasters possess ambiguity over the cross-sectional mean of idiosyncratic productivity shock. However, they are ambiguity neutral, i.e., $\lambda = 0$.

Furthermore, each island j is assumed to inhabit a professional forecaster indexed by $j \in J = [0, 1]$, who submits his forecast about period t aggregate output y_t at stage 1 of period t. In this section, we assume that professional forecaster j shares the

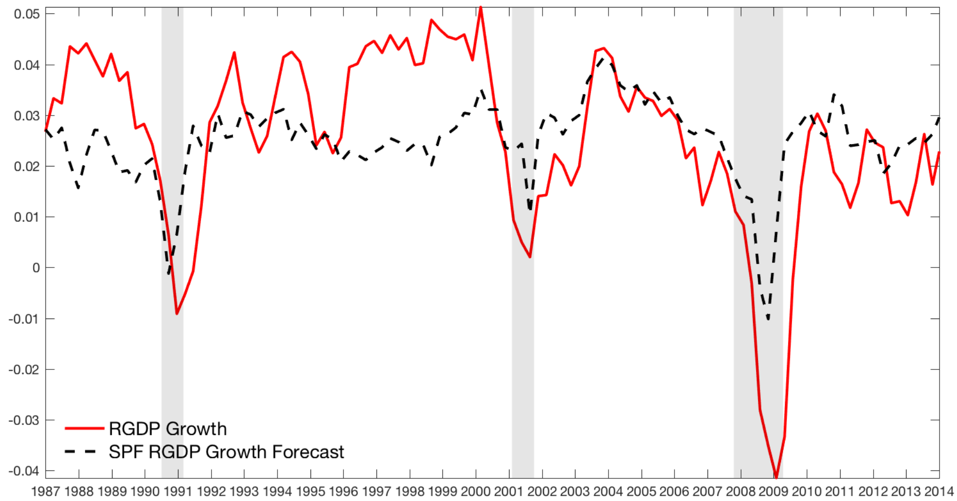

***Figure 4.** SPF Forecasts of Real GDP Growth Rate.*

Note: The figure plots real GDP growth rate (red line) and year-ahead real GDP growth rate forecast by professional forecasters (dashed black line) between 1987Q1 and 2014Q4. Forecast data is from Survey of Professional Forecasters provided by the Federal Reserve Bank of Philadelphia and real GDP growth data is from Saint-Louis Federal Reserve Economic Database.

same information as island j agents. Hence his forecast over aggregate output can be expressed into

$$E_{j,t}\left[y_t\right] = y^* + \bar{h}_y\left(\bar{\psi}\right) + \underbrace{\kappa_{ya_j}\left(\psi_t,\lambda\right)}_{\uparrow}\underbrace{\left(\frac{\sigma_\zeta^2 + e^{\psi_t}}{\sigma_\zeta^2 + e^{\psi_t} + \sigma_\iota^2}\right)}_{\uparrow} a_{j,t} + \widehat{h}_y\left(\psi_t,\lambda\right). \tag{26}$$

Therefore, belief divergence measured by the cross-sectional dispersion of ex-ante output forecast is given by

$$FD_t\left(\psi_t,\lambda\right) \equiv \int_J \left(E_{j,t}\left[y_t\right] - \bar{E}_t\left[y_t\right]\right)^2 dj = \kappa_{ya_j}^2\left(\psi_t,\lambda\right)\left(\frac{\sigma_\zeta^2 + e^{\psi_t}}{\sigma_\zeta^2 + e^{\psi_t} + \sigma_\iota^2}\right)^2 \sigma_\iota^2 \tag{27}$$

Corollary 2 summarizes the impacts of ambiguity shocks on belief divergence.

Corollary 2. (Heightened Belief Divergence) Therefore, a positive ambiguity shock that increases the amount of ambiguity ψ_t raises belief divergence in the sense that

$$\frac{\partial FD_t\left(\psi_t,\lambda\right)}{\partial \psi_t} > 0. \tag{28}$$

Proof. Straight-forward following the intuition below. ■

The intuition here is straight-forward. A positive ambiguity shock makes firms and workers on all islands believe, in their "***as if*** " subjective prior, that the aggregate fundamental is more volatile, which increases the incentive to use private information when forming expectations over own demand conditions. This maps into increased responsiveness of island output $y_{j,t}$ to island productivity $a_{j,t}$ because it is $a_{j,t}$ that serves as the private information over aggregate demand for island j agents. This raises cross-sectional dispersion of island output. Upon aggregation, we also have aggregate output y_t responds more to average productivity $\int_J a_{j,t}dj$.

From the perspective of professional forecaster j, increase in κ_{ya_j} implies that "there are more to estimate". Moreover, when he estimates the average productivity $\int_J x_{j,t}dj$, he tends to rely more on his private information $a_{j,t}$ in the sense that $\frac{\partial\left(\frac{\sigma_\zeta^2+e^{\psi_t}}{\sigma_\zeta^2+e^{\psi_t}+\sigma_\iota^2}\right)}{\partial\psi_t} > 0$. This is because he believes, in his "as if" subjective prior, the aggregate fundamental is now more volatile. These two in combine increase the responsiveness of forecaster j's forecast to private information $a_{j,t}$, which eventually leads to higher cross-sectional dispersion in output forecasts ex-ante, i.e., an increase in belief divergence. To note that, the economy itself does not become more dispersed or more volatile. It is the increased responsiveness to idiosyncratic shocks that drives up the cross-sectional dispersion. This differentiates our paper with the theory of uncertainty shock as in Bloom [2009] and Bloom et al. [2016], who take fluctuations in dispersion as model inputs rather than model output.

We provide a summary of impacts of ambiguity shock by combing Proposition 3, Corollary 1 and Corollary 2:

Proposition 4 (Impacts of Ambiguity Shock)**.** *If decision makers are ambiguity averse, i.e., $\lambda > 0$, a positive ambiguity shock that increases the amount of ambiguity ψ_t generates*

- *lower market confidence;*
- *larger belief divergence;*
- *and finally lower aggregate output on average.*

Proof. Directly following Proposition 3, Corollary 1 and Corollary 2. ∎

5.3. Discussion: incomplete information and ambiguity aversion

In 5.2, we highlight the strategic force behind the impacts of ambiguity shocks in driving aggregate fluctuations. This is closely related to incomplete information embedded in our paper. We close up the analysis of Section 5 by some additional discussions about the interplay between incomplete information and ambiguity based on the game theoretic interpretation developed in Section 5.1. By varying degree of incompleteness in information, impacts of uncertainty shocks are studied.

Incomplete information is of primary importance. When information is complete, i.e., $\sigma_\iota^2 = 0$, island j agents face no uncertainty regarding decisions of agents on other islands. Then all agents will have the perfect understanding of the whole economy. Therefore, ambiguity shocks play no role in driving aggregate fluctuations without incompleteness in information. Corollary 3 summarizes this result.

Corollary 3. Ambiguity shocks have no impacts on aggregate output $\widehat{h}_y(\psi_t, \lambda) = 0$ when information is complete $\sigma_\iota^2/\sigma_\zeta^2 = 0$.

Proof. Straight-forward following the proof of Proposition 1. ■

To note that, completeness in information only requires that there exists no private information. It does not necessarily imply perfect information[14]. Corollary 3 still holds when there presents information friction as long it is common across agents. Unlike our simplified case with $\sigma_\iota^2/\sigma_\zeta^2 = 0$, when this is the case, there still presents the distorted posterior belief over the set of possible models. However, all agents have common knowledge over Y_t under complete information. Then (16) transforms into

$$\chi Y_{j,t}^{\frac{1+\epsilon}{1-\alpha}-1+\frac{1}{\theta}} = (1-\alpha) A_{j,t}^{\frac{1+\epsilon}{1-\alpha}} Y_t^{\frac{1}{\theta}-\gamma}$$

which leaves no room for ambiguity or ambiguity shock to have any impacts. To interpret, it is the imperfect coordination across islands, possibly due to imperfect communication as in Angeletos and La'O [2013], that matters rather than imperfect information.

The question of interest naturally arises. Does more incompleteness in information amplify or dampen the impacts of ambiguity shocks? We answer these questions by varying not only σ_ι^2 but also σ_ζ^2 while holding the amount of ambiguity constant at the A-SS level.

There are two forces at work. On the one hand, more incompleteness in information, i.e., larger σ_ι^2 or lower σ_ζ^2, reduces the incentive in the use of private information because private information is becoming less informative not only about the aggregate state but also about the models. This marginally changes degree of pessimism since $g_\mu(\overline{\psi}, \lambda)$ is a function of both $\kappa_{J\omega}\left(\overline{\kappa}_{ya_j}, \overline{h}_y\right)$ and $\kappa_{J\omega\omega}\left(\overline{\kappa}_{ya_j}, \overline{h}_y\right)$, which are functions of $\overline{\kappa}_{ya_j}$. It can be proved that mean distortion $g_\mu(\overline{\psi}, \lambda)$ marginally decreases w.r.t degree of information incompleteness.

[14]See Angeletos and Lian [2016] for more formal discussions.

Lemma 3. Mean distortion at the A-SS $g_\mu\left(\overline{\psi},\lambda\right)$ marginally decreases in σ_ι^2 and increases in σ_ζ^2:

$$\frac{\partial g_\mu\left(\overline{\psi},\lambda\right)}{\partial\sigma_\iota^2}<0 \qquad\qquad \frac{\partial g_\mu\left(\overline{\psi},\lambda\right)}{\partial\sigma_\zeta^2}>0$$

Proof. See Appendix. ■

On the other hand, fixing the belief distortion $g_\mu\left(\psi_t,\lambda\right)=g_\mu^*$ and $g_\mu\left(\psi_t,\lambda\right)=g_\sigma^*$, more incompleteness in information implies more load of belief distortion on island and aggregate output. To see this, express island output $y_{j,t}$ into an infinite sum over a complete hierarchy of all higher-order beliefs of aggregate fundamentals $\widetilde{a}_t$:

$$y_{j,t}=\kappa_a\left(a_{j,t}+\sum_{n=1}^{\infty}\kappa_y^n E_{j,t}^n\left[\widetilde{a}_t\right]\right) \tag{29}$$

where higher-order beliefs of island j agents are defined recursively such that

$$E_{j,t}^1\left[\widetilde{a}_t\right]\equiv E_{j,t}\left[\widetilde{a}_t\right] \qquad\qquad E_{j,t}^n\left[\widetilde{a}_t\right]\equiv E_{j,t}\left[\int_J E_{j,t}^{n-1}\left[\widetilde{a}_t\right]dj\right] \quad \forall n\geq 1$$

It turns out that increased incompleteness in information increases the response of any order of belief to prior information. Observe that it is the prior of aggregate fundamental being distorted. Therefore, it must be the case that all orders of belief are exposed to more belief distortion. Therefore, incompleteness in information naturally amplify the impact of ambiguity shock at the margin when we fix the belief distortion. The above two forces interact with each other, which indicates that incomplete information is actually an amplifying mechanism for the impact of ambiguity. Proposition 5 summarizes this result.

Proposition 5. *At the ambiguous steady state (A-SS), a marginal increase in σ_ι^2 that increases the degree of information incompleteness, decreases the use of private information and amplifies the impact of ambiguity:*

$$\frac{d\overline{\kappa}_{ya_j}}{d\sigma_\iota^2}<0 \qquad\qquad \frac{d\overline{h}_y}{d\sigma_\iota^2}<0$$

On the contrary, a marginal increase in σ_ι^2 that reduces the degree of information incomplete-

ness, increases the use of private information and dampens the impact of ambiguity:

$$\frac{d\overline{\kappa}_{ya_j}}{d\sigma_\zeta^2} > 0 \qquad\qquad \frac{d\overline{h}_y}{d\sigma_\zeta^2} > 0$$

Proof. See Appendix. ■

5.4. A summary

Taking stock, this section has shown that a positive ambiguity shock generates a recession with depressed market confidence and heightened belief divergence qualitatively. In what follows, we study the impacts of ambiguity shocks quantitatively within an extended dynamic RBC model. We demonstrate that ambiguity shock in our theory can generate reasonable co-movements in quantities at the aggregate level. The availability of our theory is quantitatively evaluated by bringing the observable implications into the data.

6. The Dynamic RBC Model: Quantitative Evaluation

In this section, we illustrate the quantitative potential of our theory by studying a dynamic RBC model. We first set up the model. We then move to the discussion about our quantitative methodology in the estimation of conditional Log-Normal equilibrium (Definition 2). Finally, observable implications of ambiguity shock, for both the aggregate quantities, belief divergence and market confidence, are then assessed through a calibrated version of the model.

6.1. Model Setup

Geography, markets and timing. The economy consists of a continuum of islands, indexed by $j \in J = [0,1]$ and a mainland. On each island j, there exist a continuum of firms, indexed by $(i,j) \in I \times J = [0,1]^2$ and a continuum of workers, indexed by $(m,j) \in M \times J = [0,1]^2$. Firms on island j hire labor and capital from locally competitive factor markets for the production of island-specific commodity j. These commodities are traded in a centralized market operated on the mainland, where a continuum of consumers, indexed by $h \in H = [0,1]$ and a large number of final good producers inhabit. We assume that consumer h and a continuum of workers $\{(h,j)\,;j \in J\}$ constitute a large household indexed by $h \in H$, who owns a continuum of firms $\{(h,j)\,;j \in J\}$. As in the case of the simple model, our model admits a representative household at the mainland and a continuum of representative firms and workers on every island.

Time is discrete, indexed by $t \in \{0,1,...\}$ and each period t is decomposed into three stages. At stage zero, period t shocks are realized. At stage 1, island competitive factor markets open up. Island j firms make labor and capital demand decisions and symmetrically, the representative household sends out workers to each island, who make labor supply decisions on the basis of incomplete information over the ambiguous concurrent aggregate state of the economy. At stage 2, on the mainland, the centralized commodities market opens up. All uncertainty, either risk or ambiguity, over the concurrent aggregate state of the economy resolved. Final goods producers produce. And the representative household makes consumption and saving decisions upon receiving, capital income, labor income and all the transfers from island firms upon perceiving ambiguity over the future aggregate state of the economy. Here we assume that it is the representative household who owns the capital, therefore, saving takes the form of island-specific investments for all islands. Therefore, the capital supply of island j in period $t+1$ is pre-determined at stage 2 of period t by the representative household.

Households. Period utility of the representative household is given by:

$$u\left(C_t\right) - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj$$

where ϵ is the inverse Fisher elasticity of labor supply. Therefore, the flow budget constraint is such that

$$P_t C_t + P_t \int_J K_{j,t+1} - (1-\delta) K_{j,t} dj = \int_J W_{j,t} N_{j,t} dj + \int_J R_{j,t} K_{j,t} dj + \int_J \Pi_{j,t} dj$$

where $\int_j R_{j,t} K_{j,t} dj$ and $\int_J W_{j,t} N_{j,t} dj$ denote the total capital and labor income of all islands respectively and $\int_J \Pi_{j,t} dj$ is the transfers of realized profits from all island firms.

Island firms. Island j firms use labor and capital for the production of island j commodity. The production function is Cobb-Douglas:

$$Y_{j,t} = A_{j,t} N_{j,t}^{1-\alpha} K_{j,t}^{\alpha}$$

where $A_{j,t}$ is the island-specific productivity and the realized profit is given by

$$\Pi_{j,t} = P_{j,t} Y_{j,t} - W_{j,t} N_{j,t} - R_{j,t} K_{j,t}$$

where $W_{j,t}$ and $R_{j,t}$ denote the competitive factors prices on island j in period t and $P_{j,t}$ denotes the market price of island j commodity in period $t$to, which is determined at stage 2 when the centralized markets for commodities open up. Since it is the large representative household who owns the firm, any realized profits are to be transferred to the household for the purchase of final goods for consumption and investment.

Therefore, in the absence of any uncertainty concerns, island j firms care about the consumer valuation over their profits given by

$$\frac{u'(C_t)}{P_t}\Pi_{j,t}$$

where P_t is the aggregate price level to be normalized to 1.

Productivity and ambiguity shocks. Aggregate productivity $a_t \equiv \log A_t$ follows an AR(1) process

$$a_t = \rho a_{t-1} + \zeta_t$$

where ζ_t is the aggregate productivity shock in period t that follows a Normal distribution with mean 0 and variance σ_ζ^2.

Island-specific productivity, defined by $a_{j,t} \equiv \log A_{j,t}$, equals to aggregate productivity plus an idiosyncratic productivity shock $\iota_{j,t}$:

$$a_{j,t} = a_t + \iota_{j,t}$$

Idiosyncratic productivity shocks $\iota_{j,t}$ are assumed to be i.i.d normally distributed with mean ω_t and variance σ_ι^2. Objectively, the cross-sectional mean of idiosyncratic productivity shocks are zero for all periods, i.e. $\omega_t = 0\ \forall t > 0$. However, agents inside the economy cannot fully understand it. Instead, they possess some ambiguity over the complete set of cross-sectional means of idiosyncratic productivity shocks, i.e., $M \equiv \{\omega_t : \forall t \geq 0\}$.

At the very beginning of time, say period 0, all agents subjectively believe that all $\omega_t \in M$ are i.i.d Normally distributed with mean 0 and variance $\overline{\sigma}_\omega^2 \equiv e^{\overline{\psi}}$. Here $\overline{\sigma}_\omega^2$ or $\overline{\psi}$ measures the amount of ambiguity agents possess in the A-SS. Ambiguity in the past does not last forever. As it will become evident later, concurrent ambiguity is resolved at stage 2 of that period. Therefore, at stage 0 of any period t, agents inside the economy only possess ambiguity over concurrent and future cross-sectional means of idiosyncratic productivity shocks, i.e., $M_t \equiv \{\omega_{t+k} : \forall k \geq 0\}$. The amount of ambiguity they possess at this point, denoted by ψ_t, is time-varying and governed by an AR(1) process

$$\psi_t = (1-\rho_\psi)\overline{\psi} + \rho_\psi \psi_{t-1} + \tau_t$$

where τ_t is the ambiguity shock assumed to be Normally distributed with mean 0 and variance σ_τ^2. We close the description of the ambiguity process by explicitly specifying the common subjective prior belief of all agents over $M_t = \{\omega_{t+k} : \forall k \geq 0\}$ at stage 0 of period t:

$$\omega_{t+k} \sim i.i.d\ N\left(0, e^{\psi_{t,t+k}}\right) \quad \forall k \geq 0$$

where the amount of ambiguity agents perceived over ω_{t+k} at period t, denoted as $\psi_{t,t+k}$, is an increasing affine function of ψ_t:

$$\psi_{t,t+k} = \left(1 - \rho_\psi^k\right) \overline{\psi} + \rho_\psi^k \psi_t$$

This particular structure for prior beliefs ensures prior consistency over the future ambiguity $\omega_{t+k}, \forall k \geq 1$. It simply tells that period t prior over future ambiguity ω_{t+k} meets period $t+k$ prior over ω_{t+k} if there are no more ambiguity shocks in between. Put it differently,

$$\phi_{t,t+k} = E_t\left[\psi_{t+k}\right]$$

Ambiguity shock τ_t in the process of ambiguity ψ_t, by its nature, can be understood a changing prior process. Also, we implicitly assume that a positive ambiguity shock in period t, i.e., $\tau_t > 0$, makes agents become "more ambiguous" [15] over all the future ambiguity ω_{t+k}. However, it is period t biased in the sense that it raises concurrent ambiguity more than future ambiguity and the increase in ambiguity is mean-reverting such that for ambiguity ω_{t+k} in the very far future $k \to +\infty$, the subjective belief stays in its A-SS belief, i.e. $\lim_{k\to+\infty} \psi_{t,t+k} = \overline{\psi}$.

Information structure. Denote $\mathcal{I}_{t,0}$, $\mathcal{I}_{j,t,1}$ and $\mathcal{I}_{t,2}$ as the information sets that are available to all agents at stage 0 of period t, are only available to island j agents at stage 1 of period t and are available to all agents at stage 2 of period t, respectively. Recursively, we can define these information sets by

$$\mathcal{I}_{t,0} = \mathcal{I}_{t-1,2} \cup \{\psi_t\} \qquad \mathcal{I}_{j,t,1} = \mathcal{I}_{t,0} \cup \left\{a_{j,t}\right\} \qquad \mathcal{I}_{t,2} = \cup_j \mathcal{I}_{j,t,1} \cup \{\zeta_t\} \tag{30}$$

To note that all concurrent uncertainty, either risk (state uncertainty over a_t) or ambiguity (model uncertainty over ω_t) are resolved at stage 2 of period t. Hence consumption-saving decisions by households are made upon perceiving ambiguity over the future outlooks of the economy only. Also because idiosyncratic productivity shocks are i.i.d, $\left\{a_{j,t}\right\}_{j\in J}$ tells no more information than $\int_j a_{j,t} dj$ does regarding island j productivity in period $t+1$. Therefore, we can simplify information set at stage 2 of period t by $\mathcal{I}_{t,2} = \mathcal{I}_{t,0} \cup \left\{\int_j a_{j,t} dj, \omega_t\right\}$. To simplify notation, we further transform the information

[15] Here more ambiguous means an increase in the amount of ambiguity perceived by all agents.

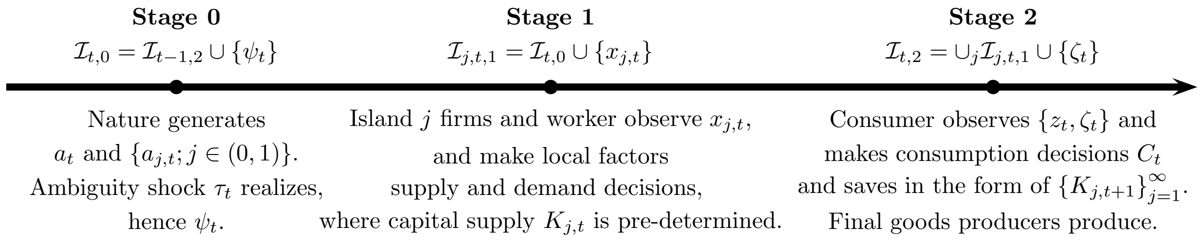

Figure 5. *Timeline for Period t*

structure into

$$\mathcal{I}_{t,0} = \mathcal{I}_{t-1,2} \cup \{\psi_t\} \qquad \mathcal{I}_{j,t,1} = \mathcal{I}_{t,0} \cup \{x_{j,t}\} \qquad \mathcal{I}_{t,2} = \mathcal{I}_{t,0} \cup \{z_t, \zeta_t\}$$

where $x_{j,t} \equiv \zeta_t + \iota_{j,t}$ denotes the de-facto private information at stage 1 over aggregate productivity shocks and $z_t = \zeta_t + \omega_t$ denotes the de-facto public information at stage 2 over aggregate productivity shocks. Figure 5 displays the timeline and information sets for period t in our dynamic RBC model.

Preference of the representative household at stage 2. Denote $s_{t+1} \equiv \mathcal{I}_{t+1,2} \backslash \mathcal{I}_{t,2}$ as the arrival of new information at stage 2 between two consecutive periods t and $t+1$. We summarize belief of the representative household at stage 2 by two corresponding Bayesian posteriors: (a) $\pi_M(s_{t+1}|\mathcal{I}_{t,2})$, the Bayesian posterior of s_{t+1} at stage 2 of period t under a particular model M, and (2) $\mu(M|\mathcal{I}_{t,2})$, the Bayesian posterior over the entire the set of possible models $M \in \mathcal{M}$.

Preference of the representative household at stage 2 of period t, therefore, can be represented by the recursive smooth model of ambiguity proposed by Klibanoff et al. [2009]:

$$V_t(C;\mathcal{I}_{t,2}) = u(C_t) + \underbrace{\beta\phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\int_{\mathcal{S}_{t+1}} V_{t+1}(C;\mathcal{I}_{t,2}, s_{t+1})\, d\pi_M(s_{t+1}|\mathcal{I}_{t,2})\right) d\mu(M|\mathcal{I}_{t,2})\right)}_{\text{Utility Equivalent of the Ambiguous Continuation Value}}$$

where $\phi(x)$ is some strictly increasing and concave function, whose curvature captures decision makers' taste for ambiguity, i.e., the degree of ambiguity aversion.[16]

Can learning over time resolve all ambiguity in the long run? Klibanoff et al. [2009] proves that if ϕ^{-1} is Lipschitz and the space of ambiguous model parameters is finite, the recursive smooth model of ambiguity will converge uniformly to expected utility

[16]Theorem 3 in Klibanoff et al. [2009] proves that under some regularity conditions, there are unique and monotonic V_t. For most of business cycle applications, those regularity conditions can be easily satisfied.

preferences with true model parameters. In our model, agents are ambiguous over an infinite parameter space $\omega \equiv \{\omega_t : \forall t \geq 0\}$. This prevents ambiguity to vanish in the long run through learning.

Moreover, the additional concavity in $\phi(x)$ is to capture the fact that ambiguous continuation value reduces the utility of the decision maker since ambiguity aversion implies that the representative household dislikes mean-preserving spread in expected continuation value due to the existence of model uncertainty. Behind recursive smooth model of ambiguity, there is the notion of consequentialism and dynamic consistency, who is the most intuitive way when we talk about decision makings within the lens of business cycles. And it admits a tractable Bellman equation formulation.

Denote value function as $J_t \equiv J\left(\left\{K_{j,t}\right\}, a_{t-1}, z_t, \zeta_t, \psi_t\right)$. Standard dynamic programming argument can be applied resulting the following Bellman equation for the representative household at stage 2:

$$J_t = \max_{C_t, \left\{K_{j,t+1}\right\}} u(C_t) + \beta\phi^{-1}\left(\int_{\mathcal{R}} \phi\left(E_{t,2}^{\omega_{t+1}}\left[J_{t+1}\right]\right) f_t(\omega_{t+1})\, d\omega_{t+1}\right) \tag{31}$$

subject to

$$P_t C_t + P_t \int_J I_{j,t} dj = \int_J W_{j,t} N_{j,t} di dj + \int_J R_{j,t} K_{j,t} dj + \int_J \Pi_{j,t} dj \tag{32}$$

and

$$I_{j,t} = K_{j,t+1} - (1-\delta) K_{j,t} \tag{33}$$

Here $E_{t,2}^{\omega_{t+1}}[\cdot]$ stands for the mathematical expectation conditioned on $\mathcal{I}_{t,2}$ under a particular model ω_{t+1} for cross-sectional mean of tomorrow's idiosyncratic productivity shocks. And $f_t(\omega_{t+1})$ stands for the probability density function for households' period t prior over tomorrow's ambiguity ω_{t+1}. Since period t knowledge does not reveal any information over tomorrow's ambiguity, prior belief over tomorrow's ambiguity ω_{t+1} at stage 2 coincides with that at stage 0.

Preference of the representative household at stage 1. Similar to the simple model, at stage 1, preference of the representative household is given by the smooth model of ambiguity

$$\int_{\Omega_t} \phi\left(E_{j,t,1}^{\omega_t}\left[J\left(\left\{K_{j,t}\right\}, a_{t-1}, z_t, \zeta_t, \psi_t\right)\right]\right) \tilde{f}_{j,t,1}^{h}(\omega_t)\, d\omega_t$$

where the interim belief follows the smooth rule of updating

$$\widetilde{f}^{h}_{j,t,1}(\omega_t) \propto \underbrace{\frac{\phi'\left(E^{\omega_t}_{j,t,0}\left[J\left(\left\{K_{j,t}\right\}, a_{t-1}, z_t, \zeta_t, \psi_t\right)\right]\right)}{\phi'\left(E^{\omega_t}_{j,t,1}\left[J\left(\left\{K_{j,t}\right\}, a_{t-1}, z_t, \zeta_t, \psi_t\right)\right]\right)}}_{\text{Weights}} \underbrace{f\left(x_{j,t}|\omega_t\right) f_t(\omega_t)}_{\text{Bayesian Kernel}}$$

Firm problem at stage 1. Firm problem is formulated in a similar fashion as the simple model:

$$\int_{\Omega_t} \phi\left(E^{\omega_t}_{j,t,1}\left[\frac{U'(C_t)}{P_t}\left(P_{j,t}Y_{j,t} - W_{j,t}N_{j,t} - R_{j,t}K_{j,t}\right)\right]\right) \widetilde{f}^{f}_{j,t,1}(\omega_t)\, d\omega_t$$

where the interim belief system satisfying the extended smooth rule of updating to ensure dynamic consistency:

$$\widetilde{f}^{w}_{j,t,1}(\omega_t) \propto \underbrace{\frac{\phi'\left(E^{\omega_t}_{j,t,0}\left[J\left(\left\{K_{j,t}\right\}, a_{t-1}, z_t, \zeta_t, \psi_t\right)\right]\right)}{\phi'\left(E^{\omega_t}_{j,t,1}\left[\ln(C_t) - \chi\int_J \frac{N^{1+\epsilon}_{j,t}}{1+\epsilon} dj\right]\right)}}_{\text{Weights}} \underbrace{f\left(x_{j,t}|\omega_t\right) f_t(\omega_t)}_{\text{Bayesian Kernel}}$$

To close up the description of the model, we make the following assumptions on functional forms

Assumption 4. (Log-Exponential) We assume $u(C_t) = \ln C_t$ and $\phi(x) = -\frac{1}{\lambda}e^{-\lambda x}$ with $\lambda \geq 0$.

In what follows, we leave out the optimality conditions to Appendix. And directly move to the discussion over our quantitative methodology in the approximation of conditional log-normal equilibrium, which is closely related to what we have done in Section 4.3.

6.2. Quantitative Methodology

The key feature of the smooth model of ambiguity is that decision makers would invoke a distorted (relative to Bayesian posterior) posterior belief over the set of the possible model when evaluating marginal effects of factors supply and demand. In our RBC extension, the belief distortion at stage 1 is given by

$$M_{t,1}(\omega_t) = e^{-\lambda E^{\omega_t}_{t,0}[J_t]} \tag{34}$$

and the belief distortion at stage 2 is given by

$$M_{t,2}\left(\omega_{t+1}\right) = e^{-\lambda E_{t,2}^{\omega_{t+1}}\left[J_{t+1}\right]} \tag{35}$$

To quantitatively pin down the whole equilibrium, expected value functions as functions of concurrent and tomorrow's ambiguity, $\left\{E_{t,0}^{\omega_t}\left[J_t\right], E_{t,2}^{\omega_{t+1}}\left[J_{t+1}\right]\right\}$, have to be approximated jointly with policy rules.

For any variable S_t of interest, we use hatted-lower-case $\widehat{s}_t$ to denote the log-deviation from its ambiguous steady state (A-SS):

$$\widehat{s}_t \equiv \ln\left(S_t\right) - \ln\left(S^*\right) - \overline{h}_s\left(\overline{\psi}\right)$$

where S^* stands for the deterministic steady state (D-SS) level of S_t and $\overline{h}_s\left(\overline{\psi}\right)$ is a function of $\overline{\psi}$, which takes into account the impacts of ambiguity on S_t at A-SS.

Focusing on conditional Log-Normal equilibrium (Definition 2), we propose the following policy rules for island employment $\widehat{n}_{j,t}$, output $\widehat{y}_{j,t}$, wage rate $\widehat{w}_{j,t}$ and rental rental rate of capital $\widehat{r}_{j,t}$ at stage 1 of period t:

$$\begin{aligned}
\widehat{y}_{j,t} &= \kappa_{yk}\widehat{k}_t + \kappa_{ya}a_{t-1} + \kappa_{yx}\left(\psi_t\right)x_{j,t} + \widehat{h}_y\left(\psi_t\right) \\
\widehat{n}_{j,t} &= \kappa_{nk}\widehat{k}_t + \kappa_{na}a_{t-1} + \kappa_{nx}\left(\psi_t\right)x_{j,t} + \widehat{h}_n\left(\psi_t\right) \\
\widehat{w}_{j,t} &= \kappa_{wk}\widehat{k}_t + \kappa_{wa}a_{t-1} + \kappa_{wx}\left(\psi_t\right)x_{j,t} + \widehat{h}_w\left(\psi_t\right) \\
\widehat{r}_{j,t} &= \kappa_{rk}\widehat{k}_t + \kappa_{ra}a_{t-1} + \kappa_{rx}\left(\psi_t\right)x_{j,t} + \widehat{h}_r\left(\psi_t\right)
\end{aligned}$$

and the following policy rules for consumption $\widehat{c}_t$, investment $\widehat{i}_t$ and capital stock tomorrow $\widehat{k}_{t+1}$ at stage 2 of period t:

$$\begin{aligned}
\widehat{c}_t &= \kappa_{ck}\widehat{k}_t + \kappa_{ca}a_{t-1} + \kappa_{cz}\left(\psi_t\right)z_t + \kappa_{c\zeta}\left(\psi_t\right)\zeta_t + \widehat{h}_c\left(\psi_t\right) \\
\widehat{i}_t &= \kappa_{ik}\widehat{k}_t + \kappa_{ia}a_{t-1} + \kappa_{iz}\left(\psi_t\right)z_t + \kappa_{i\zeta}\left(\psi_t\right)\zeta_t + \widehat{h}_i\left(\psi_t\right) \\
\widehat{k}_{t+1} &= \kappa_{kk}\widehat{k}_t + \kappa_{ka}a_{t-1} + \kappa_{kz}\left(\psi_t\right)z_t + \kappa_{k\zeta}\left(\psi_t\right)\zeta_t + \widehat{h}_k\left(\psi_t\right)
\end{aligned}$$

To elaborate a bit, first of all, the log-deviations of variables of interest are assumed to be from the A-SS instead of the D-SS. This is because the amount of ambiguity at A-SS $\overline{\psi}$ has a non-negligible first-moment impact on allocations when decision makers are ambiguity averse. Secondly, stage 1 variables are measurable with respect to stage 1 information sets. Since idiosyncratic productivity shocks are i.i.d across time and across islands, past information can be effectively summarized into $\left\{\widehat{k}_t, a_{t-1}, \psi_t\right\}$. Therefore stage 1 variables, i.e., $\left\{\widehat{y}_{j,t}, \widehat{n}_{j,t}, \widehat{w}_{j,t}, \widehat{r}_{j,t}\right\}$, are functions of $\left\{\widehat{k}_t, a_{t-1}\psi_t, x_{j,t},\right\}$ only. Similar arguments apply to stage 2 variables, i.e., $\left\{\widehat{c}_t, \widehat{i}_t, \widehat{k}_{t+1}\right\}$. Thirdly, fixing the amount of ambiguity ψ_t, the policy rules are linear in productivity shocks either the aggregate or the idiosyncratic. This corresponds to the standard log-linearization when there is no ambiguity shock at all, i.e. $\psi_t = \overline{\psi}$ for $\forall t$. When there presents ambiguity shock, we allow it to interact with productivity shocks $x_{j,t}$, z_t and ζ_t in a possibly non-linear way. This reflects the fact that ambiguity shock is capable of generating time-varying response to productivity shocks, i.e. $\{\kappa_{*x}(\psi_t), \kappa_{*z}(\psi_t), \kappa_{*\zeta}(\psi_t)\}$. In addition, if decision makers are ambiguity averse, ambiguity shock has the first moment impacts manifested by the possibly non-linear functions $\widehat{h}_*(\psi_t)$. This reflects fluctuations in degree of pessimism over the short-run outlooks of the economy. In sum, the proposed policy rules can be understood as a semi-linear perturbation around the ambiguous steady state where we allow for interactions between productivity and ambiguity shocks as well as all higher-order terms of ambiguity shocks.

To approximate the conditional Log-Normal equilibrium with the proposed policy rules, we first implement a quadratic approximation of the value function

$$\begin{aligned} J_t =& J^* + \overline{h}_J + \kappa_{Jk}\widehat{k}_t + \kappa_{Ja}a_{t-1} + \kappa_{Jz,t}z_t + \kappa_{J\zeta,t}\zeta_t \\ &+ \kappa_{Jka}\widehat{k}_t a_{t-1} + \kappa_{Jkz,t}\widehat{k}_t z_t + \kappa_{Jk\zeta,t}\widehat{k}_t\zeta_t + \kappa_{Jaz,t}a_{t-1}z_t + \kappa_{Ja\zeta,t}a_{t-1}\zeta_t + \kappa_{Jz\zeta}z_t\zeta_t \\ &+ \frac{1}{2}\kappa_{Jkk}\widehat{k}_t^2 + \frac{1}{2}\kappa_{Jaa}a_{t-1}^2 + \frac{1}{2}\kappa_{Jzz,t}z_t^2 + \frac{1}{2}\kappa_{J\zeta\zeta,t}\zeta_t^2 + \widehat{h}_J(\psi_t) \end{aligned}$$

Plug this back to the value function recursion (31), we can approximate expected value functions by

$$E_{t,0}^{\omega_t}[J_t] \approx constant_t + \kappa_{Jz,t}(\psi_t)\,\omega_t + \frac{1}{2}\kappa_{Jzz,t}(\psi_t)\,\omega_t^2$$

and

$$\begin{aligned} E_{t,2}^{\omega_{t+1}}[\beta J_{t+1}] \approx constant_t &+ \beta\left(\int_{\mathcal{R}}\kappa_{Jz,t}(\psi_{t+1})\,dF(\psi_{t+1}|\psi_t)\right)\omega_{t+1} \\ &+ \frac{1}{2}\left(\int_{\mathcal{R}}\kappa_{Jz,t}(\psi_{t+1})\,dF(\psi_{t+1}|\psi_t)\right)\omega_{t+1}^2 \end{aligned}$$

where $\kappa_{Jz,t}$ and $\kappa_{Jzz,t}$ are functions of undetermined coefficients in the above proposed policy rules. A direct implication of the quadratic forms in belief distortions $M_{t,1}(\omega_t)$ and $M_{t,2}(\omega_{t+1})$ is Normality in posterior belief over the set of possible models both at stage 1 and stage 2.

In the next step, we log-linearize the optimality conditions derived in Appendix around the A-SS. While doing this, we seriously take into account that at stage 1 and 2 posterior belief over cross-sectional mean of idiosyncratic productivity shocks are distorted in a way consistent with the estimated belief distortions $M_{t,1}(\omega_t)$ and

$M_{t,2}(\omega_{t+1})$. By doing this, we share the same spirit with Ilut and Schneider [2014] and Ilut and Saijo [2016] in dealing with log-linearization with distorted subjective belief. Plugging the proposed policy rules into the log-linearized optimality conditions, we arrive at a large system of undetermined coefficients. To note that in the practical implementation, we discretize the AR(1) process of ambiguity to transform functions over ψ_t into a finite number of coefficients, each of which corresponds to the value of the function at a specific value for ψ_t. Such a huge system of undetermined coefficients can be solved quantitatively with the restriction that $\kappa_{kk} < 1$ to ensure TVC is not violated. Detailed math can be found in the Appendix.

6.3. Calibration

Table 1 summarizes the parameters used in the calibrated version of our baseline model. Discount factor β is 0.99; Frisch elasticity of labor supply is 2; capital share in production is 0.36, and depreciation rate of capital is 0.025. θ is chosen to be 1 corresponding to Cobb-Douglas aggregation technology over island commodities, which implies $y_t = \int_J y_{j,t} dj$. Finally, χ is chosen to be 4.47 to ensure that 1/3 of time is devoted to working in the deterministic steady state.

Table 1. *Model Parameters*

Parameters	Role	Value
β	discount factor	0.99
ϵ	inverse Frisch elasticity	0.5
α	capital share	0.36
δ	depreciation rate	0.025
θ	Cobb-Douglas aggregation	1
χ	1/3 hours at D-SS	4.47
ρ	persistence of agg. productivity shock	0.95
ρ_ψ	persistence of ambiguity shock	0.75
$100\sigma_\zeta$	std. dev. of agg. productivity shock	0.78
$100\sigma_\iota$	std. dev. of island productivity shock	9.0
σ_τ	std. dev. of ambiguity shock	0.47
$\overline{\psi}$	amount of ambiguity $e^{\overline{\psi}}$ at A-SS	-5.05
λ	Degree of ambiguity aversion	12.5

The persistence of aggregate productivity shock ρ is chosen to be 0.95, a conventional value in the literature. Following Angeletos et al. [2016b], the persistence of ambiguity shock ρ_ψ is 0.75 indicating a 2.5 quarters half-life of ambiguity shock, which resembles aggregate demand shock in Blanchard and Quah [1989]. It remains to specify the standard deviations of the three exogenous shocks: the aggregate productivity shock σ_ζ, the idiosyncratic productivity shock σ_ι and the ambiguity shock σ_τ, as well as the amount of ambiguity at A-SS $\overline{\psi}$ and the degree of ambiguity aversion λ.

Following Angeletos et al. [2016b], we select these 5 parameters to minimize the distance between the model implied standard deviations of output, consumption, hours, investment and labor productivity and their data counterparts, where the distances of each variable are weighted by the precision of model-based estimators. We arrive at the calibration such that $\sigma_\zeta = 0.0078$ and $\sigma_\iota = 0.090$. The process of the ambiguity shock has a long-run mean $e^{\overline{\psi}} = 0.0064$ and standard deviation $\sigma_\tau = 0.47$. Finally, the degree of ambiguity aversion is 12.5.[17]

6.4. Business-Cycle Moments and Aggregate Co-movements

Business-cycle moments. Table 2 summarizes key moments of aggregate variables in the US data over the period of 1971Q1-2014Q4 (column 1) and in our calibrated baseline model (column 2). The overall empirical fit of the calibrated baseline model is quite well.

At the core of such overall fit are the balancing roles between the two aggregate shocks, i.e., aggregate productivity shock ζ_t and ambiguity shock ψ_t. Column 3 and 4 report the business cycle moments for aggregate variables when there are only aggregate productivity shocks by setting $\sigma_\tau = 0$ or ambiguity shocks by setting $\sigma_\zeta = 0$ respectively. As in the case of standard RBC, when there are only aggregate productivity shocks, the model fails to generate enough fluctuations in hours and predicts counterfactually high positive correlations between output (or hours) and labor productivity. On the contrary, when there are only ambiguity shocks, the model generates too much volatility in hours and predicts almost perfectly negative correlations between output or hours and labor productivity. In combination, the overall fit is achieved through a natural balancing by our calibrated baseline model.

Aggregate co-movements. Impulse response functions of key aggregate variables to a positive ambiguity shock are reported in . The aggregate co-movement patterns are akin to those of the confidence shocks in Angeletos et al. [2016b], Huo and Takayama [2015] and Ilut and Saijo [2016], where a positive ambiguity shock generates a drop of aggregate quantities, i.e., output, consumption, hours and investment, while, at the same time, an increase in labor productivity in a way consistent with

[17]To note that, even though there are 5 parameters to match 5 moments, we cannot ensure perfect matching. This is because we are not doing the unconstrained minimization. We are minimizing the objective under the constraint that $\lambda e^{\overline{\psi}}$ is bounded above. Such a constraint implicitly assumes there should not be too much deviation in belief from objective one at A-SS and also ensures the uniqueness of A-SS.

Table 2. Bandpass-Filtered Moments of Aggregate Variables

	Data(1971Q1-2014Q4)	Baseline Model	A Only	ψ Only
Standard Deviations				
$stddev(y)$	1.45	1.72	1.28	1.17
$stddev(c)$	0.87	0.72	0.38	0.63
$stddev(n)$	1.76	1.93	0.60	1.83
$stddev(i)$	5.45	4.51	3.75	2.63
$stddev(y/n)$	0.84	0.96	0.69	0.67
Correlations				
$corr(c, y)$	0.88	0.92	0.92	0.98
$corr(n, y)$	0.88	0.87	0.99	0.99
$corr(i, y)$	0.95	0.99	0.99	0.99
$corr(c, n)$	0.85	0.93	0.90	0.97
$corr(c, i)$	0.80	0.84	0.88	0.95
$corr(i, n)$	0.85	0.80	0.99	0.99
Correlations with Productivity				
$corr(y, y/n)$	-0.11	0.05	0.99	-0.99
$corr(n, y/n)$	-0.57	-0.45	0.97	-0.99

Note: The first column reports moments for US data from 1971Q1 to 2014Q4. The second column reports moments in our baseline model. Column 3 and 4 report moments generated by some models when there are only aggregate productivity shocks or ambiguity shocks respectively. All moments are band-pass filtered at frequencies 6-32 quarters.

interpretation of aggregate demand shocks.

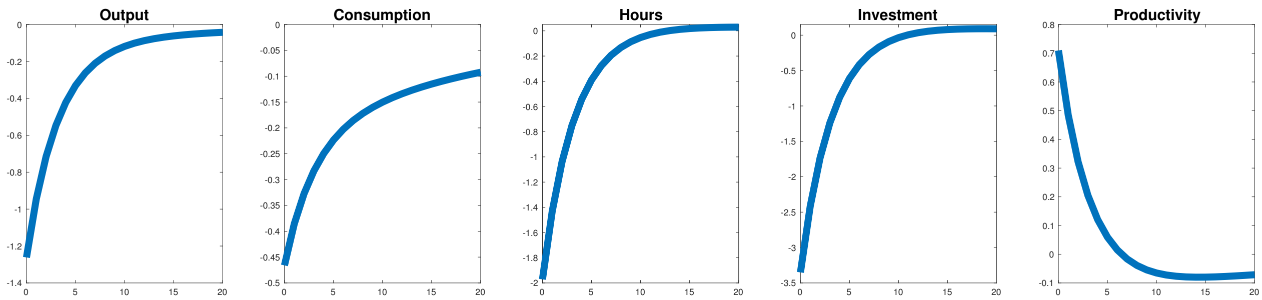

Figure 6. *Impulse responses to one standard deviation of positive ambiguity shock*

What drives the co-movements pattern behind the these IRFs is the fluctuations of in the degree of the pessimism over the short-run outlooks of the economy. By construction, a positive ambiguity shock deepens the degree of the pessimism of all agents over the cross-sectional mean of idiosyncratic productivity shocks for all periods onwards. From the perspective of the firms, such an increased pessimism means into a depressed expectation over the demand of own commodities. In response, firms reduce demand for labor and capital, generating downward pressure on factors prices. From the perspective of the households, this implies a modest decrease in expected permanent income. In response, consumption drops. At the same time, the modest drop in expected permanent income, unlike the case for aggregate productivity shocks, restricts the strength of wealth effect. Given the fact that households understand the drop in factors prices only last for the near future, hours and investment decrease in equilibrium since the relevant substitution effect dominates the opposing wealth effect. In sum, ambiguity shocks generate aggregate co-movements patterns depicted in Figure 6.

Labor wedges. Our calibrated baseline model does a considerably good job in capturing features of data on hours. We interpret such a goodness of fit through the lens of labor wedge analysis in line with Chari et al. [2007]. The key idea is to interpret aggregate data on quantities and prices as wedges in optimality conditions of a text-book RBC model. Then the question of interest is the following: how will the data generated by our calibrated baseline model be translated into wedges, especially labor wedges? It turns out that ambiguity shocks generate empirical relevant countercyclical labor wedges. To economize space, detailed math regarding the calculation of labor wedge are moved to the Appendix.

Table 3 compares moments of labor wedge estimated from US data over the period of 1971Q1-2014Q4 with their model counterparts. It turns out our calibrated baseline model does a considerable good job in capturing cyclical behaviors in labor wedge. It is the exactly the aggregate demand shock nature embedded in ambiguity shock that helps us in explaining labor market dynamics to a large extent. A positive ambiguity shock makes the households more pessimistic over current economic condition hence expecting consumption to be lower. This drives down the expected marginal rate of intra-temporal substitution between leisure and consumption, providing more incentives for households to supply labor. Therefore, a positive ambiguity shock acts as a subsidy on labor supply. On the contrary, a positive ambiguity shock makes island j firms believe demand conditions are turning bad and therefore generates downward pressure on labor demand. In this sense, a positive ambiguity shock acts as a tax on labor demand. In combine, firm side effect dominates household side effect. Therefore, a positive ambiguity shock act as a total labor wedge on island j. This implies that, on average, a positive ambiguity shock generates larger labor wedge hence decreases hours in equilibrium.

6.5. Belief Divergence

Following Assumption 3, professional forecasters in the model are assumed to be ambiguity neutral. Also, to capture the fact that these professional forecasters have better information position than private agents inside the economy do, we provide with the

Table 3. Bandpass-filtered Moments: Labor Wedges

	Data(1971Q1-2014Q4)	Baseline Model
Stddev	2.17	2.20
Correlation with y	-0.75	-0.66

Note: The first column reports moments of labor wedge for US data from 1971Q1 to 2014Q4. The second column reports corresponding moments in our baseline model. All moments are band-pass filtered at frequencies 6-32 quarters. Details can be found in Appendix.

professional forecasters in our model with one additional private information over average productivity $\int_J x_{j,t}dj$:

$$s_{j,t} = \int_J x_{j,t}dj + \xi_{j,t} \quad \text{with} \quad \xi_{j,t} \sim N\left(0, \sigma_\xi^2\right)$$

We calibrate the standard deviation of this additional private information σ_ξ^2 to match the standard deviation of the cross-sectional dispersion in SPF data over the period of 1987Q1-2014Q4 resulting in $100\sigma_\xi = 0.52$.

Table 4 reports the standard deviation and correlation with output y for the US data. Over the period of 1987Q1-2014Q4, the starting point of time of which corresponds to the beginning of great moderation, there displays little variation in belief divergence such that standard deviation is only 0.05 percentage points. However, it is significantly negative correlated with output (-0.39). Our calibrated baseline model captures exactly this pattern. To note that none of our parameters are chosen to match the correlation with output in Table 4. And the cyclical pattern of this moment is solely due to ambiguity shock in a way such that a positive ambiguity is predicted to drive up belief divergence. Therefore, we can conclude that ambiguity shocks capture salient features of data in belief divergence pretty well.

Table 4. Bandpass-Filtered Moments of Belief Divergence

	Data(1987Q1-2014Q4)	Baseline Model
Stddev	0.05	0.05
Correlation with y	-0.39	-0.67

Note: The first column reports moments of belief divergence for US data from 1987Q1 to 2014Q4. The second column reports corresponding moments in our baseline model. All moments are band-pass filtered at frequencies 6-32 quarters.

6.6. Estimated Market Confidence v.s. Sentiment Index

To further validate our theory, we address the following question: can our calibrated baseline model replicate the movements in market confidence as proxied by Sentiment

Index in the data?

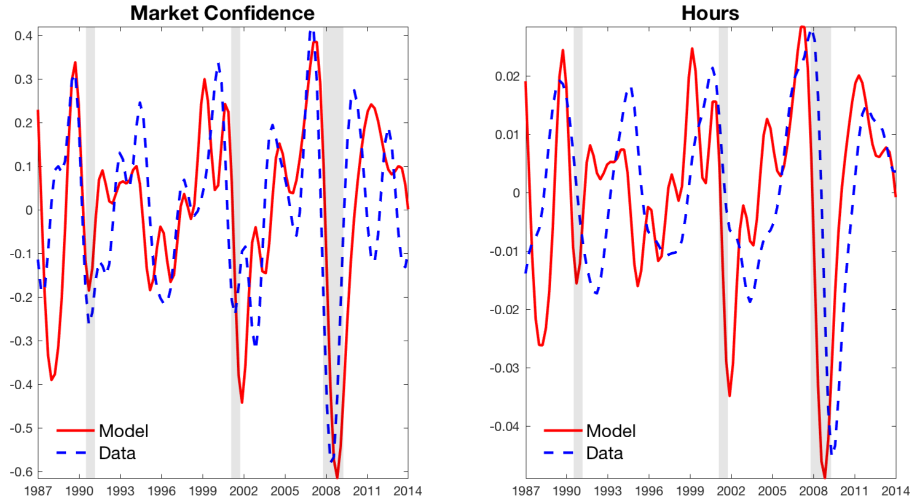

Figure 7. *Estimated Times-Series vs Empirical Proxies*
Note: The figure plots time-series for market confidence and hours estimated from our model and their data counterparts between 1987Q1-2014Q4. In the data, market confidence is proxied by Consumer Sentiment Index from Michigan Survey of Consumer. All moments are band-pass filtered at frequencies 6-32 quarters.

Figure 7 compares the estimated time-series for market confidence, hours, belief divergence and output together with their data counterparts between 1987Q1-2014Q4. To construct these estimated time-series, we first select the sequence of ambiguity and aggregate TFP shocks that can perfectly back-out output and the cross-sectional dispersion of output forecast in SPF. When we backout these shocks we assume that the economy initially stays at the ambiguous steady state.[18] We then construct a sequence of simulated market confidence following Definition 3. Notably, the simulated market confidence ψ_t closely tracks Sentiment Index in Michigan Survey of Consumers, especially for the periods corresponding to NBER recessions. Recall that we do not use any information on Sentiment Index in the construction of estimated time-series for ambiguity and aggregate TFP shocks. Such an empirical fit provides additional validation to our theory. Also note that, simulated time-series of hours also closely tracks its data counterpart.

7. Conclusion

We develop a novel theory of uncertainty-driven business cycles that contributes to accommodate the notion of non-inflationary aggregate demand shock out of varia-

[18]These initial conditions are consistent with data that 1985 is neither a boom or recession. Since the first 8 observations will be dropped after band-pass filtering, our findings are actually quite robust to the choice of initial conditions.

tions in uncertainty and also contributes to explain the co-movements across market confidence, belief divergence, and the aggregate economy. We work on a standard RBC model with multiple firms and multiple commodities and further extend it with ambiguity averse preference represented by the smooth model of ambiguity. Within the smooth model of ambiguity, we contribute to the literature with a Bayesian formulation of ambiguity shock, namely shock to the variance of agents' prior belief over possible models.

Within a simple model without capital, we demonstrate that a positive ambiguity shock makes all agents, who are ambiguity averse, behave as if they believe the aggregate fundamental is turning bad and becoming more volatile. Such dual impacts of ambiguity shock generate endogenous movements across confidence and uncertainty. When the economy features imperfect coordination due to incomplete information, dual impacts of a positive ambiguity shock translates into depressed belief over aggregate demand and the increased incentives to use private information both when making output decisions or output forecasts. The former maps into depressed market confidence and the latter maps into heightened belief divergence. And finally, aggregate output falls due to the increase in the economy-wide pessimism over aggregate demand. In this sense, ambiguity shock in our paper is nothing more than a particular formulation of the aggregate demand shock. In combination, a positive ambiguity shock generates recession with depressed market confidence and heightened belief divergence.

We further explore the quantitative potential of our theory within a dynamic RBC model. Ambiguity shock is shown to be capable of generating co-movements across real quantities together with counter-cyclical labor productivity and labor wedge. Our model is also capable of capturing cyclicalities in belief divergence as measured by the cross-sectional dispersion in output forecast in SPF dataset. Also, the estimated time series of market confidence closely tracks Sentiment Index in Michigan Survey of Consumer. Therefore, we conclude that fluctuations in market confidence, belief divergence, and the aggregate economy are nothing more than the many shades of ambiguity shock, not only qualitatively but also quantitatively.

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Appendix

A. Derivations and Proofs

Derivation of Equation (15). FOC for the island j workers' problem is such that

$$\int_{\mathcal{R}} \phi' \left(E^{\omega_t}_{j,t,1} \left[\frac{C_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right] \right) E^{\omega_t}_{j,t,1} \left[C_t^{-\gamma} W_{j,t} - \chi N_{j,t}^{\epsilon} \right] \widetilde{f}^h_{j,t,1} (\omega_t) \, d\omega_t = 0$$

Plugging in the expression for $\widetilde{f}^h_{j,t,1} (\omega_t)$ given by (6), we arrive at (12) where distorted posterior belief over the set of possible models can be shown given by (14). Similar procedures lead to (13). Then in the last step, combining (12) and (13) together leads to (15). ■

Proof of Lemma 1. Under conditional log-normal equilibrium, we have that

$$\begin{aligned} y_{j,t} &= y^* + \overline{h}_y + \kappa_{yx,t} x_{j,t} + \widehat{h}_y (\psi_t) \\ n_{j,t} &= n^* + \overline{h}_n + \kappa_{nx,t} x_{j,t} + \widehat{h}_n (\psi_t) \\ y_t &= \int_J y_{j,t} dj + \frac{1}{2} \left(1 - \frac{1}{\theta} \right) d^2_{y,j} \end{aligned}$$

where $d_{y,j} \equiv \kappa^2_{yx,t} \sigma^2_\iota$ denotes the cross-sectional dispersion in island outputs. We ignore $d_{y,j}$ in the approximation without loss of generality since they are of second order impacts at the aggregates and have no impacts at all on the cyclical behaviours of belief divergence.

Define $\overline{S} = e^{s^* + \overline{h}_s}$ for any variable of interest S. Quadratic approximation over period utility of the representative household is given by

$$\begin{aligned} & \frac{Y_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \\ \approx & \frac{\overline{Y}^{1-\gamma}}{1-\gamma} \left[1 + (1-\gamma) \widehat{y}_t + \frac{1}{2} (1-\gamma)^2 \widehat{y}_t^2 \right] - \frac{1}{1-\gamma} - \chi \frac{\overline{N}^{1+\epsilon}}{1+\epsilon} \int_J \left(1 + (1+\epsilon) \widehat{n}_{j,t} + \frac{1}{2} (1+\epsilon)^2 \widehat{n}^2_{j,t} \right) dj \\ = & Const. + \overline{Y}^{1-\gamma} \widehat{y}_t + \frac{1}{2} (1-\gamma) \overline{Y}^{1-\gamma} \widehat{y}_t - \int_J \left(\chi \overline{N}^{1+\epsilon} \widehat{n}_{j,t} + \frac{1}{2} (1+\epsilon) \chi \overline{N}^{1+\epsilon} \widehat{n}^2_{j,t} \right) dj \end{aligned}$$

Further define ex-ante (stage 0) expected utility given a particular model ω_t as $\overline{J}_t (\omega_t)$ such that

$$\overline{J}_t (\omega_t) \equiv E^{\omega_t}_{t,0} \left[\frac{Y_t^{1-\gamma} - 1}{1-\gamma} - \chi \int_J \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon} dj \right]$$

It turns out $\overline{J}_t(\omega_t)$ can be quadratically approximated by

$$\overline{J}_t(\omega_t) \approx Const_t + \kappa_{J\omega}(\psi_t, \lambda)\,\omega_t + \frac{1}{2}\kappa_{J\omega\omega}(\psi_t, \lambda)\,\omega_t^2$$

where coefficients of linear and quadratic terms are given by

$$\begin{aligned}
\kappa_{J\omega}(\psi_t, \lambda) &= (Y^*)^{1-\gamma}\left(1 + (1-\gamma)\,\overline{h}_y\right)\kappa_{ya_j}(\psi_t, \lambda) - \chi\,(N^*)^{1+\epsilon}\left(1 + (1+\epsilon)\,\overline{h}_n\right)\kappa_{na_j}(\psi_t, \lambda)\\
\kappa_{J\omega\omega}(\psi_t, \lambda) &= (Y^*)^{1-\gamma}\left(1 + (1-\gamma)\,\overline{h}_y\right)(1-\gamma)\,\kappa^2_{ya_jt}(\psi_t, \lambda)\\
&\quad - \chi\,(N^*)^{1+\epsilon}\left(1 + (1+\epsilon)\,\overline{h}_n\right)(1+\epsilon)\,\kappa^2_{na_j}(\psi_t, \lambda)
\end{aligned}$$

To derive such an approximation, we first approximate $e^{(1-\gamma)\overline{h}_y}$ and $e^{(1+\epsilon)\overline{h}_n}$ by $1 + (1-\gamma)\,\overline{h}_n$ and $1 + (1+\epsilon)\,\overline{h}_n$ respectively. This is doable in the macro application since we want to restrict the A-SS impact of ambiguity to avoid too much statistical sophistication. Then we ignore the intersection terms between ω_t and $x_{j,t}$ or between ω_t and $\widehat{h}_s(\psi_t)$ with $s \in \{y, n\}$. This is doable since those terms are negligible comparing to $\kappa_{J\omega}\omega_t$.

Finally, quadratic approximation over $\overline{J}_t(\omega_t)$ implies that belief distortion in (17) is of exponential quadratic form. Therefore, posterior belief over the set of possible models $\widetilde{f}_{j,t,1}(\omega_t)$ is Normal since the kernel would also be quadratic in ω_t. This leads to a Normal density with the mean μ_t and variance σ_t^2 given by

$$\mu_t = \left(\frac{e^{\psi_t} + g_\sigma(\psi_t, \lambda)}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)}\right) x_{j,t} + \left(\frac{\sigma_\zeta^2 + \sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)}\right) g_\mu(\psi_t, \lambda) \quad (36)$$

and

$$\sigma_t^2 = \left(\frac{\sigma_\zeta^2 + \sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)}\right)\left(e^{\psi_t} + g_\sigma(\psi_t, \lambda)\right) \quad (37)$$

where the distortions in mean $g_\mu(\psi_t, \lambda)$ and in variance $g_\sigma(\psi_t, \lambda)$ are given by

$$\begin{aligned}
g_\mu(\psi_t, \lambda) &= -\lambda\kappa_{J\omega}(\psi_t, \lambda)\left(\frac{e^{\psi_t}}{1 + \lambda\kappa_{J\omega\omega}(\psi_t, \lambda)\,e^{\psi_t}}\right)\\
g_\sigma(\psi_t, \lambda) &= -\left(\frac{\lambda\kappa_{J\omega\omega}(\psi_t, \lambda)\,e^{\psi_t}}{1 + \lambda\kappa_{J\omega\omega}(\psi_t, \lambda)\,e^{\psi_t}}\right)e^{\psi_t}
\end{aligned}$$

In any well-defined equilibrium, it has to be the case that $e^{\psi_t} + g_\sigma(\psi_t, \lambda) > 0$. Otherwise, there would be no well defined distroted posterior belief $\widetilde{f}_{j,t,1}(\omega_t)$ in the sense that its kernel density would be explosive. ■

Proof of Proposition 1. Suppose that the conditional Log-Normal equilibrium takes the following form:

$$\begin{aligned} y_{j,t} \equiv \ln Y_{j,t} &= y^* + \overline{h}_y\left(\overline{\psi}\right) + \kappa_{ya_j}\left(\psi_t, \lambda\right) \cdot a_{j,t} + \widehat{h}_y\left(\psi_t, \lambda\right) \\ n_{j,t} \equiv \ln N_{j,t} &= n^* + \overline{h}_n\left(\overline{\psi}\right) + \kappa_{na_j}\left(\psi_t, \lambda\right) \cdot a_{j,t} + \widehat{h}_n\left(\psi_t, \lambda\right) \\ y_t \equiv \ln Y_t &= y^* + \overline{h}_y\left(\overline{\psi}\right) + \kappa_{ya_j}\left(\psi_t, \lambda\right) \cdot \int_J a_{j,t} dj + \widehat{h}_y\left(\psi_t, \lambda\right) \end{aligned}$$

where we ignore dispersion adjustment of aggregate output in the approximation without loss of generality since they are of second order impacts at the aggregates and have no impacts at all on the cyclical behaviours of belief divergence. Then at D-SS, we have the following

$$\ln\left(\chi\right) + \left(1+\epsilon\right) n^* = \ln\left(1-\alpha\right) + \left(1-\gamma\right) y^*$$

While, at A-SS, impacts of ambiguity shocks at A-SS denoted by $\overline{h}_s$ $s \in \{n, y\}$ must satisfy the following

$$\overline{h}_n = \left(1-\gamma\right)\overline{h}_y + \left(\frac{1}{\theta} - \gamma\right) H_y\left(\overline{\psi}, \lambda\right) \tag{38}$$

Here $H_y\left(\overline{\psi}, \lambda\right)$ denotes degree of pessimism of island j agents over aggregate output y_t at the A-SS. Under the proposed conditional Log-Normal equilibrium, it is given by

$$H_y\left(\overline{\psi}, \lambda\right) = \kappa_{ya_j}\left(\overline{\psi}, \lambda\right)\left(\int_{\mathcal{R}} E^{\omega_t}_{j,t,1}\left[\int_J a_{j,t} dj\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t\right)\Bigg|_{x_{j,t}=0, \psi_t = \overline{\psi}}$$

To understand why this is the case, recall that ambiguous steady state refers to the state the economy converges to (a) in the absence of any shocks, i.e., $a_{j,t} = 0$, but (b) taking into account of the existence of ambiguity, i.e., evaluating $\int_{\mathcal{R}} E^{\omega_t}_{j,t,1}\left[\int_J a_{j,t} dj\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t$ at $\psi_t = \overline{\psi} \neq -\infty$. Alternatively, we can interpret $H_y\left(\overline{\psi}\right)$ from the perspective of distorted subjective beliefs of all agents. At A-SS, the amount of ambiguity $\overline{\psi}$ plays a non-trivial role in the sense that even at A-SS, agent's subjective belief over average productivity is distorted in the mean. This mean distortion has to be respected when we evaluate the A-SS, leading to a non-zero term $H_y\left(\overline{\psi}\right)$. Similar arguments can be found in Ilut and Schneider [2014] and Ilut and Saijo [2016] in the context of "worst case" belief due to multiple prior preferences. Following (36), we have that

$$H_y\left(\overline{\psi}, \lambda\right) = \kappa_{ya_j}\left(\overline{\psi}, \lambda\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\overline{\psi}} + g_\sigma\left(\overline{\psi}, \lambda\right)}\right) g_\mu\left(\overline{\psi}, \lambda\right) \tag{39}$$

where $\kappa_{ya_j}\left(\overline{\psi}, \lambda\right)$ denotes the use of private information at the A-SS.

In the next step, we log-linearize (16) around the A-SS:

$$\left(\frac{1+\epsilon}{1-\alpha}-1+\frac{1}{\theta}\right)\widehat{y}_{j,t}=\left(\frac{1+\epsilon}{1-\alpha}\right)a_{j,t}+\left(\frac{1}{\theta}-\gamma\right)\left(\int_{\mathcal{R}}E^{\omega_t}_{j,t,1}\left[\widehat{y}_t\right]\widetilde{f}_{j,t,1}\left(\omega_t\right)d\omega_t-H_y\left(\overline{\psi},\lambda\right)\right) \tag{40}$$

Therefore, matching coefficients lead to the following two equilibrium conditions

$$\left[\left(\frac{1+\epsilon}{1-\alpha}\right)-(1-\gamma)\right]\kappa_{ya_j}\left(\psi_t,\lambda\right)=\left(\frac{1+\epsilon}{1-\alpha}\right)-\left(\frac{1}{\theta}-\gamma\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2+\sigma_\iota^2+e^{\psi_t}+g_\sigma\left(\psi_t,\lambda\right)}\right)\kappa_{ya_j}\left(\psi_t,\lambda\right) \tag{41}$$

and

$$\left[\left(\frac{1+\epsilon}{1-\alpha}\right)-(1-\gamma)\right]\widehat{h}_y\left(\psi_t,\lambda\right)=\left(\frac{1}{\theta}-\gamma\right)H_y\left(\psi_t,\lambda\right)-\left(\frac{1}{\theta}-\gamma\right)H_y\left(\overline{\psi},\lambda\right) \tag{42}$$

by using the fact that under the proposed policy rules, we have that

$$\int_{\mathcal{R}}E^{\omega_t}_{j,t,1}\left[\widehat{y}_t\right]\widetilde{f}_{j,t,1}\left(\omega_t\right)d\omega_t=\left[1-\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2+\sigma_\iota^2+e^{\psi_t}+g_\sigma\left(\psi_t,\lambda\right)}\right)\right]\kappa_{ya_j}\left(\psi_t,\lambda\right)a_{j,t}$$
$$+H_y\left(\widehat{\psi}_t,\lambda\right)-H_y\left(\overline{\psi},\lambda\right)$$

where

$$H_y\left(\psi_t,\lambda\right)=\kappa_{ya_j}\left(\psi_t,\lambda\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2+\sigma_\iota^2+e^{\psi_t}+g_\sigma\left(\psi_t,\lambda\right)}\right)g_\mu\left(\psi_t,\lambda\right) \tag{43}$$

In what follows, we first present an auxiliary lemma that will be intensively used in later on proofs. It helps us prove that there exists a unique A-SS characterized by the use of private information $\overline{\kappa}_{ya_j}\equiv\kappa_{ya_j}\left(\overline{\psi},\lambda\right)$ and A-SS impacts of ambiguity shocks $\overline{h}_y\left(\overline{\psi}\right)$. After proving the existence and uniqueness. of A-SS, we move on to prove that for any given amount of ambiguity ψ_t, there exists a unique $\kappa_{ya_j}\left(\psi_t,\lambda\right)$ manifesting the use of private information. And finally, existence and uniqueness of $\widehat{h}_y$ would be straight-forward given all the results we have established, which completes the whole proof.

Lemma 4. For any realized amount of ambiguity ψ_t, both $\kappa_{J\omega}\left(\psi_t,\lambda\right)$ and $\kappa_{J\omega\omega}\left(\psi_t,\lambda\right)$ are positive.

Proof. First of all, at the D-SS, it is straight-forward to show that $\chi\left(N^*\right)^{1+\epsilon}=(1-\alpha)\left(Y^*\right)^{1-\gamma}$.

Then at the A-SS, it has to be the case that

$$\frac{(Y^*)^{1-\gamma}\left(1+(1-\gamma)\overline{h}_y\right)}{1-\gamma}-\chi\frac{(N^*)^{1+\epsilon}\left(1+(1+\epsilon)\overline{h}_n\right)}{1+\epsilon}>0$$

Otherwise, it is better-off to be inactive by choosing $\overline{Y}=\overline{C}=\overline{N}=0$. Therefore, it must be the case that

$$\frac{\left(1+(1-\gamma)\overline{h}_y\right)}{1-\gamma}-\frac{(1-\alpha)\left(1+(1+\epsilon)\overline{h}_n\right)}{1+\epsilon}>0 \tag{44}$$

Directly following (41), we know that $\kappa_{ya_j}>0$. This implies that

$$(1+\epsilon)\kappa_{na_j}-(1-\gamma)\kappa_{ya_j}<0 \tag{45}$$

since we know

$$(1+\epsilon)\kappa_{na_j}-(1-\gamma)\kappa_{ya_j}=-\left(\frac{1}{\theta}-\gamma\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2+\sigma_\iota^2+e^{\psi_t}+g_\sigma(\psi_t,\lambda)}\right)\kappa_{ya_j}$$

Furthermore, it can be shown that, by using $\chi(N^*)^{1+\epsilon}=(1-\alpha)(Y^*)^{1-\gamma}$, we would have that

$$\kappa_{J\omega}(\psi_t,\lambda)=(Y^*)^{1-\gamma}\left[\left(1+(1-\gamma)\overline{h}_y\right)\kappa_{ya_j}(\psi_t,\lambda)-\left(1+(1+\epsilon)\overline{h}_n\right)(1-\alpha)\kappa_{na_j}(\psi_t,\lambda)\right]$$

Then it is straight-forward to prove $\kappa_{J\omega}(\psi_t,\lambda)>0$ given (44) and (45). Then we know $g_\mu(\psi_t,\lambda)<0$, hence $H_y\left(\overline{\psi},\lambda\right)<0$. Following (38), we will have that

$$(1-\gamma)\overline{h}_y>(1+\epsilon)\overline{h}_n \tag{46}$$

Since we know that

$$\kappa_{J\omega\omega}(\psi_t,\lambda)=(Y^*)^{1-\gamma}\left[\left(1+(1-\gamma)\overline{h}_y\right)(1-\gamma)\kappa_{ya_jt}^2(\psi_t,\lambda)-\left(1+(1+\epsilon)\overline{h}_n\right)(1-\alpha)(1+\epsilon)\kappa_{na_j}^2(\psi_t,\lambda)\right]$$

it is straight-forward to have $\kappa_{J\omega\omega}(\psi_t,\lambda)>0$ given (45) and (46). ∎

Unique A-SS:

Define $S=(Y^*)^{1-\gamma}$, $X=\left[\frac{1+\epsilon}{1-\alpha}-(1-\gamma)\right]$, $\Sigma^2=\frac{e^{\overline{\psi}}}{1+\lambda\overline{\kappa}_{J\omega\omega}}$ and finally $\beta_\iota=\frac{\sigma_\iota^2}{\sigma_\zeta^2+\sigma_\iota^2+\Sigma^2}$. Then A-SS can be characterized by the following four equations:

$$X\overline{\kappa}_{ya_j}=\left(\frac{1+\epsilon}{1-\alpha}\right)-\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota \tag{47}$$

$$X\overline{h}_y = -\left(\frac{1}{\theta} - \gamma\right)\overline{\kappa}_{ya_j}\beta_\iota\lambda\overline{\kappa}_{J\omega}\Sigma^2 \tag{48}$$

$$\overline{\kappa}_{J\omega} = S\left[\left(1 + (1-\gamma)\overline{h}_y\right)\overline{\kappa}_{ya_j} - \left(1 + (1+\epsilon)\overline{h}_n\right)(1-\alpha)\overline{\kappa}_{na_j}\right] \tag{49}$$

and

$$\overline{\kappa}_{J\omega\omega} = S\left[\left(1 + (1-\gamma)\overline{h}_y\right)(1-\gamma)\overline{\kappa}^2_{ya_j} - \left(1 + (1+\epsilon)\overline{h}_n\right)(1+\epsilon)(1-\alpha)\overline{\kappa}^2_{na_j}\right] \tag{50}$$

Focus on (47) first. Simple algebra leads to

$$\overline{\kappa}_{ya_j} = \frac{\frac{1+\epsilon}{1-\alpha}}{X + \left(\frac{1}{\theta} - \gamma\right)\beta_\iota} \equiv f\left(\overline{\kappa}_{ya_j}, \overline{h}_y\right) \tag{51}$$

It can be shown that

$$\frac{\partial f}{\partial \overline{\kappa}_{ya_j}} < 0 \qquad\qquad \frac{\partial f}{\partial \overline{h}_y} < 0$$

For any given $\overline{h}_y$, the LHS of (47) is increasing in $\overline{\kappa}_{ya_j}$ and the RHS is decreasing in $\overline{\kappa}_{ya_j}$. Together with the boundary conditions, by intermediate value theorem, we can prove that for any $\overline{h}_y$ there exists a unique $\overline{\kappa}^*_{ya_j}\left(\overline{h}_y\right)$ that satisfy (51). Furthermore, we can show that such a function is decreasing in the sense that

$$\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y} = \frac{\partial f / \partial \overline{h}_y}{1 - \partial f / \partial \overline{\kappa}_{ya_j}} < 0$$

Next focus on (48). Things are much more complicated here. Our target here is to show that given any identified function $\overline{\kappa}^*_{ya_j}\left(\overline{h}_y\right)$, there exists a unique $\overline{h}_y$ that satisfies (48). Existence can be easily proved by boundary conditions. Uniqueness can be ensured under some regularity conditions. Define LHS and RHS of (48) upon taking as $\overline{\kappa}^*_{ya_j}\left(\overline{h}_y\right)$ into consideration as $LHS(\overline{h}_y)$ and $RHS\left(\overline{h}_y\right)$. A sufficient condition for uniqueness is that

$$\frac{dRHS}{d\overline{h}_y} < \frac{dLHS}{d\overline{h}_y} = X \quad \text{whenever} \quad LHS = RHS$$

It can be shown that

$$\frac{dRHS}{d\overline{h}_y} = \lambda\overline{\kappa}_{J\omega}\Sigma^2\left[-\left(\frac{1}{\theta} - \gamma\right)\right]\frac{d\overline{\kappa}_{ya_j}\beta_\iota}{d\overline{h}_y} + \left[-\left(\frac{1}{\theta} - \gamma\right)\overline{\kappa}_{ya_j}\beta_\iota\right]\frac{d\lambda\overline{\kappa}_{J\omega}\Sigma^2}{d\overline{h}_y}$$

By using the fact that $\overline{\kappa}^*_{ya_j}\left(\overline{h}_y\right)$ is a solution for (47), we will have that

$$\left[-\left(\frac{1}{\theta}-\gamma\right)\right]\frac{d\overline{\kappa}_{ya_j}\beta_\iota}{d\overline{h}_y}=X\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y}$$

Further evaluating it at $LHS = RHS$ implies that

$$\left[-\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota\right]=\frac{X\overline{h}_y}{\lambda\overline{\kappa}_{J\omega}\Sigma^2}$$

Therefore, we have that

$$\begin{aligned}\frac{1}{X}\frac{dRHS}{d\overline{h}_y}=&\lambda\overline{\kappa}_{J\omega}\Sigma^2\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y}+\overline{h}_y\frac{d\log\left(\overline{\kappa}_{J\omega}\right)}{d\overline{h}_y}+\overline{h}_y\frac{d\log\left(\Sigma^2\right)}{d\overline{h}_y}\\ =&-\frac{X\overline{h}_y}{\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota}\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y}+\overline{h}_y\frac{d\log\left(\overline{\kappa}_{J\omega}\right)}{d\overline{h}_y}+\overline{h}_y\frac{d\log\left(\Sigma^2\right)}{d\overline{h}_y}\end{aligned}$$

or equivalently

$$-\frac{1}{X\overline{h}_y}\frac{dRHS}{d\overline{h}_y}=\frac{X}{\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota}\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y}-\frac{1}{\overline{\kappa}_{J\omega}}\frac{\partial\overline{\kappa}_{J\omega}}{\partial\overline{\kappa}_{ya_j}}\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y}-\frac{1}{\overline{\kappa}_{J\omega}}\frac{\partial\overline{\kappa}_{J\omega}}{\partial\overline{h}_y}+\lambda\Sigma^2\frac{\partial\overline{\kappa}_{J\omega\omega}}{\partial\overline{h}_y}+\lambda\Sigma^2\frac{\partial\overline{\kappa}_{J\omega\omega}}{\partial\overline{\kappa}_{ya_j}}\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y}$$

The followings can be easily proved:

$$\begin{aligned}\frac{\partial\overline{\kappa}_{J\omega}}{\partial\overline{\kappa}_{ya_j}}=&S\left[(1-\gamma)\overline{h}_y-(1+\epsilon)\overline{h}_n\right]=S\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota\lambda\overline{\kappa}_{J\omega}\Sigma^2\\ \frac{\partial\overline{\kappa}_{J\omega}}{\partial\overline{h}_y}=&S\left[(1-\gamma)\overline{\kappa}_{ya_j}-(1+\epsilon)\overline{\kappa}_{na_j}\right]=S\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\overline{\kappa}_{J\omega\omega}}{\partial\overline{\kappa}_{ya_j}}=&2S\left[\left(1+(1-\gamma)\overline{h}_y\right)(1-\gamma)\overline{\kappa}_{ya_j}-\left(1+(1+\epsilon)\overline{h}_n\right)(1+\epsilon)\overline{\kappa}_{na_j}\right]>0\\ \frac{\partial\overline{\kappa}_{J\omega\omega}}{\partial\overline{h}_y}=&S\left[(1-\gamma)^2\overline{\kappa}^2_{ya_j}-(1+\epsilon)^2\overline{\kappa}^2_{na_j}\right]>0\end{aligned}$$

when evaluating at A-SS, i.e., (47) and (48) are satisfied. Simple algebra would eventually imply

$$-\frac{1}{X\overline{h}_y}\frac{dRHS}{d\overline{h}_y}=A\frac{d\overline{\kappa}_{ya_j}}{d\overline{h}_y}+B$$

where we define the A and B are such that

$$\begin{aligned}
A \equiv & \frac{X}{\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota} - S\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota\lambda\Sigma^2 \\
& + 2\lambda\Sigma^2 S\left[\left(1+(1-\gamma)\overline{h}_y\right)(1-\gamma)\overline{\kappa}_{ya_j} - \left(1+(1+\epsilon)\overline{h}_n\right)(1+\epsilon)\overline{\kappa}_{na_j}\right] \\
= & \frac{X}{\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota} \\
& + \lambda\Sigma^2 S\left[2\left(1+(1-\gamma)\overline{h}_y\right)(1-\gamma)\overline{\kappa}_{ya_j} - 2\left(1+(1+\epsilon)\overline{h}_n\right)(1+\epsilon)\overline{\kappa}_{na_j} - \left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota\right] \\
= & \frac{X}{\left(\frac{1}{\theta}-\gamma\right)\overline{\kappa}_{ya_j}\beta_\iota} + \lambda\Sigma^2 S\left[2(1-\gamma)^2\overline{h}_y\overline{\kappa}_{ya_j} - 2(1+\epsilon)\overline{h}_n\overline{\kappa}_{na_j} + (1-\gamma)\overline{\kappa}_{ya_j} - (1+\epsilon)\overline{\kappa}_{na_j}\right] > 0
\end{aligned}$$

and

$$\begin{aligned}
B \equiv & -\frac{1}{\overline{\kappa}_{J\omega}}\frac{\partial\overline{\kappa}_{J\omega}}{\partial\overline{h}_y} + \lambda\Sigma^2\frac{\partial\overline{\kappa}_{J\omega\omega}}{\partial\overline{h}_y} \\
= & -\frac{1}{\overline{\kappa}_{J\omega}}\frac{\partial\overline{\kappa}_{J\omega}}{\partial\overline{h}_y} + \lambda\Sigma^2 S\left[(1-\gamma)^2\overline{\kappa}^2_{ya_j} - (1+\epsilon)^2\overline{\kappa}^2_{na_j}\right] \\
< & \frac{\lambda e^{\overline{\Psi}} S\left[(1-\gamma)^2\overline{\kappa}^2_{ya_j} - (1+\epsilon)^2\overline{\kappa}^2_{na_j}\right]}{1+\lambda e^{\overline{\Psi}}\overline{\kappa}_{J\omega\omega}}
\end{aligned}$$

Therefore, we have that

$$-\overline{h}_y B < \frac{\lambda e^{\overline{\Psi}} S\left[-(1-\gamma)^2\overline{\kappa}^2_{ya_j}\overline{h}_y + (1+\epsilon)^2\overline{\kappa}^2_{na_j}(1-\alpha)\overline{h}_n\right]}{1+\lambda e^{\overline{\Psi}}\overline{\kappa}_{J\omega\omega}}$$

Finally, we can prove

$$\begin{aligned}
& 1+\lambda e^{\overline{\Psi}}\overline{\kappa}_{J\omega\omega} - \lambda e^{\overline{\Psi}} S\left[-(1-\gamma)^2\overline{\kappa}^2_{ya_j}\overline{h}_y + (1+\epsilon)^2\overline{\kappa}^2_{na_j}(1-\alpha)\overline{h}_n\right] \\
= & 1+\lambda e^{\overline{\Psi}} S\left[(1-\gamma)\overline{\kappa}^2_{ya_j} - (1+\epsilon)(1-\alpha)\overline{\kappa}^2_{na_j} + 2(1-\gamma)^2\overline{\kappa}^2_{ya_j}\overline{h}_y - 2(1+\epsilon)^2\overline{\kappa}^2_{na_j}(1-\alpha)\overline{h}_n\right] \\
= & 1+\lambda e^{\overline{\Psi}} S\left\{\left[1+2(1-\gamma)\overline{h}_y\right](1-\gamma)\overline{\kappa}^2_{ya_j} - \left[1+2(1+\epsilon)\overline{h}_n\right](1+\epsilon)(1-\alpha)\overline{\kappa}^2_{na_j}\right\} > 0
\end{aligned}$$

where the last step can be justified by the fact that $\overline{h}_y$ is relatively small for any reasonable macro-application in the sense that $1+2(1-\gamma)\overline{h}_y > 0$. Therefore, we arrive at the fact that

$$-\overline{h}_y B < 1$$

which ensures uniqueness since it directly implies that $\frac{dRHS}{d\overline{h}_y} < X$.

Unique $\kappa_{ya_j}(\psi_t, \lambda)$:

Use of private information κ_{ya_j} is determined by (41). Denote the gap between LHS and RHS of (41) as $f\left(\kappa_{ya_j}\right)$ such that

$$f\left(\kappa_{ya_j}\right) \equiv \left[\left(\frac{1+\epsilon}{1-\alpha}\right) - (1-\gamma)\right]\kappa_{ya_j} - \left(\frac{1+\epsilon}{1-\alpha}\right) + \left(\frac{1}{\theta} - \gamma\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)}\right)\kappa_{ya_j} \tag{52}$$

It can be shown that

- $f\left(\kappa_{ya_j}\right) < 0$ if $\kappa_{ya_j} < 1$
- $f\left(\kappa_{ya_j}\right) > 0$ if $\kappa_{ya_j} > 1 + \frac{(1-\gamma)(1-\alpha)}{(1+\epsilon)-(1-\gamma)(1-\alpha)}$
- $f'\left(\kappa_{ya_j}\right) > 0$

The last item follows the fact that $g_\sigma(\psi_t, \lambda)$ is decreasing in κ_{ya_j} because $g_\sigma(\psi_t, \lambda)$ decreases in $\kappa_{J\omega\omega}$, which is an increasing function of κ_{ya_j}.

Finally, in the last step of the proof, it is straight-forward to demonstrate the existence and uniqueness for $\widehat{h}_y(\psi_t, \lambda)$ from (42) given existence and uniqueness given the existence and uniqueness for A-SS and κ_{ya_j}. ■

Proof of Proposition 2 . Directly follows the comparison between Proof of Proposition 1 and the solution for the beauty contest identified in the proposition. The comparative static analysis of g_μ and g_σ can be proved by demonstrating $\kappa_{J\omega}$ is increasing and $\kappa_{J\omega\omega}$. is decreasing in ψ_t.

■

Proof of Proposition 3 . It can be shown that (52) has the following properties regarding its partial derivatives evaluated at its equilibrium point $f\left(\kappa_{ya_j}\right) = 0$:

- $\frac{\partial f\left(\kappa_{ya_j}\right)}{\partial \kappa_{ya_j}}|_{f\left(\kappa_{ya_j}\right)=0} > 0$
- $\frac{\partial f\left(\kappa_{ya_j}\right)}{\partial \psi_t}|_{f\left(\kappa_{ya_j}\right)=0} < 0$

which are all straight-forward!. Then we can prove that $\kappa_{ya_j}(\psi_t, \lambda)$ is increasing in ψ_t. Then following the (27), $FD_t(\psi_t, \lambda)$ increases with ψ_t can be proved easily.

Also it can be shown that $\kappa_{J\omega}$ is increasing in ψ_t since it is increasing in κ_{ya_j}, which is an increasing function of ψ_t. Also note that we can transform (43) into

$$H_y(\psi_t, \lambda) = -\kappa_{ya_j}(\psi_t, \lambda)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t} + g_\sigma(\psi_t, \lambda)}\right)\left(e^{\psi_t} + g_\sigma(\psi_t, \lambda)\right)\lambda\kappa_{J\omega}$$

Following (41), we know that in equilibrium it must be the case that $e^{\psi_t} + g_\sigma(\psi_t, \lambda)$ is increasing in ψ_t. Then we know $H_y(\psi_t, \lambda)$ must be decreasing in ψ_t. Put this into (41), we complete the proof.

■

Proof of Lemma 3 and Proposition 5. First of all, it can be shown that

$$\begin{aligned}
\left(\overline{\kappa}_{J\omega\omega} + \frac{1}{\lambda e^{\overline{\psi}}}\right)\frac{\partial \log \overline{\kappa}_{J\omega}}{\partial \overline{\kappa}_{ya_j}} =& \frac{1}{\overline{\kappa}_{J\omega}}\frac{\partial \overline{\kappa}_{J\omega}}{\partial \overline{\kappa}_{ya_j}}\left(\overline{\kappa}_{J\omega\omega} + \frac{1}{\lambda e^{\overline{\psi}}}\right) \\
=& S\left(\frac{1}{\theta} - \gamma\right)\overline{\kappa}_{ya_j}\beta_\iota \\
=& S\left[(1-\gamma)\kappa_{ya_j} - (1+\epsilon)\kappa_{na_j}\right] \\
<& 2S\left[\left(1 + (1-\gamma)\overline{h}_y\right)(1-\gamma)\overline{\kappa}_{ya_j} - \left(1 + (1+\epsilon)\overline{h}_n\right)(1+\epsilon)\overline{\kappa}_{na_j}\right] \\
=& \frac{\partial \overline{\kappa}_{J\omega\omega}}{\partial \kappa_{ya_j}}
\end{aligned}$$

The inequality follows the same argument as in the proof for unique A-SS. Then we can claim that

$$\frac{\partial \log \overline{\kappa}_{J\omega}}{\partial \overline{\kappa}_{ya_j}} < \frac{\lambda e^{\overline{\psi}}}{1 + \lambda \overline{\kappa}_{J\omega\omega} e^{\overline{\psi}}}\frac{\partial \overline{\kappa}_{J\omega\omega}}{\partial \overline{\kappa}_{ya_j}}$$

At A-SS, we have that

$$g_\mu(\overline{\psi}, \lambda) = -\frac{\lambda \overline{\kappa}_{J\omega} e^{\overline{\psi}}}{1 + \lambda \overline{\kappa}_{J\omega\omega} e^{\overline{\psi}}} \Rightarrow \log\left(-g_\mu(\overline{\psi}, \lambda)\right) = \log\left(\lambda \overline{\kappa}_{J\omega} e^{\overline{\psi}}\right) - \log\left(1 + \lambda \overline{\kappa}_{J\omega\omega} e^{\overline{\psi}}\right)$$

Taking derivative leads to

$$\frac{\partial \log\left(-g_\mu(\overline{\psi}, \lambda)\right)}{\partial \overline{\kappa}_{ya_j}} = \frac{\partial \log \overline{\kappa}_{J\omega}}{\partial \overline{\kappa}_{ya_j}} - \frac{\lambda e^{\overline{\psi}}}{1 + \lambda \overline{\kappa}_{J\omega\omega} e^{\overline{\psi}}}\frac{\partial \overline{\kappa}_{J\omega\omega}}{\partial \overline{\kappa}_{ya_j}} < 0$$

Therefore, we have that

$$\frac{\partial g_\mu(\overline{\psi}, \lambda)}{\partial \overline{\kappa}_{ya_j}} > 0 \tag{53}$$

At the A-SS when the amount of ambiguity is $\overline{\psi}$, equilibrium is characterized by (47) and (48). We can transform (48) by

$$X\overline{h}_y = \left[\left(\frac{1+\epsilon}{1-\alpha}\right) - X\overline{\kappa}_{ya_j}\right] g_\mu\left(\overline{\psi}, \lambda\right) \tag{54}$$

As in the proof of unique A-SS, (47) defines a decreasing function of $\kappa_{yx}\left(\overline{h}_y\right)$. And an increase in σ_ι^2 or a decrease in σ_ζ^2 increases the RHS of (47) for any given $\overline{h}_y$. This indicates that the function $\kappa_{yx}^*\left(\overline{h}_y\right)$ derived from (47) shifted to the left in response to an increase in σ_ι^2 or a decrease in σ_ζ^2. Furthermore, the RHS of (54) increases with $\overline{\kappa}_{ya_j}$ marginally at the point where LHS equals RHS. Following the proof of unique A-SS, the gap between RHS and LHS of (54) is marginally increasing in $\overline{h}_y$ at the point where LHS equals RHS. Then a marginal increase in $\overline{\kappa}_{ya_j}$ implies a marginal increase in $\overline{y}$. Therefore, (54) defines an increasing function of $\kappa_{yx}^{**}\left(\overline{h}_y\right)$. The decreasing function $\kappa_{yx}^*\left(\overline{h}_y\right)$, the increasing function of $\kappa_{yx}^{**}\left(\overline{h}_y\right)$ and the left hand side shift of $\kappa_{yx}^*\left(\overline{h}_y\right)$ in combine predict that

$$\frac{d\overline{\kappa}_{ya_j}}{d\sigma_\iota^2} < 0 \qquad \frac{d\overline{\kappa}_{ya_j}}{d\sigma_{\sigma_\zeta^2}^2} > 0 \tag{55}$$

and

$$\frac{d\overline{h}_y}{d\sigma_\iota^2} < 0 \qquad \frac{d\overline{h}_y}{d\sigma_{\sigma_\zeta^2}^2} > 0 \tag{56}$$

Finally, (53) and (55) together prove Lemma 3. ■

B. Accuracy of the Approximation

Suppose that the allocation constitutes a Log-Normal equilibrium in the sense that in equilibrium $y_{j,t} \equiv \ln\left(Y_{j,t}\right)$ takes the following forms

$$y_{j,t} = \overline{y} + \kappa_{ya_j}\left(\psi_t, \lambda\right) a_{j,t} + \widehat{h}_y\left(\psi_t, \lambda\right)$$

Then we can, implement a quadratic approximation around the ambiguous steady state (A-SS) over $\overline{J}_t\left(\omega_t\right) \equiv E_{j,t,0}^{\omega_t}\left[\frac{Y_t^{1-\gamma}-1}{1-\gamma} - \chi \int_J \frac{\left(Y_{j,t}/A_{j,t}\right)^{(1+\epsilon)/(1-\alpha)}}{1+\epsilon} dj\right]$, which gives out the following quadratic functional forms with respect to ω_t:

$$\overline{J}_t\left(\omega_t\right) \approx const_t + \kappa_{J\omega}\left(\psi_t, \lambda\right)\omega_t + \frac{1}{2}\kappa_{J\omega\omega}\left(\psi_t, \lambda\right)\omega_t^2$$

Directly following (17), the approximation implies a Normally distributed distorted posterior belief over the set of possible models. This is exactly what Lemma 1 is about. In addition, the fixed point condition (16) combined with normality in distorted belief over the set of possible models would automatically imply a conditional Log-Normal equilibrium such that

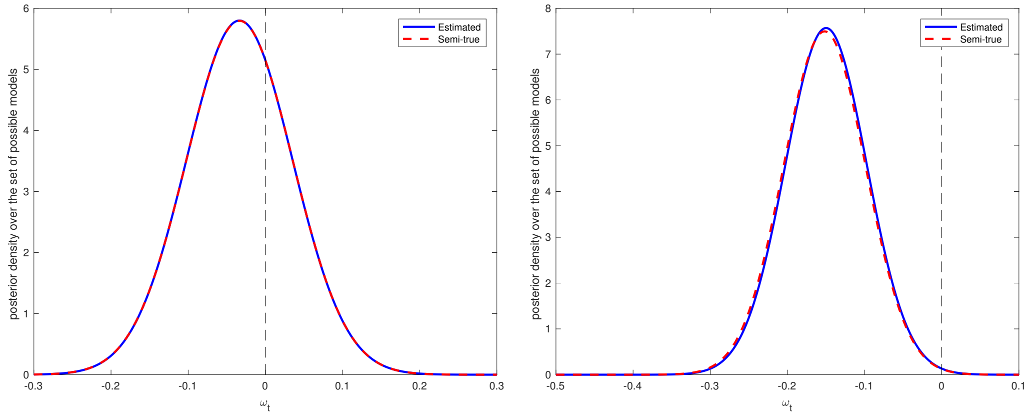
$$y_t = \bar{y} + \kappa_{ya_j}(\psi_t, \lambda) \int_J a_{j,t} dj + \widehat{h}_y(\psi_t, \lambda) + \underbrace{R_t + A_t + D_t}_{\text{uncertainty adjustment}}$$

where uncertainty adjustment consists of (a) risk adjustment R_t, (b) ambiguity adjustment and finally (c) dispersion adjustment. All of these adjustments are of second order impacts at the aggregates and have no impacts at all on the cyclical behaviours of belief divergence. Therefore, we ignore these adjustments in the approximated conditional Log-Normal equilibrium without loss of generality. Equivalently, this corresponds to the usual log-linearation of (16), but around the ambiguous steady state (A-SS) instead of deterministic steady state (D-SS). In sum, to approximate the conditional Log-Normal equilibrium, we implement a second order approximation over ex-ante expected value function of the representative household and at the same time a log-linearization of optimality conditions, both around the A-SS.

How accurate is our approximation? We address this issue into two steps. First of all, we highlight at what points of the analysis approximations are used. Then we analyze in details what are the nature of the approximations and conduct some evaluations.

During the characterization of the approximated conditional Log-Normal equilibrium, we first conduct a quadratic approximation of the ex-ante expected utility $\bar{J}_t(\omega_t)$ under a particular model ω_t so as to express the belief distortion in (17) into exponential-quadratic form. This implies approximated Normality for $\widetilde{f}_{j,t,1}(\omega_t)$ the distorted posterior over the set of possible models when evaluating marginal effects of an act. Second, given the approximated normality posterior, we log-linearize the optimality conditions, consisting of (1) and (15), around the A-SS. By doing so, we ignore uncertainty adjustments who are by nature a couple of second-order terms having negligible impacts on allocations and no impacts at all on belief divergence.

For the second approximation, it is fairly standard in the literature. As a matter of fact, the ignored uncertainty adjustments are of magnitude $\sigma_\zeta^2 + \sigma_\iota^2 + e^{\psi_t}$. However, the first moment impacts of ambiguity shocks are of magnitude λe^{ψ_t}, which dominates uncertainty adjustments given the calibrated degree of ambiguity aversion λ. While, for the first approximation, we deal with this issue by comparing the estimated posterior density as in (17) and a semi-true posterior density numerically. To compute



(a) Baseline Calibration $\lambda = 12.5$ (b) Extreme Calibration $\lambda = 100$

Figure 8. *Accuracy of the Approximation*

the semi-true posterior density, we take the estimated policy functions (18) and (19) as given and compute (17) analytically without using quadratic approximation over $\overline{J}_t(\omega_t)$. Figure 8 demonstrates such comparison where parameters are chosen such that $\theta = 1$, $\epsilon = 0.5$, $\alpha = 0.36$, $\overline{\psi} = -5.05$, $\sigma_\zeta = 0.0078$, $\sigma_\iota = 0.090$ and $\sigma_\tau = 0.47$. These are the same as our quantitative evaluations in Section 6. χ is chosen to be 1.895 to have hours in D-SS being 1/3, which is the same standard when we calibrate χ in Section 6. Following Angeletos and La'O [2009], γ is chosen to be 0.2 to ensure an empirically plausible income effect of labor supply. Comparisons over estimated- and semi-true-posteriors are conducted under two parameterizations for degree of ambiguity aversion λ: one baseline calibration $\lambda = 12.5$ (the left-panel) and one counterfactually extreme calibration $\lambda = 100$ (the right-panel). Finally, we normalize the realization of island productivity by setting $a_{j,t} = 0$. This indicates the Bayesian posterior should be mean zero. Therefore, the leftwards shifts of all the four posterior densities manifest degree of pessimism due to ambiguity aversion. Finally, it turns out the approximation of the distorted posterior over the set of possible models is fairly accurate, not only for the baseline but also for the extreme calibration. Moreover, once we move to the extreme calibration, our approximation turns out to under-estimates both degree of pessimism and volatility in beliefs over the set of possible models when comparing to the semi-true counter-part. This further indicates that our approximation is quite conservative under extreme calibrations.

C. Labor Wedges

Define the marginal rate of intra-temporal substitution between leisure and consumption $MRSN_{j,t}$ and labor productivity $MPL_{j,t}$ on island j by

$$MRSN_{j,t} = \epsilon \widehat{n}_{j,t} + \widehat{c}_t$$

and

$$MPL_{j,t} = \widehat{y}_{j,t} - \widehat{n}_{j,t}$$

Then we can define island j labor wedge from the perspective of household by the gap between real wage $\widehat{w}_{j,t}$ and island j marginal rate of intra-temporal substitution between leisure and consumption $MRSN_{j,t}$:

$$\tau_{j,t}^{nh} \equiv \widehat{w}_{j,t} - MRSN_{j,t}$$

and island j labor wedge from the perspective of firm as the gap between island j labor productivity $MPL_{j,t}$ and real wage $\widehat{w}_{j,t}$:

$$\tau_{j,t}^{nf} \equiv MPL_{j,t} - \widehat{w}_{j,t}$$

Finally, we define the total labor wedge in the economy by the cross-sectional average over the sum of the two:

$$\tau_t^n \equiv \int_J \left(\tau_{j,t}^{nh} + \tau_{j,t}^{nh}\right) dj$$

Plugging in optimality conditions for labor, model implied wedges are such that

$$\begin{aligned} \tau_{j,t}^{nh} &= \int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[\widehat{c}_t\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t - \widehat{c}_t \\ \tau_{j,t}^{nf} &= \frac{1}{\theta}\left[\widehat{y}_{j,t} - \int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[\widehat{y}_t\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t\right] \end{aligned}$$

D. Details for Quantitative Analysis

D.1. Equilibrium Conditions

Equilibrium can be characterized by the standard TVC and following conditions:

- labor optimality condition

$$\chi N_{j,t}^{\epsilon} = \left(\int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[\frac{1}{C_t}\left(\frac{Y_{j,t}}{Y_t}\right)^{-\frac{1}{\theta}}\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t\right) \left((1-\alpha)\frac{Y_{j,t}}{N_{j,t}}\right)$$

- optimal capital demand condition

$$R_{j,t} \int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[\frac{1}{C_t}\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t = \left(\int_{\mathcal{R}} E_{j,t,1}^{\omega_t} \left[\frac{1}{C_t} P_{j,t}\right] \widetilde{f}_{j,t,1}\left(\omega_t\right) d\omega_t\right) \left(\alpha \frac{Y_{j,t}}{K_{j,t}}\right)$$

- Euler equation

$$\frac{1}{C_t} = \beta \int_{\mathcal{R}} E_{t,2}^{\omega_{t+1}} \left[\frac{1}{C_{t+1}} \left((1-\delta) + R_{j,t+1} \right) \right] \widetilde{f}_{t,2} \left(\omega_{t+1} \right) d\omega_{t+1} \tag{57}$$

- budget constraint

$$Y_t = C_t + \int_J I_{j,t} dj \tag{58}$$

- capital accumulation

$$K_{j,t+1} = (1-\delta) K_{j,t} + I_{j,t} \tag{59}$$

- production function for island commodities

$$Y_{j,t} = A_{j,t} K_{j,t}^{\alpha} N_{j,t}^{1-\alpha} \tag{60}$$

- production function for final goods

$$\log Y_t = \int_J \log Y_{j,t} dj \tag{61}$$

- value function recursion for $J_t \equiv J\left(\left\{K_{j,t}\right\}, a_{t-1}, z_t, \zeta_t, \psi_t\right)$:

$$J_t = \max_{C_t, \left\{K_{j,t+1}\right\}} \ln\left(C_t\right) + \beta \left(-\frac{1}{\lambda}\right) \ln\left(\int_{\mathcal{R}} e^{-\lambda E_{t,2}^{\omega_{t+1}}[J_{t+1}]} f_t\left(\omega_{t+1}\right) d\omega_{t+1} \right)$$

- distorted beliefs $\left\{ \widetilde{f}_{j,t,1}\left(\omega_t\right), \widetilde{f}_{t,2}\left(\omega_{t+1}\right) \right\}$ at stage 1 and at stage 2 respectively

$$\widetilde{f}_{j,t,1}\left(\omega_t\right) = e^{-\lambda E_{t,0}^{\omega_t}[J_t]} f\left(x_{j,t} | \omega_t\right) f_t\left(\omega_t\right)$$

  and

$$\widetilde{f}_{t,2}\left(\omega_{t+1}\right) = e^{-\lambda E_{t,2}^{\omega_t}[J_{t+1}]} f_t\left(\omega_{t+1}\right)$$

Observe that in equilibrium, it will be the case that $K_{j,t} = K_t$ for all $t > 0$. This is because at stage 2 there exists no heterogeneity in belief over capital return $r_{j,t}$ across islands. Therefore, capital supply exhibits no heterogeneity. In what follows, we replace all $K_{j,t}$ with K_t for simplicity.

D.2. Solution Method

We propose the following semi-linear policy rules of conditional log-normal equilibrium for island employment $\widehat{n}_{j,t}$, output $\widehat{y}_{j,t}$, wage rate $\widehat{w}_{j,t}$ and rental rental rate of capital $\widehat{r}_{j,t}$ at stage 1 of period t:

$$\begin{aligned}
\widehat{y}_{j,t} &= \kappa_{yk}\widehat{k}_t + \kappa_{ya}a_{t-1} + \kappa_{yx}(\psi_t)\,x_{j,t} + \widehat{h}_y(\psi_t) \\
\widehat{n}_{j,t} &= \kappa_{nk}\widehat{k}_t + \kappa_{na}a_{t-1} + \kappa_{nx}(\psi_t)\,x_{j,t} + \widehat{h}_n(\psi_t) \\
\widehat{w}_{j,t} &= \kappa_{wk}\widehat{k}_t + \kappa_{wa}a_{t-1} + \kappa_{wx}(\psi_t)\,x_{j,t} + \widehat{h}_w(\psi_t) \\
\widehat{r}_{j,t} &= \kappa_{rk}\widehat{k}_t + \kappa_{ra}a_{t-1} + \kappa_{rx}(\psi_t)\,x_{j,t} + \widehat{h}_r(\psi_t)
\end{aligned}$$

and the for consumption $\widehat{c}_t$, investment $\widehat{i}_t$ and capital stock tomorrow $\widehat{k}_{t+1}$ at stage 2 of period t:

$$\begin{aligned}
\widehat{c}_t &= \kappa_{ck}\widehat{k}_t + \kappa_{ca}a_{t-1} + \kappa_{cz}(\psi_t)\,z_t + \kappa_{c\zeta}(\psi_t)\,\zeta_t + \widehat{h}_c(\psi_t) \\
\widehat{i}_t &= \kappa_{ik}\widehat{k}_t + \kappa_{ia}a_{t-1} + \kappa_{iz}(\psi_t)\,z_t + \kappa_{i\zeta}(\psi_t)\,\zeta_t + \widehat{h}_i(\psi_t) \\
\widehat{k}_{t+1} &= \kappa_{kk}\widehat{k}_t + \kappa_{ka}a_{t-1} + \kappa_{kz}(\psi_t)\,z_t + \kappa_{k\zeta}(\psi_t)\,\zeta_t + \widehat{h}_k(\psi_t)
\end{aligned}$$

D.2.1. Quadratic approximation of value function

Under the proposed policy rules, quadratic approximation of household period utility around the A-SS is such that

$$\ln(C_t) - \chi\int_j \frac{N_{j,t}^{1+\epsilon}}{1+\epsilon}dj \tag{62}$$

$$\approx c^* + \overline{h}_c + \kappa_{ck}\widehat{k}_t + \kappa_{ca}a_{t-1} + \kappa_{cz,t}z_t + \kappa_{c\zeta,t}\zeta_t + \widehat{h}_c(\psi_t) \tag{63}$$

$$-\chi\frac{\overline{N}^{1+\epsilon}}{1+\epsilon}\int_j\Big[1 + (1+\epsilon)\Big(\kappa_{nk}\widehat{k}_t + \kappa_{na}a_{t-1} + \kappa_{nx,t}x_{j,t} + \widehat{h}_n(\psi_t)\Big) \tag{64}$$

$$+\frac{1}{2}(1+\epsilon)^2\Big(\kappa_{nk}\widehat{k}_t + \kappa_{na}a_{t-1} + \kappa_{nx,t}x_{j,t} + \widehat{h}_n(\psi_t)\Big)^2\Big]\,dj \tag{65}$$

$$\tag{66}$$

Guess that J_t is given by

$$J_t = J^* + \overline{h}_J + \kappa_{Jk}\widehat{k}_t + \kappa_{Ja}a_{t-1} + \kappa_{Jz,t}z_t + \kappa_{J\zeta,t}\zeta_t \tag{67}$$

$$+\kappa_{Jka}\widehat{k}_t a_{t-1} + \kappa_{Jkz,t}\widehat{k}_t z_t + \kappa_{Jk\zeta,t}\widehat{k}_t\zeta_t + \kappa_{Jaz,t}a_{t-1}z_t + \kappa_{Ja\zeta,t}a_{t-1}\zeta_t + \kappa_{Jz\zeta}z_t\zeta_t \tag{68}$$

$$+\frac{1}{2}\kappa_{Jkk}\widehat{k}_t^2 + \frac{1}{2}\kappa_{Jaa}a_{t-1}^2 + \frac{1}{2}\kappa_{Jzz,t}z_t^2 + \frac{1}{2}\kappa_{J\zeta\zeta,t}\zeta_t^2 + \widehat{h}_J(\psi_t) \tag{69}$$

Hence we have

$$\left(-\frac{1}{\lambda}\right)\ln\left(\int_{\omega_{t+1}} e^{-\lambda E_{t,2}^{\omega_{t+1}}[J_{t+1}]}\widetilde{f}_{t,2}^{c}\left(\omega_{t+1}\right)d\omega_{t+1}\right) \tag{70}$$

$$=constant_t+\kappa_{Jk}\widehat{k}_{t+1}+\kappa_{Ja}a_t+\kappa_{Jka}\widehat{k}_{t+1}a_t+\frac{1}{2}\kappa_{Jkk}\widehat{k}_{t+1}^2+\frac{1}{2}\kappa_{Jaa}a_t^2 \tag{71}$$

Value function recursion implies that

$$E_{t,0}^{\omega_t}\left[J_t\right]=\left(J^*+\widetilde{\overline{h}_J+\widehat{h}_J\left(\psi_t\right)}\right)+\kappa_{Jk}\widehat{k}_t+\kappa_{Ja}a_{t-1}+\kappa_{Jz,t}\omega_t+\kappa_{Jka}\widehat{k}_ta_{t-1}+\kappa_{Jkz,t}\widehat{k}_t\omega_t+\kappa_{Jaz,t}a_{t-1}\omega_t \tag{72}$$

$$+\frac{1}{2}\kappa_{Jkk}\widehat{k}_t^2+\frac{1}{2}\kappa_{Jaa}a_{t-1}^2+\frac{1}{2}\kappa_{Jzz,t}\omega_t^2 \tag{73}$$

$$\approx constant_t+\kappa_{Jz,t}\omega_t+\frac{1}{2}\kappa_{Jzz,t}\omega_t^2 \tag{74}$$

where the following matching coefficients leads to

$$\kappa_{Jz,t}=\kappa_{cz,t}-\chi N^*\left[1+\left(1+\epsilon\right)\overline{h}_n\right]\kappa_{nx,t}+\beta\kappa_{Jk}\kappa_{kz,t} \tag{75}$$

$$\kappa_{Jk}=\kappa_{ck}-\chi N^*\left[1+\left(1+\epsilon\right)\overline{h}_n\right]\kappa_{nk}+\beta\kappa_{Jk}\kappa_{kk} \tag{76}$$

$$\kappa_{Jzz,t}=-\left(1+\epsilon\right)\chi N^*\left[1+\left(1+\epsilon\right)\overline{h}_n\right]\kappa_{nx,t}^2+\beta\kappa_{Jkk}\kappa_{kz,t}^2 \tag{77}$$

$$\kappa_{Jkk}=-\left(1+\epsilon\right)\chi N^*\left[1+\left(1+\epsilon\right)\overline{h}_n\right]\kappa_{nk}^2+\beta\kappa_{Jkk}\kappa_{kk}^2 \tag{78}$$

To arrive these expression, we first approximate $e^{(1+\epsilon)\overline{h}_n}$ by $1+\left(1+\epsilon\right)\overline{h}_n$ and by ignoring a couple of higher order terms without loss of generality.

Furthermore, we can approximate by

$$E_{t,2}^{\omega_t}\left[J_{t+1}\right]\approx constant_t+\overline{\kappa_{Jz,t+1}}\omega_t+\frac{1}{2}\overline{\kappa_{Jzz,t+1}}\omega_t^2$$

where term with over-line denotes period t expectation over that term in period $t+1$.

D.2.2. Distorted posterior beliefs

Distorted beliefs are normal with the following kernels

$$\widetilde{f}_{j,t,1}\left(\omega_t\right)=e^{-\lambda\left(\kappa_{Jz,t}\omega_t+\frac{1}{2}\kappa_{Jzz,t}\omega_t^2\right)}f\left(x_{j,t}|\omega_t\right)f_t\left(\omega_t\right)$$

and

$$\widetilde{f}_{t,2}\left(\omega_{t+1}\right)=e^{-\lambda\left(\overline{\kappa_{Jz,t+1}}\omega_t+\frac{1}{2}\overline{\kappa_{Jzz,t+1}}\omega_t^2\right)}f_t\left(\omega_{t+1}\right)$$

D.2.3. Ambiguous Steady State

A-SS can be characterized by

$$\begin{aligned}
\log(\chi) + (1+\epsilon) n^* + (1+\epsilon) \overline{h}_n =& \log(1-\alpha) + y^* - c^* + \overline{h}_y - \overline{h}_c + H_n(\overline{\psi}) \\
y^* + \overline{h}_y =& (1-\alpha)\left(n^* + \overline{h}_n\right) + \alpha\left(k^* + \overline{h}_k\right) \\
r^* + \overline{h}_r =& \log(\alpha) + y^* - k^* + \overline{h}_y - \overline{h}_k + H_r(\overline{\psi}) \\
1 =& O_1 + O_2 \\
X_c + X_i =& X_y \\
X_i =& \delta X_k
\end{aligned}$$

where terms with star denote the D-SS and the auxiliary functions are such that

$$\begin{aligned}
H_n(\psi_t) =& \left[\left(\frac{1}{\theta}\kappa_{yx,t} - \kappa_{cz,t}\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2}\right) + \kappa_{c\zeta,t}\left(\frac{\sigma_\zeta^2}{\sigma_\zeta^2 + \sigma_\iota^2}\right)\right]\left(\frac{-\lambda\kappa_{Jz,t}}{\frac{1}{\sigma_\zeta^2+\sigma_\iota^2} + \frac{1}{e^{\psi_t}} + \lambda\kappa_{Jzz,t}}\right) \\
H_r(\psi_t) =& \frac{1}{\theta}\kappa_{yx,t}\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2 + \sigma_\iota^2}\right)\left(\frac{-\lambda\kappa_{Jz,t}}{\frac{1}{\sigma_\zeta^2+\sigma_\iota^2} + \frac{1}{e^{\psi_t}} + \lambda\kappa_{Jzz,t}}\right) \\
S_1(\psi_t) \equiv& \left(\overline{\kappa_{rx,t+1} - \kappa_{cz,t+1}}\right)\left(\frac{-\lambda\overline{\kappa_{Jz,t+1}}}{\frac{1}{e^{\psi_{t,t+1}}} + \lambda\overline{\kappa_{Jzz,t+1}}}\right) + \left(\overline{\widehat{h}_r(\psi_{t+1}) - \widehat{h}_c(\psi_{t+1})}\right) \\
S_2(\psi_t) \equiv& -\left(\overline{\kappa_{cz,t+1}}\right)\left(\frac{-\lambda\overline{\kappa_{Jz,t+1}}}{\frac{1}{e^{\psi_{t,t+1}}} + \lambda\overline{\kappa_{Jzz,t+1}}}\right) - \overline{\widehat{h}_c(\psi_{t+1})} \\
O_1 =& \beta e^{r^* + \overline{h}_r + S_1(\overline{\psi})} \\
O_2 =& \beta(1-\delta)e^{S_2(\overline{\psi})}
\end{aligned}$$

and

$$X_c = \left(e^{c^* + \overline{h}_c}\right) \qquad X_i = \left(e^{i^* + \overline{h}_i}\right) \qquad X_y = \left(e^{y^* + \overline{h}_y}\right) \qquad X_k = \left(e^{k^* + \overline{h}_k}\right)$$

D.2.4. Log-linearization of optimality conditions

- labor optimality

$$(1+\epsilon)\widehat{n}_{j,t} = \left(1 - \frac{1}{\theta}\right)\widehat{y}_{j,t} + \int_{\omega_t} E_{j,t,1}^{\omega_t}\left[\frac{1}{\theta}\widehat{y}_t - \widehat{c}_t\right]\widetilde{f}_{j,t,1}(\omega_t)\, d\omega_t - H_n(\overline{\psi})$$

- optimal capital demand condition

$$\widehat{r}_{j,t} = \left(1 - \frac{1}{\theta}\right)\widehat{y}_{j,t} + \int_{\omega_t} E^{\omega_t}_{j,t,1}\left[\frac{1}{\theta}\widehat{y}_t - \widehat{k}_t\right]\widetilde{f}_{j,t,1}(\omega_t)\,d\omega_t - H_r(\overline{\psi})$$

- Euler equation

$$c_t = O_1 \int_{\mathcal{R}} E^{\omega_{t+1}}_{t,2}\left[\widehat{r}_{j,t+1} - \widehat{c}_{t+1}\right]\widetilde{f}_{t,2}(\omega_{t+1})\,d\omega_{t+1} - O_2 \int_{\mathcal{R}} E^{\omega_{t+1}}_{t,2}\left[\widehat{c}_{t+1}\right]\widetilde{f}_{t,2}(\omega_{t+1})\,d\omega_{t+1}$$

- budget constraint

$$X_c\widehat{c}_t + X_i\widehat{i}_t = X_y\widehat{y}_t$$

- capital accumulation

$$X_k\widehat{k}_{t+1} = (1-\delta)\,X_k\widehat{k}_t + X_i\widehat{i}_t$$

- production function for island commodities

$$\widehat{y}_{j,t} = \rho a_{t-1} + x_{j,t} + \alpha\widehat{k}_t + (1-\alpha)\,\widehat{n}_{j,t}$$

- production function for final goods

$$\widehat{y}_t = \int_J \widehat{y}_{j,t}dj$$

D.2.5. Systems of undetermined coefficients

Combining the proposed policy rules and log-linearized optimality conditions, we arrive at the following systems of undetermined coefficients:

- Determination of κ_{J*}

$$\begin{aligned}
\kappa_{Jz,t} &= \kappa_{cz,t} - \chi\overline{N}^{1+\epsilon}\kappa_{nx,t} + \beta\kappa_{Jk}\kappa_{kz,t} \\
\kappa_{Jk} &= \kappa_{ck} - \chi\overline{N}^{1+\epsilon}\kappa_{nk} + \beta\kappa_{Jk}\kappa_{kk} \\
\kappa_{Jzz,t} &= -(1+\epsilon)\,\chi\overline{N}^{1+\epsilon}\kappa^2_{nx,t} + \beta\kappa_{Jkk}\kappa^2_{kz,t} \\
\kappa_{Jkk} &= -(1+\epsilon)\,\chi\overline{N}^{1+\epsilon}\kappa^2_{nk} + \beta\kappa_{Jkk}\kappa^2_{kk}
\end{aligned}$$

- Determination of ambiguous steady state

$$
\begin{aligned}
\log(\chi) + (1+\epsilon) n^* + (1+\epsilon) \overline{h}_n =& \log(1-\alpha) + y^* - c^* + \overline{h}_y - \overline{h}_c + H_n(\overline{\psi}) \\
y^* + \overline{h}_y =& (1-\alpha)\left(n^* + \overline{h}_n\right) + \alpha\left(k^* + \overline{h}_k\right) \\
r^* + \overline{h}_r =& \log(\alpha) + y^* - k^* + \overline{h}_y - \overline{h}_k + H_r(\overline{\psi}) \\
1 =& O_1 + O_2 \\
X_c + X_i =& X_y \\
X_i =& \delta X_k
\end{aligned}
$$

- Determination of κ_{*k} and κ_{*a}

$$
\begin{aligned}
(1+\epsilon)\kappa_{nk} =& \left(1 - \frac{1}{\theta}\right)\kappa_{yk} + \left[\frac{1}{\theta}\kappa_{yk} - \kappa_{ck}\right] \\
(1+\epsilon)\kappa_{na} =& \left(1 - \frac{1}{\theta}\right)\kappa_{ya} + \left[\frac{1}{\theta}\kappa_{ya} - \kappa_{ca}\right] \\
\kappa_{yk} =& (1-\alpha)\kappa_{nk} + \alpha \\
\kappa_{ya} =& \rho + (1-\alpha)\kappa_{na} \\
\kappa_{rk} =& \kappa_{yk} - 1 \\
\kappa_{ra} =& \kappa_{ya} \\
-\kappa_{ck} =& (O_1\kappa_{rk} - \kappa_{ck})\kappa_{kk} \\
-\kappa_{ca} =& (O_1\kappa_{rk} - \kappa_{ck})\kappa_{ka} + (O_1\kappa_{ra} - \kappa_{ca})\rho \\
X_c\kappa_{ck} + X_i\kappa_{ik} =& X_y\kappa_{yk} \\
X_c\kappa_{ca} + X_i\kappa_{ia} =& X_y\kappa_{ya} \\
\kappa_{kk} =& (1-\delta) + \delta\kappa_{ik} \\
\kappa_{ka} =& \delta\kappa_{ia}
\end{aligned}
$$

- Determination of $\kappa_{*x,t}$, $\kappa_{*z,t}$ and $\kappa_{*\omega,t}$

$$
\begin{aligned}
(1+\epsilon)\,\kappa_{nx,t} =& \left(1-\frac{1}{\theta}\right)\kappa_{yx,t} + \left[\left(\frac{1}{\theta}\kappa_{yx,t}-\kappa_{cz,t}\right)\left(\frac{\sigma_\zeta^2}{\sigma_\zeta^2+\sigma_\iota^2}\right) - \kappa_{c\zeta,t}\left(\frac{\sigma_\zeta^2}{\sigma_\zeta^2+\sigma_\iota^2}\right)\right] \\
&+ \left[\left(\frac{1}{\theta}\kappa_{yx,t}-\kappa_{cz,t}\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2+\sigma_\iota^2}\right) + \kappa_{c\zeta,t}\left(\frac{\sigma_\zeta^2}{\sigma_\zeta^2+\sigma_\iota^2}\right)\right]\left(\frac{\frac{1}{\sigma_\zeta^2+\sigma_\iota^2}}{\frac{1}{\sigma_\zeta^2+\sigma_\iota^2}+\frac{1}{e^{\psi_t}}+\lambda\kappa_{Jzz,t}}\right) \\
\kappa_{yx,t} =& 1+(1-\alpha)\,\kappa_{nx,t} \\
\kappa_{rx,t} =& \left(1-\frac{1}{\theta}\right)\kappa_{yx,t} + \frac{1}{\theta}\kappa_{yx,t}\left(\frac{\sigma_\zeta^2}{\sigma_\zeta^2+\sigma_\iota^2}\right) \\
&+ \left(\frac{1}{\theta}\kappa_{yx,t}\right)\left(\frac{\sigma_\iota^2}{\sigma_\zeta^2+\sigma_\iota^2}\right)\left(\frac{\frac{1}{\sigma_\zeta^2+\sigma_\iota^2}}{\frac{1}{\sigma_\zeta^2+\sigma_\iota^2}+\frac{1}{e^{\psi_t}}+\lambda\kappa_{Jzz}}\right) \\
-\kappa_{cz,t} =& \left(O_1\kappa_{rk}-\kappa_{ck}\right)\kappa_{kz,t} \\
-\kappa_{c\zeta,t} =& \left(O_1\kappa_{rk}-\kappa_{ck}\right)\kappa_{k\zeta,t} + \left(O_1\kappa_{ra}-\kappa_{ca}\right) \\
X_c\kappa_{cz,t}+X_i\kappa_{iz,t} =& X_y\kappa_{yx,t} \\
X_c\kappa_{c\zeta,t}+X_i\kappa_{i\zeta,t} =& 0 \\
\kappa_{kz,t} =& \delta\kappa_{iz,t} \\
\kappa_{k\zeta,t} =& \delta\kappa_{i\zeta,t}
\end{aligned}
$$

- Determination of $\widehat{h}_*\left(\psi_t\right)$

$$
\begin{aligned}
(1+\epsilon)\,\widehat{h}_n(\psi_t) =& \left(1-\frac{1}{\theta}\right)\widehat{h}_y(\psi_t) + \left[\frac{1}{\theta}\widehat{h}_y(\psi_t) - \widehat{h}_c(\psi_t)\right] + \widehat{H}_n(\psi_t) \\
\widehat{h}_y(\psi_t) =& (1-\alpha)\,\widehat{h}_n(\psi_t) \\
\widehat{h}_r(\psi_t) =& \widehat{h}_y(\psi_t) + \widehat{H}_r(\psi_t) \\
-\widehat{h}_c(\psi_t) =& \left(O_1\kappa_{rk}-\kappa_{ck}\right)\widehat{h}_k(\psi_t) + O_1\widehat{S}_1(\psi_t) + O_2\widehat{S}_2(\psi_t) \\
X_c\widehat{h}_c(\psi_t) + X_i\widehat{h}_i(\psi_t) =& X_y\widehat{h}_y(\psi_t) \\
\widehat{h}_k(\psi_t) =& \delta\widehat{h}_i(\psi_t)
\end{aligned}
$$

where $\widehat{H}_n(\psi_t) \equiv H_n(\psi_t) - H_n(\bar{\psi})$ and $\widehat{H}_r(\psi_t) \equiv H_r(\psi_t) - H_r(\bar{\psi})$.

D.2.6. The algorithm

To quantitatively solve for the equilibrium, we first discretize the ambiguity process by a stationary Markov process with 11 states. This transforms functions over ψ_t into 11 undetermined coefficients. Then we arrive at a non-linear large system of equations regarding undetermined coefficients. We describe the algorithm to solve this system of undermined coefficients below.

1. Guess that the ambiguous steady state coincides with deterministic steady state such that $\overline{h}_S = 0$ for all $S \in \{y, c, n, i, k, r\}$. This leads to $O_1 = \beta e^{r^* + \overline{h}_r}$ and $O_2 = \beta(1-\delta)$. Further guess that $S_1(\overline{\psi}) = S_1(\overline{\psi}) = 0$.

2. Given $\{O_1, O_2, S_1, S_2, X_c, X_y, X_i, X_k\}$, solve for κ_{*k} and κ_{*a}. We restricts $\kappa_{kk} < 1$ to ensure TVC being satisfied.

3. Jointly solve κ_{J*}, $\kappa_{*x,t}$, $\kappa_{*z,t}$ and $\kappa_{*\omega,t}$ given the solution for κ_{*k} and κ_{*a}.

4. Solve $\widehat{h}_*(\psi_t)$ for the given solution for κ_{J*}, $\kappa_{*x,t}$, $\kappa_{*z,t}$, $\kappa_{*\omega,t}$, κ_{*k} and κ_{*a}.

5. Solve for the A-SS and compute new levels of $\{O_1, O_2, S_1, S_2, X_c, X_y, X_i, X_k\}$.

6. Check for convergence over $\{O_1, O_2, S_1, S_2, X_c, X_y, X_i, X_k\}$. If converge, stop. Otherwise, go back to Step 2.

Convergence can be achieved when the amount of ambiguity at A-SS $\overline{\psi}$ is not extremely high or when degree of ambiguity aversion λ is not extremely high. This is because these are the conditions to ensure unique A-SS as in the simple model without capital. Other than this constraint, the algorithm works well in the approximation of the semi-linear conditional log-normal equilibrium.