# Anticipated Financial Contagion *

Toni Ahnert[1], Gideon du Rand[2], and Co-Pierre Georg[3]

[1]Bank of Canada and CEPR
[2]University of Stellenbosch
[3]University of Cape Town and Deutsche Bundesbank

We revisit the seminal model of financial contagion of Allen and Gale (2000) by allowing aggregate liquidity shocks to occur with positive probability. We study how an ex-post shock's size and probability affect ex-ante portfolio choices and risk sharing across states and over time. We characterize a numerically approximate symmetric Nash equilibrium in the non-cooperative game between two regional banks. We describe parameter regions where contagion does and does not occur for positive probability of the aggregate liquidity shock. Our solution fully characterizes banks' ex-ante optimal portfolio choices. Additionally, we present novel benchmarks where optimal risk sharing with observable types involves (i) full default after a large but sufficiently unlikely aggregate liquidity shock; (ii) holding excess liquidity when the shock is relatively likely; (iii) partial liquidation of investment after a small and unlikely shock; and (iv) both excess liquidity and partial liquidation for shocks of intermediate size and probability. We obtain a number of additional benchmark results numerically to show the robustness of our numerical approach and highlight its usefulness for the literature.

**Keywords:** contagion, interbank lending, aggregate liquidity risk

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* Toni Ahnert, Bank of Canada, E-Mail: tahnert@bank-banque-canada.ca, Gideon du Rand, University of Stellenbosch, E-Mail: gideondurand@sun.ac.za, corresponding author; Co-Pierre Georg, University of Cape Town and Deutsche Bundesbank, E-Mail: co-pierre.georg@uct.ac.za. Du Rand is supported in this research by Thuthuka grant 94130 of the National Research Foundation of South Africa, and a research support grant of the South African Reserve Bank. The views expressed in this paper do not necessarily reflect the views of Deutsche Bundesbank.

# 1 Introduction

An aggregate liquidity shock was at the heart of the recent global financial crisis. Specifically, there was an aggregate shock to the market liquidity of a number of bonds linked to subprime mortgages (Brunnermeier, 2009; Gorton and Metrick, 2011). This shock was amplified and spread across the financial sector through the frequent use of these subprime-bonds as collateral in bilateral repo transactions—one of the major sources of liquidity of US investment banks prior to the crisis (Brunnermeier and Pedersen, 2009). Even without infrequent catastrophic events such as financial crises, aggregate liquidity varies across the business cycle, as reflected in the time-varying spreads between liquid and illiquid assets (see, for example, Eisfeldt and Rampini, 2006).

We contribute to the extant theoretical literature by studying the contagious spread of liquidity shocks across financial institutions, in particular, the branch following from the model of Diamond and Dybvig (1983). In this literature, a deposit taking bank offers an ex ante welfare improving deposit contract to consumers that face idiosyncratic liquidity demand risk. The contagion literature typically adds three components to this setting: (i) more financial institutions; (ii) either institution-level or aggregate risk; and (iii) risk-free demand-deposit contracts. The last is a standard feature in this literature, which models the distinction between deposit contracts offered by a bank and other types of financial instruments: a bank deposit contract offers risk free withdrawal returns in the absence of a bank failure. We revisit the seminal models of financial contagion of Freixas et al. (2000) and in particular of Allen and Gale (2000).[1]

A common feature of this literature is the *resilient-yet-fragile* property: balance sheet linkages provide a buffer against small shocks but contagiously spread larger shocks[2]. Therefore, the implications for financial fragility are ambiguous. Contagion ensues if the initial shock is large enough relative to the ex-ante optimal buffers held by banks. Therefore, the ex-ante choices of banks are paramount to understand the risk of contagion. The model of Allen and Gale (2000) considers an aggregate liquidity shock that occurs with zero probability, so contagion does not affect the ex-ante choices of banks and, thus, does not matter for ex-ante expected welfare.

Our model follows Allen and Gale (2000) as closely as possible, but we deviate by augmenting the model in one important aspect: we introduce aggregate liquidity shocks with *positive* probability. This allows us to study how the probability and size of aggregate liquidity risk

[1] Both models yield very similar results and were published at the same time. We focus on the model of Allen and Gale (2000) as benchmark model because it yields marginally easier analytical solutions. Our methodology, however, could just as well be applied to the equally appealing model of Freixas et al. (2000).

[2] See, for example Gai and Kapadia (2010), Elliott et al. (2014), Acemoglu et al. (2015) (and the extensive literature that follows) who study contagion in more realistic network settings.

affects ex-ante portfolio choices and risk sharing across states and over time.

The economy extends over three dates and there are investors with preferences exactly as in Diamond and Dybvig (1983): a proportion of them wishes to consume at the interim date and the remainder wishes to consume at the final date. The consumption preferences of investors are initially unknown, and a bank provides insurance against such idiosyncratic liquidity risk by offering a demand-deposit contract. The bank chooses its portfolio at the initial date, allocating its endowment between a short asset, a long asset and an interbank deposit. The short asset is simply a storage technology with zero net return in all periods. The long asset has a positive net return if held to maturity (until the final date) but a negative net return if liquidated early (at the interim date). The interbank deposit has an endogenous return depending on the equilibrium choices of the bank and the state of nature that realizes. There are two regions and, in the absence of an aggregate liquidity shock, some regional variation in liquidity demand, which motivates insurance across regions. When an aggregate liquidity shock occurs, by contrast, there is a higher aggregate proportion of investors that wish to consume at the interim date, which cannot be diversified away across regions.

The analysis of Allen and Gale (2000) shows that, in the absence of aggregate liquidity shocks, the first best allocation is deterministic (both early and late consumers face no consumption risk), and can be feasibly decentralized if banks hold interbank deposits in each other large enough to cover the necessary regional liquidity transfer to support the first best allocation. Next, they show that if an aggregate liquidity demand shock (which is large enough and unanticipated (i.e. probability zero)) hits one bank, it will be unable to offer an incentive compatible allocation to late consumers, hence face a run, default and fully liquidate. Lastly, dependent on the structure of the network of cross holdings of interbank deposits, this shock can cause the default in one region to spill over to cause default in the next region that did not experience the aggregate liquidity shock. This contagion occurs only when the network is sufficiently incomplete.

Our key contribution is the characterization of strategic interbank market determination and contagion in a situation where the probability of the aggregate liquidity shock is positive. This is novel in the literature: other studies of interbank market contagion either assume no interbank deposit market (e.g. Diamond and Rajan (2005) and Diamond and Rajan (2011) where banks are exposed to each other via information or fire sale asset price channels only), a perfectly competitive interbank market (e.g. Allen et al. (2009), although their assumed returns to the interbank position are quite different from the standard Allen and Gale (2000) setup), or study the feasibility of decentralization of the global optimum (Allen and Gale (2000) and Dasgupta (2004)) via interbank market deposit exchanges.

We study symmetric Nash equilibria in a non-cooperative game subject to regional and aggregate liquidity risk. There are two regional banks[3] that independently choose their asset portfolios (short asset, long asset and interbank deposit), as well as the deposit return on early withdrawals (which is subject to a state independence constraint unless default occurs). As a bank cannot observe the type of consumer, it must offer an incentive compatible contract in any state in which it wants to avoid a run. This problem is extremely difficult to study analytically, mainly because of the structure of the best response functions and in particular the various constraints faced by the banks. Hence, we numerically approximate the symmetric Nash equilibrium of the full strategic game between the two banks for the full range of the probability of an aggregate liquidity shock.

There are three possible types of symmetric equilibria that can arise: one in which neither bank ever defaults, one in which only a bank that experiences the aggregate liquidity shock defaults, or a situation where both banks default whenever either bank experiences the aggregate liquidity shock. The last situation is the definition of contagion in our model: As in Allen and Gale (2000), the aggregate liquidity shock is localized in one region. In the contagion case, the bank that experiences the aggregate shock does not hold enough resources to offer an incentive compatible contract that satisfies the state independence constraint on deposit returns. It thus experiences a run and defaults. The consequent return on the interbank deposit of the other bank (who did not experience the aggregate liquidity shock), is now so much lower that it also cannot offer an incentive compatible return on the deposit contract, and thus faces a run and defaults. The contagion case is thus equivalent to the mutual default equilibrium type in this paper.

Our numerical approximation of the symmetric Nash equilibrium is an iteratively stable fixed point of the implied symmetric best response function of one bank to the choice vector of the other bank. We construct the best response choice of a bank (to a given choice of the other bank) by direct numerical optimization of the constrained objective function that defines the ex ante symmetric independent and simultaneous strategic banking problem. The constrained objective function of a regional bank is the regional ex ante expected utility of an arbitrary consumer in that region, subject to a set of incentive compatibility and non-negativity constraints. For each equilibrium type and parameter set independently, we use standard constrained numerical optimization routines to solve for the approximate best response of a bank to a feasible initial choice of the counter-part bank. We then replace the choice of the counter-part bank with the found best response, and iterate until the best response sequence converges to a fixed point within numerical precision. This yields a candidate symmetric Nash equilibrium for each potential equilibrium type. Finally, we select the approximate symmetric Nash equilibrium type among the three candidate types as the one that obtains maximum expected utility.

[3] This is a simplified version of the model in Allen and Gale (2000) where there are four regions (and regional banks). The difference to our setup is immaterial for the results we obtain, but studying a two player game is computationally more feasible than studying a four player game.

Our numerical approach allows us to contribute several new results:

(i) We show that as the probability of aggregate risk approaches zero, the symmetric Nash equilibrium converges on the deterministic first best allocation, confirming the argument in Allen and Gale (2000).

(ii) We identify parameter sets where contagion never occurs for positive probability of the aggregate liquidity shock (specifically: when the return of the long asset at maturity is small relative to its early liquidation return).

(iii) We show that, if parameters are such that contagion can occur at positive probability of the aggregate liquidity shock, the symmetric Nash equilibrium can have three different characterizations: if the probability of the aggregate liquidity shock is small enough, the equilibrium is characterized by both banks defaulting if the aggregate shock realizes (i.e. contagion occurs). If the probability is intermediate, the equilibrium is characterized by only one bank (the one hit by the aggregate liquidity shock) defaults (i.e. default without contagion occurs). If the probability of the aggregate liquidity shock is high enough, the banks choose portfolios and contracts such that neither bank ever defaults.

(iv) We show that, if parameters are such that contagion can occur at positive probability of the aggregate liquidity shock, optimal choices may be discontinuous functions of the parameterization: At the parameter boundary where one equilibrium type switches to another, there is a discontinuity in the optimal choice, and within each region of a specific equilibrium type, the characterization (comparative statics) of the optimal choice is unique. This mirrors similar discontinuities and unique characterizations in the aggregate benchmark allocations when the solution switches from one type to another.

We also contribute to the literature by providing a fully analytic characterization of two aggregate benchmarks novel to the literature on liquidity risk, augmented by numerically obtained representations of these characterizations. These additional analytical results that can be numerically replicated are important tests for our numerical algorithm to ensure that we converge to known solutions wherever possible.

The key novelty in these benchmarks is that they embed the standard feature of models in this literature: to model deposit contracts that are risk free in the absence of a bank failure, the deposit return offered on early withdrawals in the model is assumed to be state independent in the absence of a run on the bank. This means that our aggregate benchmarks may also optimally choose an allocation that implies full early liquidation of all resources in states with high aggregate liquidity demand, as this is the only mechanism that transfers some consumption risk from late to early consumers.

We start by studying the optimal risk sharing arrangement when the types of investors are observed. Our convention is to call this the *social planner* allocation. This is our first aggregate benchmark. The assumption of state independent early withdrawal return on the deposit contract (in the absence of a run) means that the first-best is not attainable under aggregate (liquidity) risk in general[4].

We show that, for an aggregate liquidity shock that is *both* sufficiently large *and* sufficiently unlikely, the planner optimally chooses not to take precautions ex ante and just fully liquidates all assets in the event of such a shock, and distributes all liquidation returns proportionally. Effectively, this is the only method that allows the planner to avoid the constraint that a deposit contract must provide risk free deposit returns at the interim date, which can be quite costly ex ante. This result resonates with the insights of Allen and Gale (1998) who emphasize the value of bank default as a tool to manage aggregate solvency shocks.

In contrast, when the planner does not fully liquidate investment after an aggregate liquidity demand shock, it uses two tools to balance the marginal utility of early and late consumers: excess holdings of liquidity (short asset) in the absence of an aggregate liquidity shock (i.e. some of the short asset is used to finance late consumption), or partial liquidation of investment (long asset) after such a shock occurs (i.e. some of the long asset is used to finance early consumption). Their usage depend on the probability of an aggregate liquidity and the relative cost of foregone higher investment return to the cost of premature liquidation. For a likely aggregate liquidity shock, the planner only holds excess liquidity. For a small and unlikely shock, however, only partial liquidation is used. For shocks of intermediate size and probability, in turn, the planner uses both excess liquidity and partial liquidation. This is different from Allen and Gale (2000) where, ex ante, the short asset is used exclusively to finance early consumption, and the long asset is used exclusively to finance late consumption.

Our second aggregate benchmark is that of a *global bank* that operates in both regions. This is an entity that also aims to maximize per capita welfare in both regions simultaneously, like the planner, but cannot observe the types of investors. Hence, unlike the social planner, the global bank is subject to providing incentive compatible contracts: to avoid a run, the return to late consumers must be weakly larger than the state independent deposit return to early consumers. Our results show that due to the incentive compatibility constraints, a global bank relies more on excess liquidity and less on partial liquidation – both on the intensive margin (levels) and on the extensive margin (for a larger range of parameters). As a result, a global bank can provide less liquidity to investors (lower insurance against idiosyncratic liquidity risk).

The paper proceeds as follows: section 2 sets out the environment that defines the econ-

[4]This approach contrasts with Castiglionesi et al. (2015), a recent important contribution in the literature in which the early and late consumption levels of bank depositors are allowed to vary across aggregate states.

omy subject to liquidity risk. Section 3 provides our analytical results on the aggregate benchmarks in this economy. Section 4 sets out the strategic problem of independent regional banks, discusses the nature of the interbank market and defines the Nash equilibrium. Section 5 presents our numerical approach and results for both the aggregate benchmarks and our numerical approach and characterization of the numerically approximate symmetric Nash equilibrium of the strategic interbank problem. Section 6 concludes.

# 2 Environment

There are three dates $t = 0,1,2$ and a single good for consumption and investment. Two safe constant-returns-to scale technologies are universally available. Storage $y$ (short asset) yields a unit return at the subsequent date. Investment $x$ in a productive asset (long asset) at $t = 0$ yields a return of $R > 1$ at $t = 2$ and a liquidation value of $0 < r < 1$ at $t = 1$.

There are two regions, $k = A, B$, each inhabited by a unit mass of investors. Consumers have a unit endowment at $t = 0$ and are initially identical. At $t = 1$, they privately learn their consumption preference, an idiosyncratic liquidity shock, as in Diamond and Dybvig (1983). A fraction of investors $\nu_{ks} \in (0,1)$ in region $k$ and state $s$ values consumption at $t = 1$ ('early'), and the remainder values consumption at $t = 2$ ('late'):

$$U(c_{1ks}, c_{2ks}) = \begin{cases} u(c_{1ks}) & \text{w.p. } \nu_{ks} \\ u(c_{2ks}) & \text{w.p. } 1-\nu_{ks} \end{cases} \tag{1}$$

where $c_{tks}$ denotes consumption at date $t$ in region $k$ and state $s$. The period utility function $u(c)$ is twice continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$.

Table 1 shows the distribution of regional liquidity demand $\nu_{ks}$. There are four possible states $s \in \{1,2,3,4\}$, similar to Table 3 in Allen and Gale (2000). In states 1 and 2, there is a regional liquidity shock $\varepsilon > 0$ that is symmetric and perfectly negatively correlated across regions. This regional liquidity shock generates an insurance motive across regions, for example in the form of interbank deposits. In states 3 and 4, an aggregate liquidity shock hits one of the regions, resulting in possible contagion across regions if such links are formed. The size of the aggregate liquidity shock is $\alpha \in \left(0, \frac{1-\gamma}{2}\right)$ and it occurs with probability $p \in (0,1)$. The Allen and Gale model obtains as special case for $p \to 0$.

At $t = 1$, the state is publicly revealed and each investor privately learns her preference (early or late). Without loss of generality, our convention is that a planner observes the types of investors, while a bank does not. A late investor can therefore pretend to be an early investor,

Table 1: Distribution of regional liquidity demand $\nu_{ks}$.

| State $s$ | Probability $\pi_s$ | Region A | Region B |
|---|---|---|---|
| 1 | $\frac{1-p}{2}$ | $\nu_{A1} = \gamma - \varepsilon$ | $\nu_{B1} = \gamma + \varepsilon$ |
| 2 | $\frac{1-p}{2}$ | $\nu_{A2} = \gamma + \varepsilon$ | $\nu_{B2} = \gamma - \varepsilon$ |
| 3 | $\frac{p}{2}$ | $\nu_{A3} = \gamma$ | $\nu_{B3} = \gamma + 2\alpha$ |
| 4 | $\frac{p}{2}$ | $\nu_{A4} = \gamma + 2\alpha$ | $\nu_{B4} = \gamma$ |

so banks have to offer a contract that is incentive compatible, $c_{1ks} \leq c_{2ks}$. In section 3, we derive the allocations chosen by the planner and a global bank (one bank in the entire economy comprising both regions). We show that a global bank achieves an allocation inferior to the planner's when incentive compatibility is violated in the planner's allocation. As in Allen and Gale (2000), all allocations are not state contingent at the interim date, $c_{1ks} \equiv c_{1k}$, unless full liquidation of investment occurs in states 3 or 4.[5]

# 3 Benchmarks

We study two novel global benchmark allocations to contrast with our results on the decentralized interbank market in Section 4: a (constrained) social planner and a global bank allocation.[6] The social planner allocation is the best allocation that can be chosen by a planner that (a) can observe the type of consumer (early or late) and hence can offer non-incentive compatible contracts without allowing a bank run, but (b) differs from the first best in that the social planner is constrained to offer only non-state contingent early consumption in the absence of full liquidation. The constraint of non-state contingent early consumption is a fundamental feature of banking problems in the literature following Diamond and Dybvig (1983), employed to model deposit contracts that are risk free unless a bank fails. The global bank allocation differs from the social planner allocation in that a global bank cannot observe consumer types, and hence

[5] Similar to Allen and Gale (1998), full liquidation allows for greater aggregate risk sharing between late and early consumers. Allen and Gale (1998) consider investment risk, while liquidity risk is studied here.

[6] There are two additional benchmarks of interest: autarky and the first best. Autarky (See appendix B) considers the optimal choices of a consumer without access to a bank (which characterizes the lower bound of welfare that an individually rational deposit contract must provide). The characterization of the first best (which allows state contingent early consumption) is formally equivalent to the results in Castiglionesi et al. (2015).

must offer an incentive compatible contract to avoid a bank run.[7]

We present the analytical characterization of these benchmarks in this section, and a numerical implementation with concrete examples for these allocations is presented in Section 5.1.

## 3.1 Social planner

The planner observes the types of investors and maximizes their expected utility. Since the planner can freely allocate resources across regions, it will treat all investors identically: $c^*_{tks} = c^*_{ts}$. Therefore, the planner faces two (aggregate) states of nature: $s_H$, where the aggregate per capita liquidity demand is high $(\gamma + \alpha)$ with probability $p$, or $s_L$ where it is low $(\gamma)$ with probability $1 - p$. At $t = 0$, the planner chooses its portfolio and the consumption levels at each date, subject to the aggregate resource constraint at each date. The utility is strictly increasing, so all resource and budget constraints bind at the solution.

A key result is that the planner may wish to fully liquidate when aggregate liquidity demand is high: Full liquidation is the only tool that allows the planner to shift consumption risk from late to early investors. This is due to the defining constraint that consumption levels at $t = 1$ cannot be state-contingent, unless full liquidation occurs at $t = 1$. Full liquidation may thus be optimal since the constraint of non-state contingent consumption at $t = 1$, $c_{1H} = c_{1L} \equiv c_1$, is costly for a low probability of the aggregate liquidity shock (low $p$).

Therefore, we solve the planner's problem in two steps. First, we study the case in which the planner does not liquidate fully in state $s_H$ at $t = 1$. Second, we study the case in which full liquidation occurs. Let the value functions of these two problems be $V_{NFL}$ and $V_{FL}$ for no full liquidation (NFL) and full liquidation (FL), respectively. The complete problem of the social planner (P1) is thus characterized by the combined value function:

$$V = \max\{V_{NFL}, V_{FL}\}. \quad \text{(P1)}$$

### 3.1.1 No full liquidation after aggregate liquidity shock

Suppose the planner does not fully liquidate in state $s_H$. It is clearly optimal at $t = 0$ to hold sufficient liquidity to avoid certain early liquidation of the long asset at $t = 1$: since $r < 1$, it is more efficient to use only the short asset to provide for early consumption in the low liquidity

[7] The global bank allocation is formally equivalent to a situation where regional banks perfectly cooperate to maximize global per capita expected utility.

demand state. Likewise, it is suboptimal to hold more liquidity than required in the case of high aggregate liquidity demand (since $R > 1$). Taken together, optimality requires

$$\gamma c_1^* \leq y^* \leq (\gamma + \alpha) c_1^*. \tag{2}$$

Therefore, the planner's problem without full liquidation in state $s_H$ (P1a) reduces to:

$$V_{NFL} \equiv \max_{\{x,y,c_1,c_{2L},c_{2H}\}} (1-p)[\gamma u(c_1) + (1-\gamma)u(c_{2L})] + p[(\gamma+\alpha)u(c_1) + (1-\gamma-\alpha)u(c_{2H})] \tag{P1a}$$

$$\text{s. t.} \quad x + y = 1$$

$$c_{2L} = \frac{y - \gamma c_1 + Rx}{1-\gamma}$$

$$c_{2H} = \frac{R}{1-\gamma-\alpha}\left(x - \frac{(\gamma+\alpha)c_1 - y}{r}\right)$$

Instead of solving the problem P1a in terms of the portfolio choice $y$ and the interim-date consumption level $c_1$ (which affect welfare in both states), it is more convenient to solve it in terms of the levels of excess liquidity (which can occur only in state $s_L$), $e \equiv y - \gamma c_1 \geq 0$ and partial liquidation (which can occur only in state $s_H$), $\lambda \equiv \frac{(\gamma+\alpha)c_1 - y}{r} \in [0, x]$.

By separating the states with and without aggregate liquidity shock, we obtain strong monotonicity of the optimal choices in the probability of the aggregate liquidity shock $p$.

**Proposition 1.** *Suppose the planner does not fully liquidate in state $s_H$. The optimal risk sharing allocation with observable types is characterized by two unique thresholds of the probability of an aggregate liquidity shock $(\underline{p}, \overline{p})$ with $0 < \underline{p} < \overline{p} < 1$ that yield three cases:*

*(i) For a sufficiently probable aggregate liquidity shock, $p \geq \overline{p}$, no partial liquidation occurs, $\lambda^* = 0$. There exists a unique level of excess liquidity $e^* > 0$ that increases in the probability of the aggregate liquidity shock, $\frac{de^*}{dp} > 0$.*

*(ii) For a sufficiently improbable aggregate liquidity shock, $p \leq \underline{p}$, no excess liquidity is held, $e^* = 0$. There exists a unique level of partial liquidation $\lambda^* > 0$ that decreases in the probability of the aggregate liquidity shock, $\frac{d\lambda^*}{dp} < 0$.*

*(iii) For an intermediate level of the probability of the aggregate liquidity shock, $\underline{p} < p < \overline{p}$, there exists a unique interior solution in which both excess liquidity is held and partial liquidation occurs, $\lambda^* > 0$ and $e^* > 0$, with $\frac{de^*}{dp} > 0$ and $\frac{d\lambda^*}{dp} < 0$.*

**Proof.** See Appendix A.1. ■

With observable types and no full liquidation in state $s_H$, the planner uses two instruments to balance the marginal utility of investors across states, excess liquidity and partial liquidation. The relative attractiveness of each depends on parameters, especially the probability and size of the aggregate liquidity shock $(p, \alpha)$. It is sometimes optimal to use just one of these tools and sometimes to use both tools.

To obtain intuition for the monotonicity of optimal choices in the probability of the aggregate liquidity shock, we consider two cases. First, when the state $s_L$ is realized, any excess liquidity is inefficient ex post. More resources could have been invested, while whatever level of partial liquidation chosen for state $s_H$ does not affect ex post efficiency. Second, when state $s_H$ is realized, any partial liquidation is inefficient ex post, since more liquidity would have been desirable. Hence, an increase in the probability of state $s_H$ makes allowing for excess liquidity in state $s_L$ relatively more desirable than allowing for partial liquidation in state $s_H$.

The same argument about ex post efficiency separation does not apply to the choice of total liquidity $y$, which explains our focus on excess liquidity in state $s_L$ and partial liquidation in state $s_H$. When $p$ is low, no excess liquidity is held. As $p$ increases, it becomes optimal to transfer more resources to state $s_H$. Since $c_{2H}$ is the lowest consumption level when $p$ is near zero, it has the highest marginal utility and is therefore the targeted consumption level to be increased. Since $e^* = 0$ in this region, however, this happens by *reducing* total liquidity (hence increasing the long asset holding), which induces a more rapid increase in $c_{2H}$ than can be attained by only reducing partial liquidation (holding the long asset holding constant). As $p$ increases further, it is eventually optimal to use excess liquidity and it is possible to transfer resources to state $s_H$ by *increasing* total liquidity along with excess liquidity. As a result, the optimal choice of total liquidity is not monotonic.

### 3.1.2 Full liquidation after aggregate liquidity shock

Next, suppose the planner fully liquidates in state $s_H$. The implied consumption levels are $c_{1H} = c_{2H} \equiv c_H = y + rx$. It will never be optimal to use partial liquidation in state $s_L$, since certain liquidation can never be optimal ex ante. However, some excess liquidity $e \equiv y - \gamma c_{1L} \geq 0$ may be held, which implies a consumption level $c_{2L} = \frac{Rx+e}{1-\gamma}$.[8]

The planner's problem conditional on full liquidation in state $s_H$ (P1b) is:

$$V_{FL} \equiv \max_{\{x,y,c_{1L},c_{2L},c_H\}} (1-p)\Big[\gamma u(c_{1L}) + (1-\gamma)u(c_{2L})\Big] + pu(c_H) \tag{P1b}$$

s. t.

[8] In the case of full liquidation in state $s_H$, the consumption levels at date 1 can differ across states. Abusing notation somewhat, we label the consumption levels in $s_H$ as $c_{1H} = c_{2H} \equiv c_H$ and early consumption in state $s_L$ as $c_{1L}$ in this section, instead of $c_1$ as in the previous subsection.

$$x + y = 1$$
$$c_{1L} \leq \frac{y}{\gamma}$$
$$c_{2L} = \frac{Rx + y - \gamma c_{1L}}{1-\gamma}$$
$$c_H = y + rx$$

Using the budget constraints at $t = 0$ and $t = 2$, we obtain an unconstrained problem in the choice variables: short asset holding ($y$) and excess liquidity in state $s_L$ ($e$).

**Proposition 2.** *Suppose the planner fully liquidates in state $s_H$. The optimal risk sharing with observable types is characterized by two unique thresholds of the probability of an aggregate liquidity shock $(\underline{p}_{FL}, \overline{p}_{FL})$ with $0 < \underline{p}_{FL} < \overline{p}_{FL} < 1$ that yield three cases:*

*(i) For a sufficiently probable aggregate liquidity shock, $p \geq \overline{p}_{FL}$, all resources are kept in the short asset, $y^*_{FL} = 1 = c^*_{1L} = c^*_{2L} = c^*_H$ and $e^*_{FL} = 1 - \gamma$.*

*(ii) For a sufficiently improbable aggregate liquidity shock, $p \leq \underline{p}_{FL}$, the planner chooses holds no excess liquidity in state $s_L$, $e^*_{FL} = 0$, and there exists a unique interior solution $0 < y^*_{FL} < 1$ with $\frac{dy^*_{FL}}{dp} > 0$ and associated consumption levels $c^*_{2L} > c^*_{1L} > c^*_H$.*

*(iii) For an intermediate level of the probability of the aggregate liquidity shock, $\underline{p}_{FL} < p < \overline{p}_{FL}$, there exists a unique interior solution $0 < y^*_{FL} < 1$ with $\frac{dy^*_{FL}}{dp} > 0$, and some excess liquidity held in state $s_L$, $e^*_{FL} = y^*_{FL}(1 + (R-1)\gamma) - R\gamma$. The associated consumption levels are $c^*_{2L} = c^*_{1L} > c^*_H$.*

**Proof.** See Appendix A.2. ■

When the probability of the aggregate shock is low (state $s_L$ is the more probable), it is inefficient to allow for excess liquidity in state $s_L$, since it is costly from an ex-ante perspective. As the probability of state $s_H$ increases, the expected utility places more weight on state $s_H$, so $c^*_H$ , the consumption level in this state increases, which requires an increase in liquidity $y^*$. As liquidity increases, $c^*_{1L}$ increases and $c^*_{2L}$ falls, eventually reaching equality. Since it cannot be efficient to allow $c_{2L} < c_{1L}$ (because of strict risk aversion) the equality $c^*_{2L} = c^*_{1L}$ arises for a sufficiently likely aggregate liquidity shock. As $p$ increases further, excess liquidity is held in state $s_L$ to maintain this equality. Eventually, facing full liquidation in state $s_H$ becomes so probable that it is efficient not to invest in the long asset at all in order to avoid costly liquidation of investment.

#### 3.1.3 Complete problem

Finally, we describe the solution to the complete social planner problem that combines the previous two cases and describe the parameter set that yields optimal full liquidation in state $s_H$.

**Proposition 3.** *There exists a unique upper bound on the probability of an aggregate liquidity shock, $\check{p} \in [0,1)$, whereby the social planner fully liquidates in state $s_H$ if and only if $p \leq \check{p}$. Moreover, there are bounds $(\check{\alpha}, \check{r}, \check{\gamma})$ that define a set $\Theta \equiv \{\alpha > \check{\alpha} \cap \gamma > \check{\gamma} \cap r < \check{r}\}$ such that $\check{p} > 0$ for all levels of the investment return $R$ if and only if $(\alpha, r, \gamma) \in \Theta$.*

**Proof.** See Appendix A.3. ■

The intuition for the complete problem lies in the welfare cost of keeping the promised consumption non-state-contingent (so early withdrawals are risk free) without full liquidation, $c_{1H} = c_{1L} = c_1$. When the aggregate liquidity shock is unlikely, there is more weight on welfare in state $s_L$ and thus the average consumption levels in $s_L$ should be higher than in $s_H$. However, non-state contingency implies that $c^*_{1H}$ is also high, which can only be achieved with high partial liquidation in state $s_H$, resulting in low levels of $c^*_{2H}$. As a result, the allocation in state $s_H$ is highly inefficient ex post. With a positive weight on state $s_H$, this ex-post inefficiency is spread across both states, resulting in an ex-post inefficient allocation in state $s_L$ as well, which yields a wider-than-efficient spread between early and late consumption in $s_L$.

The planner has one tool to mitigate the constraint of non-state contingency: full liquidation with equal payouts to all investors. This tool is equivalent to forcing the planner to accept a large degree of ex-post inefficiency in state $s_H$ which, in turn, implies a more ex-post efficient allocation in state $s_L$ (characterized by a smaller spread). Since a low probability of an aggregate liquidity shock implies a high weight on $s_L$, the planner chooses full liquidation in $s_H$ – with associated reduction in ex-post inefficiency in $s_L$ relative to the no full liquidation allocation – for sufficiently low probabilities of the aggregate shock, $p < \check{p}$.

The region of the probability of the aggregate liquidity shock where full liquidation is optimal depends on other parameters of the problem in an intuitive way. For a small aggregate productivity shock (low $\alpha$) or a low cost of liquidation (large $r$), the non-state contingency of consumption levels at the interim date is not costly.[9] There is only a small additional early demand after a shock or early liquidation of investment is carries only a small penalty. As a result, the ex-post inefficiency without full liquidation allocation is small for any given $p$, thus there may be no positive $p > 0$ for which full liquidation is optimal, $\check{p} = 0$. Equivalently, $\check{p} > 0$ whenever the size of the aggregate liquidity $\alpha$ (and $\gamma$) are large enough and $r$ is small enough.

[9] The fraction of early investors in state $s_L$, $\gamma$ enters the problem analytically in a similar way to $\alpha$, hence the impact on the solution character of $\gamma$ is similar to that of $\alpha$, although the intuition is not as direct.

## 3.2 Global bank

A global bank cannot observe the types of investors at date $t = 1$, so it offers incentive-compatible contracts:

$$c_{2H} \geq c_1, \qquad c_{2L} \geq c_1. \tag{3}$$

Otherwise, the global bank is identical to the planner. Because of free entry, it maximizes the expected utility of investors. Its problem also has two parts: either there is full liquidation in state $s_H$, or there is not. The problem of the global bank (P3) is characterized by the solution to the combined, constrained value function:

$$\begin{aligned} V^{GB} &= \max\{V^{GB}_{NFL}, V^{GB}_{FL}\} \\ V^{GB}_{NFL} &\equiv \{V_{NFL} | \, c_{2L}, c_{2H} \geq c_1\} \\ V^{GB}_{FL} &\equiv \{V_{FL} | \, c_{2L} \geq c_{1L}\} \end{aligned} \tag{P3}$$

where $V_{NFL}$ is defined in problem P1a and $V_{FL}$ is defined in problem P1b.

Proposition 4 states ranges of parameters in which the planner's allocation is incentive compatible and thus equivalent to that of the global bank.

**Proposition 4.** *The allocation of the planner is incentive compatible in the following cases:*

*(i) if $p \leq \check{p}$, that is when the planner chooses to fully liquidate in state $s_H$;*

*(ii) if $\hat{p} \leq p < \check{p}$, where $\hat{p} \in [0, \frac{(R-1)r}{R-r})$ is unique.*

*(iii) if state $s_L$ is realized.*

**Proof.** See Appendix A.4. ■

Whenever the planner chooses to fully liquidate in state $s_H$ ($p \leq \check{p}$), the allocation is incentive compatible for two reasons. First, in $s_H$, full liquidation ensures incentive compatibility by definition, $c_{1H} = c_{2H} = y + rx$. Second, given $c_{1H} = c_{2H}$, it is never optimal for $c_{2L} < c_{1L}$ as $c_{2L} = c_{1L}$ would also be feasible and is strictly preferred due to risk aversion.

Suppose the planner chooses not to fully liquidate in state $s_H$ ($p > \check{p}$). If the probability of an aggregate liquidity shock is low enough, the average consumption levels in state $s_L$ are larger than in state $s_H$. Since the non-state contingency constraint applies without full liquidation, the consumption level $c_1$ in state $s_H$ has to be large relative to the average consumption in state $s_H$ if $p$ is low. This can only be achieved via high partial liquidation, resulting in a very low level of $c_{2H}$. Taken together, for a sufficiently low probability of the aggregate liquidity shock, $p < \hat{p}$, the planner's allocation violates incentive compatibility, $c^*_{2H} < c^*_1$. Lastly, we obtain $\hat{p} < \frac{(R-1)r}{R-r}$

since $c^*_{2H} > c^*_{2L}$ whenever $p > \frac{(R-1)r}{R-r}$, and the proof of proposition A.4 shows that there is never a violation in $s_L$.

As a result, there are parameter regions where $\hat{p} > \check{p}$, whereby a global bank's problem is distinct from the planner's problem. Proposition 5 characterizes the optimal choice of a global bank when the allocation chosen by the planner is not incentive compatible.

**Proposition 5.** *If $p \in (\check{p}, \hat{p})$, the planner's allocation is not incentive compatible. There are two cases that characterize the closest incentive-compatible allocation of the global bank, separated by a unique boundary $\underline{p}_{GB} \in (\check{p}, \hat{p})$:*

*(i) If $\check{p} < p \le \underline{p}_{GB}$, a global bank chooses a portfolio with partial liquidation only $(e^* = 0)$, and the optimal allocation is:*

$$\begin{aligned} y^* &= \frac{\gamma r R}{\gamma r R + \alpha R + r(1-\gamma-\alpha)} \\ c^*_1 = c^*_{2H} &= \frac{y^*}{\gamma} \\ c^*_{2L} &= \frac{R(1-y^*)}{1-\gamma}. \end{aligned}$$

*Moreover, $\underline{p}_{GB} < \underline{p}$, the bound below which the social planner uses only partial liquidation.*

*(ii) If $\underline{p}_{GB} < p < \hat{p}$, a global bank chooses a portfolio with excess liquidity and partial liquidation with optimal allocation $c_{2L}(y^*) > c_{2H}(y^*) = c_1(y^*)$ where $y^*$ is the unique solution to:*

$$u'(c_{2H}(y^*)) = \frac{(1-p)(r(R(1-\alpha-2\gamma)-(1-\alpha-\gamma)) + R(R(\alpha+\gamma)-\alpha))}{(1-r)R(\gamma+(1-\gamma)p)} u'(c_{2L}(y^*)).$$

**Proof.** See Appendix A.5. ■

Whenever the planner's allocation is not incentive compatible, a global bank optimally picks the closest incentive compatible allocation to that of the planner. Since a global bank must offer $c_{2H} = c_1$ when the planner offers $c_{2H} < c_1$ in state $s_H$, a global bank optimally uses less partial liquidation than the planner. For the same reasons as in the planner case, when $p$ is low, ex post inefficiency in $s_H$ is less costly than in $s_L$, so a global bank uses only partial liquidation in a neighborhood of $p = 0$. However, since the global bank (GB) always uses less partial liquidation than the planner (here denoted with a subscript $P$), as $p$ increases, it becomes efficient to start using some excess liquidity sooner for a global bank than for the planner:

$$\underline{p}_{GB} < \underline{p}, \qquad e^*_{GB} \ge e^*_P. \tag{4}$$

Since a global bank uses less partial liquidation in $s_H$ than a planner, it follows that $c^*_{1,GB} \leq c^*_{1,P}$ and $c^*_{2H,GB} \geq c^*_{2H,P}$. But $c^*_{1,GB} \leq c^*_{1,P}$ also implies that a global bank has a lower outlay for early consumption than the planner in $s_L$, so we conclude that $c^*_{2L,GB} \geq c^*_{2L,P}$. Thus, when their allocations differ, the global bank can provide less liquidity risk insurance to risk-averse investors than the social planner.

# 4 Strategic Regional Banks

In each region, there is a representative bank indexed by $A$ and $B$. Because of free entry within a region, each bank maximizes the expected utility of regional consumers subject to non-negative profits and, as a result, all consumers deposit their endowment at their regional bank in $t = 0$. Banks simultaneously choose the deposit return $d_A$, a portfolio of long and short assets $(x_A, y_A)$, and the amount of interbank deposits $z_A$, subject to the date 0 budget constraint $x_A + y_A + z_A = 1 + z_B$. At $t = 1$, state $s$ realizes, consumers learn their type, all early consumers withdraw, and banks liquidate assets to serve withdrawals. Banks must choose between using the short asset, withdrawing their interbank position (denoted $w_{As}$), and liquidating the long asset. All choices are non-cooperative and we study Nash equilibria, focusing on symmetric equilibria throughout.

Late consumers withdraw early (in period 1) if the implied contract in state $s$ is not incentive compatible ($c_{2As} < d_A$), in which case the bank defaults and fully liquidates. Otherwise, the remaining long asset matures in $t = 2$, the remaining interbank position is withdrawn and all resources are paid out to depositors who comprise late consumers and the other bank. Therefore, a run in state $s$ is avoided if the implied allocation is incentive compatible. In case of default, we assume pro-rata resolution with equal seniority of banks and investors. We also assume that the no-run equilibrium is played whenever multiple equilibria exist, so all bank runs are efficient as in Allen and Gale (1998) and (2000).

## 4.1 Interbank Market and Contracts

The interbank market consists of non-cooperative choices of *interbank deposits* at the other bank. Each bank treats the deposit received from regional consumers and the other bank identically, so that the interbank deposit contract and the consumer deposit contract have the same payoff structure and are thus treated identically by the bank. There is surprisingly little empirical guidance as to the exact nature of interbank loan contracts. We simplify by assuming that interbank lending is in the form of deposits, but given that about 90% of interbank loans have a maturity of only one day, they decision not to roll over an existing loan is effectively equivalent to holding demand deposits.[10] Our simplified interbank loan contract also provides analytical tractability and conforms with the setup in Allen and Gale (2000).

### 4.1.1 Optimal Pecking Order

As in Allen and Gale (2000), we maintain the assumption that a bank has a preferred sequence (*pecking order*) in which to liquidate assets to meet early demand: first the short asset, then the interbank deposit, then the long asset. This pecking order will be optimal for bank $A$ as long as the returns on the assets obey the following trade-offs in returns between period 1 and 2:

$$1 \leq \frac{c_{2Bs}}{c_{1Bs}} \leq \frac{R}{r} \tag{5}$$

where $c_{tBs}$ is the return to any depositor at bank B withdrawing in period $t$ in state $s$, thus $\frac{c_{2Bs}}{c_{1Bs}}$ is the inter-temporal trade off of the interbank deposit that bank $A$ made in bank $B$.

This is not a strong assumption: The first inequality follows directly from incentive compatibility of the contract offered by bank $B$; and the second inequality must be a feature of any deposit contract that provides valued ex ante liquidity demand insurance, since a consumer without access to a bank would also choose a consumption allocation that satisfies this inequality (see appendix B.1).

### 4.1.2 Indeterminacy in the interbank position in symmetric equilibria

The interbank position in any symmetric allocation does not affect the resource constraint. The regional resource constraints will bind in equilibrium: $x_j + y_j + z_j = 1 + z_k$. Combined with symmetry this implies that the interbank positions cancels out of this constraint. This induces an indeterminacy that requires an imposed resolution, to avoid e.g. infinite interbank positions, which are not observed empirically.

Consider any symmetric pair of decentralized choices that supports a no default equilibrium, with early withdrawal return $d^*$ and interbank position $z^*$. In such an allocation, any interbank position larger than this lower bound $z > z^*$ also supports the equilibrium: by construction, the "excessively large" interbank position is sufficient to allow each bank to service

[10] That the vast majority of interbank loans has a maturity of only one day seems to hold true across countries. For example, Arciero et al. (2016) study the euro area large value payment system Target2 and find that *"From June 2008, one-day transactions (overnight, tomorrow-next, spot-next) accounted for more than 90% of total transactions"*. This is well in line with Furfine (2003) who studies the US fed funds market and finds that *"[...] according to a Federal Reserve Bank of New York (1987) survey, overnight transactions account for 96% of the fed funds market"*.

early withdrawals of $d^*$ in period 1. The remainder of the interbank position will then be withdrawn by *both* banks in period 2 to be paid out to late consumers (and counter-part bank). However, by symmetry of the consumption allocations, these excessive positions will cancel out and not affect the size of the late withdrawal return to regional consumers.

We, therefore, restrict by assumption the possible interbank positions to be no greater than the larger of the regional or per capita aggregate shock: $z_{\max} = \max\{\varepsilon, \alpha\}$. This is motivated by fact that that this is the largest per capita transfer that the social planner or global bank would require to support an allocation without liquidation.

There are a number of possible economic motivations to bound the choice of the interbank position, even if arbitrarily large cross-holdings are allowed from an accounting perspective. First, linearity and perfect enforceability of contracts implies that banks can costlessly cross-lend any amount. Any real world situation which makes larger interbank holdings more costly than smaller ones will induce banks to choose the smallest necessary interbank position. Second, we abstract from banks holding equity to keep our analytic focus on the liquidity insurance role of banks. In the presence of equity holders, however, the interbank market serves purely as a method to effectively insure against regional and aggregate liquidity shocks, as it yields lower return than the long asset, which equity holders would prefer. Thus, equity holders would choose only as much interbank deposit so as to ensure the feasibility of the desired Nash equilibrium. Third, interbank loans carry a small, but often positive risk weight so that capital or value-at-risk constraints will bind eventually. Fourth, unmodelled liquidation costs increase in a bank's balance sheet size. This line of reasoning is in line with the empirical observation that banks net out excessive interbank positions of equal maturity.

#### 4.1.3 Optimal Interbank Withdrawal

The optimal pecking order of assets, combined with our resolution of the indeterminacy of the interbank positions in symmetric equilibrium, implies the following optimal withdrawal behaviour: a bank facing its lowest regional demand withdraws nothing of its interbank position at the other bank. This is in state 1 for bank $A$ and state 2 for bank $B$, so $w_{A1} = w_{B2} = 0$. A bank that faces its highest regional demand will fully withdraw (since we assume that it will only hold as much interbank deposits as is necessary to service its early liquidity demand needs). This happens in state 4 for bank $A$ and state 3 for bank $B$ so $w_{A4} = z_A$ and $w_{B3} = z_B$.

In intermediate states, we allow each bank to withdraw what is necessary to service its early liquidity demand, whether from consumer or counter-party bank. For bank $A$ in state 2 (symmetric for bank $B$ in state 1), required payout to regional consumers is $d_A(\gamma + \varepsilon)$ and there is no demand from bank $B$ ($w_{B2} = 0$); regional liquidity available to service this demand is $y_A$, so the shortfall is $(d_A(\gamma+\varepsilon)-y_A)$. The return on the deposit contract at bank $B$ is $d_B$ so the total required withdrawal (if positive) is: $\frac{d_A(\gamma+\varepsilon)-y_A}{d_B}$.

For bank $A$ in state 3 (symmetric for bank $B$ in state 4), required payout to regional consumers is $d_A\gamma$ and bank $B$ fully withdraws ($w_{B3} = z_B$), so total required payout is $d_A(\gamma + z_B)$. Regional liquidity available to service this demand is $y_A$, so the shortfall is $(d_A(\gamma+z_B)-y_A)$. The return on the deposit contract at bank $B$ is $d_B$ so the total required withdrawal (if positive) is: $\frac{d_A(\gamma+z_B)-y_A}{d_B}$.

We summarize the values of withdrawals in terms of choices and parameters of the model for bank $A$ in equation (6) (symmetric for bank $B$, except for the different states in which the regional liquidity demands are the same):

$$\begin{aligned} w_{As_1} &= 0 \\ w_{As_2} &= \frac{d_A(\gamma+\varepsilon)-y_A}{d_B} \\ w_{As_3} &= \frac{d_A(\gamma+z_B)-y_A}{d_B} \\ w_{As_4} &= z_A \end{aligned} \tag{6}$$

## 4.2 Consumption Levels

These assumptions allow us to characterize the consumption and welfare outcomes of any choice pair, which we use to characterize the strategic solution to the problem. Formally, let the vector of choice variables be denoted $\theta_k \equiv \{d_k, x_k, y_k, z_k,\}$. Each pair of choices $\{\theta_A, \theta_B\}$ implies a unique state dependent consumption allocation for the two regions $\{c_{1As}, c_{2As}, c_{1Bs}, c_{2Bs}\}_{s=1}^4$. We derive the consumption levels in arbitrary state $s$ at bank $A$ in two cases: first without default and then with default.

### 4.2.1 Case 1: No default

Suppose bank $A$ does not default. This requires that the implied consumption allocation is incentive compatible in all states so that no run occurs.

The total early liquidity demand facing bank $A$ is $d_A(v_{As} + w_{Bs})$, the sum of the measure of its regional early consumers and the measure of the withdrawal from bank $B$ times the promised rate of return. The available resources to cover early demand given regional liquidity ($y_A$), its own early interbank withdrawal, ($w_{As}$), and partial liquidation of the long asset ($\lambda_{As}$), is $y_A + d_B w_{As} + r\lambda_{As}$. Bank $A$ may have excess liquidity: $e_{As} = y_A + d_B w_{As} - d_A(v_{As} + w_{Bs})$

which will be paid out to late consumers and bank $B$ in $t = 2$, or have to partially liquidate $\lambda_{As} = \frac{d_A(v_{As}+w_{Bs})-(y_A+d_B w_{As})}{r}$ of its long asset to cover early demand, so that only $R(x_A - \lambda_{As})$ remain to be paid out to late consumers and bank $B$. Either excess liquidity or partial liquidation may occur in any given state, but not both. This yields the following general statement of the consumption allocation without default:

$$\begin{aligned} c_{1As}(\theta_A,\theta_B) &= d_A \\ c_{2As}(\theta_A,\theta_B) &= \frac{\overbrace{e_{As}}^{\text{residual short asset}} + \overbrace{(z_A - w_{As})}^{\text{residual interbank asset}} c_{2Bs}(\theta_A,\theta_B) + \overbrace{(x_A - \lambda_{As})}^{\text{residual long asset}} R}{\underbrace{(z_B - w_{Bs})}_{\text{residual interbank liability}} + \underbrace{(1-v_{As})}_{\text{regional late consumers}}}. \end{aligned}$$

### 4.2.2 Case 2: Default

Suppose that bank $A$ does default. This means that bank $A$ fully liquidates all its assets in $t = 1$. Its interbank deposit yields a total inflow of $c_{1Bs}(\theta_B,\theta_A)z_A$ (which is general to whether bank $B$ also defaults or not) and its long asset position yields $r x_A$. Thus the payout per unit of claim to all claimants is:[11]

$$c_{1As}(\theta_A,\theta_B) = c_{2As}(\theta_A,\theta_B) = \frac{y_A + r x_A + c_{1Bs}(\theta_B,\theta_A)z_A}{1+z_B}.$$

### 4.2.3 Consumption Identities

For any pair of choices $\{\theta_A,\theta_B\}$, the full set of all possible consumption allocations in arbitrary state $s$ are thus:

$$\begin{aligned} c_{1As}(\theta_A,\theta_B) &= \begin{cases} d_A & \text{if } A \text{ does not default} \\ \frac{y_A+r x_A+c_{1Bs}(\theta_B,\theta_A)z_A}{1+z_B} & \text{if } A \text{ defaults} \end{cases} \\ c_{2As}(\theta_A,\theta_B) &= \begin{cases} \frac{e_{As}+R(x_A-\lambda_{As})+c_{2Bs}(\theta_B,\theta_A)(z_A-w_{As})}{(z_B-w_{Bs})+1-v_{As}} & \text{if } A \text{ does not default} \\ \frac{y_A+r x_A+c_{1Bs}(\theta_B,\theta_A)z_A}{1+z_B} & \text{if } A \text{ defaults} \end{cases} \end{aligned} \quad (7)$$

These accounting identities are fully general to all possible outcomes (no default, single default or mutual default) for both bank $A$ and bank $B$ (where the identities for bank $B$ are symmetric to those above), for any promised early return, any asset portfolio and any pattern of early interbank withdrawal by the two banks.

[11] These values correspond to the *liquidation values* in Allen and Gale (2000).

## 4.3 Strategic Regional Bank Problem

We focus on a pair representative banks, one in each region. Due to free entry in each region, each bank maximizes the ex ante expected utility of consumers in its region subject to non-negative profits. A regional bank cannot observe the types of investors at date $t = 1$, so it offers incentive-compatible contracts ($c_{2ks}(\theta_A, \theta_B) \geq d_k$) in states where it wishes to avoid a bank run and default. A bank may choose to optimally default in one or more states, in which case all depositors receive their pro-rata share in the liquidation value of the bank, and hence the allocation is incentive compatible by construction. Thus the incentive compatibility constraint $c_{2ks}(\theta_A, \theta_B) \geq c_{1ks}(\theta_A, \theta_B)$ is general to all cases.

The strategic problem of bank $A$, (P4), is to choose regional deposit return, portfolio and withdrawal behavior $\theta_A$, taking as given $\theta_B$, to maximize ex ante expected utility of regional consumers:

$$
\begin{aligned}
V_A^{SB} \equiv \max_{\theta_A} \sum_{s=1}^{4} \pi_s \big[ \nu_{As} u(c_{1As}(\theta_A, \theta_B)) + (1-\nu_{As}) u(c_{2As}(\theta_A, \theta_B)) \big] \quad & \text{(P4)} \\
\text{s.t.} \quad & x_A + y_A + z_A = 1 + z_B \\
& e_{As}, \lambda_{As} \geq 0, \\
& c_{2As}(\theta_A, \theta_B) \geq c_{1As}(\theta_A, \theta_B) \\
& \text{equations (6) and (7).}
\end{aligned}
$$

## 4.4 Nash Equilibrium

Since the environment, choices and outcomes of banks $A$ and $B$ are fully symmetric ex ante, so is problem P4. This problem, solved simultaneously and independently by each bank choosing $\theta$, taking as given the choice of the other bank, defines a game $\mathscr{G}$ with a pair of symmetric best response functions that maps a compact subspace $U \subset \mathbb{R}_+^3$ (the set of feasible choices) onto itself for each bank: $\theta^{br} : U \to U$.

**Definition 1.** *A Nash equilibrium of $\mathscr{G}$ is a pair of choices $\{\theta_A^*, \theta_B^*\}$ that are best responses to each other: i.e. $\theta_A^*$ maximizes ex ante expected utility in region A conditional on bank B choosing $\theta_B^*$; and $\theta_B^*$ maximizes ex ante expected utility in region B conditional on bank A choosing $\theta_A^*$. In terms of the best response functions, a Nash equilibrium can be stated as:*

$$
\begin{aligned}
\theta_A^* &= \theta^{br}(\theta_B^*), \textit{ and} \\
\theta_B^* &= \theta^{br}(\theta_A^*)
\end{aligned}
$$

**Definition 2.** *A symmetric Nash equilibrium of $\mathcal{G}$ is a single choice $\theta^*$ that is a best response to itself: i.e. $\theta^*$ maximizes ex ante expected utility in region $A$ conditional on bank $B$ choosing $\theta^*$ and vice versa. Equivalently, $\theta^*$ is a fixed point of the symmetric best response function:*

$$\theta^* = \theta^{br}(\theta^*)$$

While it is possible that asymmetric equilibria might exist in this setting, we focus on symmetric equilibria as the most natural solution object of interest, given the ex ante symmetry of the problem and the symmetry of the global benchmarks.

As in the planner and global bank solutions, there are discrete equilibrium types that may occur. In the decentralized case, there are three possible types of ex ante symmetric equilibrium allocations: *no default* - where neither bank defaults in any state; *single default* - where only one bank defaults when the aggregate liquidity shock realizes in its region; and *mutual default* - where both banks default whenever the aggregate liquidity shock realizes in either of the regions.

# 5 Numerical Implementation

In this section we discuss the need for and results from numerical implementations of the various allocations studied in this paper. While the benchmarks in this paper can be fully characterized analytically, intuition on the many corner solutions is easiest to build with a set of fully solved parameterized examples. We present our approach and results on the benchmarks in subsection 5.1. The strategic regional bank problem cannot be feasibly characterized analytically, and thus requires a fully numerical analysis. In subsection 5.2 we motivate the need for a numerical approach and present our solution algorithm and main results.

## 5.1 Numerical Implementation of Benchmarks

The problem of a consumer without access to a bank can be solved in closed form for any constant relative or absolute risk aversion utility function (see appendix B.1). The other benchmark problems, however, do not often have closed form solutions (due to additive components in utility functions and three part marginal utilities in optimality conditions - see appendix A).

All the benchmark problems are simple constrained maximization problems, however, so are readily solved by standard constrained numerical optimization routines directly maximizing the expected utility function.

**Definition 3.** *A numerical solution to a benchmark problem is the result of a direct numerical optimization of the appropriate expected utility function subject to constraints on the choice space*

### 5.1.1 The social planner allocation revisited

In this section we present the results of our numerical solutions to the social planner problem that illuminate the intuition behind the propositions that characterize the nature of the solution analytically.

Figure 1 illustrates the results in Proposition 1. If the social planner does not choose to fully liquidate, there are three regions where the solution has different characterizations. Only excess liquidity in state $s_L$ (when the probability of the aggregate liquidity shock is high enough: $p \geq \bar{p}$), only partial liquidation in state $s_H$ (when the probability of the aggregate liquidity shock is low enough: $p \leq \underline{p}$), or both excess liquidity in state $s_L$ and partial liquidation in state $s_H$ (when the probability of the aggregate liquidity shock is intermediate: $\underline{p} \leq p \leq \bar{p}$).The figure additionally shows that the region in $p$ where interior solutions obtain is increasing in $\alpha$.

Figure 2 shows parameter regions in the probability ($p$) and size ($\alpha$) of the aggregate liquidity in which the social planner chooses either full liquidation or no full liquidation. The boundary between these regions, $\check{p}(\alpha)$, is characterized in Proposition 3. From the figure it is clear that for full liquidation to be optimal, the aggregate liquidity shock needs to be large and improbable enough. If the shock is too small, there is no probability at which the social planner chooses to fully liquidate.

### 5.1.2 The global bank allocation revisited

Figure 3 illustrates the results in Proposition 4: If the social planner does not choose to fully liquidate, there are three distinct orderings to the consumption levels across states. The payout in $s_H$ will not be incentive compatible $c_{2H} < c_1$ if $p < \hat{p}(\alpha)$. Otherwise, pay-out in both states is incentive compatible with $c_{2L} > c_{2H} > c_1$ if $\hat{p}(\alpha) < p < \frac{(R-1)r}{R-r}$, or $c_{2H} > c_{2L} > c_1$ if $p > \frac{(R-1)r}{R-r}$. The boundaries between these regions are characterized in Proposition 4.

## 5.2 Numerical Characterization of Regional Bank Problem

### 5.2.1 Motivating the need for a Numerical Characterization

Solving the strategic game between two regional banks requires a numerical approach as a fully analytical solution is not feasible. To see why this is the case, consider the impact of the various—simultaneously binding—constraints in the strategic problem.

#### Non-negativity Constraints

In the social planner allocation without full liquidation, with two possible states, we show that the non-negativity constraints on excess liquidity (possible only in state $s_L$) and partial liquidation (possible only in state $s_H$) induce a three-part solution (one interior and two corner solutions). This is analytically still manageable.

In the four-state strategic problem, however, there are many more possible outcomes. In any state, as in the aggregate benchmarks, a bank may optimally choose to fund late consumption out of excess liquidity (if the realized liquidity demand is low relative to its ex ante expectation), or to fund early consumption by partially liquidating the long asset (if the realized liquidity demand is high relative to its expectation). Consider bank $A$ in a no default allocation: it may choose to hold excess liquidity ($e_{As}$) in no state, only state 1, states 1 and 3, or states 1, 2 and 3. Similarly there may be partial liquidation ($\lambda_{As}$) in no state, in state 4, states 4 and 2 or states 2, 3 and 4. It will never be optimal to have excess liquidity in the state where the largest liquidity demand occurs, nor to fund the lowest liquidity demand with partial liquidation. There are thus 6 relevant non-negativity constraints:

$$\{e_{As} \geq 0\}_{s=1}^{3}, \{\lambda_{As} \geq 0\}_{s=2}^{3} \tag{8}$$

First, this motivates our preference for a numerical approach: These non-negativity constraints may or may not bind in a number of patterns, which induce potentially 6 different characterizations of an optimal choice of bank $A$, given the choice of bank $B$. The full set of possible outcomes across two banks in the no default allocation will thus have $6^2$ potentially different characterizations. Adding the single and mutual default allocations (with similarly many solution types) renders a concise fully analytic characterization infeasible.

Second, it motivates our choice to cast the numerical version of the problem into the three choice variables liquidity ($y$), interbank deposit ($z$), and deposit return ($d$), rather than in terms of excess liquidity and partial liquidation, and impose these non-negativity constraints as numerical constraints to our problem.

**Incentive Compatibility Constraints**

Our global bank problem is equivalent to the social planner allocation augmented with an incentive compatibility constraint. Above, we show analytically and numerically that this induces further regions in which the character of the solution is unique. Similarly, there will be parameter regions where distinct solution types exist in the decentralized problem: in any state where no default occurs, the optimal allocation may be at a point where the incentive compatibility constraint for that state ($c_{2As} \geq d_A$) binds or not.

This means the problem has up to four potentially binding incentive compatibility constraints for a specific bank $A$:

$$\{c_{2As} \geq d_A\}_{s=1}^{4} \tag{9}$$

Moreover, the incentive compatible set of choices of bank $A$ depends on the choice of bank $B$ being conditioned on. Figure 4 illustrates how the region of incentive compatibility choices for bank $A$ varies by the choice of bank $B$. In the top row, we fix the choice of bank $B$ at the first best in this parameterization. The columns show the set of liquidity and interbank deposit choices that yield incentive compatible consumption allocations given different levels of the chosen deposit return. In the bottom row, we reduce only the interbank holding of bank $B$ to zero, to show how the regions of incentive compatible choices available to bank $A$ change. We note that the incentive compatible region in two variables can be a convex set, a non-convex set or disjoint sets. This will further multiply the number of distinct types of characterization required by a fully analytic approach: optimal allocations where the incentive compatibility constraint binds will have a different characterization to those where it is slack. A numerical approach can account for this simply by directly imposing them as numerical constraints on the allowable search space, thus greatly improving the tractability of the problem.

### 5.2.2 Numerical Approximation of Symmetric Nash Equilibrium

Our solution concept of analysis is a symmetric Nash Equilibrium of the strategic simultaneous move game $\mathcal{G}$ defined in section (4.4). This corresponds to a fixed point of the symmetric best response function that defines the solution to either bank's problem.

We present results from a numerical approximation of this solution concept. For this, fix a search space $\Theta$ for the choice variables in $\theta$:

$$\Theta \equiv \{y, z, d | y \in [0,1], z \in [0, \max\{\alpha, \varepsilon\}] \text{ and } d \in [r, R]\}$$

**Definition 4.** *A numerically approximate symmetric Nash Equilibrium of game $\mathcal{G}$ is an iteratively stable, fixed point $\theta^* \in \Theta$ of the numerical best response function $\hat{\theta}^{br}(\theta)$*

$$\theta^* \quad = \quad \hat{\theta}^{br}(\theta^*)$$

*were the iterations $\theta_{i+1} = \hat{\theta}^{br}(\theta_i)$, are computed by direct numerical optimization of the constrained expected utility function over choice variables.*

$$\begin{aligned}
\hat{\theta}^{br}(\theta_i) \quad &\equiv \quad \underset{\theta_{i+1} \in \Theta}{\arg\max} \sum_{s=1}^{4} \pi_s \left[ \nu_{As} u(c_{1As}(\theta_{i+1}, \theta_i)) + (1 - \nu_{As}) u(c_{2As}(\theta_{i+1}, \theta_i)) \right] \\
& \qquad \textit{s.t.} \qquad x_A + y_A + z_A = 1 + z_B \\
& \qquad \qquad \quad\;\; e_{As}(\theta_{i+1}, \theta_i), \lambda_{As}(\theta_{i+1}, \theta_i) \geq 0, \\
& \qquad \qquad \quad\;\; c_{2As}(\theta_{i+1}, \theta_i) \geq c_{1As}(\theta_{i+1}, \theta_i), \\
& \qquad \qquad \quad\;\; \textit{and equations (6) and (7).}
\end{aligned}$$

Consider the global welfare maximization problem: by our results on the global bank, this allocation is necessarily symmetric. If we are to establish whether or not this is a Nash equilibrium of the strategic problem, we need to show that is individually optimal for each bank against all possible individual deviations, which immediately implies asymmetry is necessary in the available deviation space. The same applies to any symmetric Nash equilibrium we propose.

A fully agnostic search over the three choice variables is, however, numerically problematic due to the indeterminacy in the interbank deposit (see the discussion in Section 4.1.2).

We therefore restrict the generality of the search algorithm in one dimension: We impose that the interbank deposits that the two banks choose is symmetric (although we investigate the full range of possible values of this symmetric interbank deposit). The next section fully sets out how this restriction is implemented. Deviations in the other two choices—liquidity and deposit return promised—is left completely unrestricted. Thus, the symmetric equilibria we find are robust to asymmetric deviations in these two dimensions.

We introduce the following notation to account for our restriction: consider an arbitrary level of interbank deposit $z_s$. The remaining free choices for each bank are liquidity ($y$) and deposit return ($d$), thus the remaining search space is:

$$\tilde{\Theta}(z_s) \equiv \{y, z, d | y \in [0,1], z = z_s \text{ and } d \in [r, R]\}$$

### 5.2.3 Search Algorithm

The search algorithm is an iteration over the numerically constructed best response function (with symmetric interbank deposits). In each iteration, the best response of bank $A$ to the choice of bank $B$ is obtained via a global search method with multiple starting points.[12]

There is no guarantee that this iteration will converge to a fixed point, and we observe three patterns, which we use to define exit criteria for our algorithm. First, we call an iteration *converged* if the distance between choices in each iteration eventually falls below a tolerance level ($\delta^{crit} = 10^{-5}$). Second, we also observe iterations that fell into a *cycle*: i.e. the iterations converged to a single pair of distinct choices $\{\theta_{(1)}, \theta_{(2)}\}$ where $\theta_{(1)}$ is an iteratively stable best response to $\theta_{(2)}$ and vice versa. And third, we observe iterations that do not converge to any iteratively stable choice within $i_{max} = 2^6$ iterations. We label these *not converged*.

To simplify notation, denote the constrained ex ante expected utility to a consumer that deposits her unit endowment at bank $A$ from an arbitrary pair of regional choices $\theta_A, \theta_B \in \Theta$ by:

$$\begin{aligned} \mathbb{E}[u(\theta_A, \theta_B)] \quad &\equiv \quad \sum_{s=1}^{4} \pi_s \big[ \nu_{As} u(c_{1As}(\theta_A, \theta_B)) + (1 - \nu_{As}) u(c_{2As}(\theta_A, \theta_B)) \big] \\ &\text{s.t.} \quad x_A + y_A + z_A = 1 + z_B \\ &\qquad\quad e_{As}(\theta_A, \theta_B), \lambda_{As}(\theta_A, \theta_B) \geq 0, \\ &\qquad\quad c_{2As}(\theta_A, \theta_B) \geq c_{1As}(\theta_A, \theta_B), \\ &\qquad\quad \text{and equations (6) and (7).} \end{aligned}$$

In numerical implementation, we set $\mathbb{E}[u(\theta_A | \theta_B)]$ to a large negative value if any constraint is violated, thus encoding the constraints of the problem directly into the objective function.

The algorithm proceeds in the following steps, independently executed for each allocation type (Figure 5 schematically presents steps 1 through 5 of the algorithm):

Step 1: Partition the search space of the interbank deposit into $n_z$ points: $\{z_1, \dots, z_s, \dots, z_{n_z}\}$.

[12] Computations were performed using the constrained multivariate optimization methods implemented in Matlab (2017) on the Stellenbosch University's High Performance Cluster 1 (Rhasatsha): http://www.sun.ac.za/hpc

Step 2: For each point in the partition, $z_s$, set both banks' choice of interbank deposit to this value.

Step 3: Initialize the choice of bank $B$ at a feasible symmetric and incentive compatible choice: $\theta_0|z_s = [y_0, z_s, d_0]$.

Step 4: Find the best response of bank $A$, $\theta_{i+1}|z_s = [y_{i+1}, z_s, d_{i+1}]$, to the choice of bank $B$, $\theta_i|z_s = [y_i, z_s, d_i]$:

$$\theta_{i+1}|z_s \equiv \underset{\theta\in\tilde{\Theta}(z_s)}{\arg\max}\mathbb{E}[u(\theta,\theta_i|z_s)].$$

Step 5 : Replace the choice of bank $B$ ($\theta_i|z_s$) with the found best response ($\theta_{i+1}|z_s$) and iterate step 4 until one of the exit criteria is met:

1. Convergence to symmetric equilibrium: if the $sup$-norm of the change from one choice to the next in the iteration (denoted $\delta$ in Figure 5) falls below the convergence tolerance $\delta_{crit}$. This is the ideal case as it suggests an iteratively stable numerical fixed point to the best response function which corresponds to a pure strategy Nash equilibrium: $\{\theta^*_A, \theta^*_B|z_s\} = \{\theta_{i+1}, \theta_{i+1}|z_s\}$
2. Convergence to a cycle between two choices: For some parameter values, the algorithm converged to a cycle across a pair of choices $\theta_{(1)}, \theta_{(2)}$. Within the convergence criterion, if the choice of bank $B$ is $\theta_{(2)}$, the best response of bank $A$ is $\theta_{(1)}$; and if the choice of bank $B$ is $\theta_{(1)}$, the best response of bank $A$ is $\theta_{(2)}$. The convergence criterion for this situation is when the $sup$-norm of the two apparently alternating pair of choices falls below the convergence tolerance $\delta_{crit}$ (see Figure 5 for details). We treat these pairs of choice vectors as the two elements of a mixed strategy equilibrium with equal probability on each. Note, though, that in practice the convergence tolerance is extremely small, so that the difference between the bank choices could stem from finite numerical precision. We treat this case separately to err on the conservative side. We denote the result of this exit criterion as: $\{\theta^*_A, \theta^*_B|z_s\} = \{\theta_{i+1}, \theta_i|z_s\}$
3. Exceeding the maximum number of iterations (denoted $i_{max}$ in Figure 5): For some parameter values, the iteration neither converged to a single value nor a cycle. In this situation, which we label "not converged", we select the closest pair of choices in the iterations as a notional candidate allocation. In many cases the norm of the difference between closest pair are not far above the convergence criterion, but we maintain this label as the iterations often diverge from such close pairs. Nevertheless, for expected utility comparisons, we still treat these pairs of choice vectors as the two elements of a mixed strategy equilibrium with equal probability on each. We denote the result of this exit criterion as: $\{\theta^*_A, \theta^*_B|z_s\} = \underset{\{\theta_{i+1},\theta_i|z_s\}}{\arg\min}\, \delta_{i+1}$

Step 6: Selection of equilibrium interbank deposit: step 5 yields a sequence of candidate equilibrium choices (or pairs of choices) of liquidity and deposit return promise for every point in the interbank deposit partition ($\{\theta_A^*, \theta_B^* | z_s\}_{s=1}^{n_z}$). We select the final Nash equilibrium as the choice (or pair of choices),$\{\theta_A^*, \theta_B^*\}$, that maximizes expected utility across the partition of symmetric interbank positions:

$$\{\theta_A^*, \theta_B^*\} = \underset{\{z_s\}_{s=1}^{nz}}{\arg\max} \mathbb{E}[u(\theta_A, \theta_B | z_s)]$$

Finally, the algorithm yields a candidate Nash equilibrium for each equilibrium type and we select the ultimate Nash equilibrium allocation as the one that maximizes expected utility across equilibrium types.

Table 2 summarizes the parameters that control our optimization routines.

Table 2: Optimization control parameters

| Parameter Name | Value |
|---|---|
| Convergence Tolerance ($\delta_{crit}$) | $10^{-4}$ |
| Maximum Iterations ($i_{max}$) | 64 |
| Number of Starting Points for Global Search | 2500 |

## 5.3 Results

In this section we present our main results for the case of regional banks. The parameter we are most interested in is the probability ($p$) of the aggregate liquidity shock, so we show most results as we vary the shock probability from 0 (the Allen and Gale (2000) case) to 1. Most of the interesting dynamics happens for positive but small shock probabilities, so in our simulations we put particular emphasis on this parameter range. Table 3 summarizes all model parameters. We also have to specify a utility function and throughout this section we will use log utility $u(c) = \ln(c)$.

Table 3: Model parameters

| Parameter Name | Symbol | Value |
|---|---|---|
| Utility Function | $u(c)$ | $\ln(c)$ |
| Long asset return at maturity | $R$ | $\{5, 1.1\}$ |
| Long asset return at early liquidation | $r$ | $\{0.1, 0.9\}$ |
| Average liquidity demand | $\gamma$ | 0.5 |
| Regional liquidity demand shock | $\varepsilon$ | 0.1 |
| Aggregate liquidity demand shock size | $\alpha$ | 0.15 |
| Aggregate liquidity demand shock probability | $p$ | $\in [0, 1]$ |

Our first result characterizes how the type of equilibrium (no default, single default or mutual default) depends on parameters. We consider the type of equilibrium across the full range of probability of the aggregate liquidity shock $p \in [0, 1]$ in two cases: When the relative return on the long asset held to maturity ($R$) as opposed to when liquidated early ($r$) is large $\frac{R}{r} = \frac{5}{0.1}$ (figure 6) and small $\frac{R}{r} = \frac{1.1}{0.9}$ (figure 7).

This dimension yields interesting results as it is a measure of the relative cost of using the tools of excess liquidity (i.e. using the short asset with return 1 to fund late consumption rather than the long asset with return $R > 1$) or (partial) liquidation (i.e. using the long asset with return $r < 1$ to fund early consumption rather than the short asset with return 1). The smaller the ratio of returns $\frac{R}{r}$, the lower the efficiency cost of using either excess liquidity or partial liquidation. This means that the no default equilibrium is more likely to be superior, as it always uses some combination of excess liquidity and/or partial liquidation in some state to ensure early withdrawals that are state independent. As the ratio of returns $\frac{R}{r}$ increases, the cost of maintaining state independent early consumption with excess liquidity and partial liquidation grows. Thus we would expect parameters where switching to the single or mutual default allocations is more efficient.

First: for the case where the relative return on the long asset held to maturity ($R$) as opposed to when liquidated early ($r$) is large $\frac{R}{r} = \frac{5}{0.1}$, Figure 6 shows the expected utility obtained for our numerical approximation of the symmetric Nash equilibrium for each type of equilibrium (no default, single default or mutual default). We note that each equilibrium type is superior for some $p$: When $p$ is low enough, the mutual default equilibrium, when $p$ is intermediate, the single default equilibrium, and the no default equilibrium for $p$ large enough.

Second: For the case where the relative return on the long asset held to maturity ($R$) as opposed to when liquidated early ($r$) is small $\frac{R}{r} = \frac{5}{0.1}$, Figure 6 shows that no default equilibrium is (weakly) superior in the entire probability range (withing numerical precision, it is strictly superior $\forall\, p > 0$ and equal to the mutual default equilibrium at $p = 0$).

We summarize the results from these two parameter cases in the following result:

**Result 1.** *Regions of different types of Nash equilibrium:*

1. *There exists parameter regions (e.g. when $\frac{R}{r}$ is sufficiently small) where the no default equilibrium is weakly superior $\forall\, p \in [0,1]$. I.e. the mutual default equilibrium (contagion) occurs can only occur with probability* $0$.

2. *There exists parameter regions (e.g. when $\frac{R}{r}$ is sufficiently large) where the equilibrium type depends on the the probability of the aggregate liquidity shock.*

    *(i) When the probability of the aggregate liquidity shock is low enough, the mutual default equilibrium is superior to the single or no-default equilibrium.*

    *(ii) When the probability of the aggregate liquidity shock is intermediate, the single default equilibrium is superior to the mutual or no default equilibrium.*

    *(iii) When the probability of the aggregate liquidity shock is high enough, the no default equilibrium is superior to the single or mutual default equilibrium.*

The mutual default equilibrium is also called the contagion case because the decision of late depositors in one region to run on their bank, induces the late depositors in the other region to also run on their bank. We show that contagion only occurs for low enough probability of the aggregate liquidity shock. For higher shock probability, in the single default allocation, banks internalize the aggregate liquidity risk to the extent that a run by late depositors of one bank no longer induces a run by late depositors of the other. For high probability of the aggregate liquidity shock, banks fully internalize the ex-post risk of a shock ex-ante and the equilibrium choice is such that default never occurs in equilibrium.

From here on, we focus on the parameter set where there are multiple equilibrium types across $p$ (i.e. for large $\frac{R}{r}$), as the more interesting case.

Because of its importance as a benchmark, it is worthwhile to study the Allen and Gale (2000) benchmark of $p = 0$ in more detail. This leads to our second result.

**Result 2.** *We obtain the allocation of Allen and Gale (2000) in our numerical approximation of a symmetric Nash equilibrium when the probability of the aggregate liquidity shock is* 0*. Furthermore,*

*(i) Figure 6 shows that, at $p = 0$, the mutual default equilibrium is superior and obtains the expected utility of the Allen and Gale allocation within numerical precision.*

*(ii) Figure 8 shows that, at $p = 0$, the mutual default equilibrium is superior and obtains the deposit return of the Allen and Gale allocation $d = 1$ with numerical precision.*

*(iii) Figure 9 shows that, at $p = 0$, the mutual default equilibrium is superior and obtains the liquidity of the Allen and Gale allocation $y = \gamma$ with numerical precision.*

*(iv) Figure 10 shows that, at $p = 0$, the mutual default equilibrium is superior and the symmetric equilibrium interbank deposit is larger than $\varepsilon$.*

With regards to point $(iv)$, the Allen and Gale (2000) allocation predicts that the equilibrium interbank deposit should be equal to $\varepsilon$. However, as outlined in the discussion on the indeterminacy in the interbank deposit (see Section 4.1.2 above), a larger-than-necessary interbank deposit is sufficient to support the Allen and Gale (2000) allocation.

Figure 8 shows the deposit return ($d$) across the probability of the aggregate liquidity shock ($p$) at our numerical approximation of the symmetric Nash equilibrium for each type of equilibrium, as well as overall equilibrium deposit return, taking into account which type of equilibrium is superior. This figure can be summarized in the following result.

**Result 3.** *The symmetric Nash equilibrium deposit return ($d^*$) is discontinuous across equilibrium types and can be non-linear and non-monotonic.*

*(i) When the probability of the aggregate liquidity shock is low enough, the mutual default equilibrium is superior, and $d^* = 1$ for a positive interval in $p$. As $p$ increases, $d^*$ eventually increases until the equilibrium type switches to single default.*

*(ii) When the probability of the aggregate liquidity shock is intermediate, the single default equilibrium is superior, and $d^*$ decreases discontinuously at the probability boundary between equilibrium types. As $p$ increases, eventually the equilibrium type switches to no default.*

*(iii) When the probability of the aggregate liquidity shock is high enough, the no-default equilibrium is superior, and $d^*$ decreases discontinuously at the probability boundary between equilibrium types. As $p$ increases, $d^*$ first falls and then increases until $p = 1$.*

The discontinuities in the optimal deposit return (as in the optimal liquidity and interbank position discussed below) obtain when the equilibrium switches from one type of dominant equilibrium (e.g. mutual default) to another (e.g. single default). These discontinuities are features of our solution rather than remnants of the numerical approximation. First, Figures 8,9 and 10 show that the symmetric equilibrium choices of an arbitrary bank are defined for each equilibrium type across the probability range: for each equilibrium type, any specific choice has a distinct, continuous (up to numerical precision) character, and, while some intersect, these intersections are never occur at parameter point where a switch between equilibrium. Second, the discontinuities at switches between equilibrium types are many orders of magnitude larger than any remaining numerical imprecision.

Figure 9 shows the liquidity ($y$) across the probability of the aggregate liquidity shock ($p$) at our approximation of the symmetric Nash equilibrium for each type of equilibrium (no default, single default or mutual default) as well overall equilibrium liquidity, which takes into account which type of equilibrium is superior. Our findings can be summarized in the following result.

**Result 4.** *The symmetric Nash equilibrium liquidity ($y^*$) is discontinuous across equilibrium types and can be non-linear and non-monotonic.*

*(i) When the probability of the aggregate liquidity shock is low enough, the mutual default equilibrium is superior, and $y^* = \gamma$ for a positive interval in $p$. As $p$ increases, $y^*$ eventually increases until the equilibrium type switches to single default.*

*(ii) When the probability of the aggregate liquidity shock is intermediate, the single default equilibrium is superior, and $y^*$ increases discontinuously at the probability boundary between equilibrium types. As $p$ increases, eventually the equilibrium type switches to no default.*

*(iii) When the probability of the aggregate liquidity shock is high enough, the no-default equilibrium is superior, and $y^*$ increases discontinuously at the probability boundary between equilibrium types. As $p$ increases, $y^*$ first falls and then increases until $p = 1$.*

Figure 10 shows the interbank deposit ($z$) across the probability of the aggregate liquidity shock ($p$) at our approximation of the symmetric Nash equilibrium for each type of equilibrium (no default, single default or mutual default) as well overall equilibrium interbank deposit, which takes into account which type of equilibrium is superior. We summarize our conclusions in the following result.

**Result 5.** *The symmetric Nash equilibrium interbank deposit ($z^*$) is discontinuous across equilibrium types.*

*(i) When the probability of the aggregate liquidity shock is low enough, the mutual default equilibrium is superior, and $z^* \geq \varepsilon \forall p$ until the equilibrium type switches to single default.*

*(ii) When the probability of the aggregate liquidity shock is intermediate, the single default equilibrium is superior, and $z^* < \varepsilon$ until the equilibrium type switches to no default.*

*(iii) When the probability of the aggregate liquidity shock is high enough, the no default equilibrium is superior, and $z^* > \varepsilon$. As $p$ increases, $z^*$ increases to equal $\alpha$ and remains there until $p = 1$.*

# 6 Conclusion

In this paper we study a model where two regional banks face regional and aggregate liquidity demand shocks. We present several novel results in the literature on contagion among financial institutions, specifically the strand focusing on contagion among deposit taking banks following Diamond and Dybvig (1983), Allen and Gale (2000), and Freixas et al. (2000).

First, we present two aggregate benchmarks: a social planner and a global bank, both of which are novel in that they impose a central constraint on allowable allocations: in the absence of default (or full liquidation), deposit contracts must be risk-free. This is modeled by imposing early withdrawal returns that are state independent unless default or full liquidation occurs. This means full liquidation or default becomes the only tool that can transfer liquidity risk from late to early consumers.

A social planner maximizes per capita expected utility across regions, but is allowed to offer non-incentive compatible allocations, as we assume it can observe the true type (early or late) of a consumer, so that runs are not allowed by construction. Our analytic and numeric results show that the state-independent early withdrawal constraint is sufficient to make full liquidation optimal (when the aggregate liquidity shock realizes) for some parameter regions. Moreover, in the absence of full liquidation, the social planner optimally uses one or both of two tools to achieve an optimal consumption allocation by balancing ex-post inefficiencies: excess liquidity in the absence of large aggregate liquidity demand (where late consumption is funded out of liquidity rather than the more efficient long asset) or partial liquidation of the long asset when the aggregate liquidity demand is large (where early consumption is funded out of the less efficient early liquidation of the long asset rather than out of liquidity).

A global bank is identical to the social planner, except that we assume it is not be able to observe the types of consumers, and hence must offer incentive compatible deposit contracts in order to avoid a bank run and default. We characterize the parameter regions where the global bank allocation is different from the social planner allocation analytically and numerically, and show that the global bank can offer less liquidity insurance than the social planner in parameter regions where they are different.

Second, we present a numerical characterization of the symmetric Nash equilibrium of the game between two regional banks that make decisions independently and simultaneously. Our numerical results over the full range of the probability of an aggregate liquidity shock (for two cases in the rest of the parameters) is sufficient to provide the following novel results to the literature.

First, We show that, as the probability of aggregate risk approaches zero, the symmetric Nash equilibrium converges on the deterministic first best allocation within numerical precision (as in Allen and Gale (2000)).

Second, we identify parameter sets where contagion never occurs with positive probability of the aggregate liquidity shock (specifically when the return of the long asset at maturity is small relative to its early liquidation return).

Third, we show that, if parameters are such that contagion can occur at positive probability of the aggregate liquidity shock, the symmetric Nash equilibrium can have three different characterizations: if the probability of the aggregate liquidity shock is small enough, the equilibrium is characterized by both banks defaulting if the aggregate shock realizes (i.e. contagion occurs). If the probability is intermediate, the equilibrium is characterized by only one bank (the one hit by the aggregate liquidity shock) default (i.e. default without contagion occurs). If the probability of the aggregate liquidity shock is high enough, the banks choose portfolios and contracts such that neither bank ever defaults.

And fourth, if parameters are such that contagion can occur at positive probability of the aggregate liquidity shock, optimal choices may be discontinuous functions of the parameterization: At the parameter boundary where one equilibrium type switches to another, there is a discontinuity in the optimal choice, and within each region of a specific equilibrium type, the characterization (comparative statics) of the optimal choice are unique. This mirrors similar discontinuities in the aggregate benchmark allocations when the solution switches from one type to another.

Our paper is the first to use an explicit numerical solution to study the ex-ante internalization of the risk of ex-post shocks. The problem has proven to be elusive for analytical solutions and our method opens a range of new and interesting problems in the literature that currently cannot be studied.

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# A Proofs of Benchmark Propositions

## A.1 Proof of Proposition 1

Using the accounting identities, $e \equiv y - \gamma c_1$ and $\lambda \equiv \frac{(\gamma+\alpha)c_1 - y}{r}$, we solve for the portfolio choice $y = \frac{\gamma+\alpha}{\alpha}e + \frac{\gamma r}{\alpha}\lambda$ and express the consumption levels $(c_1, c_{2L}, c_{2H})$ as (linear) functions of $(e, \lambda)$:

$$
\begin{aligned}
c_1(e,\lambda) &= \frac{e + r\lambda}{\alpha} && (10)\\
c_{2L}(e,\lambda) &= \frac{e + (1-y)R}{1-\gamma} = \frac{R}{1-\gamma} - \frac{\left(R\left(\frac{\gamma+\alpha}{\alpha}\right) - 1\right)}{1-\gamma}e - \frac{rR\gamma}{\alpha(1-\gamma)}\lambda && (11)\\
c_{2H}(e,\lambda) &= \frac{(1-y-\lambda)R}{1-\gamma-\alpha} = \frac{R}{1-\gamma-\alpha} - \frac{\left(\frac{\gamma+\alpha}{\alpha}\right)R}{1-\gamma-\alpha}e - \frac{\left(\frac{r\gamma+\alpha}{\alpha}\right)R}{1-\gamma-\alpha}\lambda. && (12)
\end{aligned}
$$

As a result, we can also express the expected utility in terms of $(e, \lambda)$:

$$
W_{NFL}(e,\lambda) = (\gamma + p\alpha)u(c_1(e,\lambda)) + (1-p)(1-\gamma)u(c_{2L}(e,\lambda)) + p(1-\gamma-\alpha)u(c_{2H}(e,\lambda)). \qquad (13)
$$

Inserting the consumption levels yields an unconstrained problem in two choice variables $(e, \lambda)$ with Kuhn-Tucker first-order conditions of $\frac{dW}{de}(e^*, \lambda^*) \leqslant 0$ as $e^* \geqslant 0$ and $\frac{\partial W}{\partial \lambda}(e^*, \lambda^*) \leqslant 0$ as $\lambda^* \geqslant 0$:

$$
\begin{aligned}
\frac{dW}{de} &= (\gamma + p\alpha)u'(c_1) - (1-p)\big(R(\gamma+\alpha) - \alpha\big)u'(c_{2L}) - p(\gamma+\alpha)Ru'(c_{2H}) && (14)\\
\frac{dW}{d\lambda} &= (\gamma + p\alpha)u'(c_1) - (1-p)R\gamma u'(c_{2L}) - p\frac{\alpha + r\gamma}{r}Ru'(c_{2H}). && (15)
\end{aligned}
$$

### A.1.1 Optimum at p=0

At $p = 0$, the first-order conditions become

$$
\begin{aligned}
\gamma u'(c_1) &\leq \big(\gamma R + (R-1)\alpha\big)u'(c_{2L}) && (16)\\
\gamma u'(c_1) &\leq \gamma R u'(c_{2L}) && (17)
\end{aligned}
$$

Since $0 < (R-1)\alpha$, condition (16) is slack when 17 binds. Thus $e^* = 0$ and $\frac{\partial e^*}{\partial p}(p=0) = 0$. The optimum is characterized by a binding condition (17) evaluated at $e^* = 0$ and $\lambda^*$:

$$
u'\left(\frac{r\lambda^*}{\alpha}\right) = Ru'\left(\frac{R}{1-\gamma}\left[1 - \frac{r\gamma}{\alpha}\lambda^*\right]\right) \qquad (18)
$$

Because of the Inada conditions and since the left-hand side increases in $\lambda$, while the right-hand side decreases in $\lambda$, there exists a unique solution $\lambda^* \in \left(0, \frac{\alpha}{r\gamma}\right)$.

The envelope theorem implies that $e^*$ is continuous in $p$ so $\frac{\partial e^*}{\partial p} = 0$ in a neighborhood of $p = 0$, hence there exists a $\underline{p} \in (0,1]$ such that the optimum is characterized by $e^* = 0$ and $\lambda^* > 0$ if $0 \le p \le \underline{p}$. This optimum is characterized by the following first-order condition:

$$(\gamma + p\alpha)u'\left(\frac{r\lambda^*}{\alpha}\right) = (1-p)R\gamma u'\left(\frac{R}{1-\gamma}\left[1 - \frac{r\gamma}{\alpha}\lambda^*\right]\right) + pR\left(\gamma + \frac{\alpha}{r}\right)u'\left(\frac{R}{1-\gamma-\alpha}\left[1 - \left(1 + \frac{r\gamma}{\alpha}\right)\lambda^*\right]\right), \tag{19}$$

which implies the existence of a unique solution $\lambda^* \in \left(0, \frac{\alpha}{\alpha + r\gamma}\right)$.

### A.1.2 Optimum at p=1

At $p = 1$, the first-order conditions become

$$(\gamma + \alpha)u'(c_1) \le (\gamma + \alpha)Ru'(c_{2H}) \tag{20}$$

$$(\gamma + \alpha)u'(c_1) \le \left(\gamma + \frac{\alpha}{r}\right)Ru'(c_{2H}) \tag{21}$$

Since $r < 1$, condition (21) is slack when condition 20 binds. Thus $\lambda^* = 0$ and $\frac{\partial \lambda^*}{\partial p}(p = 1) = 0$. The optimum is characterized by the binding condition (20) evaluated at $\lambda^* = 0$ and $e^*$:

$$u'\left(\frac{e^*}{\alpha}\right) = Ru'\left(\frac{R}{1-\gamma-\alpha}\left[1 - \left(\frac{\gamma+\alpha}{\alpha}\right)e^*\right]\right) \tag{22}$$

Because of the Inada conditions and since the left-hand side increases in $e$, while the right-hand side decreases in $e$, there exists a unique solution $e^* \in \left(0, \frac{\alpha}{\gamma+\alpha}\right)$.

The envelope theorem implies that $\lambda^*$ is continuous in $p$ with $\frac{\partial \lambda^*}{\partial p} = 0$ in a neighborhood of $p = 1$, hence there exists a $\overline{p} \in [0,1)$ such that the optimum is characterized by $\lambda^* = 0$ and $e^* > 0$ if $\overline{p} \le p \le 1$. This optimum is characterized by the following first-order condition:

$$\begin{aligned}(\gamma + p\alpha)u'\left(\frac{e^*}{\alpha}\right) &= (1-p)\left[R(\alpha+\gamma) - \alpha\right]u'\left(\frac{1}{1-\gamma}\left[R - \left(R\left(1 + \frac{\gamma}{\alpha}\right) - 1\right)e^*\right]\right) \\ &\quad + pR(\gamma+\alpha)u'\left(\frac{R}{1-\gamma-\alpha}\left[1 - \left(1 + \frac{\gamma}{\alpha}\right)e^*\right]\right),\end{aligned} \tag{23}$$

which implies the existence of a unique solution $e^* \in \left(0, \frac{\alpha}{\alpha+\gamma}\right)$.

### A.1.3 Unique bounds on the probability of an aggregate liquidity shock

Next, we establish $\frac{\partial e^*}{\partial p} \geq 0$ and $\frac{\partial \lambda^*}{\partial p} \leq 0$, with strict inequality for positive levels of the choice variables. These monotonicity results imply the uniqueness of $\underline{p}$ and $\overline{p}$.

**Case 1:** $e^* > 0$ **and** $\lambda^* = 0$

Total differentiation of the first-order condition in (14) w.r.t. $p$ implies that $\frac{\partial e^*}{\partial p} > 0$ whenever

$$u'(c_{2L}) \quad > \quad \frac{R\gamma}{\left(R\gamma + (R-1)\alpha\right)} u'(c_{2H}),$$

for which $u'(c_{2L}) > u'(c_{2H})$ is sufficient. This sufficient condition is satisfied whenever

$$\begin{aligned} c_{2L}^* \quad &< \quad c_{2H}^* \\ \frac{R}{1-\gamma} - \frac{\left(R\left(\frac{\gamma+\alpha}{\alpha}\right)-1\right)}{1-\gamma} e^* \quad &< \quad \frac{R}{1-\gamma-\alpha} - \frac{\left(\frac{\gamma+\alpha}{\alpha}\right)R}{1-\gamma-\alpha} e^* \\ e^* \quad &< \quad \tilde{e} \equiv \frac{\alpha R}{(R-1)(\gamma+\alpha)+1} \end{aligned}$$

At $e = \tilde{e}$, we have $c_1(\tilde{e}) = c_{2L}(\tilde{e}) = c_{2H}(\tilde{e}) = \tilde{c}$ and $\frac{dW}{de}(e = \tilde{e}, \lambda = 0) = -\frac{(\gamma+\alpha)(R-1)}{\alpha} u'(\tilde{c}) < 0$. Thus $e^* < \tilde{e}$ whenever $\lambda^* = 0$ and $\frac{\partial e^*}{\partial p} > 0$ in case 1.

**Case 2:** $\lambda^* > 0$ **and** $e^* = 0$

Total differentiation of the first-order condition in (15) implies that $\frac{\partial \lambda^*}{\partial p} < 0$ whenever

$$u'(c_{2H}) > \frac{r(\gamma+\alpha)}{r\gamma+\alpha} u'(c_{2L}), \tag{24}$$

for which $u'\left(c_{2H}^*\right) > u'\left(c_{2L}^*\right)$ is sufficient. This sufficient condition is satisfied whenever

$$\begin{aligned} c_{2H}^* \quad &< \quad c_{2L}^* \\ \frac{R}{1-\gamma-\alpha} - \frac{\left(\frac{r\gamma+\alpha}{\alpha}\right)R}{1-\gamma-\alpha}\lambda^* \quad &< \quad \frac{R}{1-\gamma} - \frac{rR\gamma}{\alpha\left(1-\gamma\right)}\lambda^* \\ 1-\gamma-\alpha+r\gamma\lambda^* \quad &> \quad 0, \end{aligned}$$

which always holds since $\lambda^* \geq 0$ and $1-\gamma-\alpha > 0$. Thus, $\frac{\partial \lambda^*}{\partial p} < 0$ in case 2.

**Case 3:** $e^* > 0$ **and** $\lambda^* > 0$

Total differentiation of the first order conditions in (14) and (15) with respect to $p$ yield, after some rearrangement:

$$\frac{d\lambda^*}{dp} = \frac{N^\lambda}{D} < 0$$
$$\frac{de^*}{dp} = \frac{N^e}{D} > 0,$$

since the common denominator is negative:

$$\begin{aligned}
D_1 &= u''(c_1)(\gamma+\alpha p)(1-p)r^2(R-1)^2(1-\alpha-\gamma)u''(c_{2L}) > 0 \\
D_2 &= u''(c_1)(\gamma+\alpha p)(1-\gamma)p(1-r)^2R^2u''(c_{2H}) > 0 \\
D_3 &= (1-p)pR^2u''(c_{2H})u''(c_{2L})(\gamma(R-r)+\alpha(R-1))^2 > 0 \\
D &= -p(1-r)(\gamma+\alpha p)\,(D_1+D_2+D_3) < 0,
\end{aligned}$$

while the numerator of $\frac{\partial e^*}{\partial p}$ is negative:

$$\begin{aligned}
N_1^e &= -(1-\gamma)p^2(1-r)R(\alpha+\gamma)u''(c_{2H})(\alpha+\gamma r) > 0 \\
N_2^e &= -\gamma^2(1-p)^2r^2(R-1)(1-\alpha-\gamma)u''(c_{2L}) > 0 \\
N_3^e &= -(1-\gamma)(1-r)r^2(R-1)(1-\alpha-\gamma)u''(c_1)(\gamma+\alpha p)^2 > 0 \\
N^e &= -u'(c_{2L})\left(R(\gamma(R-r)+\alpha(R-1))\left(N_1^e+N_2^e\right)+N_3^e\right) < 0,
\end{aligned}$$

and the numerator of $\frac{\partial\lambda^*}{\partial p}$ is positive:

$$\begin{aligned}
N_1^\lambda &= -(1-\gamma)p^2(1-r)R^2(\alpha+\gamma)^2u''(c_{2H}) > 0 \\
N_2^\lambda &= -\gamma(1-p)^2r(R-1)(1-\alpha-\gamma)u''(c_{2L})(\alpha(R-1)+\gamma R) > 0 \\
N_3^\lambda &= -(1-\gamma)(1-r)r(R-1)(1-\alpha-\gamma)(\gamma+\alpha p)^2u''(c_1) > 0 \\
N^\lambda &= u'(c_{2L})\Big((\gamma(R-r)+\alpha(R-1))(N_1^\lambda+N_2^\lambda)+N_3^\lambda\Big) > 0.
\end{aligned}$$

Given these monotonicity results, we can define unique bounds on the probability of the aggregate liquidity shock:

$$\underline{p} \equiv \max\left\{p|e^*=0\right\}$$

$$\overline{p} \equiv \min\left\{p|\lambda^*=0\right\}$$

Finally, we show that $\overline{p} > \underline{p}$. Proof by contradiction. If $\underline{p} \geq \overline{p}$, then there exists a $p'' \in \left[\overline{p},\underline{p}\right]$ with corresponding optimal choice of $e^*=0$ and $\lambda^*=0$. However, this implies $c_1^*=0$ which contradicts optimality. Thus, $\overline{p} > \underline{p}$. Then, for $p \in \left(\underline{p},\overline{p}\right)$, the optimum is characterized by $e^*>0$ and $\lambda^*>0$ that jointly solve conditions (14) and (15) with equality.

Figure 11 shows the value functions of the different components. It aids the intuition on the nature of the three-part solution as a graphical illustration of an application of the envelope theorem. The interior solution is closed form in terms of parameters, where the two boundary conditions are one-dimensional but not in closed form, so they are solved by simple line search.

## A.2 Proof of Proposition 2

The problem facing a planner with *per capita* resources of 1 who is constrained to full liquidation and *pro rata* pay out in state $s_H$ is to choose $y \in [0,1]$ and $e \in [0,y]$ to maximize *ex ante* expected utility of an arbitrary investor.

Using the definition of excess liquidity in state $s_L$ as $e = y - \gamma c_{1L} \geq 0$, we can express all levels of consumption in terms of the portfolio choice $y$ and excess liquidity $e$:

$$\begin{aligned} c_{1L} &= \frac{y-e}{\gamma} \\ c_{2L} &= \frac{e+R(1-y)}{1-\gamma} \\ c_H &= y + r(1-y) \end{aligned}$$

As a result, we can also express the expected utility in terms of $(e, y)$:

$$W_{FL}(y,e) = (1-p)\Big[\gamma u(c_{1L}(y,e)) + (1-\gamma)u(c_{2L}(y,e))\Big] + pu(c_H(y,e)).$$

Note that $y = 0$ implies $e = 0$ and thus $c_{1L} = 0$ which cannot be optimal. This yields three remaining cases that characterize the optimal allocation with full liquidation in state $s_H$:

**Case 1: short asset only** $(y^* = 1, e^* > 0)$**:**

Since $\frac{dW}{dy}\big|_{p=1} = (1-r)\,u'\big(y + r(1-y)\big) > 0\ \forall y \in [0,1]$, there must exist a non-zero measure interval $[\overline{p}_{FL}, 1]$ such that $y^*_{FL}(p \geq \overline{p}_{FL}) = 1$. This in turn implies $c^*_{2L} = c^*_{1L} = c^*_H = 1$ and thus $e^*(p \geq \overline{p}_{FL}) = 1-\gamma > 0$ in this case.

**Now suppose** $p < \overline{p}_{FL}$**:**

The Kuhn-Tucker first-order conditions for optimality are as follows: First, for $y$, the optimality condition is $\frac{dW_{FL}}{dy}(y^*_{FL}, e^*_{FL}) = 0$, whereby a unique interior solution $y^*_{FL} \in (0,1)$ exists since the objective function is continuous in liquidity, strictly concave, and satisfies the Inada conditions.

$y^*_{FL} \in (0,1)$ is characterized by:

$$(1-p)\,u'\left(\frac{y^*_{FL}-e^*_{FL}}{\gamma}\right)+p\,(1-r)\,u'\big(y^*_{FL}+r(1-y^*_{FL})\big)=(1-p)\,Ru'\left(\frac{e^*_{FL}+R(1-y^*_{FL})}{1-\gamma}\right) \tag{25}$$

Second, for $e$, we have $\frac{\partial W_{FL}}{\partial e}(y^*_{FL}, e^*_{FL}) \lessgtr 0$ as $e \gtrless 0$, thus $e^*$ is characterized by:

$$u'\left(\frac{e^*_{FL}+R(1-y^*_{FL})}{1-\gamma}\right) \leqslant u'\left(\frac{y^*_{FL}-e^*_{FL}}{\gamma}\right) \tag{26}$$

**Case 2: No excess liquidity** $(0 < y^* < 1, e^* = 0)$

At $p = 0$, condition $\frac{dW_{FL}}{dy}(y^*_{FL}, e^*_{FL}) = 0$ yields:

$$u'(c_{1L}(y^*,e^*)) = R\,u'(c_{2L}(y^*,e^*))$$

Since $R > 1$, this implies $u'(c_{1L}(y^*,e^*)) > u'(c_{2L}(y^*,e^*))$ so condition (26) is slack. Hence: $e^*(p=0) = 0$ and $\frac{\partial e^*}{\partial p}(p=0) = 0$. Since $e^*$ is continuous in $p$, by the theorem of the maximum, there must exist a non-zero measure neighborhood $[0, \underline{p}_{FL}]$ such that $e^*(p < \underline{p}_{FL}) = 0$. Moreover, $0 < y < 1$, $e = 0$ and $\gamma < 1$ implies $c_{2L}(y,0) > c_{1L}(y,0) > c_H(y,0)$.

**Case 3: interior short asset and excess liquidity** $(y^* \in (0,1), e^* > 0)$

Whenever $e^*_{FL} > 0$ is optimal, which from the above is whenever $p > \underline{p}_{FL}$, condition 26 holds with equality, which implies:

$$\begin{aligned} c_{1L}(y^*_{FL}, e^*_{FL}) = c_{2L}(y^*_{FL}, e^*_{FL}) = c^*_L &= y^*_{FL} + \big(1-y^*_{FL}\big)R \\ e^*_{FL} &= y^*_{FL}\big(1+(R-1)\gamma\big) - R\gamma \end{aligned}$$

Moreover, $0 < y^* < 1$,$e^* > 0$ and $c_{2L}(y^*,e^*) = c_{1L}(y^*,e^*)$ implies $c_{2L}(y^*,e^*) = c_{1L}(y^*,e^*) = y^* + R(1-y^*) > 1 > y^* + r(1-y^*) = c_H(y^*,e^*)$.

**Establishing uniqueness of bounds:**

Total differentiation of condition 25 with respect to $p$ where $e^*_{FL} > 0$ yields:

$$\begin{aligned} \frac{dy^*}{dp} &= -\frac{(1-r)\,u'(c^*_H) + (R-1)\,u'(c^*_L)}{p(1-r)^2 u''(c^*_H) + (1-p)(R-1)^2 u''(c^*_L)} > 0 \\ &\Rightarrow \frac{de^*}{dp} > 0 \end{aligned}$$

Thus $\overline{p}_{FL} > \underline{p}_{FL}$ and each is unique.

## A.3 Proof of Proposition 3

Let $c_1^{NFL}$, $c_{2L}^{NFL}$ and $c_{2H}^{NFL}$ be the optimal allocation that solves the No-Full-Liquidation (NFL) problem, so we can state the NFL value function as:

$$V_{NFL} \equiv \max(1-p)\left[\gamma u(c_1^{NFL}) + (1-\gamma)u(c_{2L}^{NFL})\right] + p\left[(\gamma+\alpha)u(c_1^{NFL}) + (1-\gamma-\alpha)u(c_{2H}^{NFL})\right]$$

Let $c_{1L}^{FL}$, $c_{2L}^{FL}$ and $c_H^{FL}$ be the optimal allocation that solves the Full-Liquidation (FL) problem. Thus we can state the FL value function as:

$$V_{FL} \equiv \max(1-p)\left[\gamma u(c_{1L}^{FL}) + (1-\gamma)u(c_{2L}^{FL})\right] + pu(c_H^{FL})$$

At $p = 0$, the planner can do no better than the full liquidation allocation, since it is less constrained than the No Full Liquidation allocation (even though the outcome in state $s_L$ has zero weight, the definition of the NFL allocation still requires that early consumption be constant across the two states. This imposes a shadow cost on the NFL allocation). Thus: $V_{FL}\left(p=0\right) \geq V_{NFL}\left(p=0\right)$

Similarly, the planner will not choose the full liquidation allocation when $p = 1$, as this will imply certain liquidation thus. $V_{FL}\left(p=1\right) < V_{NFL}\left(p=1\right)$

Both $V_{NFL}$ and $V_{FL}$ are continuous in $p$ by the theorem of the maximum.

By the Envelope theorem that applies in this situation (Milgrom and Segal (2002)):

$$\begin{aligned} \frac{\partial V_{NFL}}{\partial p} &= \left[(\gamma+\alpha)u(c_1^{NFL}) + (1-\gamma-\alpha)u(c_{2H}^{NFL})\right] - \left[\gamma u(c_1^{NFL}) + (1-\gamma)u(c_{2L}^{NFL})\right] \\ \frac{\partial V_{FL}}{\partial p} &= u(c_H^{FL}) - \left[\gamma u(c_{1L}^{FL}) + (1-\gamma)u(c_{2L}^{FL})\right] \end{aligned}$$

For any $\check{p}$ defined by $V_{FL}\left(\check{p}\right) = V_{NFL}\left(\check{p}\right)$, we have:

(i) $u(c_H^{FL}) < \left[(\gamma+\alpha)u(c_1^{NFL}) + (1-\gamma-\alpha)u(c_{2H}^{NFL})\right]$, since partial liquidation must yield higher utility than full liquidation in state $s_H$.

(ii) $\left[\gamma u(c_{1L}^{FL}) + (1-\gamma)u(c_{2L}^{FL})\right] \geq \left[\gamma u(c_1^{NFL}) + (1-\gamma)u(c_{2L}^{NFL})\right]$ since, in state $s_L$ the $FL$ allocation is less constrained than the $NFL$ allocation which is constrained by facing a trade-off with the promised allocation in state $s_H$.

Thus we conclude that $\frac{\partial V_{FL}}{\partial p} < \frac{\partial V_{NFL}}{\partial p}$ at any $\check{p}$ where $V_{FL}(\check{p}) = V_{NFL}(\check{p})$, thus $\check{p}$ is unique. Figure 12 illustrates this result.

**Characterizing $\check{p}$ as a function of $\alpha$, $r$ and $\gamma$**

At $\alpha = 0$, for any $\gamma \in [0,1]$, $r \leq 1$ and $R > 1$, the no full liquidation allocation must be weakly better than the full liquidation allocation. This is so because the consumption allocation in the full liquidation problem in state $s_H$ is bounded above by 1, while the $s_H$ allocation in the no full liquidation problem is effectively no more constrained than in $s_L$: in the (near) zero aggregate risk case, it must be that $c_{2L} \geq c_{2H} > c_1$ otherwise the long asset is not optimally exploited, because (almost) any allocation that satisfies $R > c_{2L} = c_{2H} \geq c_1 \geq r$ is feasible and weakly better than $R > c_{2L} \geq c_{2H} = c_1 = r$. Hence, there must exist a non-zero measure neighborhood around $\alpha = 0$ where full liquidation is not optimal. Similarly, when $\alpha = 1 - \gamma$ (or in a neighbourhood), there are (almost) no late investors to provide for in $s_H$, hence the shadow cost of maintaining constant early consumption whilst providing for some $c_{2H}$ is very high. As such the full liquidation dominates in this region for $p$ low enough. Similar arguments go through for $\gamma$. Thus we can define: $\check{\alpha} \equiv \max\{\alpha | V_{NFL} \geq V_{FL}\}$ and $\check{\gamma} \equiv \max\{\gamma | V_{NFL} \geq V_{FL}\}$.

At $r = 1$ (or in a neighborhood), partial liquidation of the long asset is (almost) without penalty relative to the short asset. Hence $c_{2L}, c_{2H} > c_1$ can be supported in the no full liquidation allocation, which thus dominates the full liquidation allocation. On the other hand, when $r$ is in a neighbourhood of 0, maintaining the non-state contingent early payout in the no full liquidation allocation is very costly, and thus the full liquidation allocation dominates (for low enough $p$ and high enough $\alpha$ and $\gamma$). We can thus define $\check{r} \equiv \min\{r | V_{NFL} \geq V_{FL}\}$.

## A.4 Proof of Proposition 4

First, suppose that the parameters are such that the full liquidation in state $s_H$ is optimal for the social planner. The allocation in state $s_H$ is trivially incentive compatible and $c^*_{2L} \geq c^*_{1L}$. This follows directly from the characterization in proof of Proposition 2:

(i) If $p \geq \overline{p}_{FL}$, $c^*_{2L} = c^*_{1L} = 1$;

(ii) If $\underline{p}_{FL} \leq p < \overline{p}_{FL}$, $c^*_{2L} = c^*_{1L} > 1$;

(iii) If $p < \underline{p}_{FL}$, $c^*_{2L} > c^*_{1L}$.

Second, suppose that the parameters are such that the full liquidation in state $s_H$ is not optimal for the social planner (i.e. Proposition 1 fully characterizes the planner problem). There are three cases to consider

**If $p \geq \overline{p}$ so that $\lambda^* = 0$ and $e^* > 0$**

Proposition 1 establishes that the optimal level of excess liquidity $e^*$ in this case satisfies $e^* < \tilde{e}$, where at $\tilde{e}$, we have $c_1(\tilde{e}) = c_{2L}(\tilde{e}) = c_{2H}(\tilde{e})$.

Since, by construction, $\frac{\partial c_1}{\partial e} > 0$, $\frac{\partial c_{2L}}{\partial e} < 0$ and $\frac{\partial c_{2H}}{\partial e} < 0$, $e^* < \tilde{e}$ implies $c^*_{2L}, c^*_{2H} > c^*_1$ in this case.

Hence $p \geq \overline{p}$ is sufficient for incentive compatibility.

**If $\underline{p} < p < \overline{p}$ so that $e^* > 0$ and $\lambda^* > 0$**

Conditions 14 and 15 holding with equality implies:

$$u'\left(c^*_1\right) = \left(1-p\right)\frac{\left(\alpha\left(R-1\right)+\gamma\left(R-r\right)\right)}{\left(1-r\right)\left(\gamma+p\alpha\right)}u'\left(c^*_{2L}\right) \tag{27}$$

and

$$u'\left(c^*_1\right) = p\frac{R}{r}\frac{\left(\alpha\left(R-1\right)+\gamma\left(R-r\right)\right)}{\left(R-1\right)\left(\gamma+p\alpha\right)}u'\left(c^*_{2H}\right) \tag{28}$$

Condition 27 implies $c^*_{2L} \geq c^*_1$ if:

$$R \geq \frac{1-pr}{1-p}$$

Since $\frac{1-pr}{1-p} \leq 1$ and $R \geq 1$ this condition is always satisfied so $c^*_{2L} \geq c^*_1$ in this case.

Condition 28 implies that $c^*_{2H} < c^*_1$ in this case only if:

$$p < \hat{p}^{(1)} \equiv \frac{\gamma r(R-1)}{(R-r)(\alpha(R-1)+\gamma R)}$$

Hence $p \geq \hat{p}^{(1)}$ is necessary and sufficient for incentive compatibility this case.

**If $p < \underline{p}$ so that $\lambda^* > 0$ and $e^* = 0$**

Proposition 1 establishes that $c^*_{2L} > c^*_{2H}$ in this case. Combined with $c^*_1 < \max\left\{c^*_{2L}, c^*_{2H}\right\}$

this implies and $c_{2L}^* > c_1^*$ in this case.

Given that $e^* = 0$, the accounting identities imply that $c_{2H}^* < c_1^*$ if:

$$\lambda^* > \hat{\lambda} \equiv \frac{\alpha R}{r(\gamma R + 1 - \alpha - \gamma) + \alpha R}.$$

Since Proposition 1 establishes that $\frac{\partial \lambda^*}{\partial p} < 0$ in this case, and by construction $\frac{\partial c_1}{\partial \lambda} > 0$ and $\frac{\partial c_{2H}}{\partial \lambda} < 0$, there exists a unique $\hat{p}^{(2)} \equiv \left\{p \middle| \lambda^*(p) = \hat{\lambda}\right\}$ such that the consumption allocation is incentive compatible if and only if $p \geq \hat{p}^{(2)}$.

Last, we establish continuity of the incentive compatibility boundary across different cases: i.e. we show $\lim_{p \nearrow \underline{p}} \hat{p}^{(2)} = \lim_{p \searrow \underline{p}} \hat{p}^{(1)}$,

Suppose by contradiction that $\lim_{p \nearrow \underline{p}} \hat{p}^{(2)} > \lim_{p \searrow \underline{p}} \hat{p}^{(1)}$. Then there must exist a sequence $p'_n < \hat{p}^{(2)}$ converging to $\dot{p} = \underline{p} < \hat{p}^{(2)}$ from below, so that $c_{2H}^*(p'_n) < c_1^*(p'_n)$ for all $p'_n$. There must also be a sequence $p''_n > \hat{p}^{(1)}$ converging to $\dot{p} = \underline{p} > \hat{p}^{(1)}$ from above, so that $c_{2H}^*(p''_n) > c_1^*(p''_n)$ for all $p''_n$. But then there must be a discontinuity in the consumption allocation (and hence in the optimal portfolio) at $\dot{p}$, which contradicts the result from the theorem of the maximum that the maximizers of a continuous problem must be continuous. A similar argument would hold for the opposite inequality.

Thus we conclude that there is a unique, continuous boundary $\hat{p}$ where the optimal allocation is characterized by $c_{2H}^*(\hat{p}) = c_1^*(\hat{p})$, where

$$\hat{p} = \begin{cases} \hat{p}^{(1)} & \text{if } p > \underline{p} \\ \hat{p}^{(2)} & \text{if } p \leq \underline{p} \end{cases}$$

## A.5 Proof of Proposition 5

The allocation chosen by a global bank is identical to the allocation chosen by the planner whenever the latter is incentive compatible.

Proposition 4 establishes that the optimal allocation chosen by the planner is always incentive compatible when full liquidation is optimal (i.e. when $p \leq \check{p}$), so we only need to consider the no full liquidation case.

The no full liquidation problem of the global bank in general is given by:

$$
\begin{aligned}
V_{NFL} \quad \equiv \quad & \max_{\{x,y,c_1,c_{2L},c_{2H}\}} (1-p)\left[\gamma u(c_1)+(1-\gamma)u(c_{2L})\right]+p\left[(\gamma+\alpha)u(c_1)+(1-\gamma-\alpha)u(c_{2H})\right] \\
\text{s.t.} \quad & x+y=1 \\
& c_{2L}=\frac{y-\gamma c_1+Rx}{1-\gamma} \\
& c_{2H}=\frac{R}{1-\gamma-\alpha}\left(x-\frac{(\gamma+\alpha)c_1-y}{r}\right) \\
& c_{2L},c_{2H}\geq c_1
\end{aligned}
$$

As in the planner case, it is convenient to state the problem in terms of excess liquidity in state $s_L$ and partial liquidation in state $s_H$. In principle, the global bank allocation will have the same three cases as the planner problem in terms of whether either or both of these choice variables are positive. However, Proposition 4 also shows that never a violation of incentive compatibility in the planner problem when zero partial liquidation is optimal (i.e. when $p \geq \overline{p}$), so we only need to consider the two other cases (i.e. $p \leq \underline{p}$ and $\underline{p} < p < \overline{p}$).

Proposition 4 shows that incentive compatibility is violated in this case if and only if $p < \hat{p}$. Consider parameter values where this is the case.

When incentive compatibility is violated in state $s_H$ in the planner allocation i.e. when $c_{2H} < c_1$, the closest weakly incentive compatible allocation to the planner's that the global bank can achieve is characterized by $c_{2H} = c_1 \equiv c_H$, in which case the expected utility is given by (the same in both cases discussed below):

$$W=\left(p+\left(1-p\right)\gamma\right)u\left(c_H\right)+\left(1-p\right)\left(1-\gamma\right)u\left(c_{2L}\right)$$

The character of the solution differs between two cases identified by bound $\underline{p}_{GB} \in (0, \underline{p})$:

**If $p \leq \underline{p}_{GB}, \hat{p}$, $\lambda^* > 0$ and $e^* = 0$:**

Since we are considering a situation without full liquidation or excess liquidity in $s_L$, the only way ensure constant consumption in $t=1$ is for $\lambda = \frac{\alpha}{r\gamma}y$. Adding the additional constraint that $c_1 = \frac{y+r\lambda}{\gamma+\alpha} = \frac{(1-y-\lambda)R}{1-\gamma-\alpha} = c_{2H}$ fully determines the solution independent of the utility function:

$$\begin{aligned} y^* &= \frac{\gamma r R}{\gamma r R + \alpha R + r(1-\gamma-\alpha)} \\ c_1^* = c_{2H}^* &= \frac{y^*}{\gamma} \\ c_{2L}^* &= \frac{R(1-y^*)}{1-\gamma} \end{aligned}$$

**If $\underline{p}_{GB} < p < \hat{p}$, $e^* > 0$ and $\lambda^* > 0$**

Using $c_1 = c_{2H}$, the problem can be stated in terms of $y$ alone:

$$\begin{aligned} e &= \frac{y(r(\gamma R + 1 - \alpha - \gamma) + \alpha R) - \gamma r R}{r(1-\alpha-\gamma) + R(\alpha+\gamma)} \\ \lambda &= \frac{R(\alpha+\gamma) - y((R-1)(\alpha+\gamma)+1)}{r(1-\alpha-\gamma) + R(\alpha+\gamma)} \\ c_H &= \frac{(1-r)Ry + rR}{r(1-\alpha-\gamma) + R(\alpha+\gamma)} \\ c_{2L} &= \frac{rR(1-\alpha-2\gamma) + R^2(\alpha+\gamma)}{(1-\gamma)(r(1-\alpha-\gamma) + R(\alpha+\gamma))} \\ &\quad - \frac{y\left(rR(1-\alpha-2\gamma) + R^2(\alpha+\gamma) - \alpha R - r(1-\alpha-\gamma)\right)}{(1-\gamma)(r(1-\alpha-\gamma) + R(\alpha+\gamma))} \end{aligned}$$

yielding the optimality condition:

$$u'(c_H) = \frac{(1-p)(r(R(1-\alpha-2\gamma) - (1-\alpha-\gamma)) + R(R(\alpha+\gamma)-\alpha))}{(1-r)R(\gamma + (1-\gamma)p)} u'(c_{2L})$$

Since $\frac{\partial c_H}{\partial \lambda} < 0$ and $\frac{\partial c_{2L}}{\partial \lambda} > 0$ in this situation, there exists a unique $\lambda^* > 0$ that solves this problem.

The proof of the existence and uniqueness of bound $\underline{p}_{GB}$ is similar to the proof for the planner and is omitted.

Since a global bank must offer $c_{2H} = c_1$ when the planner offers $c_{2H} < c_1$, a global bank optimally chooses less partial liquidation than the planner. For the same reasons as in the planner case, when $p$ is low, ex post inefficiency in $s_H$ is less costly than in $s_L$, so a global bank uses only partial liquidation in a neighborhood of $p = 0$. However, since the global bank uses less partial liquidation than the planner, as $p$ increases, it becomes efficient to start using some excess liquidity sooner than the planner does, so $\underline{p}_{GB} < \underline{p}$.

# B Additional Benchmarks

## B.1 Autarky

Without banks, the idiosyncratic liquidity risk cannot be pooled away. The investor splits her endowment between the short asset $y$ and the long asset $x$ at $t = 0$, before she learns her type at $t = 1$. Her consumption levels are $c_1 = y + rx$ if early and $c_2 = y + Rx$ if late. By strict monotonicity, all endowment is invested, $y^* = 1 - x^*$, and the consumption levels are a function of investment in the long asset only, $c_1(x) = 1 - (1-r)x$ and $c_2(x) = 1 + (R-1)x$.

Since the effective probability at $t = 0$ of being an early investor is $p' \equiv \gamma + p\alpha$, the problem of the investor is to choose her portfolio to maximize her expected utility in autarky:

$$\max_{x \in [0,1]} p'\, u(c_1(x)) + \left(1 - p'\right) u(c_2(x))$$

**Proposition 6.** *The optimal portfolio choice in autarky is determined by the effective probability of facing the idiosyncratic liquidity shock, $p'$. There are three cases:*

*(i) For a sufficiently high effective probability, $p' \geq \overline{p}' \equiv \frac{R-1}{R-r} \in (0,1)$, no investment occurs, $x^{Aut} = 0$.*

*(ii) For a sufficiently low effective probability, $p' \leq \underline{p}' \equiv \frac{(R-1)u'(R)}{(R-1)u'(R)+(1-r)u'(r)} \in (0,1)$, full investment occurs, $x^{Aut} = 1$.*

*(iii) For intermediate levels, $\underline{p}' < p' < \overline{p}'$, there exists a unique interior portfolio choice $0 < x^{Aut} < 1$:*

$$p'(1-r)\, u'\big(c_1(x^{Aut})\big) = (1-p')(R-1)\, u'\big(c_2(x^{Aut})\big)$$

**Proof.**

If, $\forall x \in [0,1]$ it is the case that $p'(1-r)\, u'\big(c_1(x)\big) > (1-p')(R-1)\, u'\big(c_2(x)\big)$ it is optimal for the investor not to invest in the long asset, this implies $c_1^{Aut} = c_2^{Aut} = 1$, and the optimality condition reduces to a inequality constraint on the effective probability: $p' > \overline{p}' \equiv \frac{R-1}{R-r} \in (0,1)$.

If, $\forall x \in [0,1]$ it is the case that $p'(1-r)\, u'\big(c_1(x)\big) < (1-p')(R-1)\, u'\big(c_2(x)\big)$ it is optimal for the investor to fully invest in the long asset, this implies $c_1^{Aut} = r$ and $c_2^{Aut} = R$, and the optimality condition again reduces to a inequality constraint on the effective probability: $p' < \underline{p}' \equiv \frac{(R-1)u'(R)}{(R-1)u'(R)+(1-r)u'(r)} \in (0,1)$.

Otherwise, there must exist an $x \in (0,1)$ where $p'(1-r)\,u'\big(c_1(x)\big) = (1-p')(R-1)\,u'\big(c_2(x)\big)$ which implicitly defines the optimal level of long asset investment.

■

## B.2 First Best at $p = 0$ - the Allen and Gale (2000) allocation

In the absence of aggregate risk, the stochastic specification of the environment is described in table 4.

Table 4: Distribution of regional liquidity demand $v_{ks}$

| State | Probability | Region A | Region B |
|---|---|---|---|
| $s_1$ | $\frac{1}{2}$ | $v_{A1} = \gamma - \varepsilon$ | $v_{B1} = \gamma + \varepsilon$ |
| $s_2$ | $\frac{1}{2}$ | $v_{A2} = \gamma + \varepsilon$ | $v_{B2} = \gamma - \varepsilon$ |

Since the regional shocks cancel out when aggregated across regions, the global risk sharing problem is deterministic. Allen and Gale (2000) show that the first best risk sharing arrangement is characterized by the first order condition: $u'(c_1^{AG}) = Ru'(c_2^{AG})$, and is feasibly decentralized as follows:

Each bank holds interbank position $z_A = z_B = z^{AG} = \varepsilon$ and just enough short asset to satisfy the average global early liquidity demand $y_A = y_B = y^{AG} = \gamma c_1^{AG}$. If $s_1$ realizes, bank $B$ needs to withdraw its full interbank holding as it faces regional consumer demand $(\gamma+\varepsilon)c_1^{AG}$ but only has liquidity to cover $\gamma c_1^{AG}$. Bank $A$ does not need to withdraw as the sum of regional demand $(\gamma-\varepsilon)c_1^{AG}$ and demand from bank $B$ of $\varepsilon\, c_1^{AG}$ is equal to liquidity available locally.

If the utility function is of the standard constant relative risk aversion form: $u(c) = \frac{c^{\rho-1}-1}{\rho-1}$, the optimal choices can be solved as:

$$\begin{aligned} y^{AG} &= \frac{\gamma R}{\gamma R + (1-\gamma)R^{\frac{1}{\rho}}} \\ z^{AG} &= \varepsilon \\ d^{AG} &= \frac{R}{\gamma R + (1-\gamma)R^{\frac{1}{\rho}}} \end{aligned}$$

In the limit as $\rho \to 1$, $u(c) = \ln(c)$ and the optimal choices reduce to:

$$
\begin{aligned}
y^{AG} &= \gamma \\
z^{AG} &= \varepsilon \\
d^{AG} &= 1
\end{aligned}
$$

# C Figures

Figure 1: Optimal risk sharing with observable types and without full liquidation in state $s_H$: the usage of partial liquidation and excess liquidity varies with the size and the probability of the aggregate liquidity shock. Parameters: $u(c) = \frac{c^{1-\rho}-1}{1-\rho}$, $\rho = 2$, $R = 2$, $r = 0.5$, and $\gamma = 0.3$.

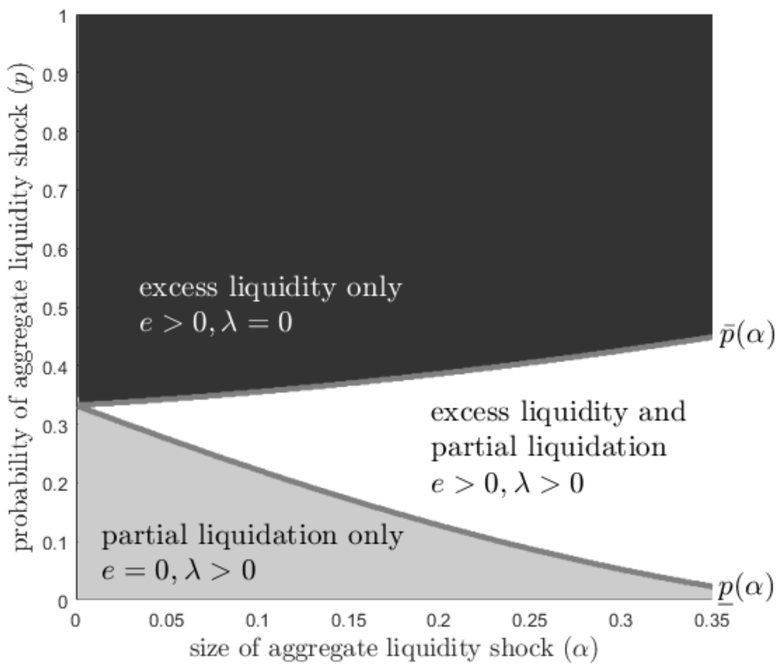

Figure 2: Optimal risk sharing with observable types: the complete problem. Full liquidation occurs when the probability of the aggregate liquidity shock is low enough, and the size of the aggregate liquidity shock large enough. Parameters: $u(c) = \frac{c^{1-\rho}-1}{1-\rho}$, $\rho = 2$, $R = 2$, $r = 0.5$, and $\gamma = 0.3$.

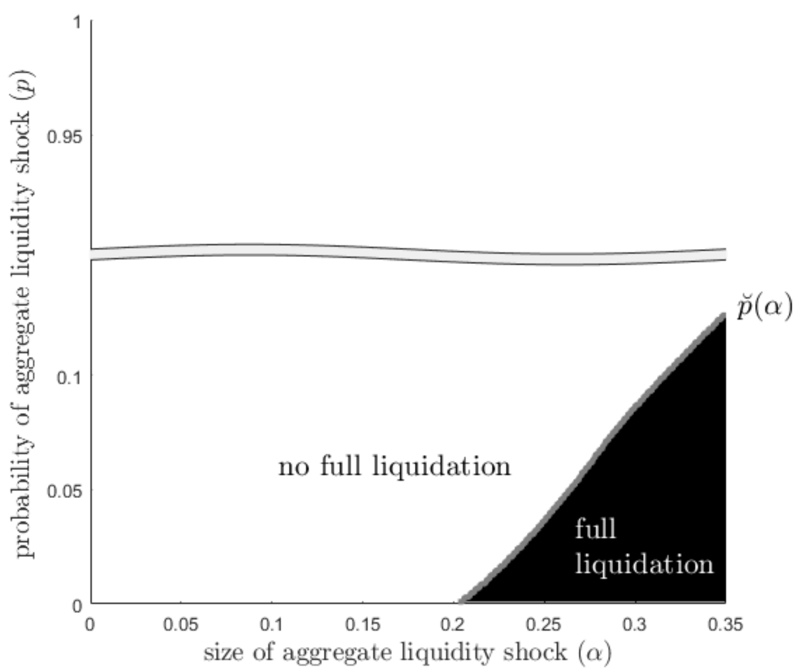

Figure 3: Optimal risk sharing with observable types and without full liquidation in state $s_H$: the optimal allocation in state $s_H$ is non-incentive compatible if the probability of the aggregate liquidity shock is small enough, and the size of the aggregate liquidity shock is large enough. Parameters: $u(c) = \frac{c^{1-\rho}-1}{1-\rho}$, $\rho = 2$, $R = 2$, $r = 0.5$, and $\gamma = 0.3$.

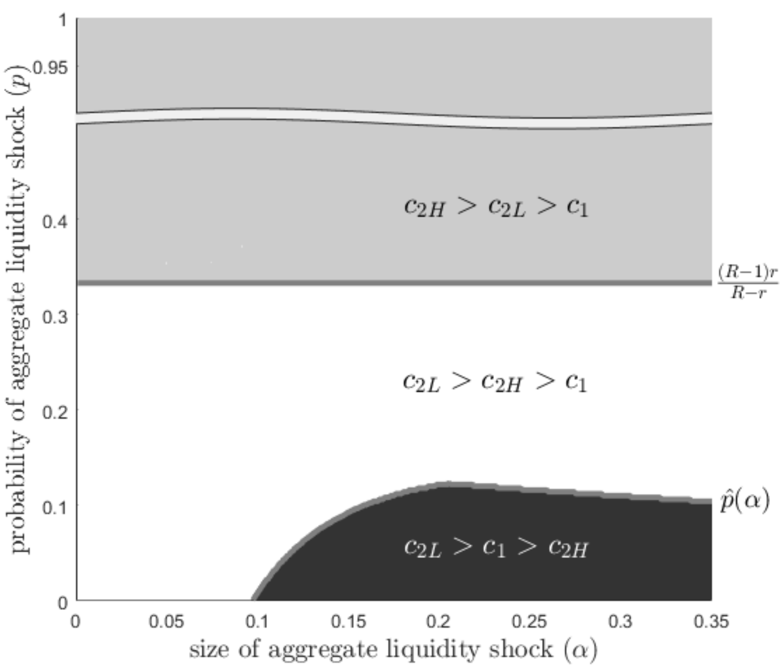

Figure 4: Incentive compatible regions for the liquidity ($y$) and interbank deposit choices ($z$) of bank $A$ for 3 different choices of deposit return ($d$) of bank A in the mutual default allocation. In the top panel, bank $B$'s choices are fixed at $[y_B = 0.4, z_B = 0.05, d_B = 1]$. In the bottom panel bank $B$'s choices are fixed at $[y_B = 0.4, z_B = 0, d_B = 1]$. Parameters: $u(c) = \ln(c)$, $p = 0$, $R = 2$, $r = 0.1$, $\gamma = 0.4$, $\alpha = 0.1$, $\varepsilon = 0.05$.

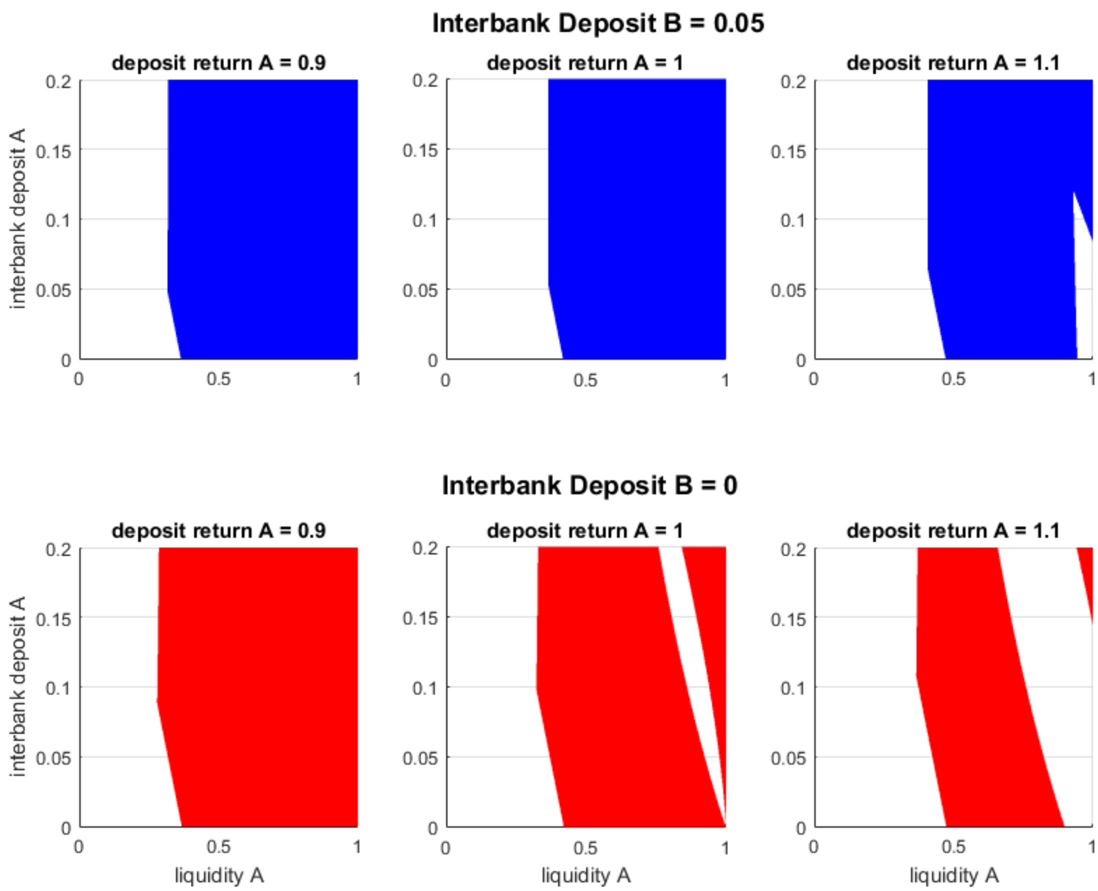

Figure 5: Schematic of search algorithm.

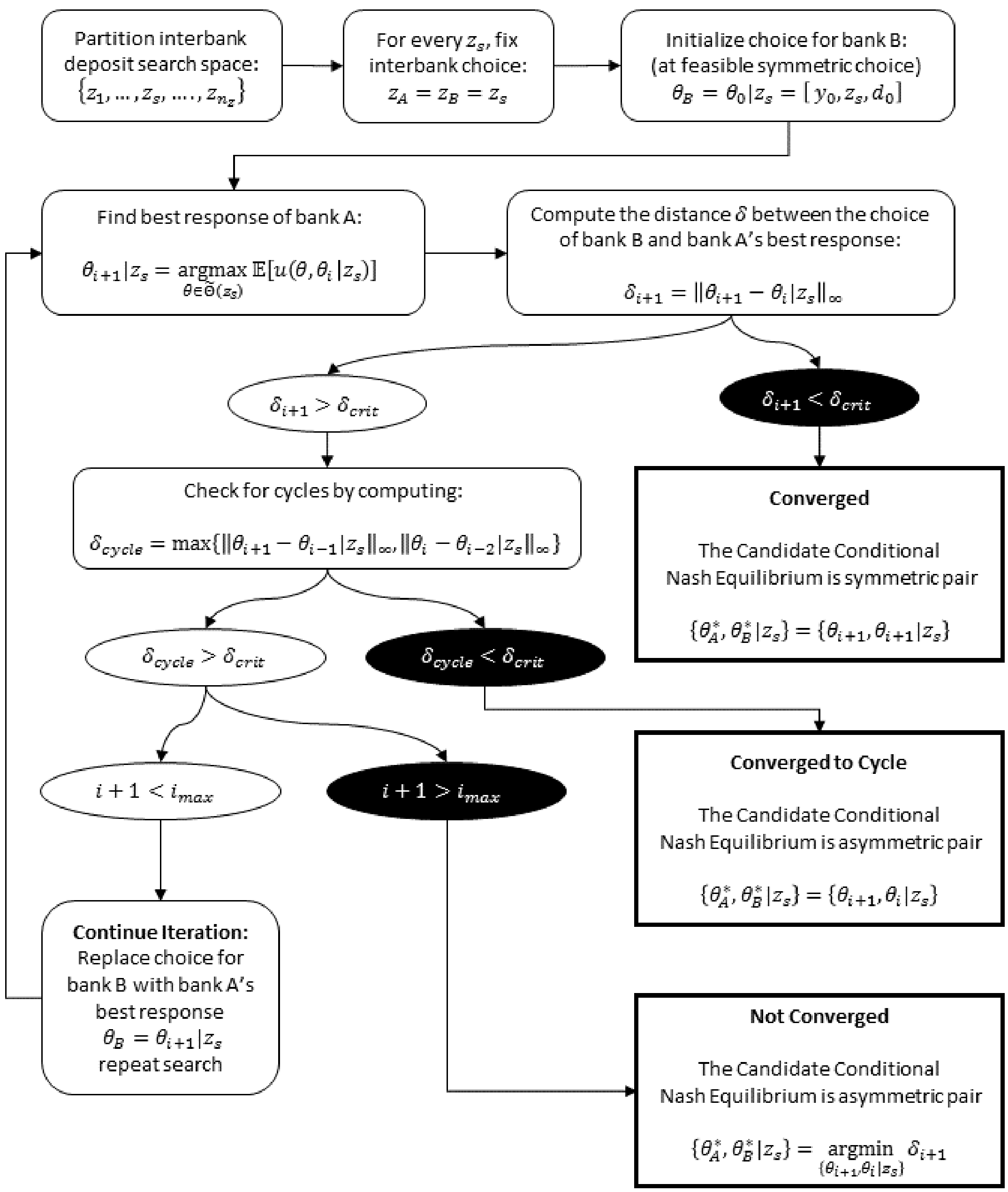

Figure 6: Expected utility at approximate symmetric Nash equilibrium of the regional bank problem across the probability of the aggregate liquidity shock. This parameter set, where the spread between the maturity and early liquidation return of the long asset is large: $\frac{R}{r} = \frac{5}{0.1}$ (relative to $\frac{R}{r} = \frac{1.1}{0.9}$ in figure 7 below), yields three probability regions with different equilibrium types. When the probability is low enough ($p \in [0, 0.067)$), the mutual default equilibrium is superior. For intermediate probability values ($p \in [0.067, 0.1467]$)), the single default equilibrium is superior, and when the probability is high enough ($p \in (0.1467, 1]$), the no default allocation is superior. Moreover, when the probability is zero, the equilibrium attains the same expected utility as the allocation in Allen and Gale (2000) within numerical precision. Parameters: $u(c) = \ln(c)$, $R = 5$, $r = 0.1$, $\gamma = 0.5$, $\alpha = 0.15$ and $\varepsilon = 0.1$.

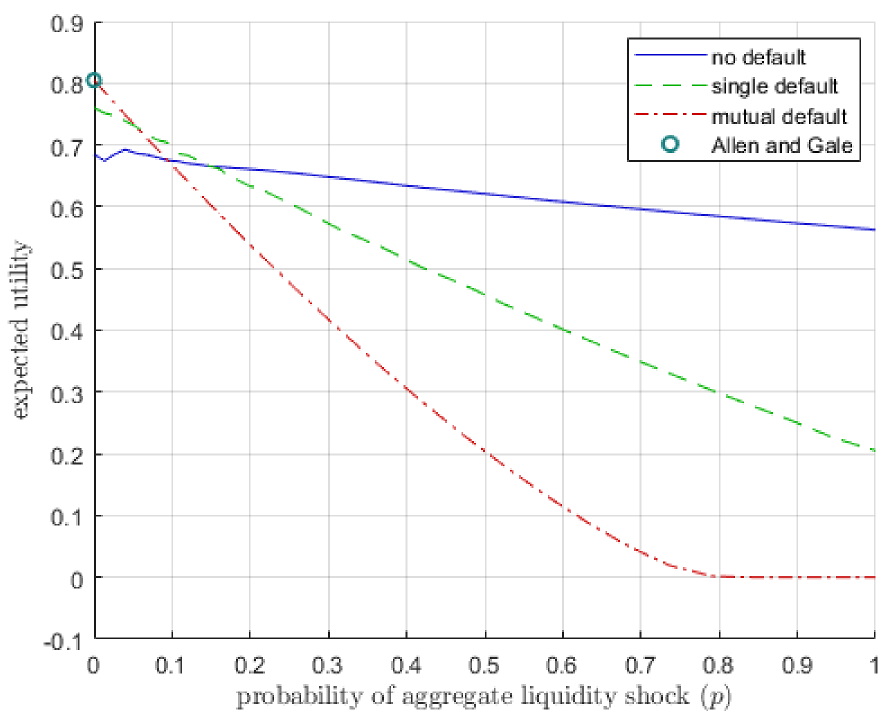

Figure 7: Expected utility at approximate symmetric Nash equilibrium of the regional bank problem across the probability of the aggregate liquidity shock. In this parameter, where the spread between the maturity and early liquidation return of the long asset is small $\frac{R}{r} = \frac{1.1}{0.9}$ (relative to $\frac{R}{r} = \frac{5}{0.1}$ in figure 6 above), there is effectively only one equilibrium type: the no default equilibrium is superior $\forall p \in [0,1]$. Thus: there is no positive probability at which mutual default (or contagion) occurs. However, when the probability is zero, the equilibrium still attains the same expected utility as the allocation in Allen and Gale (2000) within numerical precision, in both the no default and mutual default equilibria. Parameters: $u(c) = \ln(c)$, $R = 5$, $r = 0.1$, $\gamma = 0.5$, $\alpha = 0.15$ and $\varepsilon = 0.1$.

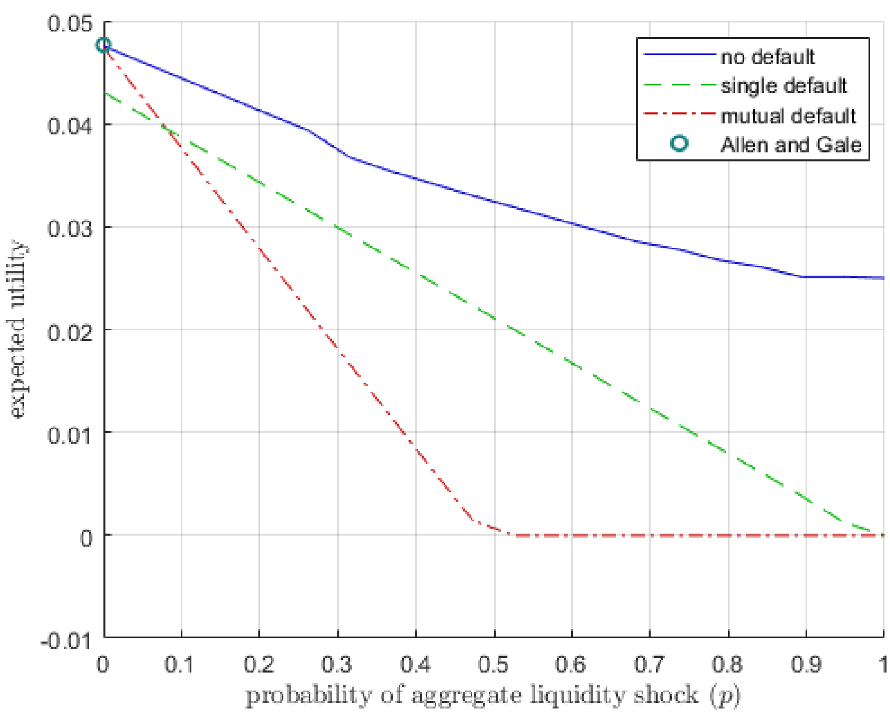

Figure 8: Deposit return at approximate symmetric Nash equilibrium of regional bank problem across the probability of the aggregate liquidity shock. For a positive measure interval around $p = 0$, the mutual default equilibrium is superior and the symmetric equilibrium deposit return is 1, as in the allocation in Allen and Gale (2000). For intermediate probability values ($p \in [0.067, 0.1467]$)), the single default equilibrium is superior, and when the probability is high enough ($p \in (0.1467, 1]$), the no default allocation is superior. There is a discrete downward jump in the equilibrium deposit return between the mutual and single default equilibrium regions, and between the single and no default regions. Parameters: $u(c) = \ln(c)$, $R = 5$, $r = 0.1$, $\gamma = 0.5$, $\alpha = 0.15$ and $\varepsilon = 0.1$.

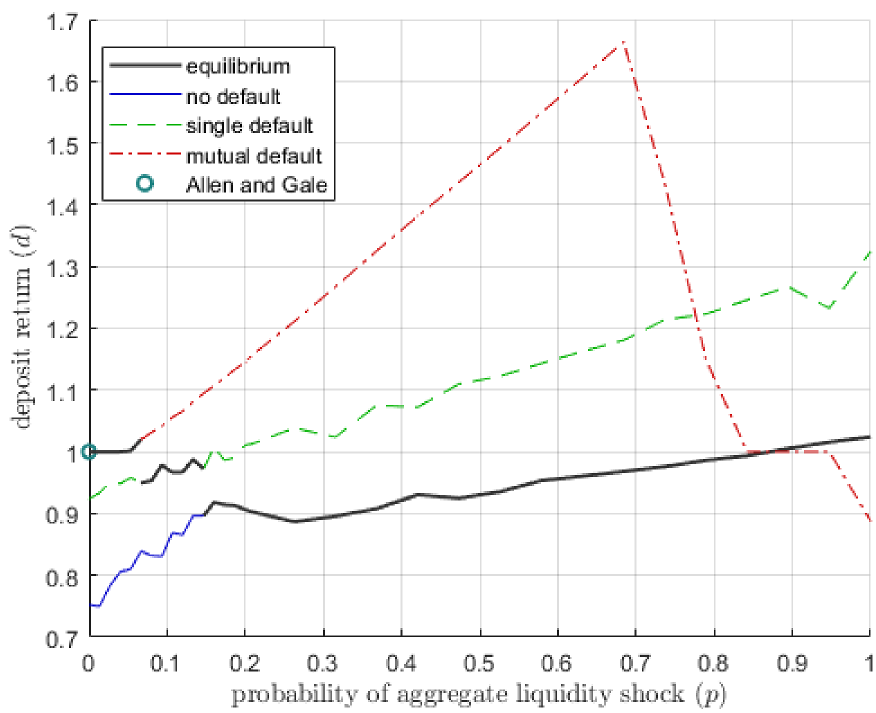

Figure 9: Liquidity at approximate symmetric Nash equilibrium of regional bank problem across the probability of the aggregate liquidity shock. For a positive measure interval around $p = 0$, the mutual default equilibrium is superior and the symmetric equilibrium liquidity is equal to $\gamma = 0.5$, as in the allocation in Allen and Gale (2000). For intermediate probability values ($p \in [0.067, 0.1467]$)), the single default equilibrium is superior, and when the probability is high enough ($p \in (0.1467, 1]$), the no default allocation is superior. There is a discrete upward jump in the equilibrium liquidity between the mutual and single default equilibrium regions, and between the single and no default regions. Parameters: $u(c) = \ln(c)$, $R = 5$, $r = 0.1$, $\gamma = 0.5$, $\alpha = 0.15$ and $\varepsilon = 0.1$.

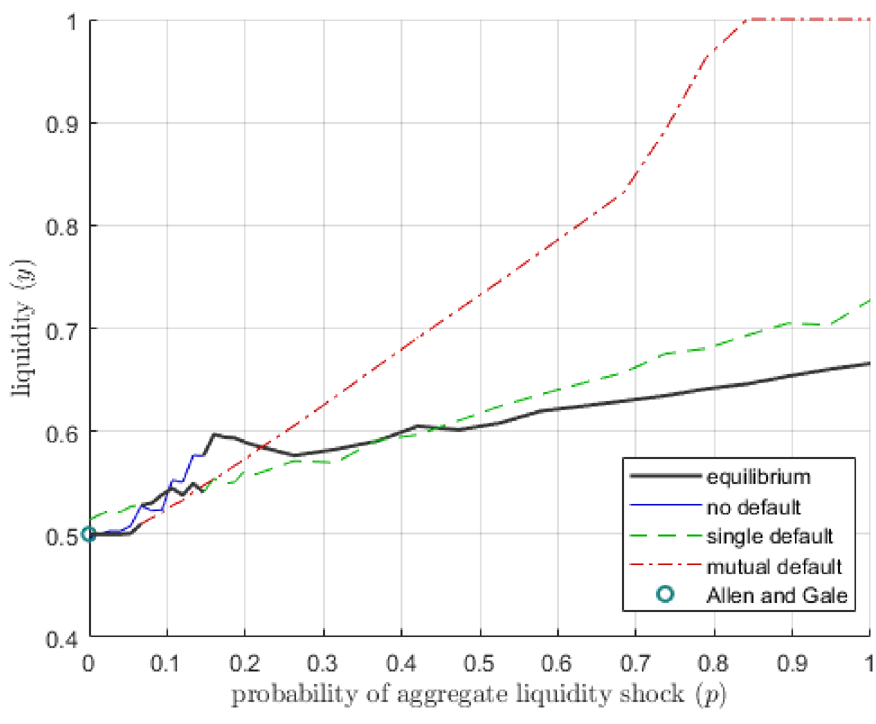

Figure 10: Interbank deposit at approximate symmetric Nash equilibrium of regional bank problem across the probability of the aggregate liquidity shock. When the probability is low enough ($p \in [0, 0.067)$), the mutual default equilibrium is superior and the symmetric equilibrium interbank deposit is equal to or larger than the regional liquidity shock $\varepsilon = 0.1$, which is sufficient to implement the allocation in Allen and Gale (2000). For intermediate probability values ($p \in [0.067, 0.1467]$)), the single default equilibrium is superior and the equilibrium interbank deposit lower than $\varepsilon$. When the probability is high enough ($p \in (0.1467, 1]$), the no default allocation is superior and the equilibrium interbank deposit is equal to the aggregate liquidity shock, $\alpha$. Parameters: $u(c) = \ln(c)$, $R = 5$, $r = 0.1$, $\gamma = 0.5$, $\alpha = 0.15$ and $\varepsilon = 0.1$.

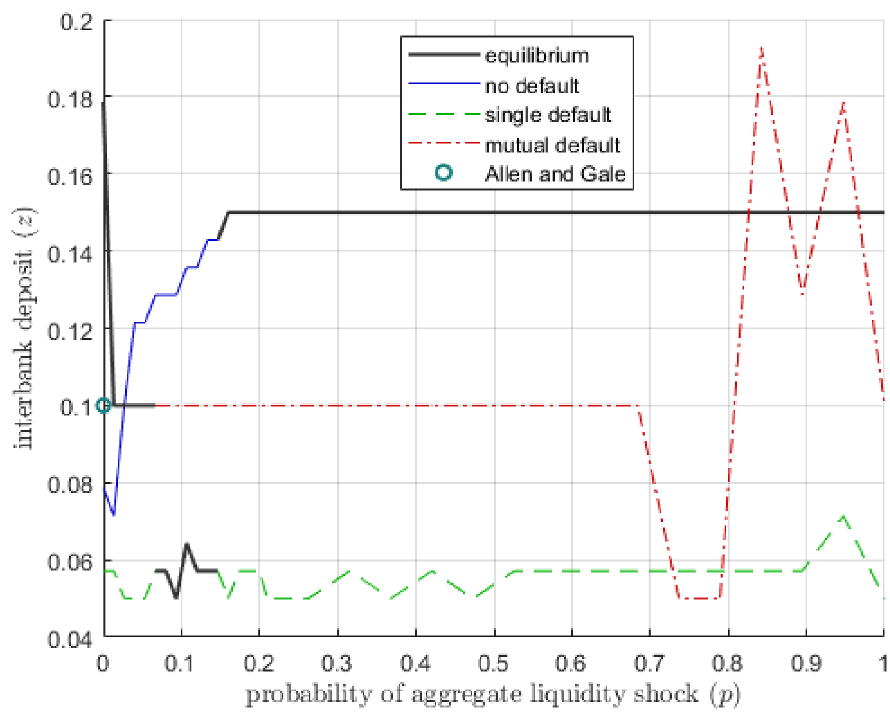

Figure 11: The different value functions in $p$ whose upper envelope forms the $NFL$ value function. Parameters: $u(c) = \ln(c)$, $R = 2$, $r = 0.5$, $\gamma = 0.4$, $\alpha = 0.5$.

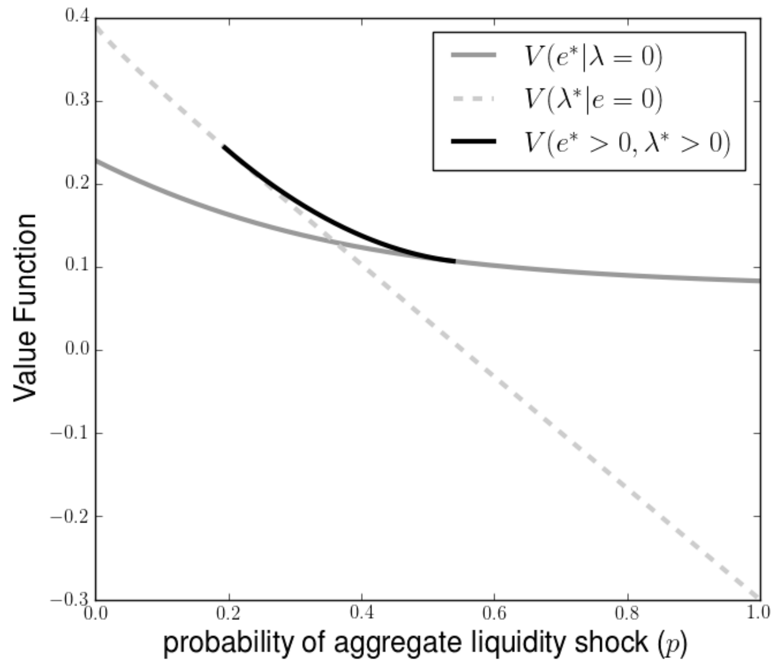


Figure 12: Comparing the $NFL$ and $FL$ value functions in $p$ for two values of the size of the aggregate shock. When the the aggregate liquidity shock is small (left panel), there is no value of the probability where full liquidation is optimal. When the aggregate liquidity shock is large (right panel), there exists a range of probabilities where full liquidation is optimal. In both panels, the slope of the full liquidation value function is steeper in probability than that of the no full liquidation value function (at point of intersection). Parameters: $u(c) = \ln(c)$, $R = 2$, $r = 0.5$ and $\gamma = 0.4$).