Public Debt, Redistribution, and Growth

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Abstract

We study the implications of economic growth for the generosity and the financing of the welfare state. In a simple model without savings, we first derive some benchmark conditions under which both the generosity of the welfare state and tax progressivity are independent of the level of economic development. This homothetic benchmark extends to the case of external public debt if the interest rate equals a threshold which depends not only on preference parameters, but also on the growth rate. When growth rates are high and interest rates are below this threshold, governments of growing economies should issue public debt to finance a generous welfare state initially; tax progressivity will then increase eventually to finance the service of the debt. We show that this force is quantitatively large. Finally, homotheticity also extends to internal savings between the government and the private agents, as long as there is no initial wealth inequality across agents: the optimal welfare state and tax progressivity are constant and the endogenous interest rate is such that there is no savings.

Next, we break homotheticity by introducing *subsistence* levels. We analytically show that in autarky, positive subsistence levels imply a more generous welfare state at earlier stages of development, financed with more progressive taxes. When allowing for external borrowing, even at the interest rate threshold for which there is no motive for borrowing in the homothetic benchmark, subsistence levels translate into interesting dynamics. The welfare state should be more generous initially, but this should solely be financed with public debt. The standard tax smoothing result remains. With internal debt, the government should use public debt to finance an initially more generous welfare state. However, optimal tax progressivity initially increases. The increase in tax progressivity is desirable because it improves the terms of trade for the government.

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Finally, we study the effects of initial *asset inequality*. Preliminary analytical results suggest that wealth inequality generate a force for increasing tax progressivity to improve the terms of trade for the poor households. The magnitude of this price manipulation mechanism depends on initial and future growth.

1 Introduction

The goal of this paper is to understand how the presence of long-run growth alters the optimal design of the welfare state. In particular, should the welfare state become more or less generous with economic growth? How should welfare systems look like in developing economies? Then, what is the legacy for developed economies? As a subsequent topic, we are also interested in the role of public debt in financing the welfare state. In particular, do dynamics of the welfare state generate interesting dynamics for the level of public debt? How does the tradeoff between internal and external debt evolve with growth?

The literature on the optimal design of the welfare state is large, but typically focuses on stationary environment. There are roughly two approaches. First the Mirleesian approach studies the optimal efficiency-redistribution/insurance trade-off under non-linear taxation schemes (Mirrlees 1971, Saez 2001). Recently, this approach has been extended to dynamic settings, see e.g. Werning (2007) or Farhi and Werning (2013) and Golosov, Troshkin, and Tsyvinski (2013). On the other hand, some papers have analyzed that same tradeoff in Ramsey environments with heterogeneous agents – see for instance Bhandari, Evans, Golosov, and Sargent (2017b) for a theoretical analysis, and Bhandari, Evans, Golosov, and Sargent (2017a) or Dyrda and Pedroni (2017) for a quantitative investigation. All these studies have in common that the implications of long-run economic growth on the generosity of the welfare state are ignored. Our contribution is to extend this literature on redistributive taxation by adding long-run growth. We ask (i) how does the generosity of the welfare state change over time as the economy grows and (ii) to what extent does the government use public debt to finance the welfare state.

Our starting point is a static model with two types of individuals, high skilled and low skilled type. As in Sheshinski (1972), the government redistributes income through an affine tax system: a linear tax on labor income, and a lump-sum rebate. In this static set-up, we can only study the first question: how does the generosity of the welfare state depend on the level of economic development? We model this change in economic development as a change in labor productivity that leaves inequality in gross incomes unchanged. Under some fairly mild conditions on the curvature of the utility function – the constant relative risk aversion case – we do find that the generosity of the welfare state and tax progressivity are independent of the level of economic development. The tax progressivity, defined as the linear tax rate, and the generosity of the welfare state, defined as the ratio of the lump sum transfer over GDP per capita, are both constant.

This result breaks down when non-homotheticicities in the utility function are introduced. We study an environment where preferences exhibit a subsistence level – a modification to the benchmark environment which does break homotheticity. For this case, we find that both tax progressivity and the generosity of the welfare state decrease in the level of economic development.

We then move to a dynamic extension of this model and consider external and internal public debts as alternatives means of financing the welfare state. For external debt, we consider the case where the government can borrow abroad at some risk free rate but households cannot. We derive the level of this risk-free rate at which the government would be indifferent between issuing government debt or not. This level of the risk free rate depends on the rate of growth, the curvature of the utility function, the discount factor of individuals and the labor supply elasticity. For reasonable calibrations, this threshold is rather high. This implies that governments would typically like to borrow. Further, this debt-issuing implies that the generosity of the welfare state decreases over time, while tax progressivity, on the other hand, increases over time. We show that, absent default concerns, this force is quantitatively large.

Finally, we study the case of internal debt. As recently shown by Bhandari, Evans, Golosov, and Sargent (2017b), the level of internal government debt is not determined: only inequality in bond holdings is. If the starting point of the economy is such that all agents hold zero initial assets, then we find that the generosity of the welfare state is constant and no agents (neither individuals nor the government) have non-zero assets. The equilibrium interest rate is the same as the threshold level derived in the case with external debt. Adding initial wealth inequality generates standard one-period price effects, with taxes initially increasing when the high-productivity household is also wealthier. The price effect also translates into increasing benefits. Finally, we show that in the presence of subsistence levels, and even without wealth inequality, the government should use public debt to finance an initially more generous welfare state. However, optimal tax progressivity increases over time. The increase in tax progressivity is desirable because it improves the terms of trade for the government (and decreases them for the skilled agents).

We conclude this introduction by a quick empirical discussion of the long-run dynamics of the welfare state in a set of countries. Section 2 present the results in autarky. Section 3 and 4 present analytical results in a two-period set up for external and internal bond markets, respectively. Section 5 extend the analysis to the infinite-horizon case, and Section 6 concludes.

1.1 Evidence

To motivate further the question investigated in this paper, we first show preliminary evidence on the dynamics of the welfare state. In particular, we document the fact that the size of the welfare state is typically not constant: in most developed countries, it has exhibited a negative long-run trend over the past twenty years.

We measure the welfare state using a comparable statistic across countries: social assistance. Social assistance is typically defined as a standard benefit to meet basic survival needs. The statistic used below is provided in the Social Assistance and Minimum Income Protection Interim Data-Set (SAMIP), a new data-set constructed by the Swedish Institute for Social Research (Stockholm University). This data is available mostly for developed countries (mostly Europe, Japan, Australia, Canada and the United States), yearly from 1990 to 2013. It describes monthly averages of social assistance granted to single adults, below retirement age, without children; it excludes housing costs, special needs benefits (like disability expenses), or one-off payments for occasional needs.

We compare this statistic to GDP per capita across 21 countries. In particular, we normalize both variables to be equal to unity in 1990. As it can be seen in Figure 1, GDP per capita has grown faster than social assistance in almost all countries – the notable exception being Japan, which welfare state became discretely more generous in the early 2000s. While in Portugal or in Italy, GDP growth is only slightly larger than the growth in social assistance, in other cases – like in Norway or Spain – GDP per capita has increased twice as much as social assistance since 1990.

We see this as preliminary evidence that the generosity of the welfare state has typically decreased over time. We plan to document further this relationship by analyzing developing economies as well.

2 Autarky

As a first step, we elaborate the role of the level of economic development on the optimal size of the welfare state within a static model, that is, in a model without private nor public savings. Concretely, we consider a simple economy with two skill levels, high skilled individuals with ability θ_h and low skilled individuals with ability $\theta_l < \theta_h$. We model economic development as factor productivity: An individual of type $i = l, h$ that exerts l units of labor effort, does produce $(1 + g)l\theta_i = (1 + g)y_i$ of output, where g captures the level of economic development and y_i captures efficiency units of labor supply of type i. Our goal in this section is to provide comparative statics on the generosity of the welfare state with respect to g. The idea is to keep

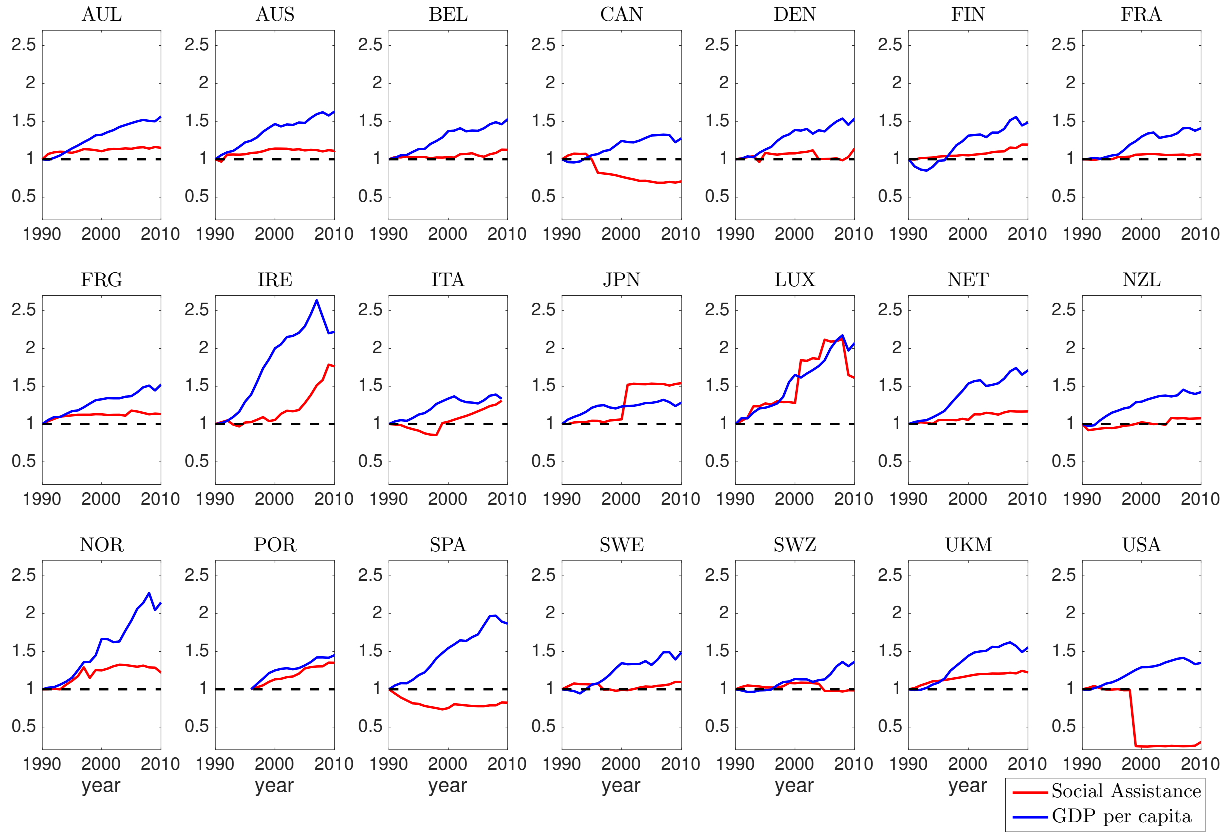

Figure 1: Evolution of the welfare state: long-run dynamics across countries

Notes: The blue line depicts GDP per capita in nominal terms, normalized by 1 in 1990 (source: PWT). The red line depicts Social Assistance in nominal terms, normalized by 1 in 1990 (source: SAMIP). Data: 1990-2010. Countries: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Japan, Luxembourg, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom, United States of America.

inequality (i.e. relative productivities) fixed but just scale up the economy.

2.1 Modeling the Welfare State

Formally, we study an optimal tax problem as in Sheshinski (1972). We assume that the government levies a linear tax rate τ on labor income. Tax revenues are redistributed through a lump sum transfer b. We define the progressivity of the tax system as τ. We define the generosity of the welfare state $\tilde{b} = b/((1+g)\bar{y})$, where $(1+g)\bar{y}$ is per capita output. In this static benchmark model without government debt, b and τ also directly determine how inequality in gross incomes translates into inequality in after tax incomes. In our more general dynamic setting, this is no longer true as government debt can also be used to finance the welfare state.

2.2 Homothetic Preferences

2.2.1 No Income Effects

We start with the most simple case where preferences do not exhibit wealth effects on labor supply, i.e. individuals preferences are of the form $c - \frac{l^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}$ and consumption of type $i = l, h$ is given by $c = (1+g)l\theta_i(1-\tau) + b$, where b is the lump sum transfer, τ is the linear tax rate and g is the level of economic development. Individual optimality conditions directly imply the following closed-form solution for optimal labor supply:

$$y_i(\tau, g) = ((1+g)(1-\tau))^{\varepsilon} \theta_i^{1+\varepsilon}. \tag{1}$$

Denote the share of low and high skilled individuals by f_l and f_h respectively and assume that $f_l + f_h = 1$. The government's assigns Pareto weights $\tilde{f}_l$ and $\tilde{f}_h$ to both types, which we normalize such that $\tilde{f}_l + \tilde{f}_h = 1$. The government then solves

$$\max_{\tau, b} \sum_{i=l,h} \left((1+g)(1-\tau) y_i(\tau, g) + b - \frac{\left(\frac{y_i(\tau,g)}{\theta_i}\right)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) \tilde{f}_i$$

subject to a government budget constraint

$$\tau_t \sum_{i=l,h} (1+g) y_i(\tau, g) f_i \geq b, \tag{2}$$

and subject to individual optimality (1). The first-order condition with respect to τ reads as

$$\frac{\partial \mathcal{L}}{\partial \tau} = -(1+g) \sum_{i=l,h} y_i(\tau, g) \tilde{f}_i + \lambda (1+g) \sum_{i=l,h} y_i(\tau, g) f_i + \lambda (1+g) \tau \sum_{i=l,h} \frac{\partial y_i(\tau, g)}{\partial \tau} f_i = 0.$$

The first order condition for b directly reveals that $\lambda = 1$, which is a standard result for the marginal value of public funds in static models with quasi-linear preferences. Simple calculations then reveal that the optimal linear income tax is given by:

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h} \left(1 - \frac{\tilde{f}_i}{f_i}\right) y_i(\tau, g) f_i}{\varepsilon \sum_{i=l,h} y_i(\tau, g) f_i}.$$

Substituting for $y_i(\tau, g)$ using (1) yields

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h} \left(1 - \frac{\tilde{f}_i}{f_i}\right) \theta_i^{1+\varepsilon} f_i}{\varepsilon \sum_{i=l,h} \theta_i^{1+\varepsilon} f_i}$$

which directly implies that the optimal linear tax rate is independent of the level of economic development g.

What is the logic behind this result? Relative wages, i.e. inequality does not change as the economy grows. Thus, the desire to redistribute is not altered by growth, neither are the efficiency costs of taxation (due to the iso-elastic specification).

Adding Curvature. We now show that the problem is not affected if curvature on the utility function is added in the following way

$$U\left(c-\frac{l^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}\right)=\frac{\left(c-\frac{l^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}\right)^{1-\gamma}}{1-\gamma}.$$

The government's problem now reads as

$$\max_{\tau,b}\sum_{i=l,h}U\left(\left(1-\tau\right)\left(1+g\right)y_i(\tau,g)+b-\frac{\left(\frac{y_i(\tau,g)}{\theta_i}\right)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}\right)\tilde{f}_i$$

subject to (2) and (1).

The optimal tax rate for this case is given by:

$$\frac{\tau}{1-\tau}=\frac{\sum_{i=l,h}\left(1-\frac{U_i'}{E(U')}\right)\theta_i^{1+\varepsilon}f_i}{\varepsilon\sum_{i=l,h}\theta_i^{1+\varepsilon}f_i} \tag{3}$$

With curvature in the utility we have

$$U_i'=\left(\frac{\left(\left(1-\tau\right)\left(1+g\right)\theta_i\right)^{1+\varepsilon}}{\varepsilon+1}+b\right)^{-\gamma}$$

Combining (1) and (2), the government's budget constraint reads as:

$$b=\tau\left(1-\tau\right)^{\varepsilon}\left(1+g\right)^{1+\varepsilon}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}$$

and hence

$$U_i'=\left(\frac{\left(\left(1-\tau\right)\left(1+g\right)\theta_i\right)^{1+\varepsilon}}{\varepsilon+1}+\tau\left(1-\tau\right)^{\varepsilon}\left(1+g\right)^{1+\varepsilon}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}$$

which can be written as

$$U_i' = \left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h} f_i\theta_i^{1+\varepsilon}\right)^{-\gamma} \tag{4}$$

which directly yields that the ratio $U'/E(U')$ is independent of the level of economic development g. As a consequence, the optimal tax rate in (16) is independent of the level of economic development.

Proposition 2.1. *If individual preferences do not exhibit income effects on labor supply, the optimal generosity of the welfare state and the optimal progressivity of the tax code are independent of the level of economic development g.*

2.2.2 Accounting for Income Effects

The model without income effects had the implication that labor supply keeps on growing with the level of economic development as there is no countervailing income effect. Whereas this made the analysis quite tractable, it may be a problematic assumption in a growth context since this ever increasing labor supply is counterfactual.

We now show that this assumption was not driving our results on the constant welfare state. For this purpose, we assume the following separable utility function with income effects:

$$U(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{\left(\frac{y}{\theta_i}\right)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}. \tag{5}$$

As we show in the appendix, the optimal tax rate satisfies

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h}\left(1-\frac{u_i'}{E(u')}\right)y_i f_i}{\sum_{i=l,h} f_i \varepsilon^i_{y,1-\tau} y_i + \varepsilon\sum_{i=l,h}\frac{\gamma}{1+\varepsilon\gamma+\frac{b}{(1-\tau)(1+g)y_i}}\sum_{i=l,h}\frac{u_i'}{E(u')}y_i f_i}. \tag{6}$$

To show that τ is again independent of g, we first assume that τ is independent and see what this would imply for the dependence of gross income and the lump sum transfer on g. We then show that these implications are consistent with τ being independent of g. It turns out pedagogical to start with the case of $\gamma = 1$, i.e. log preferences.

Log Preferences For log preferences, the first-order condition of the agent reads as

$$\frac{(1-\tau)(1+g)}{(1-\tau)(1+g)y+b} = \left(\frac{y}{\theta}\right)^{\frac{1}{\varepsilon}}\frac{1}{\theta}$$

This shows that y is independent of g if $b \propto (1+g)$. Note that $\frac{dy}{dg} = 0$ is consistent with $b \propto (1+g)$ if τ is independent of g, this can directly be seen in the government budget constraint (2). Thus, what remains to be shown is that for y_i being independent of g, the government indeed wants to set a level of τ that is independent of g. To show this, one has to verify (6) being consistent with this. In the appendix we show that the optimal τ is indeed independent of g .

CRRA Preferences For the non-log case, y does depend on g. It depends positively (negatively) on g if $\gamma < 1$ ($\gamma > 1$). For a constant τ, we have y growing at rate $1+\alpha$ (we formally define α in the appendix) and b at rate $(1+g)(1+\alpha)$. We show in the appendix that this then also implies that the size of the welfare state is independent of the level of economic development.

Proposition 2.2. *Assume that individual preferences are given by (5), then the optimal generosity of the welfare state and the optimal progressivity of the tax code are independent of the level of economic development g.*

Proof. See Appendix A.1. □

2.3 Introducing a level of subsistence income

So far we assumed homothetic preferences. However, an important issue that we cannot capture with these preferences is the concept of absolute poverty. This may be a crucial concept however if we think about countries like India or China, where a sizeable fraction of the population is very poor. We therefore now change individual preferences to capture exactly this. In the mainbody we stick to preferences without income effects, but the result holds more generally as we show in the appendix.

Concretely, we assume that preferences satisfy

$$U\left(c - \kappa - v(l)\right), \tag{7}$$

where $\kappa > 0$ and $U', -U'' > 0$. Whereas optimal individual labor supply behaviour is independent of κ and still described by (1), the ratio of marginal utilities between high skilled and low skilled agents is altered. Intuitively, the lower the level of economic development g, the stronger the desire to redistribute as the low skilled individual is closer to the subsistence level κ. We formally summarize these policy implications in the following proposition.

Proposition 2.3. *Assume that individual preferences are described by (7). The optimal generosity of the welfare state and the optimal progressivity of the tax code are both decreasing in the level of economic development, i.e.* $\frac{\partial \tau}{\partial g} < 0$ *and* $\frac{\partial \tilde{b}}{\partial g} < 0$.

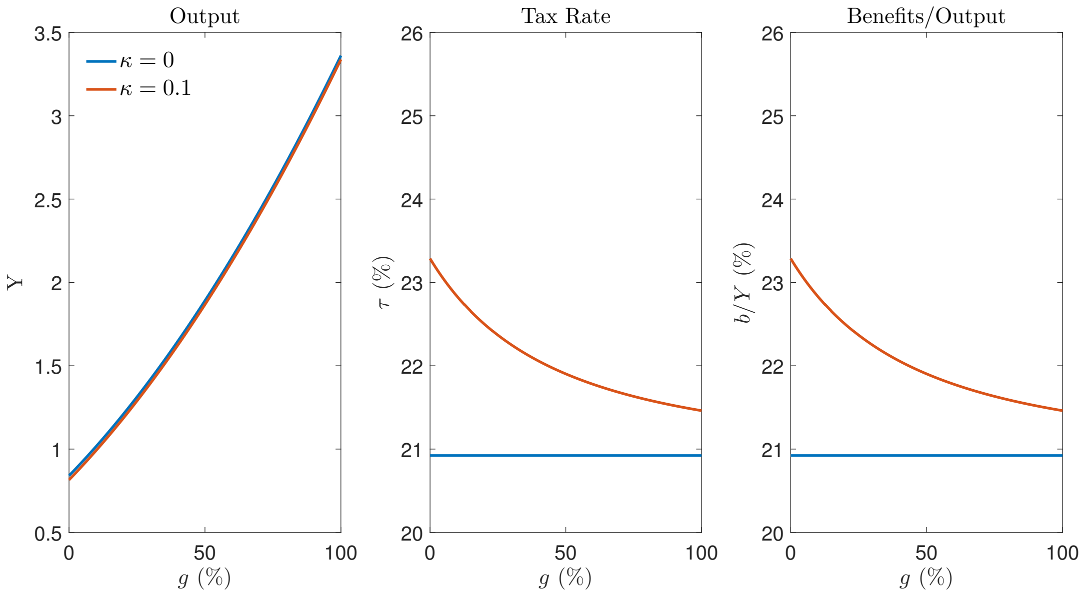

Figure 2: Welfare state and Growth without Income Effects

Notes: Optimal output, tax rate and social assistance (measured as benefits-over-output ratio), for different levels of growth g; preferences without income effects. The blue lines depict the homothetic case, while the red lines depict the case with a positive level of subsistance κ.

Proof. With a subsistence level in the utility function, the marginal utility of consumption (4) is given by

$$U_i' = \left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1} + \frac{\tau}{1-\tau}\sum_{i=l,h} f_i\theta_i^{1+\varepsilon} - \frac{\kappa}{(1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}}\right)^{-\gamma}$$

which implies that the ration $U_i'/E(U_i')$ is no longer independent of g. Instead we have that $U_l'/E(U_i')$ is decreasing in g and $U_h'/E(U_i')$ is increasing in g. Combining this with (16) directly implies $\frac{\partial\tau}{\partial g} < 0$ and $\frac{\partial b}{\partial g} < 0$. □

2.4 Numerical Illustration

We provide a very simple numerical exercise to illustrate our proposition. In particular, we compute the optimal size of the welfare state, as captured by the tax rate τ, and the social assistance over output ratio $\tilde{b}$ (these two objects being equal in a static model without debt). We adopt the following preference calibration: $\varepsilon = 1, \gamma = 2$. Figure 2 shows results for preferences without income effects: as expected, when $\kappa = 0$, the size of the welfare state is constant; when $\kappa > 0$, the size of the welfare state shrinks with growth g. Figure 10 (in Appendix) shows similar results in the presence of income effects.

3 External Public Debt

The previous section has established an important benchmark for the generosity of the welfare state being independent of the level of economic development. This static setting had the shortcoming that public debt was not considered as a means of financing the welfare state. We now overcome this shortcoming by considering a two-period model with external public debt. For now, we focus on the case without income effects.

3.1 Theoretical Results

The government's problem now reads as

$$\max_{\tau_1,\tau_2,b_1,b_2} \sum_{i=l,h} \tilde{f}_i \left(U_1^i + \beta U_2^i\right)$$

subject to a dynamic budget constraint

$$\tau_1 \sum_{i=l,h} y_{i1} + \frac{1}{R}\tau_2 \sum_{i=l,h} y_{i2} \geq b_1 + b_2 \frac{1}{R}$$

where β is the individual discount factor and R captures the interest rate on external government debt. The first-order conditions of the planning problem are given by

$$\frac{\partial \mathcal{L}}{\partial \tau_1} = -\sum_{i=l,h} y_{i1}(\tau_1) U'_{i1} \tilde{f}_i + \lambda \sum_{i=l,h} y_{i1}(\tau_1) f_i + \lambda \tau_1 \sum_{i=l,h} \frac{\partial y_{i1}(\tau_1)}{\partial \tau_1} f_i = 0.$$

$$\frac{\partial \mathcal{L}}{\partial \tau_2} = -\sum_{i=l,h} y_{i2}(\tau_2) U'_{i2} \tilde{f}_i + \frac{\lambda}{\beta R} \sum_{i=l,h} y_{i2}(\tau_2) f_i + \frac{\lambda}{\beta R}\tau_2 \sum_{i=l,h} \frac{\partial y_{i2}(\tau_2)}{\partial \tau_2} f_i = 0.$$

$$\frac{\partial \mathcal{L}}{\partial b_1} = \sum_{i=l,h} U'_{i1} \tilde{f}_i + \lambda - \lambda \tau_1 \sum_{i=l,h} \frac{\partial y_{i1}(\tau_1)}{\partial b_1} f_i = 0.$$

$$\frac{\partial \mathcal{L}}{\partial b_2} = \beta \sum_{i=l,h} U'_{i2} \tilde{f}_i - \frac{\lambda}{R} + \frac{\lambda}{R}\tau_2 \sum_{i=l,h} \frac{\partial y_{i2}(\tau_2)}{\partial b_2} f_i = 0.$$

Alternatively writing the problem with two government budget constraints and debt, we would have the following first-order condition for debt:

$$\lambda_1 = \beta R \lambda_2. \tag{8}$$

and further we have $\lambda_1 = E(U_1')$ and $\lambda_2 = E(U_2')$. Now we want to start from a situation where the government has no access to credit marktsand would set $\tau_1 = \tau_2$ and $b_2 = (1+g)b_1$. If the government then is allowed to borrow (and households are not), at which rate R would the government be indifferent to borrow?

If we start from a constant welfare state, we have

$$E(U_2')\left((1+g)^{1+\varepsilon}\right)^{\gamma} = E(U_1')$$

which follows from (4). Inserting this into (8) yields

$$E(U_2')\left((1+g)^{1+\varepsilon}\right)^{\gamma} = \beta R E(U_2')$$

which implies that zero government debt would be optimal for the government if

$$R = \bar{R}(g) \equiv \frac{\left((1+g)^{1+\varepsilon}\right)^{\gamma}}{\beta}.$$

We can therefore also conclude that the government would like to borrow if

$$R < \frac{\left((1+g)^{1+\varepsilon}\right)^{\gamma}}{\beta}. \tag{9}$$

Implications for size of the welfare state. We have established that the government would like to borrow if (9) holds. The next important step is to show what this implies for the size of the welfare state in these two periods. In the appendix, we show that the welfare state becomes more generous initially and the tax becomes less progressive. Intuitively, through public debt the government's first period budget constraint gets relaxed: the government can increase spending and decrease taxes. For the second period, the situation is vice versa.

Proposition 3.1. *If external government debt is available, then the government would like to borrow if $\beta(1+r) < ((1+g)^{1+\varepsilon})^{\gamma}$. Further, in this case, will increase τ_2 and decrease τ_1 compared to the case without borrowing, hence $\tau_2 > \tau_1$. At the same time , the government will increase b_1 decrease b_2 compared to the case without borrowing, hence $b_2 < (1+g)b_1$. Thus the generosity of the welfare state decreases over time and the tax progressivity increases over time.*

In ongoing work, we show the generalization of these results to a truly dynamic setting and also show that it extends if the economy starts with an initial level of outstanding debt.

An interesting result that is worth reporting at this point is that condition (9) simplifies to

$$R < \frac{1+g}{\beta}$$

if preferences are of the form $\log(c) - \frac{l^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}$.

3.2 Numerical Illustration

In this section, we provide a numerical illustration. We interpret one period as 10 years. We vary the annual growth rate between 0 and 10% – the higher end is motivated by the high Chinese growth rates. We do set $\beta = 0.96^{10}$, i.e. implying an annual discount factor of 0.96. We assume that half of the population is of low ability and half of the population is of high ability. We set $\frac{\theta_h}{\theta_l} - 1 = \frac{2}{3}$.

We first present numerical illustrations of the key mechanism described above, that is, how the level of the external interest rate affects the willingness to borrow or save of the government. Figure 3 presents the optimal taxes and benefits in two periods as a function of growth, in three cases. The top graphs present the case when the interest rate is exactly equal to the level at which the government is indifferent between saving and borrowing – that is, when $r^\star$ the level of interest rate, exogenously determined, is equal to $\bar{r}(g) \equiv \bar{R}(g) - 1$. As expected, public debt is optimally zero, and taxes and benefits are constant across time and independent of growth. The medium panels show the case where the interest rate is higher than the cut-off value, implying public savings between periods 1 and 2; these dynamics also translate into the fiscal variables: taxes decrease over time, while benefits increase, reflecting the willingness to backload expenses. The bottom graphs illustrate the case when the interest rate is lower than the cut-off value: the government accumulates debt between periods 1 and 2, and frontloads expenses: taxes increase over time, while benefits decrease. In Section 5 we explore the effect of growth quantitatively to show that this channel has large quantitative implications.

3.3 Subsistence

Finally, in this last subsection, we illustrate the role of subsistence when the government can borrow externally. To disentangle this force from the standard arbitrage force described above, we look at the case where the world interest rate is exactly such that, absent the positive subsistence level, the public debt would be zero. With non-homothetic preferences coming from subsistence, growth generates strong dynamics on the welfare state: the higher the growth in the second period, the higher the incentives to borrow and implement a more generous welfare state in the first period. Interestingly, the standard tax smoothing result resists the positive level of

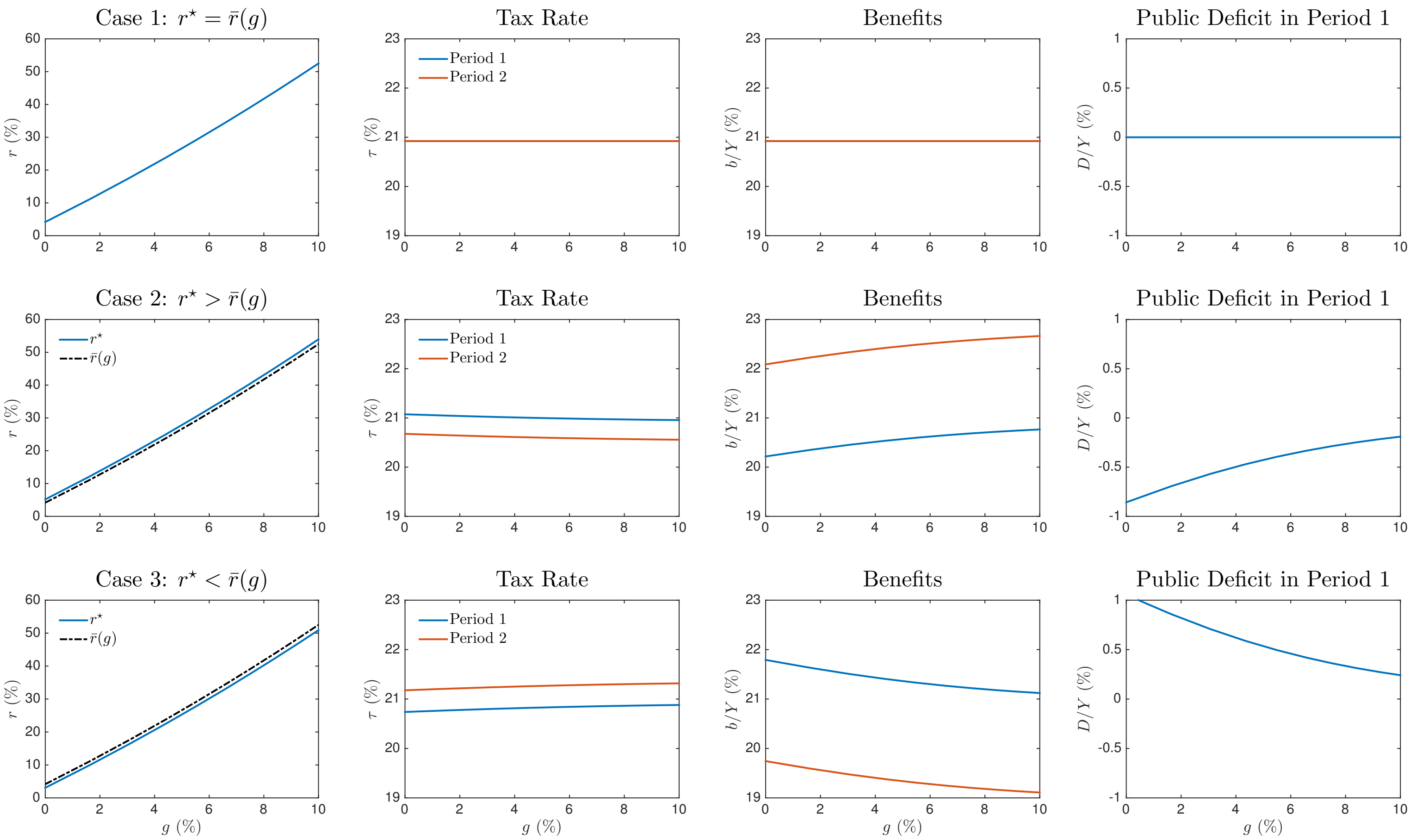

Figure 3: Dynamics of the Welfare State with External Debt: The Role of Interest Rates

Notes: Dynamics of the Welfare State and Public Debt for different values of the annual growth rate. The top line shows the case for $r^\star(g) = \bar{r}(g)$, that is, when the exogenous interest rate is such that the government does not want to borrow or save. The medium line shows the case where $r^\star(g) = 1.1\,\bar{r}(g)$. The bottom line depicts the case where $r^\star(g) = 0.9\,\bar{r}(g)$. Period length is 10 years. Growth rate g and interest rate R are expressed in annual levels.

subsistence: though the benefits decrease between the two periods, the tax rates are optimally constant. We hope to establish this result in the theoretical framework described above.

4 Internal Public Debt

We again focus on a two period model and GHH preferences. We do not assume that the government has access to external credit markets as in Section 3. But we allow for

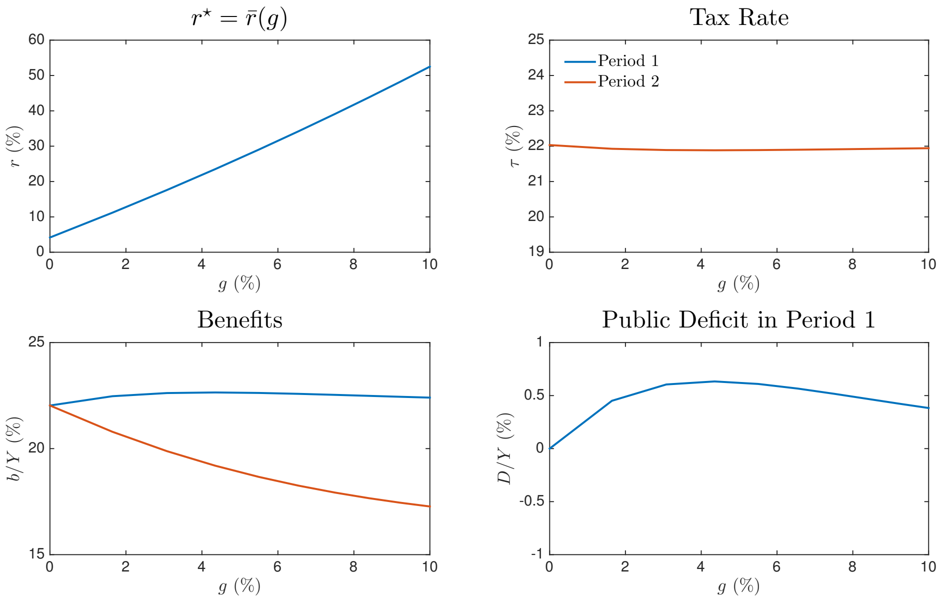

Figure 4: Dynamics of the Welfare State with External Debt: The Role of Subsistence

Notes: Dynamics of the Welfare State and Public Debt for different values of the annual growth rate, under a positive level of subsistence. The interest rate is kept fixed at its cutoff value, such that, without subsistence, the public debt would be zero. Period length is 10 years. Growth rate g and interest rate R are expressed in annual levels.

borrowing and savings arrangements between different individuals and between the government and individuals.

4.1 Ricardian Equivalence and the Distribution of Public Debt

Bhandari, Evans, Golosov, and Sargent (2017b) show how the Ricardian Equivalence holds in closed economies with heterogeneous agents: as long as the government does not face sign restrictions on lump-sum transfers, the level of (domestic) government debt is indeterminate in an optimal competitive equilibrium. Similarly, the initial level of government debt is irrelevant to the optimal allocations. On the other hand, the *distribution* of debt across households is central to determine optimal allocations. Based on this result, one option would be to focus – without loss of generality – on the case where public debt is constant and equal to zero. We choose a different route. We show the welfare state dynamics implied by assuming that the poor household cannot borrow. Varying the share of initial wealth of the richer household still allows to understand how the initial *distribution* of wealth shapes dynamics fo the welfare state, while translating into a *level* public debt uniquely pinned down. We argue that this assumption is realistic for developing economies, where poor households have little to no access to financial markets.

4.2 No initial assets

We first derive a very intuitive result: if both households start with zero debt, there is no trade; the endogenous interest rate is, in equilibrium, equal to the cutoff value $\bar{r}(g)$ defined in Section 3.

Proposition 4.1. *In a closed economy with no initial assets and no initial government debt, the equilibrium interest rate is given by* $R = \frac{((1+g)^{\varepsilon})^{-\gamma}}{\beta}$*. All individuals and the government save exactly zero.*

Proof. See Appendix A.4. □

To see this, let us recall the Euler equation of the two households: $U_{c,1,i} = \beta R U_{c,2,i}$, for $i = h, l$. It is easy to check that if initial assets positions are equal to zero, and if R fulfils the condition expressed in Proposition 4.1, then the Euler equations hold when savings from $t = 1$ to $t = 2$ are equal to zero.

4.3 How does the distribution of initial assets matter?

Now, let us analyze the more general case where household h enters with some asset a_1 – and so, by market clearing, the government l's initial debt is a_1/f_h. We call a_2 the level of saving of household h at the end of period 1, and, similarly, a_2/f_h the outstanding public debt. The budget constraints are, for household h at $t = 1, 2$:

$$c_1^h = (1-\tau_1)y_1^h + b_1 + a_1 - \frac{a_2}{1+r}$$
$$c_2^h = (1-\tau_2)y_2^h + b_2 + a_2$$

and for household l at $t = 1, 2$:

$$c_1^l = (1-\tau_1)y_1^l + b_1$$
$$c_2^l = (1-\tau_2)y_2^l + b_2.$$

Assuming GHH preferences, we have a simple formula for $y_{i,t}$, which depends only on τ_t, as already described in (1). The two Euler equations become:

$$\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1} + b_1 + a_1 - \frac{a_2}{1+r}\right)^{-\gamma} = \beta(1+r)\left(\frac{(1-\tau_2)\,(1+g)^{1+\varepsilon}\,\theta_h^{1+\varepsilon}}{\varepsilon+1} + b_2 + a_2\right)^{-\gamma} \tag{10}$$

and

$$\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1} + b_1 \right)^{-\gamma} = \beta(1+r) \left(\frac{(1-\tau_2)\,(1+g)^{1+\varepsilon}\,\theta_l^{1+\varepsilon}}{\varepsilon+1} + b_2 \right)^{-\gamma} \tag{11}$$

The government's maximization problem is now

$$\max \sum f_i \left(U_1^i + \beta U_2^i \right)$$

subject to

$$b_1 + B_1 \leq \tau_1 \left(1-\tau_1\right)^{\varepsilon} \left\{ \theta_h^{1+\varepsilon} + \theta_l^{1+\varepsilon} \right\} + \frac{B_2}{R}$$

and

$$B_2 + b_2 \leq \tau_2 \left(1-\tau_1\right)^{\varepsilon} \left(1+g\right)^{1+\varepsilon} \left\{ \theta_h^{1+\varepsilon} + \theta_l^{1+\varepsilon} \right\}$$

and subject (17) and (18).

In some ongoing work, we first derive the Generalized Euler Equation, that is, the government's Euler equation of the problem stated above. Then, using perturbation arguments, we obtain the signs of derivatives for τ_1 and τ_2 for small perturbations of a_1 around 0. We present next numerical work that supports this analysis.

4.4 Numerical Illustration

Figure 5 shows the dynamics of the welfare state under two initial wealth distributions. The top panel shows the case where the two agents have an equal distribution of welfare (normalized to zero, as we normalize the wealth of the poor household to be zero). With homothetic preferences, we retrieve the standard result that the welfare state is constant across time and across growth levels: interest rates and benefits are constant, as well as public debt.

The bottom panel is more intriguing. In that environment, we start with a positive amount of wealth for the more productive household, which holds about 24% of output. As the mass of the productive household is half, public debt is initially equal to 12%. This unequal wealth distribution generates a new non-homotheticity in this environment, which translates into rich dynamics of the welfare state. First, tax rates increase over time. This result is reminiscent of the *price manipulation* mechanism described by Lucas Jr and Stokey (1983): the indebted government has an incentive to offer a higher tax rate in the future, such that the wealthier household is more willing to save today; as such, this is a way for the government to decrease interest rates at which it can roll over its debt. This translates into an increasing benefits over output ratios over time. Finally, note that public debt decreases over time for low levels of growth, while it is increasing over time for high levels of growth. The price effect also

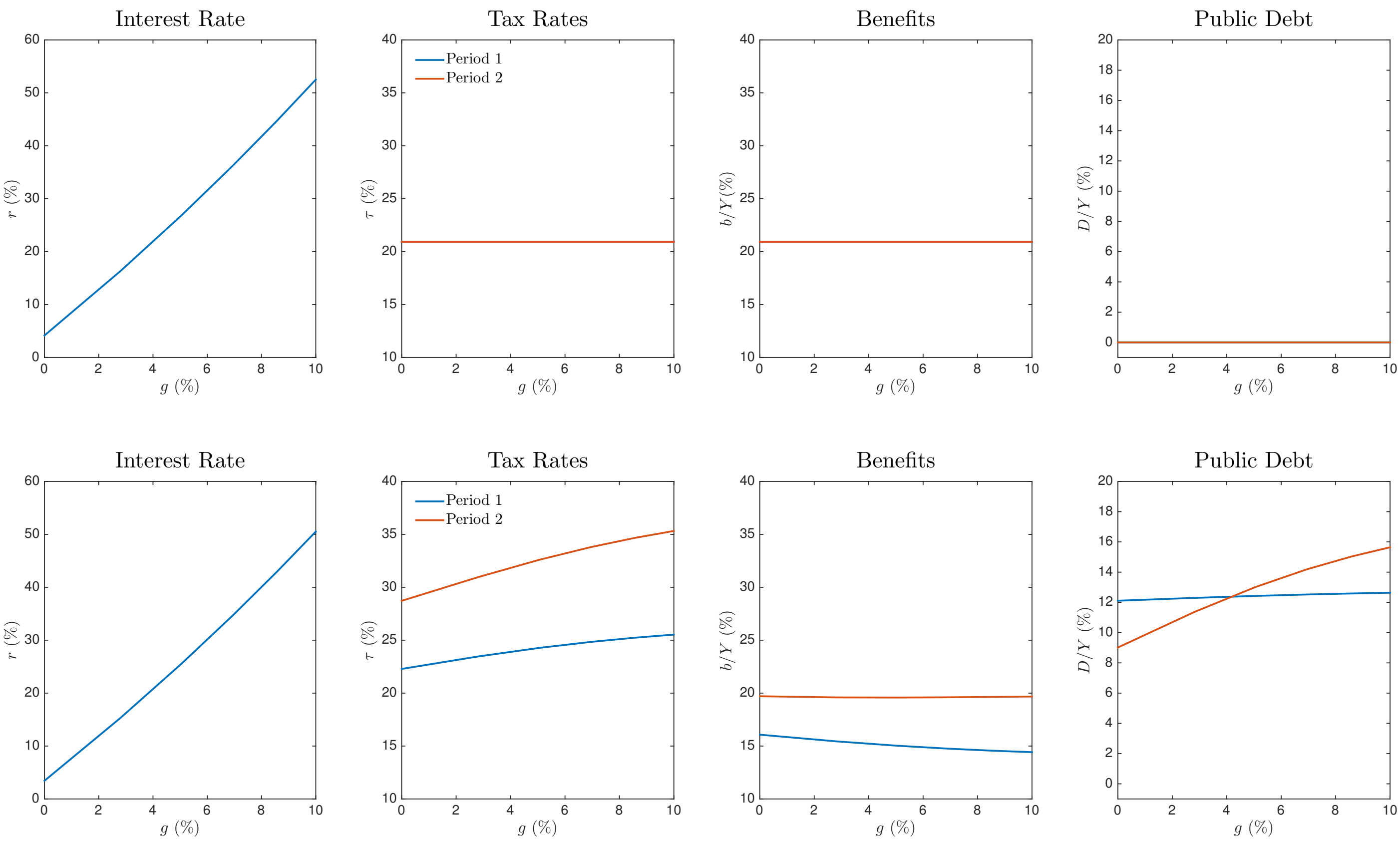

Figure 5: Dynamics of the Welfare State with Internal Debt: The Role of Wealth Distribution

Notes: Dynamics of the Welfare State and Public Debt for different values of the annual growth rate, under two levels of wealth distribution. The top panel assumes equal wealth in the first period. The second panel depicts the case when the more productive household is also more wealthy. Period length is 10 years. Growth rate g and interest rate R are expressed in annual levels. Interest rate is now endogenous.

seems reinforced with growth.

The numerical results are presented for GHH preferences, without wealth effect. However, we have extended the code to CRRA preferences: results are qualitatively unchanged.

4.5 Subsistence

We finally explore the effects of subsistence in Figure 6. Subsistence breaks homotheticity in a way which is qualitatively different from the wealth inequality mechanism

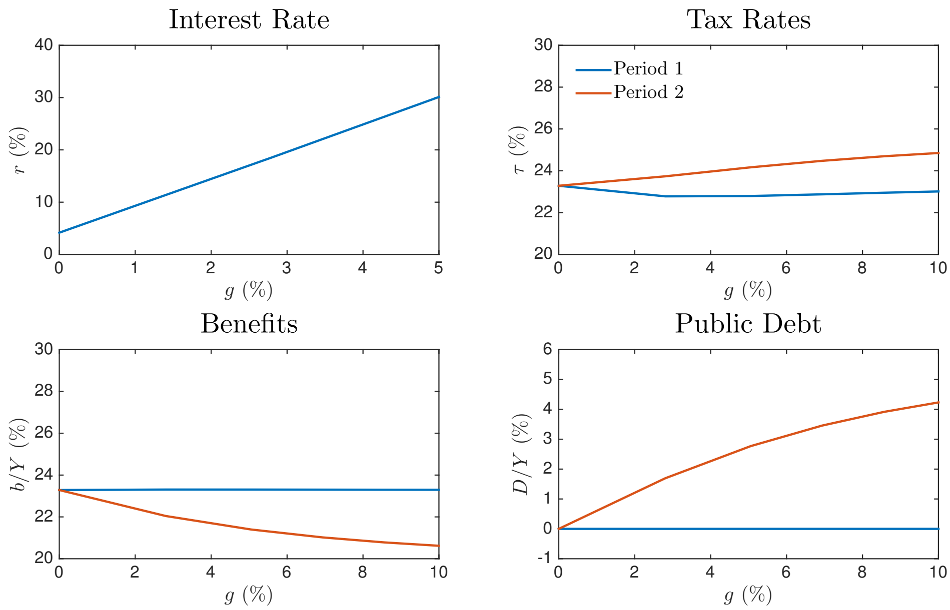

Figure 6: Dynamics of the Welfare State with Internal Debt: The Role of Subsistence

Notes: Dynamics of the Welfare State and Public Debt for different values of the annual growth rate, under a positive level of subsistence. Period length is 10 years. Growth rate g and interest rate R are expressed in annual levels. Interest rate is now endogenous.

described above. To isolate this effect, we assume again that initial wealth distribution is equalized across households: without a positive subsistence level, public debt would be constant and equal to zero, across time and growth level. With subsistence, though, the welfare state becomes more generous initially, and the stronger the growth in the second period the larger the decrease in the generosity of the welfare state across time. This also translates into an increase of tax rates over time, and therefore a positive public debt in the first period. Again, the larger the growth in the second period the larger the public debt. As public debt is equal to the asset of the more productive household through market clearing, wealth inequality across households *increases* with growth and over time.

Again, the numerical results are presented for GHH preferences, without wealth effect. However, we have extended the code to CRRA preferences: results are qualitatively unchanged.

5 Infinite-horizon

In this last section we focus on infinite horizon to conduct quantitative analysis. Importantly, we assume that growth phases-out as the economy grows, to become 0 in the long-run so that productivity converges to a level A_∞. Practically, productivity starts at $A_0 = 1$, and grows at rate g_t: $A_{t+1} = (1 + g_t)A_t$. The path for growth is assumed to be:

$$1 + g_t = 1 + \alpha_0 e^{-\alpha_1 t},$$

where α_0 captures initial growth and α_1 captures the speed of convergence to the steady state value of productivity.

5.1 Autarky

Under autarky the problem is very simple:

$$\begin{aligned}
\max_{\{\tau_t, b_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \sum_{i=\ell,h} u\left(c_t^i, \frac{y_t^i}{\theta_i A_t}\right) \tilde{f}_i \\
\text{s.t.} \quad & c_t^i = (1-\tau_t) y_t^i + b_t \quad \forall i = \ell, h \\
& u_{n,t}^i = -u_{c,t}^i (1-\tau_t)\theta_i A_t \quad \forall i = \ell, h \\
& b_t = \tau_t \sum_{i=\ell,h} f_i y_t^i
\end{aligned}$$

The three budget constraints guarantee that feasibility hold. The intratemporal first-order condition guarantees that the problem is optimal for the household.

5.2 External debt

When the government can borrow on international capital markets at exogenous interest rate R, the problem becomes:

$$\max_{\{\tau_t, b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{i=\ell,h} u\left(c_t^i, \frac{y_t^i}{\theta_i A_t}\right) \tilde{f}_i \tag{E.1}$$

$$\text{s.t.} \quad c_t^i = (1-\tau_t) y_t^i + b_t \quad \forall i = \ell, h \tag{E.1.a}$$

$$u_{n,t}^i = -u_{c,t}^i (1-\tau_t)\theta_i A_t \quad \forall i = \ell, h \tag{E.1.b}$$

$$b_t = \frac{B_{t+1}}{1+r} + \tau_t \sum_{i=\ell,h} f_i y_t^i - B_t \quad \forall t$$

A dual approach. We do not follow the standard primal approach: we express everything in terms of fiscal instruments τ and b. Note that given period-t fiscal policy (τ_t, b_t) and current level of growth g_t, period-t allocations $\{c_t^\ell, y_t^\ell, c_t^h, y_t^h\}$ are fully characterized by the two households' budget constraints (E.1.a) and the two households' first-order conditions (E.1.b). Define $\hat{c}^\ell(\tau_t, b_t, A_t)$ the consumption of the low-skill type under the period-t fiscal policy and growth (and similarly $\hat{c}^h$, $\hat{y}^\ell$, $\hat{y}^h$).[1] Finally, define $\hat{U}(\tau_t, b_t, A_t)$ as:

$$\hat{U}(\tau_t, b_t, A_t) \equiv \sum_{i=\ell,h} u\left(\hat{c}^i_t(\tau_t, b_t, A_t), \frac{\hat{y}^i_t(\tau_t, b_t, A_t)}{\theta_i A_t}\right) \tilde{f}_i.$$

We can therefore rewrite problem (E.1) as:

$$\max_{\{\tau_t, b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \hat{U}(\tau_t, b_t, A_t) \quad \text{s.t.}$$
$$B_t + b_t = \frac{B_{t+1}}{1+r} + \tau_t \sum_{i=\ell,h} f_i y^i_t \quad \forall t$$

Or, using time-0 formulation for the government's borrowing constraint, where Φ is the multiplier associated with the time-zero budget constraint:

$$\max_{\{\tau_t, b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \hat{U}_t(\tau_t, b_t) \quad \text{s.t.} \tag{E.2}$$
$$B_0 = \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t \left(\tau_t \sum_{i=\ell,h} f_i \hat{y}^i_t(\tau_t, b_t) - b_t\right) \tag{Φ}$$

This formulation makes it clear that positive growth rates will have two effects. First, it will change the value of the Lagrange multiplier Φ: higher growth reduces the cost of debt and therefore eases the financing of the welfare state. Mathematically, $\partial\Phi/\partial g_t < 0$. Second, positive growth rates might change the value of redistribution, as captured in the utility function $\hat{U}_t$. Note also that this formulation encompasses the case of subsistence, as we did not specify anything about u.

5.2.1 Recursive formulation

Under the condition that $\beta \geq R$, the economy converges to a finite debt level B (and if α_1 is small enough, subsistence might be irrelevant in the long-run). This value B will depend on initial B_0 and growth parameters (α_0, α_1).

$$V_t(B_t) = \max_{\tau_t, b_t, B_{t+1}} U_t(\tau_t, b_t) + \beta V_{t+1}(B_{t+1}) \quad \text{s.t.} \tag{E.3}$$
$$B_t = b_t + \tau_t \sum_i f_i y^i_t(\tau_t, b_t) - B_{t+1}/R$$

[1] To ease notation, define also: $x_t(\tau_t, b_t) \equiv x(\tau_t, b_t, A_t)$ whenever needed.

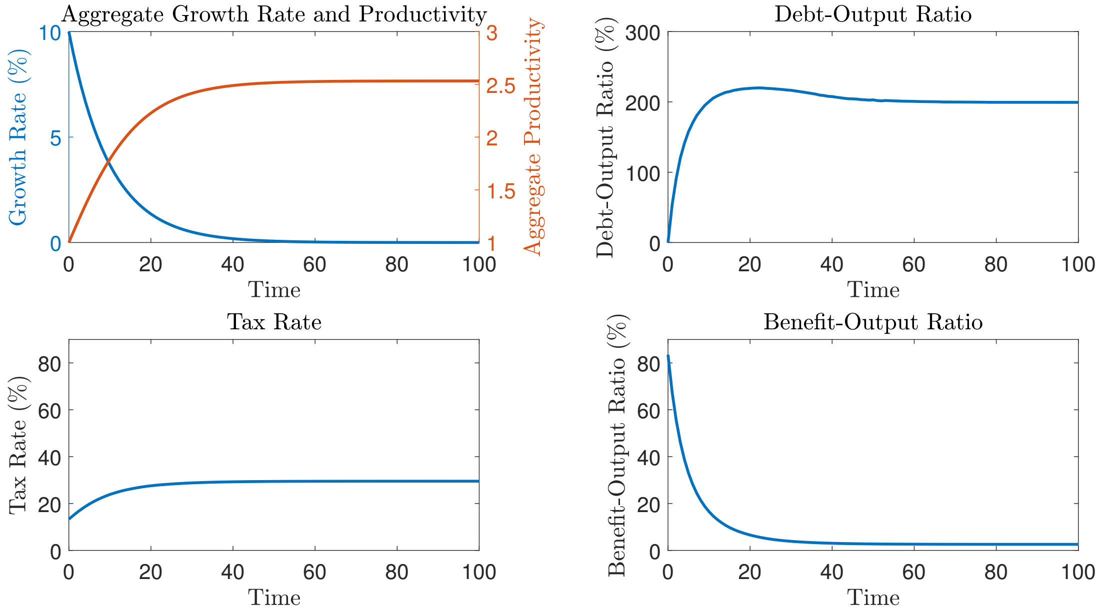

Figure 7: Dynamics of the Welfare State with External Debt: The Role of Interest Rates

Notes: Dynamics of the Welfare State and Public Debt for an economy experiencing a transition, from low productivity to high productivity. The exogenous interest rate is equal to $R = \beta^{-1}$, such that the government is neither patient nor impatient in the long run.

5.2.2 Quantitative evaluation of the mechanism

We plot the optimal fiscal plan of an economy which starts with an annual growth rate of 10%, and converges to its steady-state level in about 50 years ($\alpha_0 = \alpha_1 = 0.1$). The economy ends up with a productivity level which is about 2.5 larger than its initial one. We use GHH preferences with log-curvature; one period is one year, and the discount factor is equal to 0.9.

As we show in Figure 7, the government has a very strong incentive to take advantage of the low interest rates in the early phases of convergence. Public debt increases quickly, and stabilizes at around 200% of GDP. Tax rates are initially low, at 15%, and eventually double to reach about 30% in the long run. The size of the welfare state is very large initially, with benefits equal to 80% of output; then, it falls quickly over time, to converge to a benefit over output ratio of less than 5%. Not surprisingly, as shown in Figure 8, lower initial growth (lower α_0) or convergence to a lower productivity level (higher α_1) mitigates the dynamics of the welfare state.

Subsistence. Finally, a positive level of subsistence translates into additional dynamics: benefits-over-output ratios are higher, especially earlier in the transition, and

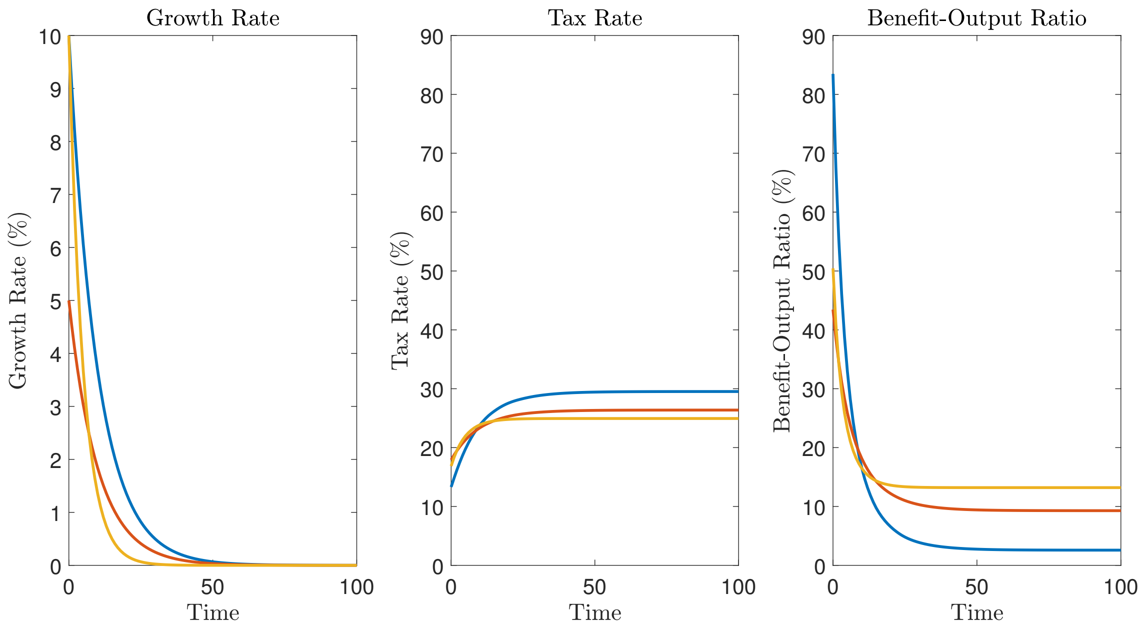

Figure 8: Dynamics of the Welfare State with External Debt: The Role of Growth

Notes: Dynamics of the Welfare State in a Growing Economy. The blue line depicts the benchmark case ($\alpha_0 = \alpha_1 = 0.1$), the red line depicts smaller initial growth α_0, the yellow line depicts faster convergence α_1.

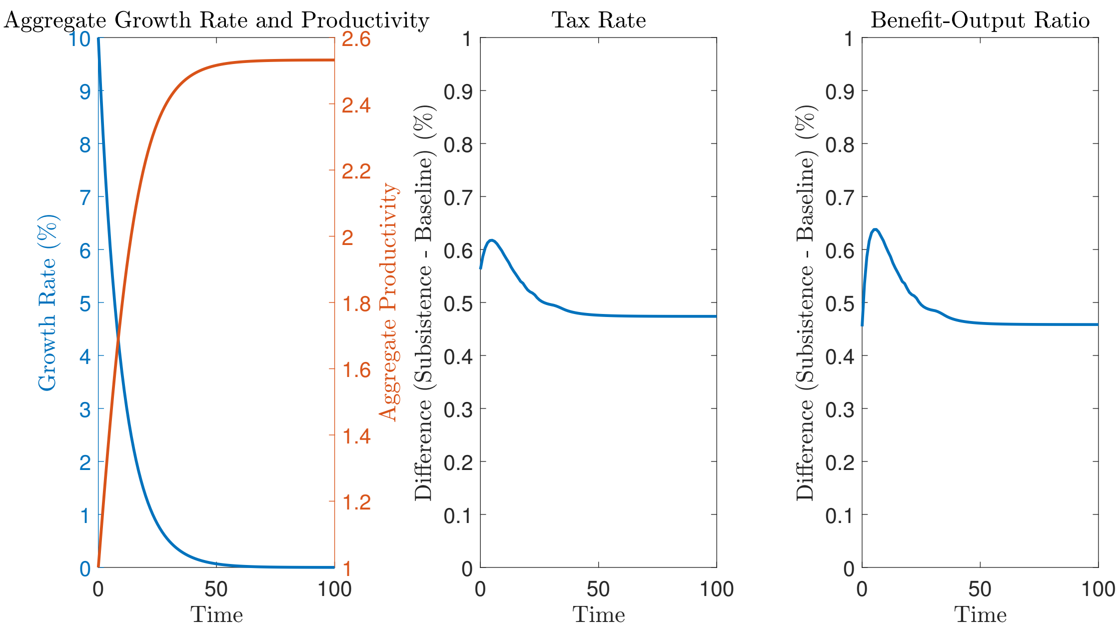

Figure 9: Dynamics of the Welfare State with External Debt: Subsistence

Notes: Dynamics of the Welfare State in a Growing Economy. Difference in tax rates and benefit-over-output ratios between positive subsistence ($\kappa > 0$) and benchmark ($\kappa = 0$).

financed partly with higher taxes. Public debt, not plotted in Figure 9, will also be larger.

5.3 Private insurance

We now turn to the the problem with private savings. Using the BEGS-Ricardian equivalence previously explained, we know that the two households' Euler equations have to hold at any point in time, but we can normalize savings of the low-skill households to be 0, such that the rich households detain all of the public debt. Public debt B_t will therefore translate in private savings $\hat{B}_t \equiv B_t/f_h$ the private savings of each high-skill household, and higher public debt will also translate into higher wealth inequality.[2]

5.3.1 Sequential formulation

The Ramsey plan is as follows. The government maximizes utilities:

$$\sum_{t=0}^{\infty} \beta^t \sum_{i=\ell,h} u\left(c_t^i, n_t^i\right) \tilde{f}_i.$$

This maximization is made under a set of 8 constraints:

1. Households' budget constraints. For the high-type, it can be written recursively:

$$\hat{B}_0 = \sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \left(\frac{1}{1+r_j}\right)\right) \left(c_t^h - (1-\tau_t)\theta_h A_t n_t^h - b_t\right)$$

   For the low-type, we have a static budget constraint without loss of generality:

$$c_t^\ell = (1-\tau_t)\theta_\ell A_t n_t^\ell + b_t$$

2. Households' intratemporal first-order conditions:

$$u_{n,t}^i = -u_{c,t}^i (1-\tau_t)\theta_i A_t \quad \forall i = \ell, h$$

3. Households' intertemporal first-order conditions:

$$u_{c,t+1}^i = -\beta(1+r_t)u_{c,t}^i \quad \forall i = \ell, h$$

[2]In this section, w.l.o.g. we use n rather than y to write down the allocations.

4. Feasibility:

$$\sum_i f_i \left(\theta_i A_t n_t^i - c_t^i\right) = 0$$

5. And the government's borrowing constraint, that we can ignore by Walras law.

We follow the Primal approach to get rid of taxes and prices in the two households' budget constraints, which become:

$$u_{c,0}^h \hat{B}_0 = \sum_{t=0}^{\infty} \beta^t \left(u_{c,t}^h (c_t^h - b_t) + u_{n,t} n_t^h\right)$$
$$0 = u_{c,t}^\ell (c_t^\ell - b_t) + u_{n,t} n_t^\ell$$

We do need to guarantee that the two households face the same interest rates and wages, that is:

$$\frac{u_{c,t}^h}{u_{c,0}^h} = \frac{u_{c,t}^\ell}{u_{c,0}^\ell}, \quad \frac{\theta^h u_{c,t}^h}{-u_{n,t}^h} = \frac{\theta^\ell u_{c,t}^\ell}{-u_{n,t}^\ell} \quad \forall t.$$

Note that we could get rid of b_t using the low-type budget constraint. We do not opt for this option as it might make the problem more complex.

Ramsey problem. Overall, we face the following problem:

$$\max_{\{\{c_t^i, y_t^i, b_t\}_{i=\ell,h}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{i=\ell,h} u\left(c_t^i, n_t^i\right) \tilde{f}_i \quad \text{s.t.} \tag{P.1}$$

$$\sum_{t=0}^{\infty} \beta^t \left(u_{c,t}^h (c_t^h - b_t) + u_{n,t}^h n_t^h\right) = u_{c,0}^h \hat{B}_0 \tag{$f_h \Phi$}$$

$$u_{c,t}^\ell (c_t^\ell - b_t) + u_{n,t} n_t^\ell = 0 \tag{$\beta^t f_\ell \nu_t$}$$

$$\frac{u_{c,t}^h}{u_{c,0}^h} - \frac{u_{c,t}^\ell}{u_{c,0}^\ell} = 0 \tag{$\beta^t \varepsilon_t$}$$

$$\frac{\theta^h u_{c,t}^h}{u_{n,t}^h} - \frac{\theta^\ell u_{c,t}^\ell}{u_{n,t}^\ell} = 0 \tag{$\beta^t \mu_t$}$$

$$\sum_i f_i \left(\theta_i (1 + g_t) n_t^i - c_t^i\right) = 0 \tag{$\beta^t \lambda_t$}$$

where variables on the right are the (scaled) Lagrangian multipliers. We first prove the standard tax smoothing result in this environment.

Proposition 5.1. *For $t \geq 1$, in the absence of growth ($g_t = 0$), allocations (and tax rates and prices) are constant for $t \geq 1$.*

Proof: See Appendix. Finally, in $t = 0$, allocations are different. Can we prove that the fiscal plan is time consistent if $\hat{B}_0 \neq 0$? Partial proof in Appendix (to be completed).

We now turn to the case of positive growth. We prove the following proposition, for CRRA and GHH preferences (proof to be extended for more general utility functions).

Proposition 5.2. *For $t \geq 1$, allocations (and tax rates and prices) are constant for $t \geq 1$ even in the case of positive growth.*

In other words, in the case of private savings, the welfare state is constant from period 1 onwards *even when there is positive growth*, as long as there is no subsistence. Growth will only change the magnitude of the price manipulation effect taking place in $t = 0$.

With subsistence. The model will feature dynamics, which will be explored numerically.

6 Conclusion

In this project, we study the implications of economic growth for the generosity of the welfare state and the mode of financing (taxes versus public debt).

In ongoing work, we do extend our two period frameworks to longer time horizons, incorporate external and internal debt jointly and aim at calibrating the model to certain countries. The latter has two purposes. On the one hand, we can evaluate the evolution of the welfare state in OECD countries over the last 50 years through the lens of our model. To what extent are the empirical patterns described in Section 1.1 consistent with the normative implications of our theory? Second, perhaps more importantly, our framework can be applied to derive policy implications for poor countries with high growth rates. Perhaps countries like India should have a more generous welfare state and partly finance this with government debt?

In our framework, so far we abstract from any political economy, default or commitment issues. Incorporating such channels will be particularly important for the study of normative implications for fast growing countries.

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A Appendix

A.1 Proofs of Section 2

Elasticity in the presence of income effects In case of income effects, the first-order condition of an individual reads as

$$\left(y\left(1-\tau\right)\left(1+g\right)+b\right)^{-\gamma}\left(1+g\right)\left(1-\tau\right)-\left(\frac{y}{\theta_i}\right)^{\frac{1}{\varepsilon}}\frac{1}{\theta_i}=0. \tag{12}$$

Based on this, our goal is to derive the elasticity of income w.r.t. to $1-\tau$. Applying the implicit function theorem, we obtain

$$\frac{\partial y_i}{\partial(1-\tau)}=-\frac{\left(y\left(1-\tau\right)\left(1+g\right)+b\right)^{-\gamma}\left(1+g\right)-\gamma\left(y\left(1-\tau\right)\left(1+g\right)+b\right)^{-\gamma-1}y\left(1-\tau\right)\left(1+g\right)^2}{-\gamma\left(y\left(1-\tau\right)\left(1+g\right)+b\right)^{-\gamma-1}\left(1-\tau\right)^2\left(1+g\right)^2-\frac{1}{\varepsilon}\left(\frac{y}{\theta_i}\right)^{\frac{1}{\varepsilon}-1}\frac{1}{(\theta_i)^2}}$$

$$\frac{\partial y_i}{\partial(1-\tau)} = \frac{c^{-\gamma}\left(1+g\right) - \gamma c^{-\gamma-1} y \left(1-\tau\right)\left(1+g\right)^2}{\gamma c^{-\gamma-1}\left(1-\tau\right)^2\left(1+g\right)^2 + \frac{1}{\varepsilon}\left(\frac{y}{\theta_i}\right)^{\frac{1}{\varepsilon}-1}\frac{1}{\theta_i^2}}$$

$$\varepsilon_{y,1-\tau} = \frac{c^{-\gamma}\left(1+g\right) - \gamma c^{-\gamma-1} y \left(1-\tau\right)\left(1+g\right)^2}{\gamma c^{-\gamma-1}\left(1-\tau\right)^2\left(1+g\right)^2 + \frac{1}{\varepsilon}\left(\frac{y}{\theta_i}\right)^{\frac{1}{\varepsilon}-1}\frac{1}{\theta_i^2}}\frac{1-\tau}{y}$$

Now divide by $c^{-\gamma}(1-\tau)\left(1+g\right) = \left(\frac{y}{\theta_i}\right)^{\frac{1}{\varepsilon}}\frac{1}{\theta_i}$

$$\varepsilon_{y,1-\tau} = \frac{1-\gamma c^{-1} y\left(1-\tau\right)\left(1+g\right)}{\gamma c^{-1} y\left(1-\tau\right)\left(1+g\right) + \frac{1}{\varepsilon}}$$

Substituting for c the individual budget constraint, we obtain

$$\varepsilon_{y,1-\tau} = \varepsilon\frac{1-\gamma\frac{y(1+g)(1-\tau)}{y(1+g)(1-\tau)+b}}{1+\varepsilon\gamma\frac{y(1+g)(1-\tau)}{y(1+g)(1-\tau)+b}} = \varepsilon\frac{1-\gamma\frac{1}{1+\frac{b}{y(1+g)(1-\tau)}}}{1+\varepsilon\gamma\frac{1}{1+\frac{b}{y(1+g)(1-\tau)}}} = \varepsilon\left(1-\frac{\gamma\left(\varepsilon+1\right)}{1+\frac{b}{y(1+g)(1-\tau)}+\varepsilon\gamma}\right)$$

alternatively we got

$$\varepsilon\frac{(1-\gamma)(1-\tau)+\frac{b}{y(1+g)}}{(1+\varepsilon\gamma)(1-\tau)+\frac{b}{y(1+g)}}$$

Lump Sum Applying the implicit function theorem, we obtain

$$\frac{\partial y_i}{\partial b} = \frac{-\gamma c^{-\gamma-1}\left(1-\tau\right)\left(1+g\right)}{\gamma c^{-\gamma-1}\left(1-\tau\right)^2\left(1+g\right)^2 + \frac{1}{\varepsilon}\left(\frac{y}{\theta_i}\right)^{\frac{1}{\varepsilon}-1}\frac{1}{\theta_i^2}}$$

Now divide by $c^{-\gamma}(1-\tau)\left(1+g\right) = \left(\frac{y}{\theta_i}\right)^{\frac{1}{\varepsilon}}\frac{1}{\theta_i}$

$$\frac{\partial y_i}{\partial b} = \frac{-\gamma c^{-1}}{\gamma c^{-1}\left(1-\tau\right)\left(1+g\right)+\frac{1}{\varepsilon}\frac{1}{y}} = -\varepsilon\frac{\gamma}{\varepsilon\gamma\left(1-\tau\right)\left(1+g\right)+\frac{y(1+g)(1-\tau)+b}{y}}$$

$$\frac{\partial y_i}{\partial b} = -\varepsilon\frac{\gamma}{\left(1+\varepsilon\gamma\right)\left(1-\tau\right)\left(1+g\right)+\frac{b}{y}}$$

Government's problem The government's problem now reads as

$$\max_{\tau,b}\sum_{i=l,h}\left(u\left(\left(1+g\right)\left(1-\tau\right)y_i(\tau,g,b)+b\right) - \frac{\left(\frac{y_i(\tau,g,b)}{\theta_i}\right)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}\right)f_i$$

$$\tau \sum_{i=l,h} (1+g)\, y_i(\tau,g,b) f_i \geq b.$$

The first order condition for b is given by

$$\frac{\partial \mathcal{L}}{\partial b} = \sum_{i=l,h} u_i' f_i - \lambda + \lambda\,(1+g)\,\tau \sum_{i=l,h} \frac{\partial y_i(\tau,g,b)}{\partial b} f_i = 0$$

and hence

$$\frac{\partial \mathcal{L}}{\partial b} = \sum_{i=l,h} u_i' f_i - \lambda - \lambda\,(1+g)\,\tau \sum_{i=l,h} \left(\varepsilon \frac{\gamma}{(1+\varepsilon\gamma)\,(1-\tau)\,(1+g) + \frac{b}{y}} \right) f_i = 0$$

Which gives us

$$\lambda = \frac{E(u')}{1 + \frac{\tau}{1-\tau} \sum_{i=l,h} \varepsilon \frac{\gamma}{(1+\varepsilon\gamma) + \frac{b}{(1-\tau)(1+g) y_i}}} \tag{13}$$

Next consider the first-order condition for τ :

$$\frac{\partial \mathcal{L}}{\partial \tau} = \left(-\sum_{i=l,h} y_i(\tau,g) u_i' f_i + \lambda \sum_{i=l,h} y_i(\tau,g,b) f_i + \lambda\tau \sum_{i=l,h} \frac{\partial y_i(\tau,g,b)}{\partial \tau} f_i \right) (1+g) = 0.$$

which yields

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h} \left(1 - \frac{u_i'}{\lambda}\right) y_i f_i}{\sum_{i=l,h} f_i \varepsilon^i_{y,1-\tau} y_i}$$

Using (13), this implies

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h} \left(1 - \frac{u_i'}{\frac{E(u')}{1+\frac{\tau}{1-\tau}\sum_{i=l,h} \varepsilon \frac{\gamma}{(1+\varepsilon\gamma) + \frac{b}{(1-\tau)(1+g)y_i}}}} \right) y_i f_i}{\sum_{i=l,h} f_i \varepsilon^i_{y,1-\tau} y_i}$$

which can be rewritten as

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h} \left(1 - \frac{u_i'}{E(u')}\right) y_i f_i}{\sum_{i=l,h} f_i \varepsilon^i_{y,1-\tau} y_i} - \frac{\sum_{i=l,h} \frac{u_i' \frac{\tau}{1-\tau} \sum_{i=l,h} \varepsilon \frac{\gamma}{1+\varepsilon\gamma + \frac{b}{(1-\tau)(1+g)y_i}}}{E(u')} y_i f_i}{\sum_{i=l,h} f_i \varepsilon^i_{y,1-\tau} y_i}$$

and hence as

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h}\left(1-\frac{u_i'}{E(u')}\right) y_i f_i}{\sum_{i=l,h} f_i \varepsilon^i_{y,1-\tau} y_i + \varepsilon \sum_{i=l,h} \frac{\gamma}{1+\varepsilon\gamma+\frac{b}{(1-\tau)(1+g)y_i}} \sum_{i=l,h} \frac{u_i'}{E(u')} y_i f_i}. \tag{14}$$

Now we want to show tat τ is independent of g. We first focus on log preferences, i.e. $\gamma = 1$. To show that, we first claim is that if τ is constant, y is constant and b grows at $1+g$. This is consistent with the individual FOC (12). The budget constraints as well. Now the only question is whether the optimality condition (14) also holds if y stays constant and b grows at $1+g$.

First look at $\frac{u'}{E(u')}$. Therefore note that

$$u' = \frac{1}{(1-\tau)(1+g)y+b} = \frac{1}{1+g}\frac{1}{(1-\tau)y+b_0}$$

which implies directly that $\frac{u'}{E(u')}$ is independent of g. As a next step, note hat the elasticity $\varepsilon^i_{y,1-\tau}$ is independent of g

$$\varepsilon^i_{y,1-\tau} = \varepsilon\frac{(1-\gamma)(1-\tau)+\frac{b_0(1+g)}{y(1+g)}}{(1+\varepsilon\gamma)(1-\tau)+\frac{b_0(1+g)}{y(1+g)}}.$$

For different values of γ, things are slightly more complicated because y will not be constant if τ is constant. If τ is constant, y grows at rate $1+\alpha$ (we derive α below) and b grows at rate $(1+g)(1+\alpha)$. The first-order conditions of the individual are given by

$$((1-\tau)(1+g)y+b)^{-\gamma}(1+g)(1-\tau) = \left(\frac{y}{\theta}\right)^{\frac{1}{\varepsilon}}\frac{1}{\theta}.$$

$$((1-\tau)(1+g)y_0(1+\alpha)+b_0(1+g)(1+\alpha))^{-\gamma}(1+g)(1-\tau) = \left(\frac{y_0(1+\alpha)}{\theta}\right)^{\frac{1}{\varepsilon}}\frac{1}{\theta}.$$

The level of α that is consistent with is given by:

$$\log(1+\alpha) = \frac{1-\gamma}{\frac{1}{\varepsilon}+\gamma}\log(1+g)$$

Apparently also the government budget constraint is consistent with that. The only things that remain to be shown are that the elasticity $\varepsilon^i_{y,1-\tau}$ is still independent of g and the ratio $\frac{u'}{E(u')}$ as well. For the elasticity wer obtain

$$\varepsilon_{y,1-\tau}^{i} = \varepsilon \frac{(1-\gamma)(1-\tau) + \frac{b_0(1+\alpha)(1+g)}{y_0(1+\alpha)(1+g)}}{(1+\varepsilon\gamma)(1-\tau) + \frac{b_0(1+\alpha)(1+g)}{y_0(1+\alpha)(1+g)}}.$$

and for the marginal utility we have

$$u' = ((1-\tau)(1+g)(1+\alpha)y_0 + (1+g)(1+\alpha)b_0)^{-\gamma} = ((1+g)(1+\alpha))^{-\gamma}((1-\tau)y_0 + b_0)^{-\gamma}$$

and hence $\frac{u'}{E(u')}$ is independent of g, which completes the proof.

A.2 Proof of Proposition 3.1

Rearranging the first-order condition w.r.t. τ_1 yields:

$$\frac{\tau_1}{1-\tau_1} = \frac{\sum_{i=l,h}\left(1 - \frac{U_i'}{E(U')}\right)\theta_i^{1+\varepsilon} f_i}{\varepsilon \sum_{i=l,h} f_i \theta_i^{1+\varepsilon}} \tag{15}$$

Now our exercise is to ask how the value of τ_1 changes if the government borrows, which is the case if

$$R < \frac{\left((1+g)^{1+\varepsilon}\right)^{\gamma}}{\beta},$$

see the discussion around equation (9). For arguments that are very similar to those in the proof of Proposition 2.3, we know that if this amount of debt is used for increasing b_1, then $U_{1l}'/E(U_1)'$ is increasing and $U_{1h}'/E(U_1)'$ is decreasing. Thus, looking at (15), this reveals that τ_1 decreases. So as a consequence, the public debt is not only used to increase b_1 but also to decrease τ_1. The opposite can be shown for τ_2 and b_2.

A.3 External Debt and Subsistence Level

Even for

$$R = \frac{\left((1+g)^{1+\varepsilon}\right)^{\gamma}}{\beta}$$

the government will borrow until

$$\sum_{i=l,h} U_{i1}' = \left((1+g)^{1+\varepsilon}\right)^{\gamma} \sum_{i=l,h} U_{i2}'$$

where in the absence of public debt we have

$$U_i' = \left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1} + \frac{\tau}{1-\tau}\sum_{i=l,h} f_i\theta_i^{1+\varepsilon} - \frac{\kappa}{(1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}}\right)^{-\gamma}$$

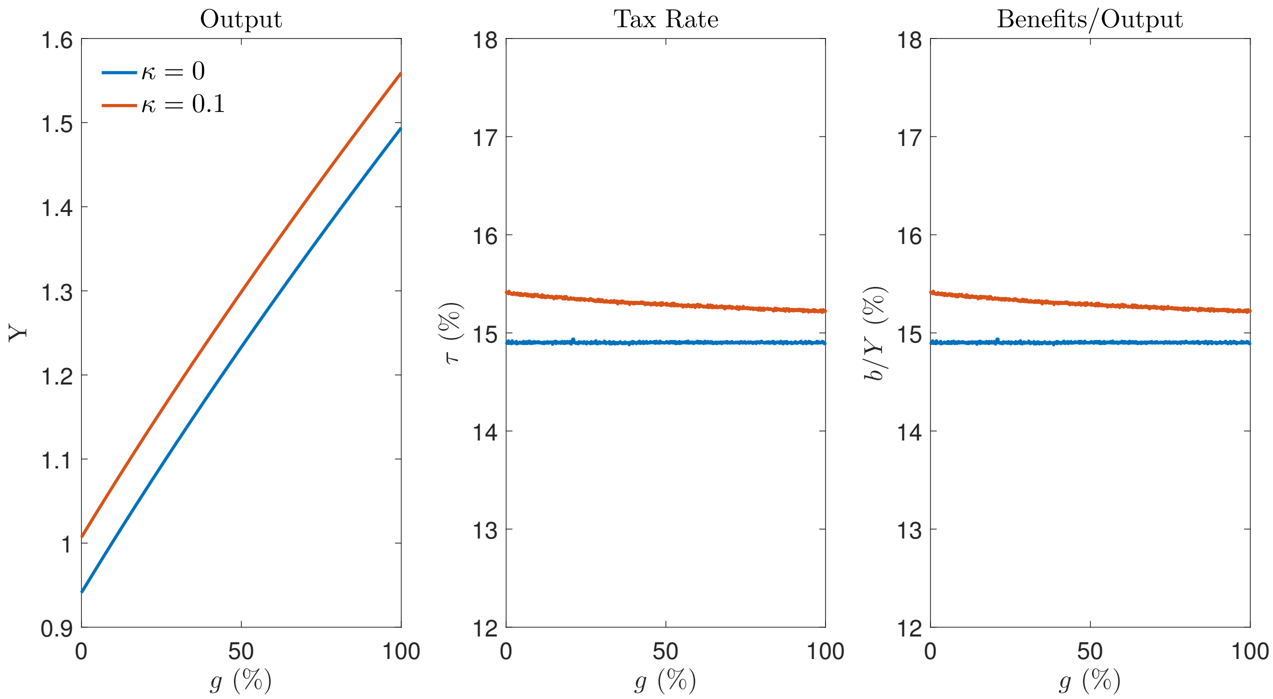

Figure 10: Welfare state and Growth with Income Effects

Notes: Optimal output, tax rate and social assistance (measured as benefits-over-output ratio), for different levels of growth g; CRRA preferences. The blue lines depict the homothetic case, while the red lines depict the case with a positive level of subsistance κ.

recall optimal tax formula:

$$\frac{\tau}{1-\tau} = \frac{\sum_{i=l,h}\left(1 - \frac{U_i'}{E(U')}\right)\theta_i^{1+\varepsilon} f_i}{\varepsilon \sum_{i=l,h} \theta_i^{1+\varepsilon} f_i}. \tag{16}$$

Now we wanna show that the taxes are time invariant. First, lets quickly see that with time invariant taxes the government would like to borrow at this rate.

$$\sum_{i=l,h} \left((1-\tau)^{1+\varepsilon}\right)^{-\gamma} \left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1} + \frac{\tau}{1-\tau}\sum_{i=l,h} f_i\theta_i^{1+\varepsilon} - \frac{\kappa}{(1-\tau)^{1+\varepsilon}}\right)^{-\gamma}$$
$$> \sum_{i=l,h} \left((1-\tau)^{1+\varepsilon}\right)^{-\gamma} \left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1} + \frac{\tau}{1-\tau}\sum_{i=l,h} f_i\theta_i^{1+\varepsilon} - \frac{\kappa}{(1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}}\right)^{-\gamma}$$

$$\sum_{i=l,h}\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}-\frac{\kappa}{(1-\tau)^{1+\varepsilon}}+D\right)^{-\gamma}$$

$$=\sum_{i=l,h}\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}-\frac{\kappa}{(1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}}-D\frac{\left((1+g)^{1+\varepsilon}\right)^{\gamma}}{\beta}\right)^{-\gamma}$$

The optimal level of debt D^* will be such that

$$-\frac{\kappa}{(1-\tau)^{1+\varepsilon}}+D^*=-\frac{\kappa}{(1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}}-D^*\frac{\left((1+g)^{1+\varepsilon}\right)^{\gamma}}{\beta}\equiv X$$

Now for the optimal tax formula we now wanna show that

$$\frac{U'_{1i}}{E(U'_1)}=\frac{U'_{2i}}{E(U'_2)}$$

Actually its

$$\frac{\left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}+X\right)^{-\gamma}}{\sum_{i=l,h}\left(\frac{\theta_i^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}+X\right)^{-\gamma}}$$

for both periods which iplies that having $\tau_1=\tau_2$ is indeed optimal.

A.4 Proof of Proposition 4.1

Our starting point is a constant constant tax rate $\tau_1=\tau_2=\tau$ and therefore b_t growing at $(1+g)$. In fact it can then easily be shown that for zero public debt or savings, the intertemporal marginal rates of substitution of both agents are equalized:

$$\frac{\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_l^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}{\left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_l^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}=\frac{\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_h^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}{\left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_h^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}$$

Hence, there will be no intertemporal trade and the market clearing interest rate is given by $R=\frac{\left((1+g)^{1+\varepsilon}\right)^{\gamma}}{\beta}$. But now introduce a_l and a_h as initial assets. Our claim is that htere is only one ratio such that they do not trade.

$$\frac{\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_l^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}{\left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_l^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}=\frac{\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_h^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}{\left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_h^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}$$

Now we introduce some inital assets

$$\frac{\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_l^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}+a_l\right)^{-\gamma}}{\left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_l^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}=\frac{\left((1-\tau)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_h^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}+a_h\right)^{-\gamma}}{\left((1-\tau)^{1+\varepsilon}(1+g)^{1+\varepsilon}\right)^{-\gamma}\left(\frac{\theta_h^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}\right)^{-\gamma}}$$

$$\frac{\partial a_l}{\partial a_h}=\frac{\frac{\theta_l^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}}{\frac{\theta_h^{1+\varepsilon}}{\varepsilon+1}+\frac{\tau}{1-\tau}\sum_{i=l,h}f_i\theta_i^{1+\varepsilon}}$$

We know from the Bandhari et al JME paper that we can set government to zero w.l.og. Now we consider the following: we start from a situation where both agents have zero initial assets. We know that then the tax rate is constant and $b_2=(1+g)(1+\alpha)b_1$. Now we perturb this whole thing in that we add a_1 as initial assets for the high type and have initial assets of $-a_1$ for the low type. This gives us then 4 budget constraints:

$$c_1^h=(1-\tau_1)y_1^h+b_1+a_1-\frac{a_2}{1+r}$$

$$c_2^h=(1-\tau_2)y_2^h+b_1(1+g)(1+\alpha)+a_2$$

$$c_1^l=(1-\tau_1)y_1^l+b_1-a_1+\frac{a_2}{1+r}$$

$$c_2^l=(1-\tau_2)y_2^l+b_1(1+g)(1+\alpha)-a_2$$

. Let's do it for GHH and look at the Euler equations. Let's start with the individual ones

$$\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+b_1+a_1-\frac{a_2}{1+r}\right)^{-\gamma}=\beta(1+r)\left(\frac{(1-\tau_2)(1+g)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+b_2+a_2\right)^{-\gamma} \tag{17}$$

$$\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+b_1-a_1+\frac{a_2}{1+r}\right)^{-\gamma}=\beta(1+r)\left(\frac{(1-\tau_2)(1+g)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+b_2-a_2\right)^{-\gamma} \tag{18}$$

So now we want to show that if $a_1=0$, then we have $a_2=0$ and $\tau_1=\tau_2$ and $b_2=b_1(1+g)(1+\alpha)$ and that the equilibrium interest rate is determined by $\beta(1+r)=(1+g)(1+\alpha)$. But for this we also have to look at the government's euler equation, the generalized euler equation.

$$\max\sum f_i\left(U_1^i+\beta U_2^i\right)$$

subject to

$$b_1\leq\tau_1(1-\tau_1)^{\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}$$

and

$$b_2\leq\tau_2(1-\tau_1)^{\varepsilon}(1+g)^{1+\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}$$

and subject (17) and (18). Let's substitute the budget constraints of the governments into the Euler equations and obtain:

$$\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+\tau_1(1-\tau_1)^{\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}+a_1-\frac{a_2}{1+r}\right)^{-\gamma} \tag{19}$$
$$=\beta(1+r)\left(\frac{(1-\tau_2)^{1+\varepsilon}(1+g)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+\tau_2(1-\tau_2)^{\varepsilon}(1+g)^{1+\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}+a_2\right)^{-\gamma}$$

$$\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+\tau_1(1-\tau_1)^{\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}-a_1+\frac{a_2}{1+r}\right)^{-\gamma} \tag{20}$$
$$=\beta(1+r)\left(\frac{(1-\tau_2)^{1+\varepsilon}(1+g)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+\tau_2(1-\tau_2)^{\varepsilon}(1+g)^{1+\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}-a_2\right)^{-\gamma}$$

Let's now write down the proper Lagrangian function:

$$\begin{aligned}
\mathcal{L}=&\frac{\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+\tau_1(1-\tau_1)^{\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}-a_1+\frac{a_2}{1+r}\right)^{1-\gamma}}{1-\gamma}\\
&+\beta\frac{\left(\frac{(1-\tau_2)^{1+\varepsilon}(1+g)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+\tau_2(1-\tau_2)^{\varepsilon}(1+g)^{1+\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}-a_2\right)^{1-\gamma}}{1-\gamma}\\
&+\frac{\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+\tau_1(1-\tau_1)^{\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}+a_1-\frac{a_2}{1+r}\right)^{1-\gamma}}{1-\gamma}\\
&+\beta\frac{\left(\frac{(1-\tau_2)^{1+\varepsilon}(1+g)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+\tau_2(1-\tau_2)^{\varepsilon}(1+g)^{1+\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}+a_2\right)^{1-\gamma}}{1-\gamma}\\
&+\mu_l\Bigg(\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+\tau_1(1-\tau_1)^{\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}-a_1+\frac{a_2}{1+r}\right)^{-\gamma}\\
&-\beta(1+r)\left(\frac{(1-\tau_2)^{1+\varepsilon}(1+g)^{1+\varepsilon}\theta_l^{1+\varepsilon}}{\varepsilon+1}+\tau_2(1-\tau_2)^{\varepsilon}(1+g)^{1+\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}-a_2\right)^{-\gamma}\Bigg)\\
&+\mu_h\Bigg(\left(\frac{(1-\tau_1)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+\tau_1(1-\tau_1)^{\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}+a_1-\frac{a_2}{1+r}\right)^{-\gamma}\\
&-\beta(1+r)\left(\frac{(1-\tau_2)^{1+\varepsilon}(1+g)^{1+\varepsilon}\theta_h^{1+\varepsilon}}{\varepsilon+1}+\tau_2(1-\tau_2)^{\varepsilon}(1+g)^{1+\varepsilon}\left\{\theta_h^{1+\varepsilon}+\theta_l^{1+\varepsilon}\right\}+a_2\right)^{-\gamma}\Bigg)
\end{aligned}$$

So for τ_1 and τ_2 we have our usual first-order conditions, plus some 'mu-terms' that should be zero if our idea is right. How can we show that these terms are zero? Look at the FOC for r first:

$$\frac{\partial\mathcal{L}}{\partial r}=-U'_{1l}\frac{a_2}{(1+r)^2}+U'_{1h}\frac{a_2}{(1+r)^2}-\beta\left(\mu_l U'_{2l}+\mu_h U'_{2h}\right)-\mu_l U''_{1l}\frac{a_2}{(1+r)^2}+\mu_h U''_{1h}\frac{a_2}{(1+r)^2}=0$$

Let's call $\mathcal{R}=-U'_{1l}\frac{a_2}{(1+r)^2}+U'_{1h}\frac{a_2}{(1+r)^2}$

$$-\mathcal{R}=-\beta\mu_h\left(U'_{2h}+\frac{SOC_h}{SOC_l}U'_{2l}\right)+\mu_h\frac{a_2}{(1+r)^2}\left(U''_{1h}-U''_{1l}\frac{SOC_h}{SOC_l}\right)$$

$$-\mathcal{R} = -\beta\mu_h \left(U'_{2h} + \frac{U''_{1h}}{U''_{1l}} U'_{2l} \right) + \mu_h \frac{a_2}{(1+r)^2} \left(U''_{1h} - U''_{1l} \frac{U''_{1h}}{U''_{1l}} \right)$$

$$\mathcal{R} = \beta\mu_h \left(U'_{2h} + \frac{U''_{1h}}{U''_{1l}} U'_{2l} \right)$$

$$\mu_h = \frac{a_2}{(1+r)^2} \frac{-U'_{1l} + U'_{1h}}{\beta \left(U'_{2h} + \frac{U''_{1h}}{U''_{1l}} U'_{2l} \right)}$$

equivlanet to

$$a_2 = (1+r)^2 \beta \frac{\mu_l U'_{2l} + \mu_h U'_{2h}}{U'_{1h} + \mu_h U''_{1h} - U'_{1l} - \mu_l U''_{1l}}$$

This is fulfilled for $a_2 = 0$ and $\mu_l = \mu_h = 0$.

$$\frac{\partial \mathcal{L}}{\partial a_2} = U'_{1l} \frac{1}{1+r} - \beta U'_{2l} - U'_{1h} \frac{1}{1+r} + \beta U'_{2h} + \mu_l SOC_l - \mu_h SOC_h = 0$$

which of course implies

$$\frac{\partial \mathcal{L}}{\partial a_2} = \mu_l SOC_l - \mu_h SOC_h = 0$$

from which we can conclude that μ_l and μ_h are of same sign. So the FOC for r only holds if also $\mu_l = \mu_h = 0$.

Take the following "snapshot" of the Lagrangian:.

$$\mathcal{L} = \mu_l \left(U'_{1l} \frac{1}{1+r} + \kappa|_{\kappa=0} - \beta U'_{2l} \right)$$

$$\frac{\partial \mathcal{L}}{\partial \kappa} = \mu_l.$$

What do we think economicylla if κ is varied?

$$U'_{1l} \frac{1}{1+r} + \kappa = \beta U'_{2l}$$

progress March 22 Part 1: Show that $a_1 = 0$ implies $a_2 = 0$ and therefore constant welfare state

- We know that the sign of μ_l and μ_h have to be equal
- If $\mu_l = \mu_h = 0$, then we get from the FOC w.r.t r that $a_2 = 0$
- The Euler equations can then only by fulfilled for both jointly (given that they face the saem interest rate) if $a_1 = 0$.
- Finally, note that $a_2 = 0$ implies $\mu_l = \mu_h = 0$,
- This seems ciruclar so let's say it differently: if the exogenous parameter a_1, then all necessary conditions are fullfilled if $a_2 = 0$ and $\mu_l = \mu_h = 0$

Part 2: what happens if $a_1 > 0$

- First question: what about the signs of μ_l and μ_h?
- So recall the snapshot lagrangian

$$\mathcal{L} = \mu_l \left(U'_{1l} \frac{1}{1+r} + \kappa|_{\kappa=0} - \beta U'_{2l} \right)$$

$$\frac{\partial \mathcal{L}}{\partial \kappa} = \mu_l.$$

What do we think economicylla if κ is varied?

$$U'_{1l} \frac{1}{1+r} + \kappa = \beta U'_{2l}$$

- Increasing κ implies that individual incentives are such that they are happy with less savings.
- Less desire to save, however, implies a higher intrest rates
- The government wants exactly the opposite if $a_1 > 0$. Hence, we have to have $\mu_h < 0$ and therefore also $\mu_l < 0$

Let's know first get the optimality conditno for τ_1 and τ_2 before we think about ordering μ_h and μ_l

$$\frac{\partial \mathcal{L}}{\partial \tau_1} = -\sum_{i=l,h} y_{1i}(\tau, g)\tilde{f}_i + \lambda \sum_{i=l,h} y_{1i}(\tau, g) f_i + \lambda \tau_1 \sum_{i=l,h} \frac{\partial y_{1i}(\tau, g)}{\partial \tau_1} f_i + \sum_{i=l,h} \mu_i \frac{\partial U'_{i1}}{\partial \tau_1} = 0.$$

What about this additional term?

$$\mu_l \frac{\partial U'_{l1}}{\partial \tau_1} + \mu_h \frac{\partial U'_{h1}}{\partial \tau_1} = \mu_l U''_{l1} \frac{\partial \tilde{c}_{l1}}{\partial \tau_1} + \mu_h U''_{h1} \frac{\partial \tilde{c}_{h1}}{\partial \tau_1}$$

Now use

$$\mu_l = \mu_h \frac{SOC_h}{SOC_l} = 0$$

hence the additional term becomes

$$\mu_h \left(U''_{l1} \frac{\partial \tilde{c}_{l1}}{\partial \tau_1} \frac{SOC_h}{SOC_l} + U''_{h1} \frac{\partial \tilde{c}_{h1}}{\partial \tau_1} \right)$$

or

$$\mu_h SOC_h \left(U''_{l1} \frac{\partial \tilde{c}_{l1}}{\partial \tau_1} \frac{1}{SOC_l} + U''_{h1} \frac{\partial \tilde{c}_{h1}}{\partial \tau_1} \frac{1}{SOC_h} \right)$$

Now what is

$$\frac{U''_{i1}}{SOC_i}$$

$$U'_{i1} - \beta(1+r)U'_{i2} = 0$$

so, we obtain

$$\frac{\partial a_{2i}}{\partial b_1} = \frac{U''_{i1}}{SOC_i}$$

$$\mu_h SOC_h \left(\frac{\partial a_{2l}}{\partial b_1} \frac{\partial \tilde{c}_{l1}}{\partial \tau_1} + \frac{\partial a_{2h}}{\partial b_1} \frac{\partial \tilde{c}_{h1}}{\partial \tau_1} \right)$$

Now we can show that

$$|\frac{\partial \tilde{c}_{h1}}{\partial \tau_1}| > |\frac{\partial \tilde{c}_{l1}}{\partial \tau_1}|$$

Now next thing we show is that

$$\frac{\partial a_{2h}}{\partial b_1} = \frac{\partial a_{2l}}{\partial b_1}$$

This is because

$$\frac{U''_{i1}}{SOC_i} = \frac{1}{1 + \beta(1+r)^2 \frac{U''_{i2}}{U''_{i1}}}$$

This is the same for $i = l, h$ if

$$\frac{\tilde{c}_{l1}}{\tilde{c}_{l2}} = \frac{\tilde{c}_{h1}}{\tilde{c}_{h2}}$$

which holds if the Euler equations holds. From this we can conclude that the additional term in the FOC τ_1

$$\mu_h SOC_h \frac{\partial a_{2l}}{\partial b_1} \left(\frac{\partial \tilde{c}_{l1}}{\partial \tau_1} + \frac{\partial \tilde{c}_{h1}}{\partial \tau_1} \right) < 0$$

which implies that we have a lower marignal tax rate in period 1. Next thing to show would be the opposite for period 2. What is more tricky and we need to understand: what about b_1 and b_2? Actually it is simple if we have zero government debt. Maybe we first show it for that and then make the transformation to the respective transfers if the poor would not borrow and the government would do that job. Let's quickly go back to

$$\mu_h SOC_h \frac{\partial a_{2l}}{\partial b_1} \left(\frac{\partial \tilde{c}_{l1}}{\partial \tau_1} + \frac{\partial \tilde{c}_{h1}}{\partial \tau_1} \right) < 0$$

We have closed forms for the stuff in brackets. Further we know $\frac{\partial a_{2l}}{\partial b_1} = \frac{\partial a_{2h}}{\partial b_1}$ and hence

$$SOC_h \frac{\partial a_{2h}}{\partial b_1} = U''_{1h}$$

$$\mu_h U''_{1h} \left(\frac{\partial \tilde{c}_{l1}}{\partial \tau_1} + \frac{\partial \tilde{c}_{h1}}{\partial \tau_1} \right) < 0$$

We know that

$$\frac{\partial \tilde{c}_{l1}}{\partial \tau_1} + \frac{\partial \tilde{c}_{h1}}{\partial \tau_1} = \frac{1 - \tau_1(1+\varepsilon)}{1 - \tau_1} \sum \theta_i^{1+\varepsilon}$$

May 2nd;

We got.

$$\mu_h = \frac{a_2}{(1+r)^2} \frac{-U'_{1l} + U'_{1h}}{\beta \left(U'_{2h} + \frac{U''_{1h}}{U''_{1l}} U'_{2l} \right)}$$

Further, we know the additional tem in the FOC for the tax rate is

$$\mu_h SOC_h \frac{\partial AS}{\partial \tau_1}$$

where AS is 'aggregate saving'. Now we know that we can write this as

$$\frac{a_2}{(1+r)^2} \frac{-U'_{1l}+U'_{1h}}{\beta\left(U'_{2h}+\frac{U''_{1h}}{U''_{1l}}U'_{2l}\right)} \left(U''_{1h}+\beta(1+r)^2 U''_{2h}\right) \frac{\partial AS}{\partial \tau_1}$$

$$\frac{a_2}{(1+r)^2} \frac{-U'_{1l}+U'_{1h}}{\beta\left(\frac{U'_{2h}}{SOC_h}+\frac{U'_{2l}}{SOC_l}\right)}) \frac{\partial AS}{\partial \tau_1}$$

Mechanically we know that the effect of the change in aggregate savings should be

$$\frac{a_2}{(1+r)^2} \left(U_{1h}-U_{1l}\right) \times \frac{\partial r}{\partial AS} \times \frac{\partial AS}{\partial \tau_1}$$

Can we get a formula for the change in the interest rate? Probably yes.

$$\frac{\partial a_{2l}}{\partial \tau_1} d\tau_1 + \frac{\partial a_{2h}}{\partial \tau_1} d\tau_1 + \frac{\partial a_{2l}}{\partial r} dr + \frac{\partial a_{2h}}{\partial r} dr = 0$$

$$\frac{dr}{d\tau_1} = -\frac{\frac{\partial a_{2l}}{\partial \tau_1}+\frac{\partial a_{2h}}{\partial \tau_1}}{\frac{\partial a_{2l}}{\partial r}+\frac{\partial a_{2h}}{\partial r}} = -\frac{\frac{\partial a_{2l}}{\partial b_1}\left(\frac{\partial \tilde{c}_{l1}}{\partial \tau_1}+\frac{\partial \tilde{c}_{h1}}{\partial \tau_1}\right)}{\frac{\partial a_{2l}}{\partial r}+\frac{\partial a_{2h}}{\partial r}}$$

Doing the math, directly shows that this works, so the additional term (probably ex post trivial is just given by)

$$\frac{a_2}{(1+r)^2} \left(U_{1h}-U_{1l}\right) \times \frac{\partial r}{\partial \tau_1}$$

B T Periods GHH

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{i=l,h} \tilde{f}_i u\left(c_{it}, y_{it}(\tau_t)\right)$$

subject to government budget constraint

$$\forall\, t \ : B_t + b_t = \frac{1}{1+r_t} B_{t+1} + \tau_t \sum_{i=l,h} y_{it}(\tau_t) f_i$$

Euler equations

$$\forall t, \forall i : u'_{it} = \beta\left(1+r_t\right) u'_{it+1}$$

and household budget constraints

$$c_t^h + \frac{1}{1+r_t} \frac{B_{t+1}}{f_h} = \left(1-\tau_t\right) y_{ht}(\tau_t) + b_t + \frac{B_t}{f_h}$$

and

$$c_t^l = \left(1-\tau_t^l\right) y_{lt}(\tau_t) + b_t$$

$$
\begin{aligned}
\max_{\tau_t, b_t, B_t, r_t} \ \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \tilde{f}_l u \left(\left(1 - \tau_t\right) y_{lt}(\tau_t) + b_t, y_{lt}(\tau_t) \right) \\
& + \sum_{t=0}^{\infty} \beta^t \tilde{f}_h u \left(\left(1 - \tau_t\right) y_{ht}(\tau_t) + b_t + \frac{B_t}{f_h} - \frac{1}{1+r_t} \frac{B_{t+1}}{f_h}, y_{ht}(\tau_t) \right) \\
& + \sum_{t=0}^{\infty} \mu_{lt} \left(u_c \left[\left(1 - \tau_t^l\right) y_{lt}(\tau_t) + b_t, y_{it}(\tau_t) \right] - \beta \left(1 + r\right) u_c \left[\left(1 - \tau_{t+1}^l\right) y_{lt+1}(\tau_{t+1}) + b_{t+1}, y_{it+1}(\tau_{t+1}) \right] \right) \\
& + \sum_{t=0}^{\infty} \mu_{ht} \Bigg(u_c \left[\left(1 - \tau_t\right) y_{ht}(\tau_t) + b_t + \frac{B_t}{f_h} - \frac{1}{1+r_t} \frac{B_{t+1}}{f_h}, y_{ht}(\tau_t) \right] \\
& - \beta \left(1 + r_t\right) u_c \left[\left(1 - \tau_{t+1}\right) y_{ht+1}(\tau_{t+1}) + b_{t+1} + \frac{B_{t+1}}{f_h} - \frac{1}{1+r_{t+1}} \frac{B_{t+2}}{f_h}, y_{ht}(\tau_t) \right] \Bigg) \\
& + \sum_{t=0}^{\infty} \lambda_t \left(\frac{1}{1+r_t} B_{t+1} + \tau_t \sum_{i=l,h} y_{it}(\tau_t) f_i - B_t - b_t \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial b_t} = & \beta^t \tilde{f}_l u_{ct}^l + \beta^t \tilde{f}_h u_{ct}^h - \lambda_t \\
& + \mu_{lt} u_{cc,t}^l - \beta \left(1 + r_{t-1}\right) \mu_{lt-1} u_{cc,t}^l \\
& + \mu_{ht} u_{cc,t}^h - \beta \left(1 + r_{t-1}\right) \mu_{ht-1} u_{cc,t}^h = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \tau_t} = & - \beta^t \tilde{f}_l u_{ct}^l y_{lt}(\tau_t) - \beta^t \tilde{f}_h u_{ct}^h y_{lt}(\tau_t) + \lambda_t \sum_{i=l,h} \left(y_{i,t} + \tau_t \frac{\partial y_{i,t}}{\partial \tau_t} \right) \\
& - \mu_{lt} u_{cc,t}^l y_{lt}(\tau_t) + \beta \left(1 + r_t\right) \mu_{lt-1} u_{cc,t}^l y_{lt}(\tau_t) \\
& - \mu_{ht} u_{cc,t}^h y_{ht}(\tau_t) + \beta \left(1 + r_t\right) \mu_{ht-1} u_{cc,t}^h y_{ht}(\tau_t) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial B_t} = & \underbrace{\beta^t \tilde{f}_h u_{ct}^h \frac{1}{f_h} - \beta^{t-1} \tilde{f}_h u_{ct-1}^h \frac{1}{f_h} \frac{1}{1+r_t}}_{=0} \\
& - \lambda_t + \lambda_{t-1} \frac{1}{1+r_{t-1}} \\
& + \mu_{ht} u_{cc,t}^h \frac{1}{f_h} + \mu_{ht-1} \left(-u_{cc,t-1}^h \frac{1}{1+r_{t-1}} \frac{1}{f_h} - \beta \left(1 + r_{t-1}\right) u_{cc,t}^h \frac{1}{f_h} \right) \\
& + \mu_{ht-2} \beta \left(1 + r_{t-2}\right) u_{cc,t-1}^h \frac{1}{1+r_{t-1}} = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r_t} = & - \lambda_t \frac{B_{t+1}}{\left(1+r_t\right)^2} + \tilde{f}_h \beta^t u_{ct}^h \frac{B_{t+1}}{\left(1+r_t\right)^2 f_h} \\
& - \mu_{ht} \beta u_{c,t+1}^h + \mu_{ht} u_{cct}^h \frac{B_{t+1}}{\left(1+r_t\right)^2 f_h} - \mu_{ht-1} \beta \left(1 + r_{t-1}\right) u_{cc,t}^h \frac{B_{t+1}}{\left(1+r_t\right)^2 f_h} = 0
\end{aligned}
$$

So if $B_1 = 0$, then the problem should be homothetic again.

Let's look at period 1 as it is specical

$$\frac{\partial \mathcal{L}}{\partial B_1} = \tilde{f}_h u^h_{c1} \frac{1}{f_h} - \lambda_1 + \mu_{h1} u^h_{cc,1} \frac{1}{f_h} = 0$$

I am quite sure that $\beta \tilde{f}_h u^h_{c1} \frac{1}{f_h} - \lambda_1 < 0$, which implies

$$\mu_{h1} < 0.$$

Next, let's look at

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial B_2} =& \underbrace{\beta \tilde{f}_h u^h_{c2} \frac{1}{f_h} - \tilde{f}_h u^h_{c1} \frac{1}{f_h} \frac{1}{1+r_1}}_{=0} - \lambda_2 + \frac{\lambda_1}{1+r_1} \\ &+ \mu_{h2} u^h_{cc,2} \frac{1}{f_h} + \mu_{h1} \left(-u^h_{cc,1} \frac{1}{1+r_1} \frac{1}{f_h} - \beta \left(1+r_1\right) u^h_{cc,2} \frac{1}{f_h} \right) = 0\end{aligned}$$

hence

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial B_2} =& - \lambda_2 + \frac{\lambda_1}{1+r_1} \\ &+ \mu_{h2} u^h_{cc,2} \frac{1}{f_h} + \mu_{h1} \left(-u^h_{cc,1} \frac{1}{1+r_1} \frac{1}{f_h} - \beta (1+r_1) u^h_{cc,2} \frac{1}{f_h} \right) = 0\end{aligned}$$

Let's look at

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial b_1} =& \tilde{f}_l u^l_{c1} + \tilde{f}_h u^h_{c1} - \lambda_1 \\ &+ \mu_{l1} u^l_{cc,1} + \mu_{h1} u^h_{cc,1} = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial b_2} =& \beta \tilde{f}_l u^l_{c2} + \beta \tilde{f}_h u^h_{c2} - \lambda_2 \\ &+ \mu_{l2} u^l_{cc,2} - \beta \left(1+r_1\right) \mu_{l1} u^l_{cc,2} \\ &+ \mu_{h2} u^h_{cc,2} - \beta \left(1+r_1\right) \mu_{h1} u^h_{cc,2} = 0\end{aligned}$$

Inserting the two in the foc for B_2 yields:

$$\begin{aligned}&\mu_{h2} u^h_{cc,2} \frac{1}{f_h} + \mu_{h1} \left(-u^h_{cc,1} \frac{1}{1+r_1} \frac{1}{f_h} - \beta \left(1+r_1\right) u^h_{cc,2} \frac{1}{f_h} \right) + \frac{\mu_{l1} u^l_{cc,1} + \mu_{h1} u^h_{cc,1}}{1+r_1} \\ &- \mu_{l2} u^l_{cc,2} + \beta \left(1+r_1\right) \mu_{l1} u^l_{cc,2} \\ &- \mu_{h2} u^h_{cc,2} + \beta \left(1+r_1\right) \mu_{h1} u^h_{cc,2} = 0\end{aligned}$$

$$\mu_{h2}u^h_{cc,2}\left(\frac{1}{f_h}-1\right)+\mu_{h1}u^h_{cc,1}\frac{1}{1+r_1}\left(1-\frac{1}{f_h}\right)+\mu_{h1}u^h_{cc,2}\beta(1+r_1)\left(1-\frac{1}{f_h}\right)+\frac{\mu_{l1}u^l_{cc,1}}{1+r_1}$$
$$-\mu_{l2}u^l_{cc,2}+\beta\left(1+r_1\right)\mu_{l1}u^l_{cc,2}=0$$

hence

$$\mu_{h2}u^h_{cc,2}\left(\frac{1}{f_h}-1\right)+\mu_{h1}\left(1-\frac{1}{f_h}\right)SOC_{h1}+\mu_{l1}SOC_{l1}-\mu_{l2}u^l_{cc,2}=0$$

hence

$$\left(1-\frac{1}{f_h}\right)\left(\mu_{h1}SOC_{h1}-\mu_{h2}u^h_{cc,2}\right)+\left(\mu_{l1}SOC_{l1}-\mu_{l2}u^l_{cc,2}\right)=0$$

What can this mean? Let'S also derive this for the FOC for B_3 and look whether we see a pattern and at

$$\frac{\partial\mathcal{L}}{\partial\tau_1}=-\tilde{f}_l u^l_{ct}y_{l1}(\tau_1)-\tilde{f}_h u^h_{c1}y_{l1}(\tau_1)+\lambda_1\sum_{i=l,h}\left(y_{i,1}+\tau_1\frac{\partial y_{i,1}}{\partial\tau_1}\right)$$
$$-\mu_{l1}u^l_{cc,1}y_{l1}(\tau_1)-\mu_{h1}u^h_{cc,1}y_{h1}(\tau_1)=0$$

$$\frac{\partial\mathcal{L}}{\partial\tau_2}=-\tilde{f}_l u^l_{c2}y_{l2}(\tau_2)-\beta\tilde{f}_h u^h_{c2}y_{l2}(\tau_2)+\lambda_2\sum_{i=l,h}\left(y_{i,2}+\tau_2\frac{\partial y_{i,2}}{\partial\tau_2}\right)$$
$$-\mu_{l2}u^l_{cc,t}y_{l2}(\tau_2)+\beta\left(1+r_2\right)\mu_{l1}u^l_{cc,2}y_{l2}(\tau_2)$$
$$-\mu_{h2}u^h_{cc,2}y_{h2}(\tau_2)+\beta\left(1+r_2\right)\mu_{h1}u^h_{cc,2}y_{h2}(\tau_2)=0$$

B.1 Proof of Section 5.3

We now take first-order conditions. A bit more notation: $\theta_{t,i}=\theta_i(1+g_t)$ for $i=\ell,h$.

$$c^\ell_t:\quad u^\ell_{c,t}\tilde{f}_\ell+f_\ell\nu_t\left(u^\ell_{cc,t}(c^\ell_t-b_t)+u^\ell_{c,t}+u^\ell_{cn,t}n^\ell_t\right)-\varepsilon_t\frac{u^\ell_{cc,t}}{u^\ell_{c,0}}-\mu_t\theta_\ell\left(\frac{u^\ell_{cc,t}u^\ell_{n,t}-u^\ell_{cn,t}u^\ell_{c,t}}{(u^\ell_{n,t})^2}\right)=\lambda_t f_\ell$$

$$c^h_t:\quad u^h_{c,t}\tilde{f}_h+f_h\Phi\left(u^h_{cc,t}(c^h_t-b_t)+u^h_{c,t}+u^h_{cn,t}n^h_t\right)+\varepsilon_t\frac{u^h_{cc,t}}{u^h_{c,0}}+\mu_t\theta_h\left(\frac{u^h_{cc,t}u^h_{n,t}-u^h_{cn,t}u^h_{c,t}}{(u^h_{n,t})^2}\right)=\lambda_t f_h$$

$$n^\ell_t:\quad u^\ell_{n,t}\tilde{f}_\ell+f_\ell\nu_t\left(u^\ell_{cn,t}(c^\ell_t-b_t)+u^\ell_{n,t}+u^\ell_{nn,t}n^\ell_t\right)-\varepsilon_t\frac{u^\ell_{cn,t}}{u^\ell_{c,0}}-\mu_t\theta_\ell\left(\frac{u^\ell_{cn,t}u^\ell_{n,t}-u^\ell_{nn,t}u^\ell_{c,t}}{(u^\ell_{n,t})^2}\right)=-\lambda_t\theta_{\ell,t}f_\ell$$

$$n^h_t:\quad u^h_{n,t}\tilde{f}_h+f_h\Phi\left(u^h_{cn,t}(c^h_t-b_t)+u^h_{n,t}+u^h_{nn,t}n^h_t\right)+\varepsilon_t\frac{u^h_{cn,t}}{u^h_{c,0}}+\mu_t\theta_h\left(\frac{u^h_{cn,t}u^h_{n,t}-u^h_{nn,t}u^h_{c,t}}{(u^h_{n,t})^2}\right)=-\lambda_t\theta_{h,t}f_h$$

$$b_t:\quad -f_h\Phi u^h_{c,t}-f_\ell\nu_t u^\ell_{c,t}=0$$

Rearranging the last equation, and using the constraint corresponding to the Euler equation, we get:

$$-\frac{f_h}{f_\ell}\frac{\Phi}{\nu_t} = \frac{u^\ell_{c,t}}{u^h_{c,t}} = \frac{u^\ell_{c,0}}{u^h_{c,0}}$$

And therefore $\nu_t = \nu$ independent of t. This is not surprising as the low-type household also faces complete markets without borrowing constraints.

Now, we can combine first-order conditions with respect to c^ℓ_t and n^ℓ_t to get rid of λ_t and obtain $(\star_\ell)$:

$$\begin{aligned}\left(\tilde{f}_\ell + f_\ell \nu\right)(\theta_{\ell,t}u^\ell_{c,t} + u^\ell_{n,t}) + f_\ell \nu \left[(\theta_{\ell,t}u^\ell_{cc,t} + u^\ell_{cn,t})(c^\ell_t - b_t) + (\theta_{\ell,t}u^\ell_{cn,t} + u^\ell_{nn,t})n^\ell_t\right]\dots \\ - \varepsilon_t \frac{\theta_{\ell,t}u^\ell_{cc,t} + u^\ell_{cn,t}}{u^\ell_{c,0}} - \mu_t\theta_\ell\left(\frac{(\theta_{\ell,t}u^\ell_{cc,t} + u_{cn,t})u^\ell_{n,t} - (\theta_{\ell,t}u^\ell_{cn,t} + u^\ell_{nn,t})u^\ell_{c,t}}{(u^\ell_{n,t})^2}\right) = 0 \qquad (\star_\ell)\end{aligned}$$

Similarly, we can obtain $(\star_h)$ for the high-type:

$$\begin{aligned}\left(\tilde{f}_h + f_h \Phi\right)(\theta_{h,t}u^h_{c,t} + u^h_{n,t}) + f_h \Phi \left[(\theta_{h,t}u^h_{cc,t} + u^h_{cn,t})(c^h_t - b_t) + (\theta_{h,t}u^h_{cn,t} + u^h_{nn,t})n^h_t\right]\dots \\ + \varepsilon_t \frac{\theta_{h,t}u^h_{cc,t} + u^h_{cn,t}}{u^h_{c,0}} + \mu_t\theta_h\left(\frac{(\theta_{h,t}u^h_{cc,t} + u_{cn,t})u^h_{n,t} - (\theta_{h,t}u^h_{cn,t} + u^h_{nn,t})u^h_{c,t}}{(u^h_{n,t})^2}\right) = 0 \qquad (\star_h)\end{aligned}$$

And finally, we can also combine first-order conditions with respect to c^ℓ_t and c^h_t, with new notation again, $\Phi_h \equiv \Phi$ and $\Phi_\ell = -\nu$, to get $(\star_c)$:

$$\begin{aligned}u^h_{c,t}\frac{\tilde{f}_h}{f_h} - u^\ell_{c,t}\frac{\tilde{f}_\ell}{f_\ell} + \sum_{i=\ell,h}\Phi_i\left(u^i_{cc,t}(c^i_t - b_t) + u^i_{c,t} + u^i_{cn,t}n^i_t\right)\dots \\ +\varepsilon_t\sum_i \frac{1}{f_i}\frac{u^i_{cc,t}}{u^i_{c,0}} + \mu_t\sum_i\frac{\theta_i}{f_i}\left(\frac{u^i_{cc,t}u^i_{n,t} - u^i_{cn,t}u^i_{c,t}}{(u^i_{n,t})^2}\right) = 0 \qquad (\star_c)\end{aligned}$$

Finally, we could get a last equation for labor $(\star_n)$ but it should be a linear combination of $(\star_\ell)$, $(\star_h)$, and $(\star_c)$, so let's ignore it.

Last but not least: these derivations would be different in $t = 0$, because of $u^i_{c,0}$ appearing specifically in the problem (P.1). (And in time-0, we expect the right-hand-side of the high-type borrowing contraint to disappear; but still, note that $\epsilon_0 = 0$; what does this mean for the time-0 price effect?).

Proof. I have 7 variables $\{c^\ell_t, c^h_t, n^\ell_t, n^h_t, b_t, \varepsilon_t, \mu_t\}$, for each period t. I also have 7 static equations to determine these 7 variables, given $\{\Phi_\ell, \Phi_h, c^\ell_0, c^h_0\}$: $(\star_\ell)$, $(\star_h)$, and $(\star_c)$, feasibility, low-skill household's borrowing constraint, and the two constraints associated with ε_t and μ_t. Hence, history-independence (trivial in deterministic environments), and as long as $g_t = 0$, allocations are constant across time (and so are prices and taxes).

Time-zero. In time-zero, as pointed out above, allocations are different. Is it true (again, that is, as with one agent) that they are different only if $\hat{B}_0 \neq 0$? It seems less obvious now, because as $t = 0$, we also have that $\epsilon_0 = 0$.

The first-order conditions (reported here for consumption only) would become:

$$c_0^\ell: \quad u_{c,0}^\ell \tilde{f}_\ell + f_\ell \nu \left(u_{cc,0}^\ell (c_0^\ell - b_0) + u_{c,0}^\ell + u_{cn,0}^\ell n_0^\ell\right) - \mu_0 \theta_\ell \left(\frac{u_{cc,0}^\ell u_{n,0}^\ell - u_{cn,0}^\ell u_{c,0}^\ell}{(u_{n,0}^\ell)^2}\right) + \dots$$

$$+ \sum_{t=1}^{\infty} \beta^t \varepsilon_t u_{cc,0}^\ell \frac{u_{c,t}^\ell}{(u_{c,0}^\ell)^2} = \lambda_0 f_\ell$$

$$c_0^h: \quad u_{c,0}^h \tilde{f}_h + f_h \Phi \left(u_{cc,0}^h (c_0^h - b_0) + u_{c,0}^h + u_{cn,0}^h n_0^h - u_{cc,0}^h \hat{B}_0\right) + \mu_0 \theta_h \left(\frac{u_{cc,0}^h u_{n,0}^h - u_{cn,0}^h u_{c,0}^h}{(u_{n,0}^h)^2}\right) + \dots$$

$$- \sum_{t=1}^{\infty} \beta^t \varepsilon_t u_{cc,0}^h \frac{u_{c,t}^h}{(u_{c,0}^h)^2} = \lambda_0 f_h$$

From the numerical exercise, I am expecting that when $g_t = 0$ and $B_0 = 0$, then time-inconsistency disappears. Now, as $\sum_{t=1}^{\infty} \beta^t \varepsilon_t u_{cc,0}^i \frac{u_{c,t}^i}{(u_{c,0}^i)^2} = 0$ will be equal to 0 *only if* if ε_t is equal to zero. We already know that ε_t is constant across time, as we just shown.

Then, we have:

$$\begin{aligned}
\sum_{t=1}^{\infty} \beta^t \varepsilon_t u_{cc,0}^i \frac{u_{c,t}^i}{(u_{c,0}^i)^2} &= \sum_{t=1}^{\infty} \beta^t \varepsilon u_{cc,0}^i \frac{u_{c,1}^i}{(u_{c,0}^i)^2} \\
&= \varepsilon u_{cc,0}^i \frac{u_{c,1}^i}{(u_{c,0}^i)^2} \sum_{t=1}^{\infty} \beta^t \\
&= \varepsilon u_{cc,0}^i \frac{u_{c,1}^i}{(u_{c,0}^i)^2} \frac{\beta}{1-\beta}
\end{aligned}$$

If there is no price effect, then $\varepsilon_t = 0 \ \forall t$. Can we show it?

Growth without subsistence. For some utility functions, we can show that for $t \geq 1$, the tax rate is constant even when there is growth.

For instance, let us assume this standard utility function:

$$u(c, n) = \frac{c^{1-\rho}}{1-\rho} - \frac{n^{1+\varphi}}{1+\varphi}$$

Then, if n grows at rate $(1+g)^{\frac{1-\rho}{\varphi+\rho}}$ and c and b grow at rate $(1+g)^{\frac{1+\varphi}{\varphi+\rho}}$, then multiplier ε grows at rate $(1+g)^{\frac{1+\varphi}{\rho+\varphi}}$ and μ at rate $(1+g)^{\frac{-\varphi(1-\rho)}{\rho(1+\varphi)}}$, and τ is constant, b/y is constant, etc.

Something similar holds with GHH preferences