

Exchange Rates Co-movement and International Trade*

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Abstract

Nominal exchange rates strongly co-move. However, little is known about the economic source of common variation. This paper examines how international trade links nominal exchange rates. First, I document that two countries that trade more intensively with each other have more correlated exchange rates against the U.S dollar. Second, I develop a general equilibrium multi-country model, where a shock to a single country propagates to the exchange rates of its trading partners and serves as a source of common variation. In the baseline three-country model, I show that the sign and the strength of correlation between exchange rates depend on the elasticities of trade balances of countries with respect to both exchange rates. As a result, the model's prediction about the relationship between bilateral trade intensity and exchange rates correlation depends on the currency in which international prices are set. Lastly, an augmented model is calibrated to twelve countries to quantitatively assess the importance of trade linkages. I find that trade linkages alone, with uncorrelated shocks across countries, account for 50% of the empirical trade-exchange-rates-correlation slope coefficient.

Keywords: Exchange Rates, Trade Network, International Finance, Multi-Country Models

JEL classification: F31, F32, F41

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1 Introduction

Understanding forces that determine the equilibrium behavior of nominal exchange rates is essential for recommendations on the monetary policy and the analysis of the international shocks transmission. However, conventional macroeconomic models produce predictions about the exchange rate behavior that are inconsistent with the data. The long-standing puzzle in international macroeconomics is the weak relationship between the exchange rate and macroeconomic fundamentals that according to the theory should be relevant to the exchange rate determination (see Meese and Rogoff (1983) and Obstfeld and Rogoff (2000)). Nonetheless, recent research in international finance reports that the nominal exchange rates strongly co-move. In particular, Verdelhan (2018) and Lustig and Richmond (2017) construct factors that explain a large share of the variation in bilateral exchange rates. Furthermore, there exists substantial heterogeneity in loadings on common factors across exchange rates. If exchange rates do not move independently, an economic source of common variation should play an important role in the exchange rates determination.¹

In this paper, I argue that trade between countries is one of the mechanisms that cause exchange rates co-movement. First, using data for 33 countries over 25 years, I document that bilateral trade intensity, that is the gross trade between two countries normalized by the sum of their GDP, is associated with the cross-sectional dispersion in exchange rates correlations. In particular, two countries that trade more intensively with each other have more correlated exchange rates against the U.S dollar. I find that an increase in the bilateral trade intensity from the 25th percentile to the 75th percentile is associated with an increase in correlation by 0.13. This increase equals to almost half of the standard deviation of nominal exchange rates correlation and, therefore, is economically significant.

Second, I develop a three-country baseline model that admits a closed-form solution, thus allowing me to pinpoint a novel mechanism determining the cross-sectional dispersion in exchange rates correlation. In the model, three countries trade with each other and exchange rates are linked through trade flows. In particular, a shock to a single country propagates to the exchange rates of its trading partners and serves as a source of common variation. I show that the sign and the strength of correlation between exchange rates depend on the elasticities of trade balances of countries with respect to both exchange rates. These elasticities summarize all general equilibrium effects of nominal exchange rates on the trade balance due to the adjustments in international prices and quantities. Moreover, the effect of country-pair characteristics (for example, bilateral trade intensity) on exchange rates co-movement depends on how these characteristics affect elasticities of trade balances.² Due to this property, perhaps counterintuitively, the baseline model does not necessarily generate a monotone relationship between trade intensity and exchange rates correlation.

¹Hodrick and Vassalou (2002) investigate the dynamics of exchange rates using a multi-country generalization of Cox et al. (1985) class of terms of structure models. They show that in some cases the multi-country model outperforms the two-country model in explaining the dynamics of exchange rates. Berg and Mark (2015) examine third-country spillovers to bilateral exchange rates. They show that the inclusion of third-country factors into bilateral exchange rate regressions increases the adjusted R^2 .

²In the baseline model, I study the effect of the steady-state level of trade intensity that is exogenous and captured by the preference parameters. The similar approach is used in Acemoglu et al. (2016) and Richmond (2016).

The elasticities of trade balance with respect to exchange rates are determined by the sensitivity of international prices to the nominal exchange rates fluctuations. This sensitivity depends crucially on the currency in which firms charge prices for their foreign consumers. The classical assumption in international macroeconomics was that prices are set in the currency of producer, which implies the law of one price (see, e.g., Obstfeld and Rogoff (1995) and Clarida et al. (2001)). Under this assumption, the baseline model does not predict the positive association between trade intensity and exchange rates correlation. Based on the weak empirical evidence supporting the law of one price (see Burstein and Gopinath (2014) for a recent survey), Betts and Devereux (2000) proposed a model where prices are set in the currency of consumer. Additionally, motivated by the evidence that the bulk of prices are set in U.S. dollars even if none of the trading partners is the U.S., recent papers of Casas et al. (2016) and Boz et al. (2017) propose a model where all import prices are set in the dominant currency. I show that the baseline model where firms set prices either in the currency of consumer or in the dominant currency generates the positive relationship between bilateral trade intensity and exchange rates co-movement.

Lastly, I quantitatively assess the importance of trade linkages for exchange rates co-movement. I build an augmented version of the model and calibrate it to the set of 12 developed countries.³ Countries are identical except for (i) the share of firms that set prices in U.S. dollars; (ii) the relative sizes; and (iii) contributions of a particular trading partner into country's total consumption. The latter two characteristics generate heterogeneity in bilateral trade intensities. I allow for a fraction of firms to set prices in the producer's currency, a fraction of firms to set prices in the consumer's currency, and the rest of the firms to set prices in U.S. dollars. These fractions are calibrated to match estimates of pass-through of the bilateral and the U.S. dollar exchange rates into import prices by Boz et al. (2017). Exchange rates co-move because shocks to individual countries propagate to exchange rates of their trading partners through international trade. I find that trade linkages alone, with uncorrelated shocks across countries, account for 50% of the empirical slope coefficient in the regression of exchange rates correlation on trade intensity.

The baseline three-country model offers the economic insight behind exchange rates co-movement due to trade linkages. In the model, I assign one country to be the base-currency country, i.e., the United States. For the ease of exposition, I call the two other identical countries Canada and Japan. I define two nominal exchange rates of the Canadian dollar and the Japanese yen against the U.S. dollar.⁴ The baseline model features two properties which are important for the model's ability to account for the low correlation between exchange rates and other macroeconomic variables in a two-country setting. The first property is the assumption that financial markets are incomplete, and so the country's budget constraint plays a role in determining the equilibrium exchange rates (see, for example, Devereux and Engel (2002), Gabaix and Maggiori (2015), and Itskhoki and Mukhin (2017)). The second property is

³I build a N -country model with standard assumptions of NKM, except for the structure of financial markets as in Gabaix and Maggiori (2015) and the presence of international asset demand shocks additionally to monetary policy and TFP shocks.

⁴For the remainder of the paper, *the exchange rate of the Canadian dollar (Japanese yen)* means *the exchange rate of the Canadian dollar (Japanese yen) against the U.S. dollar* and *the Canadian dollar (the Japanese yen) appreciates/depreciates* means that it appreciates/depreciates against the U.S. dollar.

that a shock to demand for the international asset drives the exchange rates fluctuations.⁵

Under these two assumptions, two nominal exchange rates re-balance in response to international asset demand shocks to restore the budget constraints of Canada and Japan. The latter represents the balance of payments of a country. In the simplest version of the baseline model, the stochastic international asset demand to a country can only be absorbed by the domestic households. Then, the equilibrium behavior of exchange rates is determined by the elasticities of trade balances of Canada and Japan with respect to both nominal exchange rates. These elasticities capture all general equilibrium effects of exchange rates on the country's trade balance due to the adjustments of international prices and quantities and depend on the structure of the model.

Suppose that Canada is hit by an international asset demand. This stochastic demand is absorbed by the Canadian households if they are willing to consume more than produce and, therefore, borrow from the rest of the world. Hence, the shock requires the adjustment of the Canadian dollar exchange rate to create the trade deficit in Canada. As long as the elasticity of the trade balance of Japan with respect to the Canadian dollar exchange rate is not zero, the change in the latter disturbs the budget constraint of Japan and induces the adjustment of the Japanese yen exchange rate to restore the equilibrium. As a result, the shock to Canada induces fluctuations of both exchange rates.

The sign of exchange rates correlation depends on the signs of the trade balance elasticities and the strength of correlation depends on relative magnitudes of elasticities. Suppose that the trade balance of a country improves if the domestic currency depreciates. Then, an international asset demand shock to Canada leads to the appreciation of the Canadian dollar. If the Canadian dollar appreciation improves the trade balance of Japan, the Japanese yen appreciates in equilibrium to induce the deterioration of the trade balance. Hence, two exchange rates positively co-move in response to the financial shock to Canada. If the trade balance of Japan is weakly sensitive to the exchange rate of the Canadian dollar and highly sensitive to the exchange rate of the Japanese yen, the Japanese yen has to appreciate just a bit to restore the Japanese budget constraint. As a result, the co-movement between exchange rates is weak. The correlation between exchange rates increases as the sensitivity of the trade balance of Japan to the Canadian dollar exchange rate increases relatively to the sensitivity to the Japanese yen exchange rate. If instead, the trade balance of Japan worsens when the Canadian dollar appreciates, two exchange rates negatively co-move.

The trade intensity between Canada and Japan affects the sensitivity of the trade balances to exchange rates fluctuations through two channels. First, an increase in the former increases the sensitivity of trade balances of Canada and Japan to fluctuations in the net trade flow between two countries. In the baseline model, if prices are set in the currency of consumer or in U.S. dollars, this channel results in a positive association between trade intensity and exchange rates correlation. Second, an increase in trade intensity

⁵Itskhoki and Mukhin (2017) argue that only such type of shocks combined with a particular transmission mechanisms is able to generate a volatile exchange rate that does not propagate to local prices and quantities. Their transition mechanism consists of (i) significant home bias in consumption, (ii) pricing to market, and (iii) weak substitutability between home and foreign goods. I do not restrict countries' openness to trade, the elasticity of substitution between domestic and foreign goods. Instead, I study how results on exchange rates co-movement change depending on values on these parameters and as well as on the currency of pricing assumptions.

decreases the sensitivity of Canadian (Japanese) demand for import from Japan (Canada) to changes in its price. If prices are set in the currency of producer, for small values of trade intensity, the first channel results in the positive link between trade intensity and exchange rates correlation. However, for high values of trade intensity, exchange rates correlation decreases with trade intensity due to the second channel. As a result, more intense trade does not necessarily lead to stronger correlation of exchange rates.

Related literature. There is growing literature on the exchange rate determination (see, Itskhoki and Mukhin (2017), Gabaix and Maggiori (2015), Evans and Lyons (2012), Engel (2016), Valchev (2017), Hassan et al. (2016), Bacchetta and Van Wincoop (2006) and Bacchetta and Van Wincoop (2010)). Most of the papers focus on the model's ability to account for the stylized facts about the correlation between the exchange rate and macroeconomic fundamentals in a two-country setting. In contrast, I develop the multi-country model of exchange rates determination that explains exchange rates co-movement with trade linkages. To the best of my knowledge, this is the first paper that documents empirically and explains theoretically the link between exchange rates co-movement and international trade.

Second, this paper is related to the recent studies on exchange rates co-movement in international finance (see, for example, Verdelhan (2018), Lustig and Richmond (2017), Hassan and Mano (2014), and Mueller et al. (2017)). Verdelhan (2018) and Lustig and Richmond (2017) construct common factors that account for the significant share of individual exchange rates variations. Lustig and Richmond (2017) find that the country's distance from the U.S (i.e., physical distance, a shared language, legal origin, shared border, etc.) is the key determinant of a country's exchange rate (against the U.S. dollar) loading on the common factor. In this paper, I show that international trade is associated with cross-sectional dispersion of exchange rates correlations even after controlling for gravity variables with country-pair fixed effects. Verdelhan (2018) and Lustig and Richmond (2017) study exchange rates covariation in the reduced-form stochastic discount factor framework. In this work, I suggest one possible explanation of the existence of common factors using the structural general equilibrium framework.

This paper also contributes to the strand of literature that investigates the joint determination of exchange rates in the general equilibrium setting. Pavlova and Rigobon (2008) show that exchange rates correlation increases with the tightness of portfolio constraints of the Centre's agents in the three-country Centre-Periphery model. Berg and Mark (2015) show that exchange rates co-move due to the propagation of third-country shocks to the bilateral exchange rate through the asymmetric monetary policy rules and cross-country differences in price stickiness. Colacito et al. (2018) build a multi-country frictionless risk-sharing model with recursive preferences and multiple consumption goods to study the cross-section of currency risk premia. None of these papers discusses the role of international trade.

The paper is organized as follows. Section 2 presents the main empirical findings. Section 3 describes the assumptions of the baseline general equilibrium 3-country model that admits the closed-form solution. Next, it discusses the mechanism linking the exchange rates through international trade and reports all the analytical results of the paper. In Section 4, I build a generalized version of the model and show quantitatively that the model is consistent with the empirical findings of Section 2.

2 Empirical Results

This section demonstrates that the volume of bilateral trade between countries is associated with the cross-sectional dispersion in exchange rates co-movement. I show that there exists a significant positive association between the bilateral trade intensity and exchange rate correlations after one controls for the peg, similarities in commodity export and various fixed effects. I start with describing the data, then I present the estimation strategy, and, lastly, discuss the main empirical results.

Data. I use a sample of 33 (non-U.S.) countries.⁶ The maximum time period is from 1989 to 2016. The set of countries and the time period is limited by the data availability and by the fact that the analysis excludes periods of the hard peg. I also exclude periods of high inflation and free falling exchange rates (that would be classified periods of floating exchange rate) based on the Country Chronologies from Ilzetzi et al. (2017). Hence, I use the data for 10 countries only starting from 1999.⁷ Countries that joined the Euro Area are excluded after 1999 and the euro series start

Nominal exchange rates are the exchange rates against the U.S. dollar at the end of each period (month or quarter) reported in the Federal Reserve historical dataset. The trade data come from the IMF Direction of Trade Statistics. As a robustness check, I also use the STAN Bilateral Trade Database by Industry and End-Use data (BTDIxE) by OECD. Both datasets report the U.S. dollar value of export/import of each country to/from partners. In the datasets, the reported export from a country A to a country B is not always equal to the reported import of the country B from the country A. I use the reported data for export for the main regressions. Results are robust to using the reported import to calculate bilateral trade intensity. The trade data are annual and normalized by annual nominal GDP measured in U.S. dollars collected from the OECD Main Economic Indicators.

The peg data is from Shambaugh (2004). Shambaugh (2004) divides all countries into pegs and nonpegs in each year, based on the volatility of exchange rates. To be classified as peg the volatility of the country's nominal exchange rate return has to be less than 2% in two consecutive years. I define the nominal exchange rate as pegged if it is pegged in more than 15% of time periods used to calculate correlations and other series.

To control for similarities between countries in terms of commodity export, I use data from United Nations COMTRADE. I construct a matrix, where each vector contains the share of export of each of 45 primary commodities in total export for each country in the sample.⁸ Next, I exclude commodities that contribute less than 2% to total export (setting value to zero), as these commodities are likely to have no effect on a country's exchange rate. The final measure of the commodity-export similarity between two

⁶These countries are: Australia, Austria, Belgium, Brazil, Canada, Chile, Colombia, Czech Republic, Denmark, Euro Area, Finland, France, Germany, Greece, Hungary, Ireland, Israel, Italy, Japan, Korea, Mexico, Netherlands, New Zealand, Norway, Poland, Portugal, Russian Federation, South Africa, Spain, Switzerland, Sweden, Turkey, the United Kingdom.

⁷These countries are Brazil, Chile, Colombia, Czech Republic, Korea, Mexico, Poland, Russian Federation, South Africa, and Turkey.

⁸Commodities include alumina, aluminum, bananas, barley, beef, coal, cocoa beans, coffee, copper, cork and wood, cotton, fish, gold, groundnuts, hides, iron, lamb, lead, maize, natural gas, nickel, olive oil, oranges, palm oil, platinum, poultry, rapeseed oil, rice, rubber, shrimps, silver, soybeans, soybean oil, petroleum, sugar, sunflower oil, swine meat, tea, timber, tin, tobacco, uranium, wheat, wool, zinc.

countries is the cosine similarity between two vectors. This measure takes values between 0 and 1 and its higher value implies the higher similarity.

Exchange rates correlation and trade intensity. I first empirically investigate whether there exists a positive association between trade intensity and exchange rates correlation. I estimate the correlation of nominal exchange rate returns measured at the quarterly frequency for each pair of countries k and l , $corr(\Delta \ln \mathcal{E}^k, \Delta \ln \mathcal{E}^l)$, where \mathcal{E}^k is the nominal exchange rate of country k in terms of the U.S. dollar. Second, I compute the measure of trade intensity between countries k and l as follows:

$$Bil.Tr_{kl} = mean \left(\frac{Export_{kl} + Export_{lk}}{NGDP_k + NGDP_l} \right),$$

where $Export_{kl}$ is the export from country k to l in the U.S. dollars and $NGDP_k$ is the nominal GDP of country k in the U.S. dollars. Export and GDP data is annual and I take the mean of annual measure over the entire period. The same measure of bilateral trade intensity is used in Frankel and Rose (1998), Kose and Yi (2006), and Lustig and Richmond (2017).

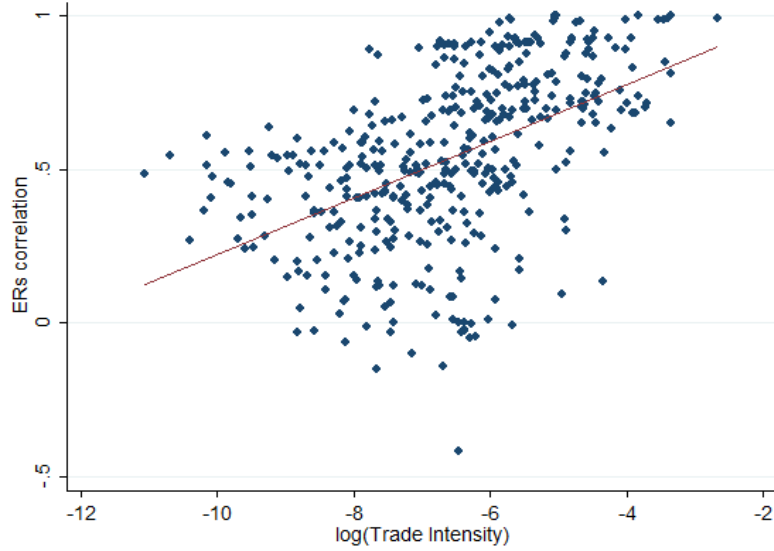


Figure 1: Plot of correlations of nominal exchange rate returns versus the logarithm of measure of bilateral trade intensity. Correlations and trade intensity are measured using time series for the period 1989-2013. Nominal exchange rate returns are quarterly. Trade data is reported export from the IMF.

Figure (1) plots the estimated exchange rates correlations against the logarithm of the constructed measure of trade intensity. The correlations and trade intensity is estimated using data for the period from 1989 to 2013.⁹ First, most of the exchange rates are positively correlated with average pairwise correlation being close to 0.5. Second, data shows a positive association between the two variables.

To examine this association formally, I run the following regression:

$$corr(\Delta \ln \mathcal{E}^k, \Delta \ln \mathcal{E}^l) = \beta_0 + \beta_1 \log(Bil.Tr_{kl}) + \beta_2 Peg_{kl} + \beta_3 Exp.Sim_{kl} + u_{kl}. \quad (1)$$

⁹The period from 1989 to 2013 consists of 25 years that later are split into 5 sets of 5 windows.

The regression includes two control variable. As exchange rates of two countries might be more correlated if both are pegged to the same currency, I include the peg dummy, Peg_{kl} . The peg dummy is equal to one if both currencies are pegged to the same currency (USD, EUR or DM before 1998) or if one currency in the pair is pegged to another currency in the pair. It is expected that the estimated coefficient β_2 is positive. Chen et al. (2010) show that there exists a link between commodity currency exchange rates and commodity prices. Therefore, the exchange rates of two countries that export the same commodities can be more correlated. As the sample of countries includes commodity exporters, I construct a measure of commodity-export similarity between countries k and l , $Exp.Sim_{kl}$. The higher values of $Exp.Sim_{kl}$ means the higher similarity, so β_3 expected to be positive.

Estimation results are shown in Table 1. The last three columns of Table 1 include country fixed effects. The results show that the coefficient in front of trade intensity is statistically significant. Given the point estimates from column (6), the increase of the bilateral trade intensity from the 25th percentile to the 75th percentile leads to the rise of the exchange rate correlation by 0.13. This effect is economically significant. The standard deviation of exchange rates correlations is 0.27, the difference between 75th and 25th percentiles is 0.33. The peg dummy is significant and has a positive effect on exchange rates correlation as expected. The measure of commodity-export similarity is not significant.

Robustness. Results are robust if: (i) monthly exchange rate returns are used instead of quarterly returns, (ii) correlations and other series are estimated using only the pre-crisis period or the period is extended up to 2016, (iii) the real exchange rates returns are used instead of nominal (monthly and quarterly), and (iv) if one uses other trade data (import data instead of export data from the IMF, or data published by OECD). See results in Appendix A.1.

As another robustness check, I construct a panel dataset creating 5 periods of 5 years each. In every time window I compute all series defined above for all country pairs. Panel dataset allows to control for the country-pair fixed effects. The estimated regression equation is

$$corr(\Delta \ln \mathcal{E}^k, \Delta \ln \mathcal{E}^l)_t = \beta_0 + \beta_1 \log(Bil.Tl_{tkl}) + \beta_2 Peg + \beta_3 Exp.Sim_{tkl} + a_{kl} + \delta_t + u.$$

where a_{kl} is a country-pair fixed effect and δ_t is a time-window fixed effect, $t = \{1, \dots, 5\}$. Lustig and Richmond (2017) find that the country's "distance" from the U.S. is the key determinant of country's exchange rate's loading on the common factor. The distance is measured using the gravity variables, i.e. physical distance, shared language, legal origin, shared border, etc. The estimation results with country-pair fixed effects allows me to test whether linkages among exchange rates due to international trade exist even if we allow for linkages due to gravity variables.

Results are shown in Table 2. Columns (1) and (2) show estimation results for nominal exchange rate returns measure at quarterly and monthly frequencies respectively. Columns (3) and (4) represent estimation results for real exchange rate returns measure at quarterly and monthly frequencies respectively. Results are similar to those in Table 1 and Tables 6-9 in Appendix A.1.

Table 1: Cross-sectional regression

Dependent variable: $corr(\Delta e^i, \Delta e^j)$						
	(1)	(2)	(3)	(4)	(5)	(6)
log(Bil.Tr)	0.0917*** (11.85)	0.0680*** (8.75)	0.0677*** (8.68)	0.0689*** (6.96)	0.0634*** (6.71)	0.0633*** (6.68)
Peg		0.274*** (8.06)	0.277*** (8.02)		0.257*** (6.33)	0.256*** (6.30)
Exp.Sim.			0.0166 (0.52)			-0.0180 (-0.43)
Country FE				Yes	Yes	Yes
Observations	407	407	407	407	407	407
Adjusted R^2	0.256	0.357	0.356	0.574	0.614	0.614

t statistics in parentheses: $p < .10$, ** $p < .05$, *** $p < .01$.

Period: 89-13. Nominal exchange rate, quarterly changes. Trade data - IMF export.

Table 2: Regression with panel data

Dependent variable: $corr(\Delta e^i, \Delta e^j)$				
	NER, Q	NER, M	RER, Q	RER, M
log(Bil.Tr)	0.10** (2.33)	0.10*** (3.03)	0.10** (2.53)	0.12*** (3.27)
Peg	0.10*** (2.98)	0.04 (1.30)	0.12*** (3.36)	0.06** (2.34)
Exp.Sim.	0.12*** (2.79)	0.03 (1.06)	0.13*** (3.00)	0.02 (0.67)
Country-Pair FE	Yes	Yes	Yes	Yes
Time Window FE	Yes	Yes	Yes	Yes
N	1092	1092	1092	893
Adj- R^2	0.384	0.448	0.382	0.456

t statistics in parentheses

* $p < .10$, ** $p < .05$, *** $p < .01$

3 Baseline model

I start with a baseline model that admits the closed form solution and carries the main intuition. As I need at least two nominal exchange rates to study exchange rates correlation, I assume that there are three countries $h \in \{1, 2, 3\}$. I assign country $h = 1$ to be a base-currency country and call this country as the United States. For the simplicity of exposition, I call country $h = 2$ Canada and country $h = 3$ Japan. Money is a unit of nominal account in each country with price normalized to 1. In the three-currency world, I study the joint determination of nominal exchange rates of the Canadian dollar and the Japanese yen with respect to the base currency, i.e., the U.S. dollar. The exchange rate between the Canadian dollar and the Japanese yen is determined as a cross rate.

Each country is populated by a representative household, continuum of firms, and a local government. Financial markets are incomplete and the household in each country has an access only to risk-free bonds denominated in the domestic currency. International trade of bonds is intermediated by two types of agents: financiers and noise traders. Financiers build their portfolio of bonds based on relative expected returns. In contrast, noise traders' demand for bonds does not depend on fundamentals. This stochastic demand is the only shock in this economy.

The demand for money is introduced through the cash-in-advance constraint. The equilibrium supply of money is determined by the monetary policy that stabilizes the level of consumption over time. Firms in each country produce identical country-specific tradable good. Prices are completely rigid, so the assumption about the currency in which prices are pre-set matters. The currency choice is exogenous and I consider separately the cases of producer currency pricing, local currency pricing, and dominant currency pricing. Naturally, the dominant currency in this world is the currency of country 1.

3.1 Model setup

Households. A representative household in country $h \in \{1, 2, 3\}$ maximizes the discounted expected utility over consumption C_t^h and labor L_t^h

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{(C_t^h)^{1-\sigma}}{1-\sigma} - \kappa \frac{(L_t^h)^{1+1/\nu}}{1+1/\nu} \right),$$

where σ is the parameter of relative risk aversion, ν is the Frisch elasticity of labor supply, and κ is the scaling parameter. The consumption basket C_t^h is defined as the CES function

$$C_t^h = \left[\sum_{i=1}^3 (a_i^h)^{\frac{1}{\theta}} (C_{it}^h)^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}}, \quad \sum_{i=1}^3 a_i^h = 1,$$

where C_{it}^h is consumption of good produced in country i by the household from country h , $a_i^h \geq 0$ are preferences parameters, and θ is the elasticity of substitution between home and foreign goods.

The consumption-based price index that corresponds to the above specification of preferences is

given by

$$P_t^h = \left(\sum_{i=1}^3 a_i^h (P_{it}^h)^{1-\theta} \right)^{\frac{1}{1-\theta}}, \quad (2)$$

where P_{it}^h is a price in country h of good produced in country i denominated in currency of country h . The optimal demand is given by

$$C_{it}^h = a_i^h \left(\frac{P_{it}^h}{P_t^h} \right)^{-\theta} C_t^h. \quad (3)$$

From the definition of the price index, it follows that $\sum_{i=1}^3 C_{it}^h P_{it}^h = P_t^h C_t^h$.

The currency of country 1, i.e., of the United States is the base currency and two exchange rates relative the U.S. dollar are simultaneously determined. The nominal exchange rate \mathcal{E}_{1t}^h , $h \in \{2, 3\}$, is the price of the currency of country h in terms of the U.S. dollars. Hence, an increase in \mathcal{E}_{1t}^h means a nominal appreciation of currency h relative to the U.S. dollar.

One purpose of the analysis is to investigate how more intense trade between Canada and Japan influences the correlation between exchange rates of the Canadian dollar, \mathcal{E}_{1t}^2 , and the Japanese yen, \mathcal{E}_{1t}^3 . In the symmetric steady state, the preference parameters a_3^2 and a_2^3 are equal to the steady-state level of the trade intensity between Canada and Japan as it is defined in the previous section, $a_3^2 = a_2^3 = \frac{\mathcal{E}_1^2 P_3^2 C_3^2 + \mathcal{E}_1^3 P_2^3 C_2^3}{\mathcal{E}_1^2 P_2^2 C_2^2 + \mathcal{E}_1^3 P_3^3 C_3^3}$. In section 3.2, I study how an increase in these parameters affects the strength of comovement between exchange rates in response to noise-traders shocks. Notice, that the price of the Canadian dollar in terms of the Japanese yen is determined as a cross rate $\mathcal{E}_{2t}^3 = \mathcal{E}_{1t}^3 / \mathcal{E}_{1t}^2$.

The parameters a_3^2 and a_2^3 affect the equilibrium allocation through two channels. First, as it can be seen from (3), the higher value of a_3^2 (a_2^3) implies that the households from Canada (Japan) consume more of goods produced in Japan (Canada) *given* relative prices and total level of consumption. Second, as (2) demonstrates, the parameters a_3^2 and a_2^3 control the responsiveness of consumer price index of Canada (Japan) to changes in import prices. The higher a_3^2 and, therefore the higher level of Canadian consumption produced in Japan *on average*, means that the consumer price index in Canada, P_t^2 , responds more to changes in the price of goods produced in Japan, P_{3t}^2 . Hence, changes in P_{3t}^2 due to the exchange rates fluctuations will be followed by changes in P_t^2 and with CES preferences Canadian demand for import from Japan responds less to changes in its price.

To determine the nominal exchange rates, I assume that money is the nominal unit of account in each country, with price normalized to 1. To create demand for money, I assume that the households choose optimal consumption and labor supply subject to the cash-in-advance constraints additionally to the budget constraints.¹⁰ At each period t , first, the state is realized. The household starts the period with some amount of the domestic currency, NI_t . The household decides how to split this amount between risk-free bonds denominated in domestic currency, B_{t+1}^h , and domestic currency holdings, M_t^h , so $B_{t+1}^h + M_t^h = NI_t$. Next, the representative household is split into a "worker" and "shopper". The worker produces consumption good and receives the salary, paid by firms in the domestic currency. The

¹⁰Similar approach to model demands for foreign exchange is used in Rey (2001).

shopper spends M_t^h on domestic and foreign goods and pays for them in the currency of producer.¹¹ The amount of the domestic currency the household starts each period with consists of the salary that the worker received in the previous period, $W_{t-1}^h L_{t-1}^h$, returns on risk-free bonds, $B_t^h R_t^h$, and the sum of profits paid by domestic good producers, Π_t^h , and financial intermediaries Pr_t^h , transfer from the government T_t^h , and money unspent in the previous period. The household's budget constraint is

$$B_{t+1}^h + M_t^h = W_{t-1}^h L_{t-1}^h + \Pi_t^h + Pr_t^h + B_t^h R_{t-1}^h + \left(M_{t-1}^h - \sum_{i=1}^3 C_{it-1}^h P_{it-1}^h \right) + T_t^h$$

where R_t^h is a risk-free rate of return.

The shopper has to pay for consumption goods in the currency of producer, therefore, she splits money holdings among the domestic currency and foreign currencies, $M_t^h = \sum_{i=1}^3 M_{it}^h / \mathcal{E}_{it}^h$, where M_{it}^h is the holding of currency i by household from country h , and \mathcal{E}_{it}^h is again the price of currency i in terms of currency h . Additionally, households are not allowed to keep saving in foreign currencies. However, they can save in the local currency. To summarize, the cash-in-advance constraints can be written as follows:

$$M_{ht}^h > P_{ht}^h C_{ht}^h, \quad M_{it}^h / \mathcal{E}_{it}^h = P_{it}^h C_{it}^h, \quad i \neq h.$$

The household's optimal consumption/saving decision is summarized by the Euler equation

$$\frac{1}{R_t^h} = \beta \mathbb{E}_t \left\{ \frac{(C_{t+1}^h)^{-\sigma} P_t^h}{(C_t^h)^{-\sigma} P_{t+1}^h} \right\}.$$

The optimal labor supply is given by the following equation:

$$\kappa (C_t^h)^\sigma (L_t^h)^{1/\nu} = \frac{W_t^h}{P_t^h R_t^h}. \quad (4)$$

The difference of the previous equation from the standard labor supply condition is that the households discount the real wage with the risk-free interest rate. I assume that the equilibrium nominal return is always greater than one, $R_t^h > 1$, so the CIA constraint always binds.

Production. Consumption goods are produced with only one input - labor, L_t^h , - with the linear production technology

$$Y_t^h = L_t^h, \quad (5)$$

where Y_t^h is the total production of tradable good in country h .

Pricing. As it was stressed in the introduction, the co-movement of nominal exchange rates crucially depends on the elasticity of trade balance of country h with respect to both exchange rates. The trade balance is the difference between total export and total import of the country. Therefore, the assumption about the currency in which firms set prices to foreign consumers plays a noticeable role in the adjustment

¹¹Another assumption that delivers the same equilibrium conditions would be that consumers pay in local currency and firms demand foreign exchange to pay wages in their domestic currency.

of the trade balance to the exchange rates changes.¹² Traditionally, international macroeconomic models assumed the producer currency pricing (PCP), under which prices are set in the currency of a producing country. The producer currency pricing implies high exchange rate pass-through, meaning that import prices (expressed in the importer's currency) should fluctuate with the nominal exchange rate between currencies of trading partners and export prices (expressed in the exporter's currency) are insensitive to exchange rates. Another assumption is that prices are set in the currency of a destination country (local currency pricing, LCP). With LCP import prices are not sensitive to exchange rate fluctuations. Based on evidences that the U.S. dollar exchange rates considerably affect international prices and trade quantities regardless of a destination or a source country (Boz et al. (2017)) and that a large share of international prices is invoiced in few currencies, the most important of which is the U.S. dollar (see Goldberg and Tille (2008) and Gopinath (2015)), Casas et al. (2016) propose the dominant currency pricing paradigm (DCP). With DCP import prices are sensitive to exchange rate of a country's currency relative to the U.S. dollar regardless of a source country.

I assume that prices are perfectly rigid and are pre-set at the steady state values at the initial date, $t = 0$, either in the currency of a producer, in the currency of a consumer, or in the dominant currency, i.e., in the U.S. dollars. This assumption can be considered as a limiting case of a Calvo pricing environment when the probability that a firm is able to reset a price goes to zero. The currency choice is modeled as exogenous. Given equilibrium exchange rates, firms produce an amount of goods that is sufficient to satisfy domestic and foreign demand.

Prices of goods produced domestically are always set in the domestic currency, this implies that P_{1t}^1 , P_{2t}^2 , and P_{3t}^3 are fixed. Three cases are considered separately:

- **Case 1 PCP:** the law of one price holds - $P_{ht}^i = P_{ht}^h / \mathcal{E}_{ht}^i$ and P_{ht}^h is constant for all t .
- **Case 2 LCP:** P_{ht}^i is fixed for all t and any $i, h \in \{1, 2, 3\}$;
- **Case 3 DCP:** $P_{ht}^i = P_{ht}^1 / \mathcal{E}_{1t}^i$ and P_{ht}^1 (the price for good produced in h expressed in the U.S. dollar) is fixed for all t and for any $i, h \in \{1, 2, 3\}$;

The total profit of the firms in country h paid to the household in period $t + 1$ is given by

$$\Pi_{t+1}^h = \sum_{i=1}^3 \mathcal{E}_{ht}^i P_{ht}^i C_{ht}^i - W_t^h L_t^h.$$

Financial intermediaries. I assume that financial markets are incomplete. The market incompleteness assumption was shown to be important for the model's ability to be consistent with the exchange rate puzzles in a two-country setting. The complete market assumption implies the perfect positive correlation between relative consumption growth and real exchange rate, which contradicts the Backus and Smith (1993) puzzle. If instead households are allowed to trade bonds denominated in different currencies directly, then the uncovered interest parity holds, the prediction that is not supported by the data

¹²Of course, this assumption is important not only for the results of this paper. Recent research emphasizes the role of this assumption for the optimal monetary policy and propagation of shocks in open economy models (see, e.g. Betts and Devereux (2000) and Devereux and Engel (2003)).

(Fama (1984)). The specific structure of the financial markets adapted here allows for obtaining the analytical solution while providing the intuitive and tractable expression for demand for bonds denominated in different currencies. It is assumed that the household in each country has an access only to domestic-currency-denominated bonds. Hence, there is a need for the intermediation to provide international lending and borrowing. There exist two types of agents that provide the intermediation: financiers and noise traders. The difference between the two is that financiers choose the portfolio of bonds based on expected returns and demand for bonds of noise traders is stochastic.

Financiers. The setting that describes the behavior of financiers is borrowed from Gabaix and Maggiori (2015). There is a unit mass of global identical financial firms that trade bonds internationally. Each financial firm is managed by a cashless financier that is randomly chosen to run a company for one period. Each financier trades bonds denominated in the U.S. dollars against bonds denominated in (only) one of two non-base currencies. The portfolio of the financier that trades the U.S. dollar-denominated bonds against h -currency-denominated bonds consists of Q_{t+1}^h of the U.S. dollar-denominated bonds and $-\frac{Q_{t+1}^h}{\mathcal{E}_{1t}^h}$ of h -currency denominated bonds. Financiers are risk-neutral and each chooses Q_{t+1}^h to maximize the expected value of her firm in the next period given by

$$V_t^h = \mathbb{E}_t \left[\beta \left(R_t^1 - R_t^h \frac{\mathcal{E}_{1t+1}^h}{\mathcal{E}_{1t}^h} \right) \right] Q_{t+1}^h, \quad h = 2, 3.$$

The positions of financier are limited by a credit constraint. After the positions are taken, in period t the financiers can withdraw a part $\Gamma^f \left| \frac{Q_{t+1}^h}{\mathcal{E}_{1t}^h} \right|$ of its position. This assumption reflects that the size of the position that can be withdrawn increases with the complexity of the position, measured by the size. The creditors of the financier anticipate such possibility and impose the following credit constraint

$$\frac{V_t^h}{\mathcal{E}_{1t}^h} \geq \left| \frac{Q_{t+1}^h}{\mathcal{E}_{1t}^h} \right| \Gamma^f \left| \frac{Q_{t+1}^h}{\mathcal{E}_{1t}^h} \right|, \quad h = 2, 3.$$

This constraint states that the expected value of the firm cannot be not smaller than the expected value of the withdrawn position. As the expected value of the firm is a linear function of excess return on the U.S.-dollar-denominated bonds, the credit constraint should be binding. Therefore, the total demand for currency h denominated bonds is

$$-\frac{Q_{t+1}^h}{\mathcal{E}_{1t}^h} = \frac{1}{\Gamma^f} \beta \left[\frac{\mathbb{E}_t \{ \mathcal{E}_{1t+1}^h \}}{\mathcal{E}_{1t}^h} R_t^h - R_t^1 \right], \quad h = 2, 3.$$

It increases with the excess returns on these bonds relative to the U.S.dollars-denominated bonds, $\left[\frac{\mathbb{E}_t \{ \mathcal{E}_{1t+1}^h \}}{\mathcal{E}_{1t}^h} R_t^h - R_t^1 \right]$. The parameter Γ^f measures the risk-taking capacity of the financiers. It determines the size of the risky position, given the expected return of this position. If $\Gamma^f = 0$, the risk-taking capacity of the financier is infinite and the uncovered interest parity condition must hold in the equilibrium, $\frac{\mathbb{E}_t \{ \mathcal{E}_{1t+1}^h \}}{\mathcal{E}_{1t}^h} R_t^h = R_t^1$. The case of $\Gamma^f \rightarrow 0$ corresponds to financial autarky (in the absence of noise

traders). The realized profit/loss of the financiers that run financial firms between periods $t - 1$ and t are assumed to be paid to the households in country h

$$\hat{P}r_t^h = \frac{Q_t^h}{\mathcal{E}_{1t}^h} \left[R_{t-1}^1 - R_{t-1}^h \frac{\mathcal{E}_{1t}^h}{\mathcal{E}_{1t-1}^h} \right], \quad h = 2, 3, \quad \hat{P}r_t^1 = 0.$$

Noise traders. There exists a unit mass of identical noise traders whose portfolio choice does not depend on fundamentals. Their portfolios consist of Z_t^h of the U.S.-dollar-denominated bonds and $-\frac{Z_{t+1}^h}{\mathcal{E}_t^h}$ of h -currency-denominated bonds. As it is shown below, this stochastic demand for bonds alters the countries' budget constraints. The similar shock can be introduced as a cross-country wedge in asset prices and returns. This type of shocks was shown to be an important source of exchange rate volatility (see Itskhoki and Mukhin (2017), Gabaix and Maggiori (2015), and Devereux and Engel (2002)).

As with the financiers, the realized profit/loss of the noise traders between periods $t - 1$ and t are assumed to be paid to the households of country h

$$\tilde{P}r_t^h = \frac{Z_t^h}{\mathcal{E}_{1t}^h} \left[R_{t-1}^1 - R_{t-1}^h \frac{\mathcal{E}_{1t}^h}{\mathcal{E}_{1t-1}^h} \right], \quad h = 2, 3, \quad \tilde{P}r_t^1 = 0.$$

Then, the total profit from the financial intermediation that goes to the pocket of households is

$$Pr_t^h = \hat{P}r_t^h + \tilde{P}r_t^h.$$

Government. The budget constrain of the government in country h is

$$R_{t-1}^h \hat{B}_t^h + T_t^h = \hat{B}_{t+1}^h + \hat{M}_t^h - \hat{M}_{t-1}^h,$$

where \hat{M}_t^h is the total supply of currency h , \hat{B}_t^h is government supply of bonds denominated in the domestic currency. To derive the analytical solution, I simplify the analysis by assuming that governments in each country adapts the monetary policy rule that fixes the level of total consumption at the steady state level. Given the fixed level of total consumption, the exchange rates fluctuations due to noise-traders shock will alter the demand for goods produced in different countries.

Equilibrium conditions. The model is closed by the equilibrium conditions. The product market equilibrium conditions ensures that the total demand for good produced in country h is equal to total supply:

$$Y_t^h = \sum_{i=1}^3 C_{ht}^i, \quad h \in \{1, 2, 3\}.$$

The labor market equilibrium condition states that labor supply L_t^h given by (4) is equal to labor demand L_t^h , determined by the production function (5). The bond market equilibrium conditions are

$$B_{t+1}^1 + \sum_{h=2}^3 (Q_{t+1}^h + Z_{t+1}^h) = \hat{B}_{t+1}^1, \quad B_{t+1}^h - (Q_{t+1}^h + Z_{t+1}^h)/\mathcal{E}_{1t}^h = \hat{B}_{t+1}^h \text{ for } h = 2, 3.$$

Lastly, the money market equilibrium conditions for $h \in \{1, 2, 3\}$ are

$$\hat{M}_t^h = (M_{ht}^h - P_{ht}^h C_{ht}^h) + \sum_{i=1}^3 M_{ht}^i.$$

Given the overall price level, the monetary authority chooses the level of money supply such that the level of consumption is constant at the steady state value.

The country budget constraint/FX markets. Combining household budget constraints, government budget constraints, bond market equilibrium conditions, and money market equilibrium conditions yields the budget constraints of Canada, the equation (6), and of Japan, the equation (7), *where both conditions are written in terms of the U.S. dollars:*¹³

$$\overbrace{P_{2t}^1 C_{2t}^1 - \mathcal{E}_{1t}^2 P_{1t}^2 C_{1t}^2 + \mathcal{E}_{1t}^3 P_{2t}^3 C_{2t}^3 - \mathcal{E}_{1t}^2 P_{3t}^2 C_{3t}^2}^{\text{Trade balance of Canada (in \$)}} = Q_{t+1}^2 - R_t^1 Q_t^2 + Z_{t+1}^2 - R_t^1 Z_t^2, \quad (6)$$

$$\overbrace{P_{3t}^1 C_{3t}^1 - \mathcal{E}_{1t}^3 P_{1t}^3 C_{1t}^3 + \mathcal{E}_{1t}^2 P_{3t}^2 C_{3t}^2 - \mathcal{E}_{1t}^3 P_{2t}^3 C_{2t}^3}^{\text{Trade balance of Japan (in \$)}} = Q_{t+1}^3 - R_t^1 Q_t^3 + Z_{t+1}^3 - R_t^1 Z_t^3. \quad (7)$$

Two conditions above can be thought as representing the balance of payments of Canada and Japan, as they state that net export is equal to capital outflow minus interest payments. Two exchange rates \mathcal{E}_{1t}^2 and \mathcal{E}_{1t}^3 are linked through the common blue term in both equations. This term is equal to net nominal trade flow between Canada and Japan and enters into equations with different sign. A stochastic supply of Canadian-dollar-denominated bonds ($Z_{t+1}^2 > 0$) requires depreciation of the Canadian dollar against the U.S. dollar, such that the Canadian total export exceeds Canadian total import and the capital outflow is balanced by trade surplus. As a results, the Canadian export to Japan exceeds the Canadian import to Japan, which distorts the Japanese budget constraint. To restore the equilibrium condition (7), \mathcal{E}_{1t}^3 has to adjust in response to an initial stochastic supply of Canadian-dollar-denominated bonds. This is how a shock to the single country propagates into an exchange rate of its trading partner.

3.2 International trade and exchange rates correlation

First, this section describes the solution (up to log-linearization) of the baseline model presented in the previous section. The solution of the model comes to reducing the whole system to two equations, that shape the evolution of exchange rates. Second, it shows how the elasticities of the trade balances of Canada and Japan with respect to both exchange rates govern the sign and the strength of correlation between two exchange rates. The effect of trade intensity between Canada and Japan on exchange rates correlation is then defined by the effect of trade intensity on the elasticities of the trade balance. Two propositions show that there exists a positive association between the trade intensity and the strength of exchange rates co-movement with LCP and DCP. However, with PCP, there is no monotone relationship between trade intensity and exchange rates correlation.

¹³The budget constraint of the U.S. is given by the sum of (6) and (7).

To obtain an analytical results, I log-linearize the equilibrium conditions around a zero-inflation, zero-trade-balance steady state, that is symmetric for Canada and Japan. In particular, at the steady state the share of Canadian consumption produced in Japan is equal to the share of Japanese consumption produced in Canada, $a_3^2 = a_2^3 = \frac{P_3^2 C_3^2}{P^2 C^2} = \frac{P_2^3 C_2^3}{P^3 C^3}$. Similarly, the share of the U.S. consumption produced in Canada is equal to the share produced in Japan, as well as to shares of Canadian and Japanese consumption produced in the U.S., i.e. $a_2^1 = a_3^1 = a_1^2 = a_1^3 = \frac{P_2^1 C_2^1}{P^1 C^1} = \frac{P_3^1 C_3^1}{P^1 C^1} = \frac{P_1^2 C_1^2}{P^2 C^2} = \frac{P_1^3 C_1^3}{P^3 C^3}$. Log-deviation of the variable from steady is denoted with small letter, $x_t = \log(X_t/X)$, except for $q_t^h = Q_t^h/P^1 C^1$, $h \in \{1, 2\}$ that denotes the linear deviation of Q_t^h from its steady state value of $Q^h = 0$ relative to nominal GDP. The small i^h defines the deviation of return from the steady state, $i_t^h = \log(R_{t+1}^h) - \log(R^h)$. To simplify the notation, I remove the subscript 1 to denote the exchange rates in terms of the U.S. dollar, i.e., $e_t^h = \log(\mathcal{E}_{1t}^h)$. I denote the normalized noise-traders shocks as $z_t^h = Z_t^h/P^1 C^1$ and I assume that $z_{t+1}^h = 1/\beta z_t^h - \tilde{z}_t^h$.

To solve for the equilibrium exchange rates, one needs to use only a part of the equilibrium system: the countries' budget constraints, the demand functions for import, the definitions of price indexes, the expressions for demand for bonds and the Euler equations. The only two shocks in the economy are noise traders shocks \tilde{z}_t^h , $h \in \{2, 3\}$, that are assumed to be identically and independently distributed, $\text{corr}(\tilde{z}_t^2, \tilde{z}_t^3) = 0$, with the variance σ_z^2 . This means that the exchange rates co-movement arises only due to international trade among countries. It follows that the relevant parameters for exchange rates correlation are the discount factor, β , the risk capacity of financiers Γ^f , the elasticity of substitution between domestic and foreign goods θ , and the steady state trade intensities, a_3^2 and a_2^1 .

The model can be reduced to two equations with two unknowns, e_t^2 and e_t^3 . To do that, it is needed to express each term of the budget constraints of Canada and Japan as a function of both exchange rates. The effect of exchange rates of import prices depends on the assumption on the currency in which prices are set. Given the effect on import prices, one can compute the responses of consumer price indexes, p_t^h , and then of consumption, c_{it}^h , to exchange rate changes. From the Euler equations, the expected CPI inflation determines the equilibrium interest rates, i_t^h , and therefore the demand for bonds denominated in different currencies, q_{t+1}^h .

The following two propositions show that with LCP and DCP, if trade intensity between Canada and Japan increases, ($a_3^2 = a_2^3 \nearrow$), the exchange rates of the Canadian dollar and the Japanese yen against the U.S. dollar co-move stronger with each other in response to noise traders shocks. However, with PCP, there is no monotone relationship between trade intensity and the strength of exchange rates co-movement. Proposition 1 considers the limiting case of $\Gamma^f \rightarrow \infty$, meaning that a noise traders shock can be absorbed only by the equilibrium trade balances. This case provides a simple intuition for the result. Proposition 2 shows that the result holds for any value of Γ^f .

Proposition 1. *In the absence of strategical bond traders ($\Gamma^f \rightarrow \infty$), if prices are set in the currency of consumer or in the U.S. dollar, for any parameter values*

$$\frac{d\text{corr}(\Delta e_t^2, \Delta e_t^3)}{da_3^2} > 0.$$

If prices are set in the currency of producer, there exist values of a_2^1 and θ , such that there is no monotone relationship between a_3^2 and $\text{corr}(\Delta e_{2t}, \Delta e_{3t})$.

An analytical proof is in Appendix A.3.

After the whole equilibrium system is reduced to the budget constraints of Canada and Japan, two exchange rates are determined as a solution of the linear system:

$$\begin{cases} de_t^3 - ce_t^2 = -\tilde{z}_t^2, \\ de_t^2 - ce_t^3 = -\tilde{z}_t^3, \end{cases} \quad (8)$$

where c and d are functions of parameters a_2^1 , a_3^2 , and θ . I define as $TB_t^h \equiv \sum_{i \neq h} \mathcal{E}_{ht}^i P_{ht}^i C_{ht}^i - \sum_{i \neq h} P_{it}^h C_{it}^h$ the trade balance of country h in terms of currency h . Due to the fact that the model is log-linearized around the zero-trade-balance steady state, it follows that $\frac{\partial TB_t^2(e_t^2, e_t^3)}{\partial e_t^2} \frac{1}{P^2 C^2} = -c$. The positive value of c means that the trade balance of Canada improves if the Canadian dollar depreciates against the U.S. dollar, keeping the Japanese yen exchange rate constant. Similarly, $\frac{\partial TB_t^2(e_t^2, e_t^3)}{\partial e_t^3} \frac{1}{P^2 C^2} = d$, which means that if $d > 0$, the trade balance of Canada improves if the Japanese yen appreciates against the U.S. dollar, keeping the Canadian dollar exchange rate constant. As the model is symmetric, it follows that $\frac{\partial TB_t^3(e_t^2, e_t^3)}{\partial e_t^2} \frac{1}{P^3 C^3} = -c$ and $\frac{\partial TB_t^3(e_t^2, e_t^3)}{\partial e_t^3} \frac{1}{P^3 C^3} = d$. Therefore, I will focus on the determinants of elasticities of the trade balance of Canada with respect to two exchange rates, keeping in mind that the same forces affect the trade balance of Japan.

It is straightforward to find the solution for both exchange rates:

$$e_t^2 = \frac{c\tilde{z}_t^2 + d\tilde{z}_t^3}{c^2 - d^2}, \quad e_t^3 = \frac{c\tilde{z}_t^3 + d\tilde{z}_t^2}{c^2 - d^2},$$

and correlation between two exchange rates changes is given by

$$\text{corr}(\Delta e_t^2, \Delta e_t^3) = \frac{2cd}{c^2 + d^2}.$$

A financial shock to a single country leads to a positive co-movement of two exchange rates if c and d have the same sign. The intuition is as follows. Assume that $c > 0$, so the depreciation of the Canadian dollar (the Japanese yen) against the U.S. dollar improves the trade balance of Canada (Japan). A stochastic supply of Canadian-dollar-denominated bonds ($\tilde{z}_t^2 < 0$) requires the depreciation of the Canadian dollar against the U.S. dollar ($e_t^2 \searrow$) to balance the capital outflows with the trade surplus. If the Canadian dollar depreciation deteriorates the trade balance of Japan ($d > 0$), the Japanese yen has to depreciate against the U.S. dollar ($e_t^3 \searrow$) to overcome the effect of the Canadian dollar depreciation on the budget constraint of Japan. The exchange rates co-move positively. If instead Canadian dollar depreciation improves the trade balance of Japan ($d < 0$), then Japanese yen has to appreciate ($e_t^3 \nearrow$), such that the improvement of the trade balance of Japan due to the Canadian dollar depreciation is balanced by the deterioration due to the Japanese yen appreciation. Two exchange rates co-move negatively.

Next, one can show that the sign of the derivative of the correlation with respect to bilateral trade

intensity, a_3^2 , is determined by two terms:

$$\text{sign} \frac{d \text{corr}(\Delta e_t^2, \Delta e_t^3)}{da_3^2} = \text{sign}(|c| - |d|) * \text{sign} \left(\frac{d}{c} \right)'_{a_3^2}. \quad (9)$$

Let us assume that: (i) the sensitivity of the trade balance of the country to the domestic exchange rate is higher than the sensitivity to the exchange rate of the trading partner ($c > d$); and (ii) a stochastic demand for Canadian-dollar-denominated bonds leads to depreciation of both exchange rates ($c > 0, d > 0$). Then for the derivative on the left hand side of (9) to be positive it is needed that $(d/c)'_{a_3^2} > 0$. The mechanics of this condition is as follows. Assume that only a shock \tilde{z}_t^2 hits the world ($\tilde{z}_t^3 = 0$). Then, the ratio of exchange rates returns $\Delta e_t^3 / \Delta e_t^2$ is equal to $d/c < 1$, which grows to 1 from below if increases, so the two exchange rates co-move stronger in response noise traders shocks. The economic intuition is that the higher ratio d/c implies that the Japanese yen exchange rate has to follow closely the Canadian exchange rate, such that the deterioration of the Japanese trade balance due to the Canadian dollar depreciation is equilibrated by the improvement of the Japanese trade balance due to the depreciation of the Japanese yen.

The elasticity of trade balance of Canada with respect to an exchange rate depends on the elasticities of quantities and prices. Therefore, values of c and d depend on the assumption about the currency of pricing. Next, I consider separately the cases of the producer currency pricing, the local currency pricing, and the dominant currency pricing.

Case 1: Producer currency pricing. Figure 2 depicts how the strength of co-movement between the exchange rates of the Canadian dollar and the Japanese yen evolves depending on the trade intensity between Canada and Japan. The blue area corresponds to the values of model parameters for which the correlation between two exchange rates increases if the trade intensity between Canada and Japan increases marginally. The left panel depicts the sing of $\frac{d \text{corr}(\Delta e_t^2, \Delta e_t^3)}{da_3^2}$ in the (a_3^2, θ) space with $a_2^1 = 0.2$. The right panel depict the same object in the (a_3^2, a_2^1) space with $\theta = 2$. One can see that there is no monotone relationship between the trade intensity and exchange rates correlation.

To understand the intuition behind this result, it is helpful to start with economic forces that define c and d and the link between a_3^2 and c and d . As it is discussed above, the effect of a_3^2 on c and d ultimately defines the effect of a_3^2 on $\text{corr}(\Delta e_t^2, \Delta e_t^3)$.

Let us start with the effect of the depreciation of the Canadian dollar against the U.S. dollar ($e_t^2 \searrow$) on the trade balance of Canada, given the fixed exchange rate of the Japanese yen ($e_t^3 = 0$), i.e., with c .

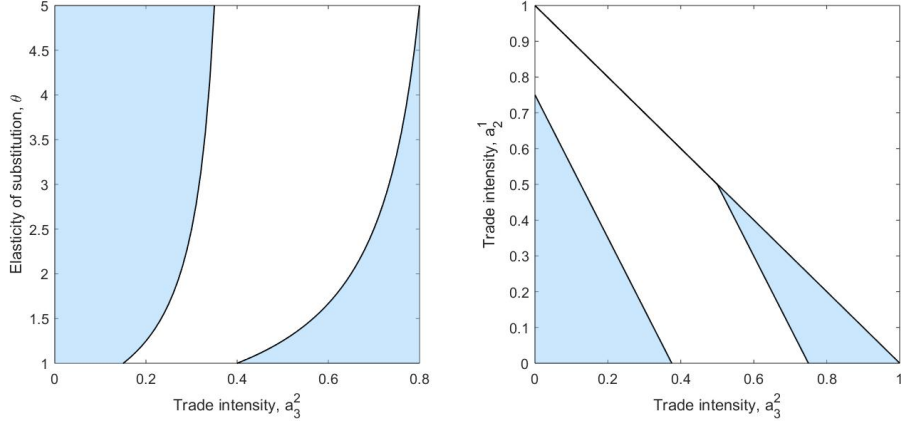


Figure 2: The sign of $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2}$ with PCP in the absence of strategical bond traders ($\Gamma^f \rightarrow \infty$). The left panel depicts the sign $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2}$ in the (a_3^2, θ) space with $a_2^1 = 0.2$. The right panel depicts the sign $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2}$ in the (a_3^2, a_2^1) with $\theta = 2$. The blue area represents parameters for which $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2} > 0$ and the white area represents parameters for which $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2} < 0$.

The value of c is given by the following equation:

$$\begin{aligned}
 c = a_2^1 & \left\{ \left(\frac{\partial p_{1t}^2}{\partial e_t^2} + \frac{\partial c_{1t}^2}{\partial e_t^2} \right) - \left(\frac{\partial p_{2t}^1}{\partial e_t^2} - \frac{\partial e_t^2}{\partial e_t^2} + \frac{\partial c_{2t}^1}{\partial e_t^2} \right) \right\} \\
 & \quad \underbrace{\left(\frac{\partial p_{1t}^2}{\partial e_t^2} + \frac{\partial c_{1t}^2}{\partial e_t^2} \right)}_{\text{net nominal trade from Japan to Canada}} \\
 & \quad \underbrace{\left(\frac{\partial p_{3t}^2}{\partial e_t^2} + \frac{\partial c_{3t}^2}{\partial e_t^2} \right)}_{\text{direct effect}} - \underbrace{\left(\frac{\partial p_{2t}^3}{\partial e_t^2} - \frac{\partial e_t^2}{\partial e_t^2} + \frac{\partial c_{2t}^3}{\partial e_t^2} \right)}_{\text{ind. eff.}}.
 \end{aligned} \tag{10}$$

As e_t^3 is fixed, when the Canadian dollar depreciates against the U.S. dollar, it depreciates against the Japanese yen as well. On the one hand, as prices are fixed in the currency of the producer, goods produced in Canada become relatively cheaper for foreign consumers, so they switch their consumption toward import from Canada. Moreover, goods produced abroad become relatively more expensive for Canadian households and they reduce their consumption of import. This consumption-switching effect of the depreciation of the Canadian dollar improves the trade balance of Canada. On the other hand, the Canadian dollar depreciation increases the price that Canadian households have to pay for each unit

of import, while Canadian firms are paid the same price for each unit of export. This price effect of the depreciation of the Canadian dollar worsens the trade balance of Canada. Hence, c is positive if the consumption-switching effect is strong enough to overcome the price effect. Otherwise, c is negative.

The strength of the consumption-switching effect is directly affected by the trade intensity. As it is described in the previous section, the higher value of a_3^2 decreases the sensitivity of demand for imported goods to exchange rate fluctuations of households from Canada and Japan. Therefore, the consumption-switching effect is strong, when a_3^2 is small, and weak when a_3^2 is large. I call this effect of a_3^2 on c as the *indirect effect*. The indirect effect implies the negative relationship between a_3^2 and c . There exists the second effect of a_3^2 on c , which I call as the *direct effect*. Let us fix the reaction of the net nominal trade from Japan to Canada to nominal exchange rate fluctuations, which is the term in the braces on the second line of equation (10). The higher trade intensity, a_3^2 , increases the sensitivity of the trade balance to fluctuations of the net nominal trade flow from Japan to Canada. In the limiting case, Canada and Japan do not trade with the U.S. and the trade balances of both countries are equal to the net export between Canada and Japan ($a_2^1 = 0$, $a_3^2 = 1$). If the term in the braces on the second line of equation (10) is positive, the direct effect implies that c increases with a_3^2 . If it is negative, the direct effect implies that c decreases with a_3^2 . The sign of the term in the braces depends on the strength of the consumption-switching effect in the same way as c does.

To summarize, for the small values of a_3^2 , the direct effect implies that c increases with a_3^2 (see the left panel of Figure (2)). Moreover, the direct effect overcomes the indirect effect and the overall effect of a_3^2 on c is positive. For the medium values of a_3^2 , the indirect effect overcomes the direct effect, and the overall effect of a_3^2 on c is negative. For the high values of a_3^2 , the consumption-switching effect becomes so insignificant that the direct effect implies that c decreases with a_3^2 , so both the direct and indirect effects lead to the negative association between a_3^2 and c .

Let us turn to the effect of the appreciation of the Japanese yen against the U.S. dollar ($e_t^3 \nearrow$), given the fixed values of the exchange rate of the Canadian dollar ($e_t^2 = 0$), i.e., to d :

$$\begin{aligned}
 d = a_2^1 & \left\{ \underbrace{\left(\frac{\partial p_{2t}^1}{\partial e_t^3} + \frac{\partial c_{2t}^1}{\partial e_t^3} \right)}_0 - \underbrace{\left(\frac{\partial p_{1t}^2}{\partial e_t^3} + \frac{\partial c_{1t}^2}{\partial e_t^3} \right)}_{\theta \underbrace{a_3^2}_{\text{ind. eff.}}} \right\} \\
 & \underbrace{\hspace{10em}}_{\text{net nominal trade from Canada to Japan}} \\
 + \underbrace{a_3^2}_{\text{direct effect}} & \left\{ \underbrace{\left(\frac{\partial p_{2t}^3}{\partial e_t^3} + \frac{\partial e_t^3}{\partial e_t^3} \right)}_0 + \underbrace{\frac{\partial c_{2t}^3}{\partial e_t^3}}_{\theta(1-a_2^1 - \underbrace{a_3^2}_{\text{ind. eff.}})} - \underbrace{\left(\frac{\partial p_{3t}^2}{\partial e_t^2} + \frac{\partial c_{3t}^2}{\partial e_t^2} \right)}_{1 - \theta(1 - \underbrace{a_3^2}_{\text{ind. eff.}})} \right\}
 \end{aligned} \tag{11}$$

On the one hand, as the Japanese yen appreciates against the U.S. dollar as well as against the Canadian dollar, with PCP goods produced in Canada become relatively cheaper than goods produced in Japan. Foreign consumers switch their consumption toward import from Canada. Additionally, goods produced in Japan become relatively more expensive for the Canadian households and they decrease the import from Japan. This consumption-switching effect leads to the improvement of the trade balance of Canada. On the other hand, Canadian households increase their consumption of goods produced in the U.S., as they also become relatively cheaper. Moreover, the Canadian households have to pay the higher price for each unit of goods imported from Japan. Both effects lead to the deterioration of the trade balance of Canada. Depending on the strength of the consumption-switching effect, the appreciation of the Canadian dollar leads either to improvement ($d > 0$) or to deterioration ($d < 0$) of the trade balance of Canada. It can be seen from equation (11) that, due to the fact that the model is log-linearized around the symmetric steady state, the effect of a_3^2 on d is the same as the effect of a_3^2 on c .

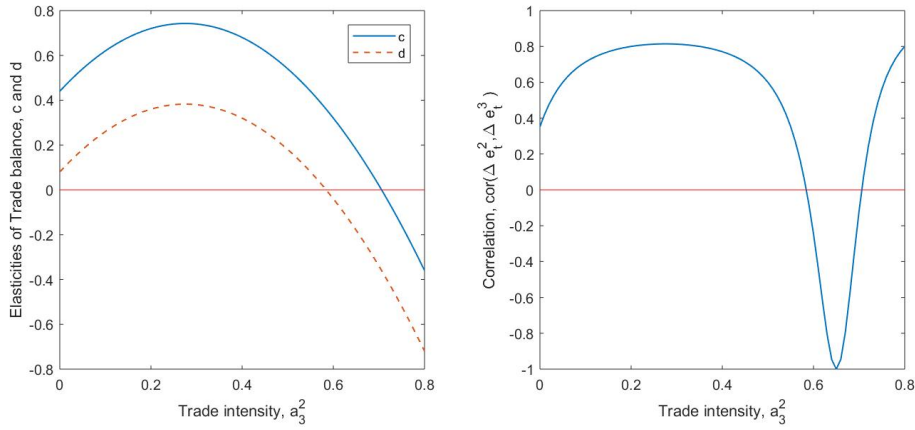


Figure 3: The bottom left panel depicts the evolution of c and d and the bottom right panel shows the evolution of $\text{corr}(\Delta e_t^2, \Delta e_t^3)$ for $a_3^2 \in [0, 0.8]$, $a_2^1 = 0.2$ and $\theta = 2$.

Let us now follow the evolution of $\text{corr}(\Delta e_t^2, \Delta e_t^3)$ as a_3^2 increases from zero to the maximum value, depicted on Figure (3) for $a_2^1 = 0.2$ and $\theta = 2$. With these parameter values $c > d$. This means that the trade balance of Canada is more sensitive to fluctuations of the Canadian dollar exchange rate than to fluctuations of the Japanese yen exchange rate. As c is greater than d and both grow/decline with a_3^2 at the same speed, the ratio d/c grows with a_3^2 if d and c grow with a_3^2 and vice versa.¹⁴

When $a_3^2 = 0$, we have $c > d > 0$, and two exchange rates are positively correlated. As a_3^2 increases, d/c grows toward one and the strength of co-movement between two exchange rates increases. As a_3^2 grows further, the contribution of the consumption-switching effect on trade balances diminishes, until c and d start decreasing with a_3^2 . Consequently, d/c starts moving away from one, and the correlation between exchange rates decreases. As the consumption-switching becomes more and more insignificant as a_3^2 grows, d becomes negative, and noise-traders shocks result in a negative correlation between two

¹⁴In particular, $(d/c)'_{a_3^2} = (c - d) * (c)'_{a_3^2}$ and $\text{sign}(d/c)'_{a_3^2} = \text{sign}(c)'_{a_3^2}$.

exchange rates. As both c and d decrease at the same pace, there is a value of a_3^2 , such that $c = -d$ and the correlation between two exchange rates becomes -1. As a_3^2 grows further, we have $|c| < |d|$, so the correlation between two exchange rates increases as a_3^2 increases towards zero. For very high values of a_3^2 , the correlation becomes positive again.

As it can be seen from Figure (3), the change of the sign of $\frac{d\text{corr}(\Delta e_t^2, \Delta e_t^3)}{da_3^2}$ happens for the relatively high values of a_3^2 . However, as Figure (4) shows below, the change of the sign of the derivative happens for the lower values of a_3^2 once the strategic bond traders are introduced into the model.

Case 2: Local currency pricing. Appendix A.3 shows that with LCP $c = a_2^1 + a_3^2$, $d = a_3^2$. Therefore,

$$c > 0, \quad d > 0, \quad |c| > |d|, \quad \left(\frac{d}{c}\right)'_{a_3^2} > 0 \text{ for any } a_2^1, a_3^2, \theta.$$

With LCP both exchange rates do not influence relative prices that consumers face and, therefore, the equilibrium demand for imported goods. The depreciation of the Canadian dollar against the U.S. dollar improves the trade balance of Canada because Canadian firms are paid more for each unit of good exported to the U.S. and Japan, so $c > 0$. The appreciation of the Japanese yen against the U.S. dollar improves the trade balance of Canada because Canadian firms are paid more for each unit of exported good to Japan, so $d > 0$. Because the depreciation of the Canadian dollar against the U.S. dollar increases revenues of firms exporting to both countries and the appreciation of the Japanese yen increases revenues only of firms exporting to Japan, $c > d$.

As demand for import does not depend on exchange rates, there is no consumption-switching effect and, therefore, no indirect effect of a_3^2 on c and d . The net export from Japan to Canada increases if the Canadian dollar depreciates or if the Japanese yen appreciates. This means that the direct effect implies that c and d increase with a_3^2 and, because the model is log-linearized around the symmetric steady state, they increase at the same speed. As $c > d$, it means that (d/c) always increases with a_3^2 and the strength of co-movement of exchange rates increases.

Case 3: Dominant currency pricing. With DCP, as shown in Appendix A.3, $c = (a_3^2 + a_2^1)\theta(1 - a_3^2 - a_2^1)$, $d = a_3^2\theta(1 - a_3^2 - a_2^1)$. Therefore,

$$c > 0, \quad d > 0, \quad |c| > |d|, \quad \left(\frac{d}{c}\right)'_{a_3^2} > 0 \text{ for any } a_2^1, a_3^2, \theta.$$

To make the interpretation of the case of DCP simpler, let me use the fact that c and d measure the elasticities of the trade balance of Canada and Japan expressed in the U.S. dollar. With DCP prices denominated in the U.S. dollar are not influenced by the exchange rates. As a result, the adjustment of the trade balances of Canada and Japan *expressed in the U.S. dollar* in response to noise-traders shocks happens through the adjustment of consumption.

When prices are rigid in the U.S. dollar, prices of imported goods in the U.S. do not respond to shocks \tilde{z}_t^2 and \tilde{z}_t^3 , and prices of imported goods in Canada (Japan) adjust one-to-one with the exchange rate of the Canadian dollar (the Japanese yen) against the U.S. dollar. Therefore, the depreciation of the Canadian

dollar makes foreign goods more expensive for Canadian households and they increase their demand for import from the U.S. and Japan. Thus, the trade balance of Canada improves, $c > 0$. The appreciation of the Japanese yen makes goods produced in Canada cheaper for the Japanese households, so they increase their demand for import from Canada. The trade balance of Canada again improves, $d > 0$. As the depreciation of the Canadian affects trade between Canada and both of its trading partners, while the appreciation of the Japanese yen affects the trade flow only between Canada and Japan, it implies that $c > d$.

It follows that, as in the case of LCP, $d/c = a_3^2/(a_2^1 + a_3^2)$, and the trade intensity between Canada and Japan a_3^2 influences the exchange rates correlation only through the importance of trade between Canada and Japan in total international trade of both countries. The exchange rate of the Japanese yen has to follow less the exchange rate of the Canadian dollar if the trade balance of Japan is relatively more sensitive to the former than to the latter. The lower value of a_3^2 implies the higher relative sensitivity of the trade balance of Japan to exchange rate of Japanese yen due to the higher share of trade between Japan and the U.S. and the lower share of trade between Japan and Canada in total trade. Despite the fact that with DCP the consumption-switching effect is present, as the model is log-linearized around the symmetric steady state, this effect of a_3^2 on d/c is canceled out equilibrium.

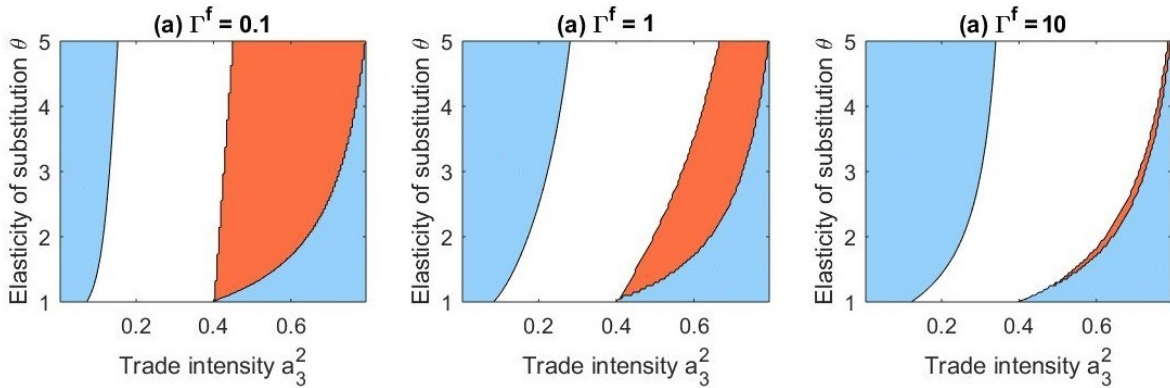


Figure 4: The sign of $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2}$ with PCP: (a) $\Gamma^f = 0.1$, (b) $\Gamma^f = 1$, and (c) $\Gamma^f = 10$. $a_2^1 = 0.2$, $\beta = 0.99$. The horizontal axis depicts values of $a_3^2 \in [0, 0.8]$, the vertical axis depicts values of $\theta \in [1, 5]$. The red area represents parameters for which the Blanchard-Khan condition is not satisfied. The blue area represents parameters for which $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2} > 0$ and the white area represents parameters for which $\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2} < 0$.

Next proposition shows that the relaxation of the assumption that capital flows are only due to noise traders, that makes the model static, does not change the result of Proposition 1.

Proposition 2. For $0 < \Gamma^f < \infty$, if prices are set in the currency of consumer or in the U.S. dollar, for any parameter values

$$\frac{dcorr(\Delta e_t^2, \Delta e_t^3)}{da_3^2} > 0.$$

If prices are set in the currency of producer, there exist values of a_2^1 and θ , such that there is no monotone

relationship between a_3^2 and $\text{corr}(\Delta e_{2t}, \Delta e_{3t})$.

Proof is in Appendix A.4. Figure 4 shows the sign of $\frac{d\text{corr}(\Delta e_t^2, \Delta e_t^3)}{da_3^2}$ under the producer currency pricing for three values of Γ^f . The illustrated pattern is very similar to the case of the financial autarky (see the right panel of Figure (3)).

4 Quantitative analysis

This section quantitatively tests the proposed mechanism that links the nominal exchange rates through trade. First, I build an N -country model that features the standard assumptions of open economy models, except for the structure of financial markets as in the baseline model and the presence of noise-traders shocks. Countries can differ in their economic/population sizes, where n_h denotes the size of country h . The first country is a dominant-currency country - the U.S. The N th country is the rest of the world. Each country is subject to TFP shocks, monetary policy shocks, and noise-traders shocks.

Following the New Keynesian literature, I do not impose the cash-in-advance constraint and consider the cashless economy. I use a classical Calvo model of sticky prices instead of assuming that prices are perfectly rigid. More precisely, firms in each country produce a continuum of differentiated goods. In each period a fraction $\lambda \in [0; 1)$ of randomly chosen producers is not allowed to change the nominal price. The remaining fraction of firms, $1 - \lambda$, chooses prices optimally by maximizing the expected discounted value of profits.

Due to the importance of the assumption about the currency in which firms set prices to foreign consumers, I do not stick to a particular pricing assumption. Instead, I allow a fraction $\theta_h^{i,p}$ of firms that export from country h to country i to set prices in the producer currency, a fraction $\theta_h^{i,l}$ of firms to set prices in the consumer currency, and the rest of firms, $\theta_h^{i,d} = 1 - \theta_h^{i,p} - \theta_h^{i,l}$, to set prices in U.S dollars.

In the baseline model, the monetary authorities stabilizes the level of total consumption. Now, I assume that monetary authorities follow the Taylor rule when setting nominal exchange rates:

$$i_t^h = \rho_i^h i_{t-1}^h + (1 - \rho_i^h)(\alpha_\pi^h \mathbb{E}_t\{\pi_{t+1}^h\} + \alpha_{\Delta y}^h (y_t^h - y_{t-1}^h) + \alpha_{\Delta\pi}^h (\pi_t^h - \pi_{t-1}^h)) + \epsilon_{m,t}^h,$$

where $i_t^h = \log(R_t^h) - \log(R)$, ρ_i^h , α_π^h , $\alpha_{\Delta\pi}^h$, and $\alpha_{\Delta y}^h$, are the monetary policy parameters, π_t^h is the CPI inflation rate, $y_t^h - y_{t-1}^h$ is the country h 's output growth and $\epsilon_{m,t}^h$ is a country-specific monetary policy shock.

I calibrate the model to a set of 12 countries and the rest of the world. Countries are identical except for the share of firms that set prices in U.S. dollars and the two following characteristics: (i) the relative sizes, and (ii) the shares of consumption produced in different countries in total consumption (given by preference parameters in the utility functions). These last two countries' characteristics generate heterogeneity in the bilateral trade intensities. Shocks are uncorrelated and nominal exchange rates are linked only due to the propagation of country-specific shocks through the trade network. I check whether the model is able to generate the positive link between trade intensity and the dispersion of exchange rates

correlation as in the data. I find that the model is able to account for 50% of the empirical trade-exchange rate-correlation slope.

The full setup of the model can be found in the Appendix A.5. The rest of this section describes the calibration strategy, reports the results and lastly the robustness checks.

4.1 Calibration

The model is calibrated to $N = 13$ countries for the period 1999 to 2013. Twelve countries are the sample of developed countries, and the 13th country represents the rest of the world.¹⁵ The model is log-linearized around the zero-inflation, zero-net-foreign-asset (NFA) position steady state. The bilateral trade balances are not restricted to be equal to zero. The detailed description of the steady state is in Appendix A.5. The complete log-linearized model is represented in Appendices A.5 and A.6.

Table (3) reports some parameter values. Following Itskhoki and Mukhin (2017), I set the quarterly discount factor $\beta = 0.99$, the intertemporal rate of substitution $\sigma = 2$, and the Frisch elasticity of labor supply $\nu = 1$. The values of the elasticity of substitution between bundles of domestic and foreign goods are typically assumed to take a value between 1.5 and 6, so I set $\theta = 4$. I assume that the elasticities of substitution between varieties produced domestically and abroad are equal to θ , i.e $\gamma = \phi = 4$. The probability of price adjustment $1 - \lambda$ is set to 0.25, so prices adjust once a year on average.

Table 3: Values of parameters

Parameter		Value
<i>From literature</i>		
β	discount factor	0.99
σ	relative risk aversion	2
ν	Frisch elasticity of labor supply	1
θ	elasticity of substitution between domestic and foreign goods	4
$\gamma (\phi)$	elasticity of substitution between domestic (foreign) varieties	4
$1 - \lambda$	probability of price adjustment	1/4
<i>Calibrated to exchange rates and trade imbalances moments</i>		
$\tilde{\Gamma}^f$	risk-taking capability of the financial intermediaries	17
$\sigma_{\tilde{z}}$	standard deviation of noise-traders shock	0.002
<i>Calibrated to Boz et al. (2017)</i>		
$\tilde{\theta}^p$	share of firms that set prices in the producer currency	0.02
$\tilde{\theta}^l$	share of firms that set prices in the local currency	0.47
$\tilde{\theta}^d$	share of firms that set prices in dominant the currency	0.51
<i>Taylor rule (Christiano et al. (2010))</i>		
ρ_i	Coeff. on lagged interest rate	0.88
α_π	Weight on inflation	1.85
$\alpha_{\Delta\pi}$	Weight on change in inflation	0.19
$\alpha_{\Delta y}$	Weight on output growth	0.32
σ_m	Standard deviation of M.P. shock	0.0013

¹⁵The list of countries is the United States, Australia, Canada, the Euro Area, Israel, Japan, Korea, New Zealand, Norway, Sweden, Switzerland, and the United Kingdom.

I assume that the shares of firms that set prices in the producer currency, the consumer currency or the dominant currency are the same regardless of the source and destination countries, i.e. $\tilde{\theta}^p \equiv \theta_h^{i,p}/n_h$, $\tilde{\theta}^l \equiv \theta_h^{i,l}/n_h$, $\tilde{\theta}^d \equiv \theta_h^{i,d}/n_h$ for all i and h . Parameters $\tilde{\theta}^p$, $\tilde{\theta}^l$, $\tilde{\theta}^d$ are calibrated to match the results of Boz et al. (2017) who estimate the U.S. dollar and bilateral exchange rates pass-through into bilateral price and volume indices on the country-pair level. They document that dollar exchange rate pass-through dominates the bilateral exchange rate pass-through. In particular they estimate the following regression:

$$\Delta p_{it}^{aj} = \lambda_{ij} + \delta_t + \sum_{k=0}^2 \alpha^k \Delta e_{jt-k}^{ai} + \sum_{k=0}^2 \beta^k \Delta(-e_{1t-k}^{aj}) + \sum_{k=0}^2 \gamma_k \Delta p_{t-k}^{ai} + \epsilon_{ijt}$$

where x_{jt}^{ai} denotes an annual average of x_{jt}^i , λ_{ij} and δ_t are country-pair and time fixed effects. The parameters $\tilde{\theta}^p$ and $\tilde{\theta}^l$ are chosen to match the estimates of the bilateral exchange rate pass-through, α_0 , and the U.S. dollar exchange rate pass-through, β_0 , on the subsample for which both trading partners are advanced economies ($\alpha_0 = 0.332$ and $\beta_0 = 0.409$). The value of $\tilde{\theta}^d$ is then given by the condition $\tilde{\theta}^p + \tilde{\theta}^l + \tilde{\theta}^d = 1$. The results of calibration imply that 51% set prices in the U.S. dollars, 47% of firms set prices in the local currency, and 2% set prices in the producer currency.

There are three types of shocks in the model: country-specific productivity shocks, monetary policy shocks, and noise traders shocks. Productivity shocks are assumed to follow AR(1) process. First, I estimate country-specific persistence parameters and country-specific variance of productivity shocks. I also estimate the correlation between country-level productivity processes:

$$a_t^h = \rho_a^h a_{t-1}^h + \epsilon_t^h, \quad std(\epsilon_t^h) = \sigma_a^h, \quad corr(\epsilon_t^h, \epsilon_t^j) \in R$$

The parameters of productivity processes and country-pair correlations are estimated to fit the log of labor productivity in each country, estimated using one-side HP-filtered data from the OECD dataset (GDP per person employed, seasonally adjusted for period 1995q1-2013q4). In the *benchmark* calibration, I assume that countries are identical except for relative sizes and consumption shares and that domestic shocks are not correlated across countries. I assign the same values to parameters of productivity process for each country, that are equal to the average of the same parameters across 12 countries, and set correlations to zero. In the *benchmark2* calibration, I assign country-specific productivity shock parameters and cross-country correlations as estimated on data. Then the persistence and variance parameters for the productivity shock for the rest of the world are assigned to be equal to the average of the same parameters for 12 countries. Productivity shock for the rest of the world is not correlated with productivity shocks in other countries.

The country-specific monetary policy shocks and noise traders shocks are i.i.d. with the same standard deviations across countries, σ_m and σ_z respectively.

The parameters of the monetary policy rule and the standard deviation of monetary policy shock are borrowed from Christiano et al. (2010), namely $\rho_i = 0.88$, $\alpha_\pi = 1.85$, $\alpha_{\Delta\pi} = 0.19$, $\alpha_{\Delta y} = 0.32$, and

$\sigma_m = 0.0013$.¹⁶

The risk-taking capacity of the financiers, Γ^f , and the standard deviation of noise traders shock, σ_z are simultaneously calibrated to match the average standard deviation of exchange rate changes across 11 currencies against U.S. dollar and the average standard deviation of the balance of trade (measured in U.S. dollars) divided by nominal GDP of the rest of the world (measured in U.S. dollars) across 12 countries.¹⁷

Lastly, I need to calibrate preference parameters α^h s, α_i^h , and parameters that determine the relative size of a country h . As the model is log-linearized around zero-NFA steady state, I calculate the country total import (= total export), $\mathcal{E}_1^h P_F^h n_h C_F^h$, as an average between actual total import and total export.¹⁸ To allocate the total import of country h between trading partners, I use the data of actual import of country h from different destinations. Then the import of country h from the rest of the world is calculated as the difference between the total import and import from other 11 countries. Given bilateral import data, it is direct to calculate the country h 's export to 11 countries, and, therefore, the export to the rest of the world. This procedure guarantees that the steady-state balance of trade of the rest of the world is equal to zero. The GDP of the rest of the world is calculated as the world GDP minus nominal GDP of 12 countries in the sample. The data are the same as the data used in the empirical part of the paper.

Parameters n_h are set to match the ratio of country's GDP to GDP of the rest of the world, $\frac{n_h}{n_N} = \frac{\mathcal{E}_1^i P^i C^i}{\mathcal{E}_1^N P^N C^N}$. To calibrate the home bias in consumption α^h , I use the fact that at the steady state $\alpha^h = \frac{\mathcal{E}_1^h P_F^h n_h C_F^h}{\mathcal{E}_1^h P^h n_h C^h}$ is equal to the share of the country i 's total import to country's total consumption. Moreover, at the steady state, $\alpha_i^h = \frac{\mathcal{E}_1^h P_F^h n_h C_F^h}{\mathcal{E}_1^h P_F^i n_h C_F^i}$ is equal to the share of the country import from country h to the total country import of country i . I compute the corresponding ratios for each year and take an average over period.

4.2 Results

Table 4 summarizes the main results of this section. It compares the moments in the data with the moments generated by the calibrated model. Column (1) presents the results of the benchmark calibration, column (2) presents the results when TFP shocks have country-specific parameters and are correlated across countries. Columns (3)-(5) presents results when only noise-traders are active, when only TFP shocks are active, and when only monetary policy shocks are active, respectively.

I estimate the regression (1) in Section 2 with country fixed effects on data for the sample of 11

¹⁶Christiano et al. (2010) estimate an augmented variant of Christiano et al. (2005) and Smets and Wouters (2003) for the Euro Area. For other countries in the sample there exist studies that estimate similar models and find similar parameters of the monetary policy rule. See, for example, Jääskelä and Nimark (2011) for Australia Argov et al. (2012) for Israel, Sugo and Ueda (Sugo and Ueda) for Japan, Kim (2013) for Korea, Kamber et al. (2016) for New Zealand, Adolfson et al. (2011) for Sweden, and Burgess et al. (2013) for the United Kingdom.

¹⁷Both the risk-taking capacity of financial intermediaries and the standard deviation of noise traders shock are adjusted by nominal GDP of the rest of the world measured in U.S. dollars. In particular, $\tilde{\Gamma}^f = \Gamma^f (\mathcal{E}_1^N P^N Y^N)$, $\tilde{z} = z / (\mathcal{E}_1^N P^N Y^N)$. See Appendix A2 for details.

¹⁸The only one exception is Norway. One obtains the negative export to the rest of the world if follows the procedure described above. This is because for Norway total export was much bigger than total import over the period. Therefore, I compute $\mathcal{E}_1^h P_F^h n_h C_F^h$ as actual total export of Norway.

countries (that generates 55 observations) I used for calibration and time period from 1999 to 2013. The estimated coefficient in front of the trade intensity is significant (at the 5% level of significance), but smaller than one estimated on data for the whole sample of countries (see Table 1). However, the economic importance of the trade intensity is the same. The dependent variable is either country-pair correlation of nominal exchange rates or country-pair correlation of real exchange rates.

Table 4: Quantitative properties

	Data	Model				
		Benchmark (1)	Benchmark2 (2)	NT (3)	TFP (4)	MP (5)
β_1^e	0.037	0.019	0.020	0.022	0.027	0.003
β_1^q	0.034	0.021	0.022	0.023	0.018	0.003
$\sigma(\Delta e_t)$	0.053	0.053	0.053	0.051	0.004	0.012
$\sigma(\Delta q_t)$	0.052	0.041	0.041	0.040	0.001	0.001
$\rho(e_t)$	0.98	0.99	0.98	0.99	0.99	0.99
$corr(\Delta e_t^k, \Delta e_t^l)$	0.41	0.14	0.14	0.11	0.55	0.41
$corr(\Delta e_t, \Delta q_t)$	0.98	0.98	0.98	0.98	0.84	0.93
$\frac{\sigma(\Delta c_t - \Delta c_t^1)}{\sigma(\Delta e_t)}$	0.148	0.15	0.15	0.067	0.23	0.44
$\frac{\sigma(\Delta i_t - \Delta i_t^1)}{\sigma(\Delta e_t)}$	0.051	0.039	0.039	0.063	0.071	0.10
$corr(\Delta c_t - \Delta c_t^1, \Delta q_t)$	-0.096	-0.18	-0.16	-0.67	-0.50	0.99
$corr(\Delta c_t - \Delta c_t^1, \Delta e_t)$	-0.11	-0.16	-0.17	-0.78	-0.80	0.93
$corr(\Delta i_t - \Delta i_t^1, \Delta e_t)$	-0.083	0.32	0.31	0.91	0.94	-0.93
$corr(\Delta c_t^k - \Delta c_t^1, \Delta c_t^l - \Delta c_t^1)$	0.25	0.30	0.29	-0.07	0.35	0.48
$corr(\Delta i_t^k - \Delta i_t^1, \Delta i_t^l - \Delta i_t^1)$	0.41	0.29	0.28	0.08	0.51	0.50

Note: The data and model moments are the average across all countries used for calibration. All exchange rates are against the U.S. dollar and all relative changes, $\Delta x_t^k - \Delta x_t^1$, are for domestic variables against the same variables in the U.S. Column (1) presents the results of the benchmark calibration, Column (2) presents when TFP shocks have country-specific parameters and are correlated across countries, Columns (3), (4) and (5) present results when only noise-traders, TFP and monetary policy shocks are active.

Given the calibrated parameter values, I simulate a sequence of 5000 shocks and using data produced

by the model. Then I estimate the following regressions:

$$\begin{aligned} \text{corr}(\Delta e_t^k, \Delta e_t^l) &= \beta_0^e + \beta_1^e \log(\text{Bil.Tr}_{kl}) + \alpha_k^e + \alpha_l^e + u^e, \\ \text{corr}(\Delta q_t^k, \Delta q_t^l) &= \beta_0^q + \beta_1^q \log(\text{Bil.Tr}_{kl}) + \alpha_k^q + \alpha_l^q + u^q. \end{aligned} \tag{12}$$

These regressions are identical to the regression in Column (4) of Table (1) in the Section 2. I exclude controls for peg and commodity-export similarity, as the model does not feature a possibility of foreign exchange interventions and commodity trade. However, as Columns (4) and (6) of Table 1 report, the inclusion of these controls does not decrease significantly the magnitude of the effect of the bilateral trade intensity.

The first two lines of Table 4 demonstrate the model's ability to generate the empirical finding of Section 2: the positive association between trade intensity and exchange rates correlation. The rest of the table compares other moments. As all exchange rates are against the U.S. dollar, all relative changes, $\Delta x_t^k - \Delta x_t^1$, are for domestic variables against the same variables in the U.S. The data and model moments are the average across all countries used for calibration.

Column (1) of Table 4 represents the results of the baseline calibration. Exchange rates are linked through trade of consumption goods and through trade of bonds by noise traders. Noise traders purchase N_j^i of i -currency-denominated bonds against $-Z_j^i/\mathcal{E}_i^j$, of j -currency denominated bonds. Therefore, the shock Z_j^i for $i \neq 1$ and $j \neq 1$ enters the budget constraints of country i and country j with different signs. Column (1) shows that the model is able to account for 50% of the empirical trade-exchange rate-correlation slope for nominal exchange rates. As demonstrated in Column (2), allowing the productivity shocks to have country-specific parameters and be correlated as in the data does not affect significantly the moment produced by the model.

The average correlation across 55 pairs of nominal exchange rates is 0.14, which is lower than the average correlation in the data. One of the reason might be that the shock Z_j^i ($i \neq 1, j \neq 1$) induces the exchange rates \mathcal{E}_{1t}^i and \mathcal{E}_{1t}^j to move in different directions. In the model, the volatility of Z_j^i is the same for any i and j . In reality probably not all bonds pairs are traded so actively as pairs that include the U.S. dollar-denominated bonds.

The average correlation of exchange rates against the U.S. dollar is low but still positive, even the U.S. is identical to other countries except for the share of firms that set prices in the U.S. dollar. However, as Figure 5 shows below, the average correlation is even higher if all firms set prices in currency of consumer ($\tilde{\theta}^p = \tilde{\theta}^d = 0, \tilde{\theta}^l = 1$). Therefore, the positive average correlation is due to the relative sizes and relative consumption shares of countries.

The baseline model accounts for the high persistence of nominal exchange rates, for the high correlation between country-level nominal and real exchange rates, and for the high volatility of the nominal exchange rates relative to consumption and interest rates. Due to the incomplete market assumption, the model also reproduces low and negative correlation of nominal and real exchange rates with relative consumption growth. This result is in line with previous papers that emphasized the importance of imperfect financial markets for the exchange rate determination. However, the model does not match the negative

correlation between nominal exchange rates and relative interest rates growth.

Columns (3)-(5) show that the positive association between trade intensity and exchange rates correlations is due to the noise-traders shock. The TFP shocks alone can generate β_1^e and β_1^q that are close to their empirical counterparts, but these shocks do not contribute much to the volatility of exchange rates. Naturally, when the model is hit by only one type of shocks the absolute value of correlation between exchange rates and other fundamentals increases.

Table 5 reports results of the robustness analysis with respect to deviations of some model parameters from their baseline values. Columns (2)-(5) of Table 5 show the sensitivity of results to the risk-capacity parameter Γ^f , and to the parameter of the elasticity of substitution between domestic and foreign goods θ . Columns (6) and (7) show the sensitivity of the model to the specification of the monetary policy rule. Last two columns of 5 show moments produced by the model if prices are flexible, $\lambda = 0$. For the flexible-price specification, I re-calibrate the parameter Γ^f to match the standard deviation of the real exchange rates. Column (8) reports the moments produced by the data when only TFP shocks hit the economy. Under this specification, the model generates the values of β_1^e and β_1^q that are close to (and even higher than) their empirical counterparts. However, the correlation between country-level nominal and real exchange rates becomes negative. Column (9) represents the moments when the economy is hit by TFP and noise-traders shocks. The model matches well the positive association between trade intensity and real exchange rates correlation, however, generate the value of β_1^e that is less than zero.

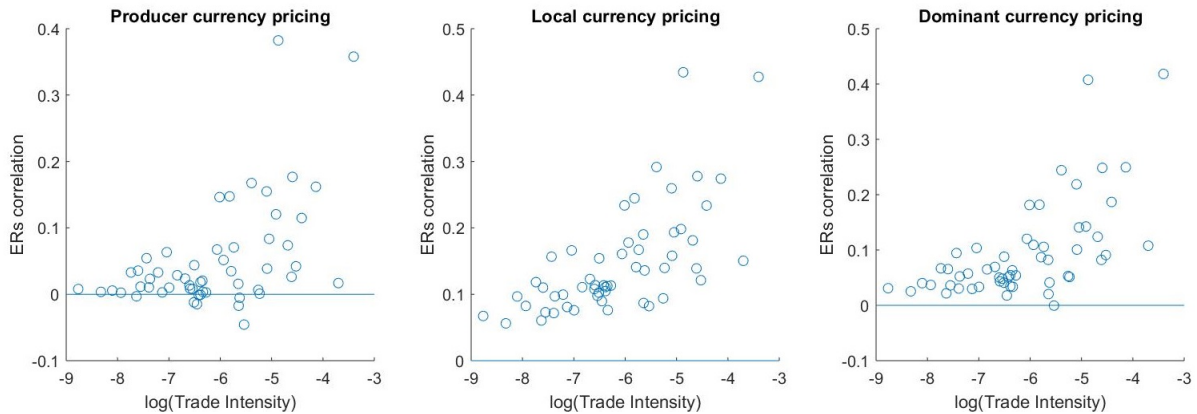


Figure 5: Plot of correlations of nominal exchange rate returns versus the logarithm of measure of bilateral trade intensity. Correlations and trade intensity are measured using time series generated by the model under benchmark calibration (except for $\tilde{\theta}^p$, $\tilde{\theta}^l$, and $\tilde{\theta}^d$) when only noise-traders shocks are active. Left panel: $\tilde{\theta}^p = 1$, $\tilde{\theta}^l = 0$, $\tilde{\theta}^d = 0$. Middle panel: $\tilde{\theta}^p = 1$, $\tilde{\theta}^l = 0$, $\tilde{\theta}^d = 0$. Right panel: $\tilde{\theta}^p = 0$, $\tilde{\theta}^l = 0$, $\tilde{\theta}^d = 1$.

The baseline model predicts that: (i) nominal exchange rates are negatively correlated in response to noise-traders shocks only if prices are set in the currency of producer; (ii) the strength of exchange rates co-movement decreases with trade intensity for the middle range of values of trade intensity. These predictions are partly confirmed by Figure 5. Figure 5 plots the correlation of nominal exchange rate returns versus the logarithm of bilateral trade intensity. Both variables are measured using time series generated by the model under benchmark calibration (except for values of $\tilde{\theta}^p$, $\tilde{\theta}^l$, and $\tilde{\theta}^d$) when only

noise-traders shocks are active. The left panel depicts the case when all firms set prices in producer currency, the middle panel corresponds to the case when all firms set prices in consumer currency and the right panel corresponds to the case when all firms set prices in U.S. dollars. The left panel shows that under producer currency pricing, for the middle range of trade intensity exchange rates correlation for some pairs becomes negative. The positive link between trade intensity and exchange rates correlation becomes looser. Interestingly, the average correlation among nominal exchange rates in the biggest under local currency pricing, and the positive association between trade intensity and exchange rates correlation is the strongest under dominant currency pricing.

Table 5: Robustness

	Data	Model								
		Bench.	Γ^f	Γ^f	θ	θ	α_{π}^h	α_{π}^h	fl.pr1	fl.pr2
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
β_1^e	0.037	0.019	0.01	0.037	0.024	0.016	0.019	0.027	0.041	-0.011
β_1^q	0.034	0.021	0.01	0.039	0.026	0.018	0.021	0.028	0.052	0.044
$\sigma(\Delta e_t)$	0.053	0.053	0.018	0.11	0.076	0.034	0.054	0.039	0.28	1.61
$\sigma(\Delta q_t)$	0.052	0.041	0.013	0.013	0.061	0.022	0.042	0.032	0.005	0.053
$\rho(e_t)$	0.98	0.99	0.99	0.97	0.99	0.99	0.99	0.97	0.99	0.99
$corr(\Delta e_t^k, \Delta e_t^l)$	0.41	0.14	0.38	0.10	0.14	0.10	0.14	0.20	0.57	0.21
$corr(\Delta e_t, \Delta q_t)$	0.98	0.98	0.94	0.99	0.98	0.95	0.98	0.99	-0.84	0.89
$\frac{\sigma(\Delta c_t - \Delta c_t^1)}{\sigma(\Delta e_t)}$	0.148	0.15	0.36	0.05	0.12	0.25	0.15	0.24	0.01	0.03
$\frac{\sigma(\Delta i_t - \Delta i_t^1)}{\sigma(\Delta e_t)}$	0.051	0.039	0.07	0.015	0.03	0.015	0.03	0.13	0.038	0.037
$corr(\Delta c_t - \Delta c_t^1, \Delta q_t)$	-0.096	-0.18	0.46	-0.11	-0.32	0.00	-0.12	-0.58	0.83	-0.92
$corr(\Delta c_t - \Delta c_t^1, \Delta e_t)$	-0.11	-0.16	0.46	-0.13	-0.31	0.03	-0.11	-0.57	-0.90	-0.95
$corr(\Delta i_t - \Delta i_t^1, \Delta e_t)$	-0.083	0.32	0.50	0.15	0.29	0.23	0.37	0.35	0.71	0.74

Note: The data and model moments are the average across all countries used for calibration. All exchange rates are against the U.S. dollar and all relative changes, $\Delta x_t^k - \Delta x_t^l$, are for domestic variables against the same variables in the U.S. The table reports robustness analysis with respect to deviations of some model parameters from their baseline values. Columns (6) and (7) correspond to the case when the monetary authority reacts only to changes in expected inflation when sets interest rate, i.e. $\alpha_{\Delta\pi}^h = \alpha_{\Delta y}^h = 0$. Columns (8) and (9) report results when $\lambda = 0$ and Γ^f is re-calibrated to match the standard deviation of the real exchange rates. Column (8) corresponds to the case when only TFP shocks are active. Column (9) corresponds to the case when TFP and noise-traders shocks are active.

5 Conclusion

This paper argues that trade between countries is one of the mechanisms that link nominal exchange rates. Based on the data for 33 countries over 25 years, I document that countries that trade more with each other have more correlated exchange rates against the U.S. dollar. This result implies that, despite the fact that the financial imperfections seem to be a necessary ingredient of the exchange-rate-determination models, the equilibrium adjustment of trade balances to exchange rate fluctuations also plays an important role.

I build a baseline three-country general equilibrium model, where a shock to a single country propagates to the exchange rates of its trading partners and serves as a source of common variation. The baseline model admits the closed-form solution and the analytical characterization of the relationship between bilateral trade intensity and exchange rates correlation. I show that this relationship depends crucially on the assumption about the currency in which firms set their prices. If prices are set in the currency of consumer or in dominant currency, i.e., U.S. dollar, the model's prediction is consistent with the empirical finding of the paper. However, if prices are set in the currency of producer, the baseline model does not generate a monotone relationship between trade intensity and exchange rates correlation.

I quantitatively assess the importance of trade linkages for exchange rates co-movement building a multi-country generalized version of the model and calibrating it to the set of 12 countries and the rest of the world. Shocks are uncorrelated across countries and exchange rates are linked through trade that makes shocks to individual countries to propagate to exchange rates of their trading partners. I find that the model is able to account for 50% of the empirical slope coefficient in regression of exchange rate correlations on trade intensity.

The underlying mechanism behind exchange rates co-movement might be important for the evaluation of optimal monetary policy and the investigation of international spillover effects. For example, the argument in favour of flexible exchange rates is that the nominal exchange rate fluctuations lead to efficient relative prices adjustments, such that households efficiently re-distribute their consumption between domestic and foreign goods. However, the results of the paper imply that the exchange rate between currencies of close trading partners is very stable, so such adjustment of consumption might be limited.

Another interesting direction for future research is to analyze the effect of bilateral financial flow on exchange rate co-movement. Unfortunately, such analysis is restricted by the data limitation for all types of bilateral capital flows (e.g. FDI, portfolio flows, currency and deposits, etc.).

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A Appendix

A.1 Robustness of Empirical Results

Table 6: Period: 89-13. Nominal exchange rate, quarterly (NER, Q) and monthly changes (NER, M), real exchange rate, quarterly (RER, Q) and monthly changes (RER, M). IMF export.

	NER, Q	NER, M	RER, Q	RER, M
log(Trade Intensity)	0.0633*** (6.68)	0.0601*** (6.82)	0.0589*** (6.04)	0.0507*** (6.10)
Peg	0.256*** (6.30)	0.243*** (6.40)	0.274*** (6.54)	0.111*** (3.27)
Com. Exp. Sim.	-0.0180 (-0.43)	-0.00849 (-0.22)	0.000936 (0.02)	0.00972 (0.26)
Country FE	Yes	Yes	Yes	Yes
Observations	407	407	407	344
Adjusted R^2	0.614	0.649	0.604	0.747

t statistics in parentheses

* $p < .10$, ** $p < .05$, *** $p < .01$

Table 7: Period 1989-2006 and 1989-2016. IMF export.

	NER, 89-06	RER, 89-06	NER, 89-16	NER, 89-16
log(Trade Intensity)	0.0884*** (8.07)	0.0738*** (6.67)	0.0639*** (6.90)	0.0602*** (6.32)
Peg	0.202*** (4.12)	0.238*** (4.80)	0.260*** (6.61)	0.277*** (6.85)
Com. Exp. Sim.	-0.0224 (-0.44)	0.00278 (0.05)	-0.0165 (-0.40)	-0.0152 (-0.36)
Country FE	Yes	Yes	Yes	Yes
Observations	407	407	407	407
Adjusted R^2	0.660	0.645	0.610	0.605

t statistics in parentheses

* $p < .10$, ** $p < .05$, *** $p < .01$

Table 8: Period: 89-13. IMF import. Nominal exchange rate (quarterly and monthly changes) and Real exchange rate (quarterly and monthly changes)

	NER, Q	NER, M	RER, Q	RER, M
log(Trade Intensity)	0.0711*** (7.04)	0.0673*** (7.15)	0.0681*** (6.56)	0.0582*** (6.63)
Peg	0.256*** (6.32)	0.242*** (6.43)	0.273*** (6.56)	0.111*** (3.29)
Com. Exp. Sim.	-0.0208 (-0.50)	-0.0104 (-0.27)	-0.00147 (-0.03)	0.00970 (0.27)
Constant	0.914*** (10.11)	0.932*** (11.05)	0.875*** (9.42)	0.772*** (7.20)
Observations	406	406	406	343
Adjusted R^2	0.617	0.653	0.609	0.752

t statistics in parentheses

* $p < .10$, ** $p < .05$, *** $p < .01$

Table 9: Period: 89-11. OECD export. Nominal exchange rate (quarterly and monthly changes) and Real exchange rate (quarterly and monthly changes)

	NER, Q	NER, M	RER, Q	RER, M
log(Trade Intensity)	0.0671*** (6.26)	0.0623*** (6.38)	0.0625*** (5.64)	0.0513*** (5.53)
Peg	0.246*** (5.27)	0.240*** (5.63)	0.268*** (5.54)	0.0993*** (2.61)
Com. Exp. Sim.	-0.00992 (-0.21)	-0.0180 (-0.42)	0.0109 (0.22)	-0.00940 (-0.24)
Constant	0.912*** (9.35)	0.917*** (10.32)	0.864*** (8.56)	1.023*** (13.00)
Observations	345	345	345	288
Adjusted R^2	0.598	0.649	0.586	0.754

t statistics in parentheses

* $p < .10$, ** $p < .05$, *** $p < .01$

A.2 The log-linearized model

The log-linearized model is given by the following set of equations

1. Labor supply for $h \in \{1, 2, 3\}$ is

$$\sigma c_t^h + \frac{1}{\nu} l_t^h = w_t^h - p_t^h - i_t^h.$$

2. Labor demand for $h \in \{1, 2, 3\}$ is

$$y_t^h = a_t^h + l_t^h.$$

3. Consumption goods market equilibrium conditions for $h \in \{1, 2, 3\}$ are

$$y_t^h = a_h^1 c_{ht}^1 + a_h^2 c_{ht}^2 + a_h^3 c_{ht}^3.$$

4. Demand equations for all $i, h \in \{1, 2, 3\}$ are

$$c_{it}^h = -\theta(p_{it}^h - p_t^h) + c_t^h.$$

5. Pricing indexes for $h \in \{1, 2, 3\}$ are

$$p_t^h = a_1^h p_{1t}^h + a_2^h p_{2t}^h + a_3^h p_{3t}^h.$$

6. Asset demand conditions, for $h \in \{1, 2, 3\}$ are

$$i_t^h = \mathbb{E}_t\{\sigma \Delta c_{t+1}^h + \Delta p_{t+1}^h\}.$$

7. Foreign exchange market equilibrium conditions

$$\begin{aligned} a_1^2(e_t^2 + p_{1t}^2 + c_{1t}^2) + a_3^2(e_t^2 + p_{3t}^2 + c_{3t}^2) - a_2^1(p_{2t}^1 + c_{2t}^1) - a_2^3(e_t^3 + p_{2t}^3 + c_{2t}^3) - z_t^2 &= q_t^2 \frac{1}{\beta} - q_{t+1}^2, \\ a_1^3(e_t^3 + p_{1t}^3 + c_{1t}^3) + a_2^3(e_t^3 + p_{2t}^3 + c_{2t}^3) - a_3^1(p_{3t}^1 + c_{3t}^1) - a_3^2(e_t^2 + p_{3t}^2 + c_{3t}^2) - z_t^3 &= q_t^3 \frac{1}{\beta} - q_{t+1}^3. \end{aligned}$$

8. Optimal bond demand

$$-q_{t+1}^h = \frac{\beta}{\Gamma^f} (\mathbb{E}_t\{e_{t+1}^h - e_t^h + i_t^h - i_t^1\}) \text{ for } h = \{2, 3\}.$$

9. Monetary policy

$$c_t^h = 0, \quad \text{for } h \in \{1, 2, 3\}.$$

10. Money supply implied by the monetary policy

$$\hat{m}_t^h = p_t^h + q_{t+1}^h - \frac{1}{\beta} q_t^h - z_{ht-1}^T$$

where

$$z_{us,t}^T = - \sum_{i=1}^t (z_i^2 + z_i^3), \quad z_{2t}^T = \sum_{i=1}^t z_t^2, \quad z_{3t}^T = \sum_{i=1}^t z_t^3.$$

11. Pricing equations

- *LCL*: $p_h^i = 0$, for $h, i \in \{1, 2, 3\}$
- *DCP*:

$$p_{1t}^2 = -e_t^2, \quad p_{3t}^2 = -e_t^2, \quad p_{2t}^3 = -e_t^3, \quad p_{1t}^3 = -e_t^3, \quad p_{2t}^1 = 0, \quad p_{3t}^1 = 0,$$

$$p_{1t}^1 = p_{2t}^2 = p_{3t}^3 = 0,$$

- *PCP*:

$$p_{2t}^1 = e_t^2, \quad p_{3t}^1 = e_t^3, \quad p_{1t}^2 = -e_t^2, \quad p_{3t}^2 = e_t^3 - e_t^2, \quad p_{1t}^3 = -e_t^3, \quad p_{2t}^3 = e_t^2 - e_t^3.$$

$$p_1^1 = 0, \quad p_2^2 = 0, \quad p_3^3 = 0.$$

A.3 Proof of Proposition 1

Given the equilibrium system described in A.1, it can be shown that:

- with *LCP*:

$$c = a_2^1 \left(1 + \underbrace{\frac{dp_{1t}^2}{de_t^2} + \frac{dc_{1t}^2}{de_t^2}}_0 \right) + a_3^2 \left(1 + \underbrace{\frac{dp_{3t}^2}{de_t^2} + \frac{dc_{3t}^2}{de_t^2}}_0 \right) - a_2^1 \left(\underbrace{\frac{dp_{2t}^1}{de_t^2} + \frac{dc_{2t}^1}{de_t^2}}_0 \right) - a_3^2 \left(\underbrace{\frac{dp_{2t}^3}{de_t^2} + \frac{dc_{2t}^3}{de_t^2}}_0 \right)$$

$$= a_2^1 + a_3^2,$$

$$d = a_2^1 \left(\underbrace{\frac{dp_{2t}^1}{de_t^3} + \frac{dc_{2t}^1}{de_t^3}}_0 \right) + a_3^2 \left(1 + \underbrace{\frac{dp_{3t}^3}{de_t^3} + \frac{dc_{3t}^3}{de_t^3}}_0 \right) - a_2^1 \left(\underbrace{\frac{dp_{1t}^2}{de_t^3} + \frac{dc_{1t}^2}{de_t^3}}_0 \right) - a_3^2 \left(\underbrace{\frac{dp_{3t}^2}{de_t^3} + \frac{dc_{3t}^2}{de_t^3}}_0 \right)$$

$$= a_3^2.$$

- with DCP:

$c =$

$$a_2^1 \left(1 + \underbrace{\frac{dp_{1t}^2}{de_t^2}}_{-1} + \underbrace{\frac{dc_{1t}^2}{de_t^2}}_{=\theta(1-a_2^1-a_3^2)} \right) + a_3^2 \left(1 + \underbrace{\frac{dp_{3t}^2}{de_t^2}}_{-1} + \underbrace{\frac{dc_{3t}^2}{de_t^2}}_{\theta(1-a_2^1-a_3^2)} \right) - a_2^1 \left(\underbrace{\frac{dp_{2t}^1}{de_t^2} + \frac{dc_{2t}^1}{de_t^2}}_0 \right) - a_3^2 \left(\underbrace{\frac{dp_{2t}^3}{de_t^2} + \frac{dc_{2t}^3}{de_t^2}}_0 \right)$$

$$= (a_3^2 + a_2^1)\theta(1 - a_3^2 - a_2^1),$$

$d =$

$$a_2^1 \left(\underbrace{\frac{dp_{2t}^1}{de_t^3} + \frac{dc_{2t}^1}{de_t^3}}_0 \right) + a_3^2 \left(1 + \underbrace{\frac{dp_{2t}^3}{de_t^3}}_{-1} + \underbrace{\frac{dc_{2t}^3}{de_t^3}}_{\theta(1-a_2^1-a_3^2)} \right) - a_2^1 \left(\underbrace{\frac{dp_{1t}^2}{de_t^3} + \frac{dc_{1t}^2}{de_t^3}}_0 \right) - a_3^2 \left(\underbrace{\frac{dp_{3t}^2}{de_t^2} + \frac{dc_{3t}^2}{de_t^2}}_0 \right)$$

$$= a_3^2\theta(1 - a_3^2 - a_2^1).$$

For PCP we can solve for the price indexes

$$p_t^1 = a_2^1 e_t^2 + a_2^1 e_t^3, \quad p_t^2 = a_3^2 e_t^3 - (a_2^1 + a_3^2) e_t^2, \quad p_t^3 = a_3^2 e_t^2 - (a_2^1 + a_3^2) e_t^3.$$

Therefore, optimal consumption is given by

$$c_{2t}^1 = -\theta(1 - a_2^1) e_t^2 + \theta a_2^1 e_t^3, \quad c_{3t}^1 = -\theta(1 - a_2^1) e_t^3 + \theta a_2^1 e_t^2,$$

$$c_{1t}^2 = \theta a_2^2 e_t^2 + \theta a_3^2 e_t^3, \quad c_{3t}^2 = \theta a_2^2 e_t^2 + \theta(a_3^2 - 1) e_t^3,$$

$$c_{1t}^3 = \theta a_2^2 e_t^3 + \theta a_3^2 e_t^2, \quad c_{2t}^3 = \theta a_2^2 e_t^3 + \theta(a_3^2 - 1) e_t^2.$$

Two forex equilibrium conditions can be written as

$$a_2^1 c_{1t}^2 + a_3^2 (e_t^3 + c_{3t}^2) - a_2^1 (e_t^2 + c_{2t}^1) - a_3^2 (e_t^2 + c_{2t}^3) = z_t^2,$$

$$a_2^1 c_{1t}^3 + a_3^2 (e_t^2 + c_{2t}^3) - a_2^1 (e_t^3 + c_{3t}^1) - a_3^2 (e_t^3 + c_{3t}^2) = z_t^3.$$

Plugging the expressions for optimal consumption into the forex markets equilibrium conditions yields the linear system of two equations with two unknowns of the form

$$\begin{cases} ce_t^2 - de_t^3 = \tilde{z}_t^2, \\ ce_t^3 - de_t^2 = \tilde{z}_t^3. \end{cases}$$

The functions of parameters c and d are defined as

$c =$

$$\begin{aligned} & a_2^1 \left(1 + \underbrace{\frac{dp_{1t}^2}{de_t^2}}_{-1} + \underbrace{\frac{dc_{1t}^2}{de_t^2}}_{\theta(1-a_2^1-a_3^2)} \right) + a_3^2 \left(1 + \underbrace{\frac{dp_{3t}^2}{de_t^2}}_{-1} + \underbrace{\frac{dc_{3t}^2}{de_t^2}}_{\theta(1-a_2^1-a_3^2)} \right) - a_2^1 \left(\underbrace{\frac{dp_{2t}^1}{de_t^2}}_1 + \underbrace{\frac{dc_{2t}^1}{de_t^2}}_{-\theta(1-a_2^1)} \right) - a_3^2 \left(\underbrace{\frac{dp_{2t}^3}{de_t^2}}_1 + \underbrace{\frac{dc_{2t}^3}{de_t^2}}_{-\theta(1-a_3^2)} \right) \\ & = (a_2^1 + a_3^2)\theta(1 - (a_2^1 + a_3^2)) - a_2^1(1 - \theta(1 - a_2^1)) - a_3^2(1 - \theta(1 - a_3^2)), \end{aligned}$$

$d =$

$$\begin{aligned} & a_2^1 \left(\underbrace{\frac{dp_{2t}^1}{de_t^3}}_0 + \underbrace{\frac{dc_{2t}^1}{de_t^3}}_{\theta a_2^1} \right) + a_3^2 \left(1 + \underbrace{\frac{dp_{2t}^3}{de_t^3}}_{-1} + \underbrace{\frac{dc_{2t}^3}{de_t^3}}_{\theta(1-a_2^1-a_3^2)} \right) - a_2^1 \left(\underbrace{\frac{dp_{1t}^2}{de_t^3}}_0 + \underbrace{\frac{dc_{1t}^2}{de_t^3}}_{\theta a_3^2} \right) - a_3^2 \left(\underbrace{\frac{dp_{3t}^2}{de_t^2}}_1 + \underbrace{\frac{dc_{3t}^2}{de_t^2}}_{-\theta(1-a_3^2)} \right) \\ & = (a_2^1)^2\theta + a_3^2\theta(1 - a_3^2 - a_2^1) - a_3^2a_2^1\theta - a_3^2(1 - \theta(1 - a_3^2)). \end{aligned}$$

or

$$\begin{aligned} c &= (2\theta y - y - 2\theta y^2) + (2\theta - 1 - 2\theta y)x - 2\theta x^2 \\ d &= \theta y^2 + (2\theta - 1 - 2\theta y)x - 2\theta x^2, \end{aligned}$$

where for the simplicity of exposition, I define $x \equiv a_3^2$ and $y \equiv a_2^1$. The correlation between exchange rate changes is equal to

$$\text{corr}(\Delta e_t^2, \Delta e_t^3) = \frac{2cd}{c^2 + d^2}$$

and a derivative with respect to x is

$$\frac{d\text{corr}(\Delta e_t^2, \Delta e_t^3)}{dx} = \frac{(c+d)(c-d)(dc'_x - dc'_x)}{(c^2 + d^2)^2}.$$

One can calculate:

$$\begin{aligned} c - d &= 2\theta y - 3\theta y^2 - y, \\ c + d &= 4\theta x - 4\theta xy - 2x - 4\theta x^2 - y - \theta y^2 + 2\theta y, \\ d'_x c - c'_x d &= (2\theta y - y - 3\theta y^2)(2\theta - 2\theta y - 4\theta x - 1) = (c - d)(2\theta - 2\theta y - 4\theta x - 1). \end{aligned}$$

Combining all expressions together yields

$$\frac{d\text{corr}(\Delta e_t^2, \Delta e_t^3)}{dx} \propto (4\theta x - 4\theta xy - 2x - 4\theta x^2 - y - \theta y^2 + 2\theta y)(2\theta - 2\theta y - 4\theta x - 1).$$

■

A.4 Proof of Proposition 2

With LCP and DCP the equilibrium system can be written in the generalized matrix form as

$$\begin{pmatrix} \mathbb{E}_t\{e_{t+1}^2\} \\ \mathbb{E}_t\{e_{t+1}^3\} \\ \hat{b}_{t+1}^2 \\ \hat{b}_{t+1}^3 \end{pmatrix} = A \begin{pmatrix} e_t^2 \\ e_t^3 \\ \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix} + \begin{pmatrix} \hat{z}_t^2 \\ \hat{z}_t^3 \\ \hat{z}_t^2 \\ \hat{z}_t^3 \end{pmatrix}, \quad A \equiv \begin{pmatrix} a+1 & -b & c & 0 \\ -b & a+1 & 0 & c \\ a & -b & c & 0 \\ -b & a & 0 & c \end{pmatrix}.$$

where functions of parameters a , b , and \hat{b}_{t+1}^h , \hat{z}_t^h are given for LPC and PCP below and $c = 1/\beta$. The matrix A has 4 eigenvalues:

$$\begin{aligned} l_1 &= \frac{a-b}{2} + \frac{c+1}{2} - \frac{[(a-b)^2 + (c-1)^2 + 2(a-b)(c+1)]^{1/2}}{2}, \\ l_2 &= \frac{a-b}{2} + \frac{c+1}{2} + \frac{[(a-b)^2 + (c-1)^2 + 2(a-b)(c+1)]^{1/2}}{2}, \\ l_3 &= \frac{a+b}{2} + \frac{c+1}{2} - \frac{[(a+b)^2 + (c-1)^2 + 2(a+b)(c+1)]^{1/2}}{2}, \\ l_4 &= \frac{a+b}{2} + \frac{c+1}{2} + \frac{[(a+b)^2 + (c-1)^2 + 2(a+b)(c+1)]^{1/2}}{2}. \end{aligned}$$

As for DCP and LCP $a > b > 0$ and $c > 1$ it follows that $l_1 < 1$, $l_2 > 1$, $l_3 < 1$, $l_4 > 1$. The corresponding left eigenvectors are

$$v_1^T = \begin{pmatrix} l_1/c - 1 \\ l_1/c - 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2^T = \begin{pmatrix} l_2/c - 1 \\ l_2/c - 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3^T = \begin{pmatrix} 1 - l_3/c \\ l_3/c - 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_4^T = \begin{pmatrix} 1 - l_4/c \\ l_4/c - 1 \\ -1 \\ 1 \end{pmatrix}.$$

By the definition $v_2 A = v_2 l_2$ and $v_4 A = v_4 l_4$. I use the following transformation

$$v_2 \begin{pmatrix} \mathbb{E}_t\{e_{t+1}^2\} \\ \mathbb{E}_t\{e_{t+1}^3\} \\ \hat{b}_{t+1}^2 \\ \hat{b}_{t+1}^3 \end{pmatrix} = \underbrace{v_2 A}_{=l_2 v_2} \begin{pmatrix} e_t^2 \\ e_t^3 \\ \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix} + v_1 \begin{pmatrix} \hat{z}_t^2 \\ \hat{z}_t^3 \\ \hat{z}_t^2 \\ \hat{z}_t^3 \end{pmatrix}.$$

We define $w_t = v_2 \begin{pmatrix} e_t^2 \\ e_t^3 \\ \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix}$. The equation above can be rewritten as $\mathbb{E}_t\{w_{t+1}\} = l_2 w_t + l_2/c(\hat{z}_t^2 + \hat{z}_t^3)$ or

$$w_t = \frac{1}{l_2} \mathbb{E}_t\{w_{t+1}\} - \frac{(\hat{z}_t^2 + \hat{z}_t^3)}{c} = \lim_{T \rightarrow \infty} \left(\frac{1}{l_2}\right)^T \mathbb{E}_t\{w_{t+T}\} - \frac{(\hat{z}_t^2 + \hat{z}_t^3)}{c} \sum_{i=0}^{\infty} \left(\frac{\rho}{l_2}\right)^i = \underbrace{\frac{l_2}{\rho - l_2} \frac{(\hat{z}_t^2 + \hat{z}_t^3)}{c}}_{\equiv x_t}$$

where the second equality comes from the assumption that both shocks follow the AR(1) process with persistency parameter ρ : $\hat{z}_t^2 = \rho \hat{z}_{t-1}^2 + \epsilon_t^2$, $\hat{z}_t^3 = \rho \hat{z}_{t-1}^3 + \epsilon_t^3$, $\mathbb{E}_t\{\epsilon_{t+1}^2\} = \mathbb{E}_t\{\epsilon_{t+1}^3\} = 0$ for any $i > 0$,

and $\text{corr}(\epsilon_t^2, \epsilon_t^3) = 0$. In the same way I define $o_t = v_4 \begin{pmatrix} e_t^2 \\ e_t^3 \\ \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix}$. Again the equilibrium system can be

rewritten as $\mathbb{E}_t\{o_{t+1}\} = l_4 o_t - l_4 \frac{(\hat{z}_t^3 - \hat{z}_t^2)}{c}$ or

$$o_t = \frac{1}{l_4} \mathbb{E}_t\{o_{t+1}\} + \frac{(\hat{z}_t^3 - \hat{z}_t^2)}{c} = \lim_{T \rightarrow \infty} \left(\frac{1}{l_4}\right)^T \mathbb{E}_t\{o_{t+T}\} + \frac{(\hat{z}_t^3 - \hat{z}_t^2)}{c} \sum_{i=0}^{\infty} \left(\frac{\rho}{l_4}\right)^i = \underbrace{\frac{l_4}{l_4 - \rho} \frac{(\hat{z}_t^3 - \hat{z}_t^2)}{c}}_{\equiv y_t}$$

Using the definitions of w_t and o_t one obtains the system, that can to be solved for e_t^2 and e_t^3 :

$$\begin{cases} \hat{b}_t^2 + \hat{b}_t^3 = x_t - (l_2/c - 1)e_t^2 - (l_2/c - 1)e_t^3, \\ \hat{b}_t^2 - \hat{b}_t^3 = -y_t + (1 - l_4/c)e_t^2 + (l_4/c - 1)e_t^3, \end{cases}$$

Solving for exchange rates yields, (from now I assume that $\rho = 0$) the following represented in matrix form

$$\begin{pmatrix} e_t^2 \\ e_t^3 \end{pmatrix} = \underbrace{\begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}}_{=P} \begin{pmatrix} \hat{b}_t^2 + \hat{z}_t^2/c \\ \hat{b}_t^3 + \hat{z}_t^3/c \end{pmatrix},$$

where P is a symmetric matrix with:

$$p_1 \equiv \frac{2 - l_2/c - l_4/c}{2(l_4/c - 1)(l_2/c - 1)}, \quad p_2 \equiv \frac{l_2/c - l_4/c}{2(l_4/c - 1)(l_2/c - 1)}$$

Therefore,

$$\begin{pmatrix} \Delta e_t^2 \\ \Delta e_t^3 \end{pmatrix} = P \begin{pmatrix} \Delta \hat{b}_t^2 \\ \Delta \hat{b}_t^3 \end{pmatrix} + P \begin{pmatrix} \Delta \hat{z}_t^2/c \\ \Delta \hat{z}_t^3/c \end{pmatrix}.$$

Then, solving for the law of motion of \hat{b}_t^2 and \hat{b}_t^3 in matrix form yields

$$\begin{pmatrix} \hat{b}_{t+1}^2 \\ \hat{b}_{t+1}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} n_1 & n_2 \\ n_2 & n_1 \end{pmatrix}}_{=N} \begin{pmatrix} \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix} + \begin{pmatrix} \omega_t^2 \\ \omega_t^3 \end{pmatrix}$$

where $n_1 = ap_1 - bp_2 + c$, $n_2 = ap_2 - bp_1$, $\omega_t^2 \equiv (ap_1 - bp_2 + c)\hat{z}_t^2/c + (ap_2 - bp_1)\hat{z}_t^3/c$, $\omega_t^3 \equiv (ap_1 - bp_2 + c)\hat{z}_t^3/c + (ap_2 - bp_1)\hat{z}_t^2/c$. Two eigenvalues of N are $\tilde{l}_1 = \frac{a-b}{1-l_2/c} + c$, $\tilde{l}_2 = \frac{a+b}{1-l_4/c} + c$. Both eigenvalues are less than one in absolute value.¹⁹ The left eigenvalue that correspond to \tilde{l}_1 is $\tilde{v}_1 = [1, 1]$, the left eigenvector corresponding to \tilde{l}_2 is $\tilde{v}_2 = [-1, 1]$ Using the same approach as before yields

$$\tilde{v}_1 \begin{pmatrix} \hat{b}_{t+1}^2 \\ \hat{b}_{t+1}^3 \end{pmatrix} = \underbrace{\tilde{v}_1 N}_{=\tilde{l}_1 \tilde{v}_1} \begin{pmatrix} \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix} + \tilde{v}_1 \begin{pmatrix} \omega_t^2 \\ \omega_t^3 \end{pmatrix}.$$

Denote $m_t = \tilde{v}_1 \begin{pmatrix} \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix}$, then the system above can be written as

$$m_{t+1} = \tilde{l}_1 m_t + (\omega_t^2 + \omega_t^3) = \sum_{i=0}^{\infty} (\tilde{l}_1)^i (\omega_{t-i}^2 + \omega_{t-i}^3),$$

where for the last equality the iteration backward was used. Using the second eigenvalue, one obtains

$$\tilde{v}_2 \begin{pmatrix} \hat{b}_{t+1}^2 \\ \hat{b}_{t+1}^3 \end{pmatrix} = \underbrace{\tilde{v}_2 N}_{=\tilde{l}_2 \tilde{v}_2} \begin{pmatrix} \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix} + \tilde{v}_2 \begin{pmatrix} \omega_t^2 \\ \omega_t^3 \end{pmatrix}.$$

Define Denote $g_t = \tilde{v}_2 \begin{pmatrix} \hat{b}_t^2 \\ \hat{b}_t^3 \end{pmatrix}$, then the same system can be written as

$$g_{t+1} = \tilde{l}_2 g_t + (\omega_t^3 - \omega_t^2) = \sum_{i=0}^{\infty} (\tilde{l}_2)^i (\omega_{t-i}^3 - \omega_{t-i}^2).$$

Therefore, one can solve for \hat{b}_t^2 and \hat{b}_t^3 from the system

$$\begin{cases} \hat{b}_t^2 + \hat{b}_t^3 = \sum_{i=0}^{\infty} (\tilde{l}_1)^i (\omega_{t-1-i}^2 + \omega_{t-1-i}^3), \\ \hat{b}_t^2 - \hat{b}_t^3 = \sum_{i=0}^{\infty} (\tilde{l}_2)^i (\omega_{t-1-i}^2 - \omega_{t-1-i}^3). \end{cases}$$

The solution for \hat{b}_t^h for $h = 1, 2$ is

$$\hat{b}_t^2 = \frac{1}{2} \left[\sum_{i=0}^{\infty} (\tilde{l}_1^i + \tilde{l}_2^i) \omega_{t-1-i}^2 + \sum_{i=0}^{\infty} (\tilde{l}_1^i - \tilde{l}_2^i) \omega_{t-1-i}^3 \right], \quad \hat{b}_t^3 = \frac{1}{2} \left[\sum_{i=0}^{\infty} (\tilde{l}_1^i + \tilde{l}_2^i) \omega_{t-1-i}^3 + \sum_{i=0}^{\infty} (\tilde{l}_1^i - \tilde{l}_2^i) \omega_{t-1-i}^2 \right].$$

¹⁹It can be shown that $\tilde{l}_1 = l_1 < 1$, $\tilde{l}_2 = l_3 < 1$.

Using definitions of ω s one gets

$$\begin{aligned}\hat{b}_t^2 &= \frac{1}{2} \sum_{i=0}^{\infty} \left\{ [(a-b)(p_1+p_2)+c]\tilde{l}_1^i + [(a+b)(p_1-p_2)+c]\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^2}{c} \\ &\quad + \frac{1}{2} \sum_{i=0}^{\infty} \left\{ [(a-b)(p_1+p_2)+c]\tilde{l}_1^i - [(a+b)(p_1-p_2)+c]\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^3}{c}, \\ \hat{b}_t^3 &= \frac{1}{2} \sum_{i=0}^{\infty} \left\{ [(a-b)(p_1+p_2)+c]\tilde{l}_1^i + [(a+b)(p_1-p_2)+c]\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^3}{c} \\ &\quad + \frac{1}{2} \sum_{i=0}^{\infty} \left\{ [(a-b)(p_1+p_2)+c]\tilde{l}_1^i - [(a+b)(p_1-p_2)+c]\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^2}{c}.\end{aligned}$$

Notice that $p_1+p_2 = \frac{1}{1-l_2/c}$ and $p_1-p_2 = \frac{1}{1-l_4/c}$. Therefore, $(a-b)(p_1+p_2)+c = \frac{a-b}{1-l_2/c} + c = \tilde{l}_1$, $(a+b)(p_1-p_2)+c = \frac{a+b}{1-l_4/c} + c = \tilde{l}_2$ and hence

$$\begin{aligned}\hat{b}_t^2 &= \frac{1}{2} \sum_{i=0}^{\infty} \left\{ \tilde{l}_1^{i+1} + \tilde{l}_2^{i+1} \right\} \frac{\hat{z}_{t-1-i}^2}{c} + \frac{1}{2} \sum_{i=0}^{\infty} \left\{ \tilde{l}_1^{i+1} - \tilde{l}_2^{i+1} \right\} \frac{\hat{z}_{t-1-i}^3}{c}, \\ \hat{b}_t^3 &= \frac{1}{2} \sum_{i=0}^{\infty} \left\{ \tilde{l}_1^{i+1} + \tilde{l}_2^{i+1} \right\} \frac{\hat{z}_{t-1-i}^3}{c} + \frac{1}{2} \sum_{i=0}^{\infty} \left\{ \tilde{l}_1^{i+1} - \tilde{l}_2^{i+1} \right\} \frac{\hat{z}_{t-1-i}^2}{c}.\end{aligned}$$

Next, it follows

$$\begin{aligned}\Delta \hat{b}_t^2 &= (ap_1 - bp_2 + c) \frac{\hat{z}_{t-1}^2}{c} + (ap_2 - bp_1) \frac{\hat{z}_{t-1}^3}{c} \\ &\quad + \frac{1}{2} \sum_{i=1}^{\infty} \left\{ (\tilde{l}_1 - 1)\tilde{l}_1^i + (\tilde{l}_2 - 1)\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^2}{c} + \frac{1}{2} \sum_{i=1}^{\infty} \left\{ (\tilde{l}_1 - 1)\tilde{l}_1^i - (\tilde{l}_2 - 1)\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^3}{c}, \\ \Delta \hat{b}_t^3 &= (ap_1 - bp_2 + c) \frac{\hat{z}_{t-1}^3}{c} + (ap_2 - bp_1) \frac{\hat{z}_{t-1}^2}{c} \\ &\quad + \frac{1}{2} \sum_{i=1}^{\infty} \left\{ (\tilde{l}_1 - 1)\tilde{l}_1^i + (\tilde{l}_2 - 1)\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^3}{c} + \frac{1}{2} \sum_{i=1}^{\infty} \left\{ (\tilde{l}_1 - 1)\tilde{l}_1^i - (\tilde{l}_2 - 1)\tilde{l}_2^i \right\} \frac{\hat{z}_{t-1-i}^2}{c}.\end{aligned}$$

Now, one can calculate the correlation between exchange rate returns, using the fact that $corr(\Delta e_t^2, \Delta e_t^3) = cov(\Delta e_t^2, \Delta e_t^3)/var(\Delta e_t^2)$ (as the model is symmetric $var(\Delta e_t^2) = var(\Delta e_t^3)$). First, we calculate $cov(\Delta e_t^2, \Delta e_t^3)$:

$$\begin{aligned}cov(\Delta e_t^2, \Delta e_t^3) &= (p_1^2 + p_2^2)cov(\Delta \hat{b}_t^2, \Delta \hat{b}_t^3) + p_1p_2(var(\Delta \hat{b}_t^2) + var(\Delta \hat{b}_t^3)) \\ &\quad + (p_1^2 + p_2^2) \underbrace{cov(\Delta \hat{z}_t^2/c, \Delta \hat{z}_t^3/c)}_{=0} + p_1p_2(var(\Delta \hat{z}_t^2/c) + var(\Delta \hat{z}_t^3/c)) \\ &\quad + 2p_1p_2(cov(\Delta \hat{b}_t^2, \Delta \hat{z}_t^2/c) + cov(\Delta \hat{b}_t^3, \Delta \hat{z}_t^3/c)) + (p_1^2 + p_2^2)(cov(\Delta \hat{b}_t^2, \Delta \hat{z}_t^3/c) + cov(\Delta \hat{b}_t^3, \Delta \hat{z}_t^2/c))\end{aligned}\tag{13}$$

Using the assumption that shocks \hat{z}_t^2 and \hat{z}_t^3 are i.i.d with $var(\hat{z}_t^2) = var\hat{z}_t^3 = \sigma_z^2$, one obtains

$$\begin{aligned}
cov(\Delta\hat{b}_t^2, \Delta\hat{z}_t^2/c) &= cov(\Delta\hat{b}_t^3, \Delta\hat{z}_t^3/c) = -(ap_1 - bp_2 + c) \frac{\sigma_z^2}{c^2}, \\
cov(\Delta\hat{b}_t^2, \Delta\hat{z}_t^3/c) &= cov(\Delta\hat{b}_t^3, \Delta\hat{z}_t^2/c) = -(ap_2 - bp_1) \frac{\sigma_z^2}{c^2}, \\
cov(\Delta\hat{b}_t^2, \Delta\hat{b}_t^3) &= 2(ap_1 - bp_2 + c)(ap_2 - bp_1) \frac{var(\hat{z}_t^2)}{c^2} + \frac{1}{2} \sum_{i=1}^{\infty} \left(\{(\tilde{l}_1 - 1)\tilde{l}_1^i\}^2 - \{(\tilde{l}_2 - 1)\tilde{l}_2^i\}^2 \right) \frac{var(\hat{z}_t^2)}{c^2} \\
&= \left\{ 2(ap_1 - bp_2 + c)(ap_2 - bp_1) + \frac{1}{2} \left[[\tilde{l}_1(\tilde{l}_1 - 1)]^2 \frac{1 - \tilde{l}_1^{2T}}{1 - \tilde{l}_1^2} - [\tilde{l}_2(\tilde{l}_2 - 1)]^2 \frac{1 - \tilde{l}_2^{2T}}{1 - \tilde{l}_2^2} \right] \right\} \frac{\sigma_z^2}{c^2}, \\
var(\Delta\hat{b}_t^2) &= [(ap_1 - bp_2 + c)^2 + (ap_2 - bp_1)^2] \frac{var(\hat{z}_t^2)}{c^2} + \frac{1}{4} \sum_{i=1}^{\infty} \left\{ (\tilde{l}_1 - 1)\tilde{l}_1^i + (\tilde{l}_2 - 1)\tilde{l}_2^i \right\}^2 \frac{var(\hat{z}_t^2)}{c^2} \\
&+ \frac{1}{4} \sum_{i=1}^{\infty} \left\{ (\tilde{l}_1 - 1)\tilde{l}_1^i - (\tilde{l}_2 - 1)\tilde{l}_2^i \right\}^2 \frac{var(\hat{z}_t^2)}{c^2} \\
&= [(ap_1 - bp_2 + c)^2 + (ap_2 - bp_1)^2] var(\hat{z}_t^2) + \frac{1}{2} \sum_{i=1}^{\infty} \left(\{(\tilde{l}_1 - 1)\tilde{l}_1^i\}^2 + \{(\tilde{l}_2 - 1)\tilde{l}_2^i\}^2 \right) \frac{var(\hat{z}_t^2)}{c^2} \\
&= \left\{ [(ap_1 - bp_2 + c)^2 + (ap_2 - bp_1)^2] + \frac{1}{2} \left[[\tilde{l}_1(\tilde{l}_1 - 1)]^2 \frac{1 - \tilde{l}_1^{2T}}{1 - \tilde{l}_1^2} + [\tilde{l}_2(\tilde{l}_2 - 1)]^2 \frac{1 - \tilde{l}_2^{2T}}{1 - \tilde{l}_2^2} \right] \right\} \frac{\sigma_z^2}{c^2}.
\end{aligned}$$

Notice, that $p_1^2 + p_2^2 = \left[\frac{1}{2(1-\tilde{l}_4/c)^2} + \frac{1}{2(1-\tilde{l}_2/c)^2} \right]$, $p_1p_2 = \left[\frac{1}{4(1-\tilde{l}_2/c)^2} - \frac{1}{4(1-\tilde{l}_4/c)^2} \right]$ Plugging the expressions above into the equation (13) yields

$$\begin{aligned}
cov(\Delta e_t^2, \Delta e_t^3) &= \{2(p_1^2 + p_2^2)(ap_1 - bp_2 + c)(ap_2 - bp_1) + 2p_1p_2[(ap_1 - bp_2 + c)^2 + (ap_2 - bp_1)^2] \\
&+ 4p_1p_2 - 4p_1p_2(ap_1 - bp_2 + c) - 2(p_1^2 + p_2^2)(ap_2 - bp_1)\} \frac{\sigma_z^2}{c^2} \\
&+ \frac{1}{2}(p_1^2 + p_2^2) \left[[\tilde{l}_1(\tilde{l}_1 - 1)]^2 \frac{1 - \tilde{l}_1^{2T}}{1 - \tilde{l}_1^2} - [\tilde{l}_2(\tilde{l}_2 - 1)]^2 \frac{1 - \tilde{l}_2^{2T}}{1 - \tilde{l}_2^2} \right] \frac{\sigma_z^2}{c^2} \\
&+ p_1p_2 \left[[\tilde{l}_1(\tilde{l}_1 - 1)]^2 \frac{1 - \tilde{l}_1^{2T}}{1 - \tilde{l}_1^2} + [\tilde{l}_2(\tilde{l}_2 - 1)]^2 \frac{1 - \tilde{l}_2^{2T}}{1 - \tilde{l}_2^2} \right] \frac{\sigma_z^2}{c^2}.
\end{aligned}$$

Notice that $\tilde{l}_1 + \tilde{l}_2 = 2(ap_1 - bp_2 + 1)$, $\tilde{l}_1 - \tilde{l}_2 = 2(ap_2 - bp_1)$. Therefore,

$$\begin{aligned}
cov(\Delta e_t^2, \Delta e_t^3) &= \left[\frac{1}{2}(p_1^2 + p_2^2)(\tilde{l}_1 + \tilde{l}_2)(\tilde{l}_1 - \tilde{l}_2) + \frac{1}{2}p_1p_2[(\tilde{l}_1 + \tilde{l}_2)^2 + (\tilde{l}_1 - \tilde{l}_2)^2] \right] \frac{\sigma_z^2}{c^2} \\
&+ [4p_1p_2 - 2p_1p_2(\tilde{l}_1 + \tilde{l}_2) - (p_1^2 + p_2^2)(\tilde{l}_1 - \tilde{l}_2)] \frac{\sigma_z^2}{c^2} \\
&+ \frac{1}{2}(p_1 + p_2)^2 \left[\tilde{l}_1(\tilde{l}_1 - 1) \right]^2 \frac{1 - \tilde{l}_1^{2T}}{1 - \tilde{l}_1^2} \sigma_z^2 - \frac{1}{2}(p_1 - p_2)^2 \left[\tilde{l}_2(\tilde{l}_2 - 1) \right]^2 \frac{1 - \tilde{l}_2^{2T}}{1 - \tilde{l}_2^2} \sigma_z^2 \\
&= 4p_1p_2 \frac{\sigma_z^2}{c^2} + \frac{1}{2}(p_1 + p_2)^2 \left[\tilde{l}_1(\tilde{l}_1 - 1) \right]^2 \frac{1 - \tilde{l}_1^{2T}}{1 - \tilde{l}_1^2} + \tilde{l}_1^2 - 2\tilde{l}_1 \right] \frac{\sigma_z^2}{c^2} \\
&- \frac{1}{2}(p_1 - p_2)^2 \left[\tilde{l}_2(\tilde{l}_2 - 1) \right]^2 \frac{1 - \tilde{l}_2^{2T}}{1 - \tilde{l}_2^2} + \tilde{l}_2^2 - 2\tilde{l}_2 \right] \frac{\sigma_z^2}{c^2}.
\end{aligned}$$

Similarly, one can calculate $var(\Delta e_t^2) = var(\Delta e_t^3)$:

$$\begin{aligned}
var(\Delta e_t^2) &= (p_1^2 + p_2^2)var(\Delta \hat{b}_t^2) + 2p_1p_2cov(\Delta \hat{b}_t^2, \Delta \hat{b}_t^3) + 2(p_1^2 + p_2^2)cov(\Delta \hat{b}_t^2, \Delta \hat{z}_t^2/c) \\
&+ 4p_1p_2cov(\Delta \hat{b}_t^2, \Delta \hat{z}_t^3/c) + 2p_1p_2cov(\Delta \hat{z}_t^2/c, \Delta \hat{z}_t^3/c) + (p_1^2 + p_2^2)var(\Delta \hat{z}_t^2/c).
\end{aligned}$$

Assume that $T \rightarrow \infty$, then

$$\begin{aligned}
cov(\Delta e_t^2, \Delta e_t^3) &= \left[4p_1p_2 - (p_1 + p_2)^2 \frac{\tilde{l}_1}{1 + \tilde{l}_1} + (p_1 - p_2)^2 \frac{\tilde{l}_2}{1 + \tilde{l}_2} \right] \frac{\sigma_z^2}{c^2}, \\
var(\Delta e_t^2) &= \left[2(p_1^2 + p_2^2) - (p_1 + p_2)^2 \frac{\tilde{l}_1}{1 + \tilde{l}_1} - (p_1 - p_2)^2 \frac{\tilde{l}_2}{1 + \tilde{l}_2} \right] \frac{\sigma_z^2}{c^2}.
\end{aligned}$$

Next step is to calculate the derivative of $cov(\Delta e_t^2, \Delta e_t^3)$ with respect to $x = a_3^2$. To do that, I use the following

$$\begin{aligned}
p_1^2 + p_2^2 &= \left[\frac{1}{2(1 - l_4/c)^2} + \frac{1}{2(1 - l_2/c)^2} \right], & p_1p_2 &= \left[\frac{1}{4(1 - l_2/c)^2} - \frac{1}{4(1 - l_4/c)^2} \right], \\
p_1 + p_2 &= \frac{1}{1 - l_2/c}, & p_1 - p_2 &= \frac{1}{1 - l_4/c}.
\end{aligned}$$

For the simplification I define:

$$\begin{aligned}
C &\equiv \frac{1}{(1 - l_4/c)^2}, & D &\equiv \frac{1}{(1 - l_2/c)^2}, \\
A &\equiv (p_1 + p_2)^2 \frac{\tilde{l}_1}{1 + \tilde{l}_1} = \frac{1}{(1 - l_2/c)^2} \frac{\tilde{l}_1}{1 + \tilde{l}_1} = D \frac{\tilde{l}_1}{1 + \tilde{l}_1}, \\
B &\equiv (p_1 - p_2)^2 \frac{\tilde{l}_2}{1 + \tilde{l}_2} = \frac{1}{(1 - l_4/c)^2} \frac{\tilde{l}_2}{1 + \tilde{l}_2} = C \frac{\tilde{l}_2}{1 + \tilde{l}_2}.
\end{aligned}$$

Then

$$cov(\Delta e_t^2, \Delta e_t^3) = [D - C - A + B], \quad var(\Delta e_t^2) = [D + C - A - B].$$

and

$$\begin{aligned}
(\text{corr}(\Delta e_t^2, \Delta e_t^3))'_x &= \frac{2[D'_x(C - B) + C'_x(A - D) + A'_x(B - C) + B'_x(D - A)]}{(D + C - A - B)^2} \\
&= \left[2C(D'_x - A'_x) \frac{1}{1 + \tilde{l}_2} + 2D(B_x - C'_x) \frac{1}{1 + \tilde{l}_1} \right] \frac{1}{(D + C - A - B)^2} \\
&= \left[\underbrace{\frac{2C}{1 + \tilde{l}_2}}_{\geq 0} \left[\frac{D'_x}{1 + \tilde{l}_1} - F'_xD \right] + \underbrace{\frac{2D}{1 + \tilde{l}_1}}_{\geq 0} \left[E'_xC - \frac{C'_x}{1 + \tilde{l}_2} \right] \right] \frac{1}{(D + C - A - B)^2}.
\end{aligned}$$

where $F \equiv \frac{\tilde{l}_1}{1 + \tilde{l}_1}$, $E \equiv \frac{\tilde{l}_2}{1 + \tilde{l}_2}$. As $\tilde{l}_1 > 0$ and $\tilde{l}_2 > 0$, $2C/(1 + \tilde{l}_2) > 0$, $2D/(1 + \tilde{l}_1) > 0$. Next, define $o(x) = a - b$, $t(x) = 1 - l_2/c$. Then

$$D = \frac{1}{t^2}, \quad D'_x = -2 \frac{t'_x}{t^3}, \quad 1 + \tilde{l}_1 = \frac{o + (1 + c)t}{t}, \quad F = \frac{o + ct}{o + (1 + c)t}, \quad F'_x = \frac{to'_x - ot'_x}{(o + (1 + c)t)^2}$$

Then

$$\frac{D'_x}{1 + \tilde{l}_1} - F'_xD = -\frac{1}{t^2} \left[\frac{2t'_x}{o + (1 + c)t} + \frac{to'_x - ot'_x}{(o + (1 + c)t)^2} \right] = -\frac{1}{t^2} \frac{2(1 + c)tt'_x + to'_x + ot'_x}{(o + (1 + c)t)^2}.$$

Similarly, define $v(x) = 1 - l_4/c$, $p(x) = a + b$. Then

$$C = \frac{1}{v^2}, \quad C'_x = -2 \frac{v'_x}{v^3}, \quad 1 + \tilde{l}_2 = \frac{p + (1 + c)v}{v}, \quad E = \frac{p + cv}{p + (1 + c)v}, \quad E'_x = \frac{vp'_x - pv'_x}{(p + (1 + c)v)^2}.$$

Then

$$E'_xC - \frac{C'_x}{1 + \tilde{l}_2} = \frac{1}{v^2} \left[\frac{vp'_x - pv'_x}{(o + (1 + c)t)^2} + \frac{2v'_x}{p + (1 + c)v} \right] = \frac{1}{v^2} \frac{2(1 + c)vv'_x + vp'_x + pv'_x}{(p + (1 + c)v)^2}.$$

Next, we need to calculate $2(1 + c)tt'_x + to'_x + ot'_x$ and $2(1 + c)vv'_x + vp'_x + pv'_x$

$$\begin{aligned}
t(x) &= 1 - \frac{l_2}{c} = 1 - \frac{1}{2} \frac{1}{c} [o + (o^2 + 2(c + 1)o + (c - 1)^2)^{0.5} + (c + 1)] =, \\
t'_x &= -\frac{1}{2} \frac{1}{c} [o'_x + (o'_x o + (1 + c)o'_x)(o^2 + 2(1 + c)o + (c - 1)^2)^{-0.5}], \\
v(x) &= 1 - l_4 = 1 - \frac{1}{2} \frac{1}{c} [p + (p^2 + 2(c + 1)p + (c - 1)^2)^{0.5} + (c + 1)], \\
v'_x &= -\frac{1}{2} \frac{1}{c} [p'_x + (p'_x p + (1 + c)p'_x)(p^2 + 2(1 + c)p + (c - 1)^2)^{-0.5}].
\end{aligned}$$

For simplicity define $h \equiv (o^2 + 2(c + 1)o + (c - 1)^2)^{0.5}$, $k \equiv o'_x o + (1 + c)o'_x$, then $t = 1 - \frac{1}{2} \frac{1}{c} [0 +$

$h^{0.5} + (c + 1)]$, $t'_x = -\frac{1}{2}\frac{1}{c}[o'_x + yx^{-0.5}]$. Using simplifying notations, one can compute

$$\begin{aligned} 2(1+c)tt'_x + to'_x + ot'_x &= \frac{1}{c^2}o'_x o + \frac{2+c+c^2}{2c^2}o'_x + \frac{1}{2c^2}okh^{-0.5} + \frac{1-c^2}{2c^2}kh^{-0.5} + \frac{1}{2c^2}o'_x h^{0.5} \\ &= \frac{1}{c^2}o'_x o + \frac{2+c+c^2}{2c^2}o'_x + \frac{1+o-c^2}{2c^2} \frac{(o'_x o + (1+c)o'_x)}{(o^2 + 2(1+c)o + (c-1)^2)^{0.5}} \\ &\quad + \frac{o'_x}{2c^2} \frac{(o^2 + 2(1+c)o + (c-1)^2)}{(o^2 + 2(1+c)o + (c-1)^2)^{0.5}}. \end{aligned}$$

Similarly, one can show that

$$\begin{aligned} 2(1+c)vv'_x + vp'_x + pv'_x &= \frac{1}{c^2}p'_x p + \frac{2+c+c^2}{2c^2}p'_x + \frac{1+p-c^2}{2c^2} \frac{(p'_x p + (1+c)p'_x)}{(p^2 + 2(1+c)p + (c-1)^2)^{0.5}} \\ &\quad + \frac{p'_x}{2c^2} \frac{(p^2 + 2(1+c)p + (c-1)^2)}{(p^2 + 2(1+c)p + (c-1)^2)^{0.5}}. \end{aligned}$$

Local currency pricing. With the LCP assumption prices that consumers face and demand for imported goods are not affected by the z shocks, so the interest rates are constant as well. Therefore the system is simplified to

$$\begin{aligned} (a_2^1 + a_3^2)e_t^2 - a_3^2 e_t^3 - z_t^2 &= \frac{\beta}{\Gamma^f}(\mathbb{E}_t\{e_{t+1}^2\} - e_t^2) - \frac{1}{\beta} \frac{\beta}{\Gamma^f}(\mathbb{E}_{t-1}\{e_t^2\} - e_{t-1}^2), \\ (a_2^1 + a_3^2)e_t^3 - a_3^2 e_t^2 - z_t^3 &= \frac{\beta}{\Gamma^f}(\mathbb{E}_t\{e_{t+1}^3\} - e_t^3) - \frac{1}{\beta} \frac{\beta}{\Gamma^f}(\mathbb{E}_{t-1}\{e_t^3\} - e_{t-1}^3). \end{aligned}$$

I define the cross-country bonds holdings are $b_{t+1}^2 \equiv -q_{t+1}^2 = \frac{\beta}{\Gamma^f}(\mathbb{E}_t\{e_{t+1}^2\} - e_t^2)$ and $b_{t+1}^3 \equiv -q_{t+1}^3 = \frac{\beta}{\Gamma^f}(\mathbb{E}_t\{e_{t+1}^3\} - e_t^3)$. To simplify notation define $\hat{b}_{t+1}^2 \equiv \frac{\Gamma^f}{\beta}b_{t+1}^2$, $\hat{z}_t^2 \equiv -\frac{\Gamma^f}{\beta}z_t^2$, $\hat{b}_{t+1}^3 \equiv \frac{\Gamma^f}{\beta}b_{t+1}^3$, $\hat{z}_t^3 \equiv -\frac{\Gamma^f}{\beta}z_t^3$, $\hat{a}_2^1 \equiv \frac{\Gamma^f}{\beta}a_2^1$, $\hat{a}_3^2 \equiv \frac{\Gamma^f}{\beta}a_3^2$. The system is rewritten as

$$\begin{aligned} \mathbb{E}_t\{e_{t+1}^2\} &= [(\hat{a}_2^1 + \hat{a}_3^2) + 1] e_t^2 - \hat{a}_3^2 e_t^3 + \frac{1}{\beta} \hat{b}_t^2 + \hat{z}_t^2, \\ \mathbb{E}_t\{e_{t+1}^3\} &= [(\hat{a}_2^1 + \hat{a}_3^2) + 1] e_t^3 - \hat{a}_2^2 e_t^2 + \frac{1}{\beta} \hat{b}_t^3 + \hat{z}_t^3, \\ \hat{b}_{t+1}^2 &= (\hat{a}_2^1 + \hat{a}_3^2)e_t^2 - \hat{a}_2^2 e_t^3 + \frac{1}{\beta} \hat{b}_t^2 + \hat{z}_t^2, \\ \hat{b}_{t+1}^3 &= (\hat{a}_2^1 + \hat{a}_3^2)e_t^3 - \hat{a}_3^2 e_t^2 + \frac{1}{\beta} \hat{b}_t^3 + \hat{z}_t^3. \end{aligned} \tag{14}$$

Hence, with LCP we define $a \equiv \hat{a}_2^1 + \hat{a}_3^2$ and $b \equiv \hat{a}_2^2$. Given the general solution above, $p(x) = a + b = \hat{a}_2^1 + \hat{a}_3^2 + \hat{a}_2^2 = \hat{a}_2^1 + 2\hat{a}_3^2 = \frac{\Gamma}{\beta}(a_2^1 + 2a_3^2) = \frac{\Gamma}{\beta}(a_2^1 + 2x) \geq 0$, $p'_x = 2\frac{\Gamma}{\beta} \geq 0$, $o(x) = a - b = \hat{a}_2^1 = \frac{\Gamma}{\beta}a_2^1$, $o'_x = 0$. As $o'_x = 0$, it follows that

$$\frac{D'_x}{1 + \tilde{l}_1} - F'_x D = 0.$$

Therefore, the sign of $dcorr(\Delta e_t^2, \Delta e_t^3)/dx$ is equal to the sign of $2(1+c)vv'_x + vp'_x + pv'_x$, which is

given by

$$\begin{aligned}
(1+c)vv'_x + vp'_x + pv'_x &= \frac{p'_x}{c^2} \frac{1}{(p^2 + 2(1+c)p + (c-1)^2)^{0.5}} \\
&* [(2p+2+c+c^2)(p^2 + 2(1+c)p + (c-1)^2)^{0.5} + p+p^2+1+c+(1+c)p - pc^2 - c^2 - c^3 \\
&+ p^2 + 2(1+c)p + (c-1)^2] \\
&> \frac{p'_x}{c^2} \frac{1}{(p^2 + 2(1+c)p + (c-1)^2)^{0.5}} \\
&* [(1+c)p - pc^2] + (2p+2+c+c^2)(c-1) + 1+c - c^2 - c^3 + c^2 - 2c + 1 = p[1+c-c^2],
\end{aligned}$$

where I use that $(p^2+2(1+c)p+(c-1)^2)^{0.5} > (c-1)$. Lastly, $[1+c-c^2] > 0$ for $c < 1.6$ or for $\beta > 0.65$, which is not a restrictive assumption for the discount factor. Therefore, $dcorr(\Delta e_t^2, \Delta e_t^3)/dx > 0$.

Dominant currency pricing. The equilibrium interest rate can be found from the combination of the monetary policy and the Euler equation $i_t^h = \mathbb{E}_t\{\sigma\Delta c_{t+1}^h + \Delta p_{t+1}^h\} = \mathbb{E}_t\{\Delta p_{t+1}^h\}$. With DCP, the cross-country bonds holdings are equal to $b_{t+1}^2 = \frac{\beta}{\Gamma^f}[(1-a_2^1-a_3^2)(\mathbb{E}_t\{e_{t+1}^2\} - e_t^2)]$ and $b_{t+1}^3 = \frac{\beta}{\Gamma^f}[(1-a_2^1-a_3^2)(\mathbb{E}_t\{e_{t+1}^3\} - e_t^3)]$. Therefore, in the equilibrium system can be written as

$$\begin{aligned}
(a_2^1 + a_3^2)a_2^2\theta e_t^2 - a_3^2a_2^2\theta e_t^3 - z_t^2 &= \frac{\beta}{\Gamma^f}a_2^2(\mathbb{E}_t\{e_{t+1}^2\} - e_t^2) - \frac{1}{\beta}\frac{\beta}{\Gamma^f}a_2^2(\mathbb{E}_{t-1}\{e_t^2\} - e_{t-1}^2), \\
(a_2^1 + a_3^2)a_2^2\theta e_t^3 - a_3^2a_2^2\theta e_t^2 - z_t^3 &= \frac{\beta}{\Gamma^f}a_2^2(\mathbb{E}_t\{e_{t+1}^3\} - e_t^3) - \frac{1}{\beta}\frac{\beta}{\Gamma^f}a_2^2(\mathbb{E}_{t-1}\{e_t^3\} - e_{t-1}^3).
\end{aligned}$$

This system can be rewritten as (14) with $\hat{b}_t^2 = \frac{\Gamma^f}{\beta}\frac{1}{a_2^2}b_t^2$, $\hat{b}_t^3 = \frac{\Gamma^f}{\beta}\frac{1}{a_2^2}b_t^3$, $\hat{a}_2^1 = \frac{\Gamma^f}{\beta}\theta a_2^1$, $\hat{a}_3^2 = \frac{\Gamma^f}{\beta}\theta a_3^2$, $\hat{z}_t^2 = -\frac{\Gamma^f}{\beta}\frac{1}{a_2^2}z_t^2$, $\hat{z}_t^3 = -\frac{\Gamma^f}{\beta}\frac{1}{a_2^2}z_t^3$. Therefore, we define $a \equiv \hat{a}_2^1 + \hat{a}_3^2$ and $b \equiv \hat{a}_3^2$.

Given the general solution above, $p(x) = a + b = \hat{a}_2^1 + \hat{a}_3^2 + \hat{a}_3^2 = \hat{a}_2^1 + 2\hat{a}_3^2 = \frac{\Gamma^f}{\beta}\theta(a_2^1 + 2a_3^2) = \frac{\Gamma^f}{\beta}\theta(a_2^1 + 2x) \geq 0$, $p'_x = 2\frac{\Gamma^f}{\beta}\theta \geq 0$, $o(x) = a - b = \hat{a}_2^1 = \frac{\Gamma^f}{\beta}a_2^1$, $o'_x = 0$. Then again $D'_x/(1+\tilde{l}_1) - F'_xD = 0$. The rest of the proof is the same as for the LCP case. ■

A.5 Quantitative model. The model setup, the steady state and the log-linearized equilibrium

A.5.1 The model setup

I build a N -country model. The world is populated with a continuum of agents of unit mass. The population in the segment $\Omega_h = [\sum_{i=0}^{h-1} n_i, \sum_{i=1}^h n_i + n_h]$ belongs to country $h \in \{1, \dots, N\}$, where n_h is the population size of a country, $n_0 = 0$ and $\sum_{i=1}^{N-1} n_i + n_N = 1$. n_h defines not only population size of the country, but also determines the relative economic size of the country. The first country is a dominant-currency country - the U.S. The N th country is the rest of the world.

A typical household $j \in \Omega_h$ in country h chooses individual levels of total consumption C_t^h , bond holdings B_{t+1}^h and labor supply L_t^h , etc. Then the country level of total consumption is $n_h C_t^h$, of bond holdings is $n_h B_{t+1}^h$, and of labor supply is $n_h L_t^h$.

Households. A household utility is the same as in the baseline model and the total consumption is defined as

$$C_t^h \equiv \left[(1 - a^h)^{\frac{1}{\theta}} (C_{ht}^h)^{\frac{\theta-1}{\theta}} + (a^h)^{\frac{1}{\theta}} (C_{Ft}^h)^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}},$$

where $1 - a^h$ is a parameter of home bias in consumption preferences. An index of imported goods, $C_{F,t}^h$, is defined as

$$C_{F,t}^h \equiv \left[\sum_{i=1, i \neq h}^N (a_i^h)^{\frac{1}{\gamma}} (C_{it}^h)^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}},$$

where preference parameters a_i^h satisfy the condition $\sum_{i \neq h} a_i^h = 1$, $\theta > 0$ is the parameter of the elasticity of substitution between domestic and foreign goods, and $\gamma > 0$ is the elasticity of substitution between domestically produced goods. In each country there is a continuum of firms each producing a specific variety of goods indexed by $\omega \in \Omega_h$. An index of goods produced in country $i \in \{1, \dots, N\}$ and consumed by a household in country h is denoted as $C_{i,t}^h$ and given by the CES function:

$$C_{it}^h \equiv \left(\left(\frac{1}{n_i} \right)^{1/\phi} \int_{j \in \Omega_i} (C_{it}^h(\omega))^{\frac{\phi-1}{\phi}} d\omega \right)^{\frac{\phi}{\phi-1}},$$

where ϕ is the parameter of substitution between goods imported from the same destination.

The optimal demand is given by

$$C_{ht}^h(\omega) = \frac{1}{n_h} \left(\frac{P_{ht}^h(\omega)}{P_{ht}^h} \right)^{-\phi} C_{ht}^h, \quad C_{it}^h(\omega) = \frac{1}{n_i} \left(\frac{P_{it}^h(\omega)}{P_{it}^h} \right)^{-\phi} C_{it}^h,$$

where P_{ht}^h is the domestic price index in country h (i.e., an index of prices of domestically produced goods), P_{it}^h is a price index for goods imported from country i (expressed in currency of country h). Furthermore, the optimal allocation of expenditures between domestic and imported goods is given by

$$C_{ht}^h = (1 - a^h) \left(\frac{P_{ht}^h}{P_t^h} \right)^{-\theta} C_t^h, \quad C_{it}^h = a^h a_i^h \left(\frac{P_{Ft}^h}{P_t^h} \right)^{-\theta} \left(\frac{P_{it}^h}{P_{Ft}^h} \right)^{-\gamma} C_t^h.$$

where P_{Ft}^h is the price index of imported goods, and P_t^h is the consumer price index (CPI).

Budget constraint and Euler equations. The household can trade only bonds denominated in domestic currency. I do not impose the cash-in-advance constraint and households in country h make optimal consumption/saving decision subject to the budget constraint:

$$B_{t+1}^h + P_t^h C_t^h = W_t^h L_t^h + B_t^h R_{t-1}^h + \frac{\Pi_t^h}{n_h} + \frac{Pr_t^h}{n_h} + T_t^h,$$

where Π_t^h/n_h is the share of the total profit paid to the individual household by the domestic goods producers, Pr_t^h/n_h is the share of the total profit paid by the financial intermediaries, and T_t^h is the transfer from the government: The optimal behavior of the households is summarized by the Euler

equation

$$\frac{1}{R_t^h} = \beta \mathbb{E}_t \left\{ \left(\frac{C_{t+1}^h}{C_t^h} \right)^{-\sigma} \frac{P_{t+1}^h}{P_t^h} \right\},$$

and the optimal labor supply condition

$$\kappa (C_t^h)^\sigma (L_t^h)^{1/\nu} = \frac{W_t^h}{P_t^h}.$$

Financial Intermediaries. International exchange of bonds denominated in different currencies is intermediated by the financial intermediaries. In section 3.1, the financial intermediaries trade bonds based on the relative return of the Canadian dollar/the Japanese yen-denominated bonds against the U.S. dollar-denominated. Here, I treat all bonds symmetrically and consider the structure of the financial market where for every possible pair of bonds $\{i, j\}$, $i \in \{1, \dots, N\}$, $j \in \{i + 1, \dots, N\}$, there exist financial intermediaries that trade that the bonds denominated in currency i against the bonds denominated in currency j . In N -country world, there exist $N(N - 1)/2$ of such markets. The financial intermediaries again consist of the strategical financiers and noise-bond traders.

Financiers. The portfolio of the financiers that trade on the $\{i, j\}$ market consists of $Q_{i,t+1}^j$ of bonds denominated in currency i and $-Q_{i,t+1}^j/\mathcal{E}_{it}^j$ of bonds denominated in currency j . The risk-neutral but constrained financial intermediaries solve the same optimization problem as one represented in the section 3.1. Their demand for j -currency-denominated bonds is

$$-\frac{Q_{i,t+1}^j}{\mathcal{E}_{it}^j} = \frac{\beta}{\Gamma^h} \left[\frac{\mathbb{E}_t\{\mathcal{E}_{i,t+1}^j\}}{\mathcal{E}_{it}^j} R_t^j - R_t^i \right].$$

Profits of financiers that trade on the $\{i, j\}$ market are assumed to be paid to the households in country i

$$Pr_{it}^j = \frac{Q_{it}^j}{\mathcal{E}_{it}^j} \left[R_{t-1}^i - R_{t-1}^j \frac{\mathcal{E}_{it}^j}{\mathcal{E}_{it-1}^j} \right].$$

Noise Traders. Additionally to the financiers there exist noise traders whose portfolio choice does not depend on fundamentals. On the $\{i, j\}$ market noise traders demand for Z_{it+1}^j of i -currency-denominated bonds and $-Z_{it+1}^j/\mathcal{E}_{it}^j$ of j -currency-denominated bonds. As for the strategic financial intermediaries, it is assumed that noise traders, that trade the $\{i, j\}$ pair, pay their profits/losses to the household in country i :

$$\tilde{P}r_{jt}^j = \frac{Z_{it}^j}{\mathcal{E}_{it}^j} \left[R_{t-1}^i - R_{t-1}^j \frac{\mathcal{E}_{it}^j}{\mathcal{E}_{it-1}^j} \right].$$

Therefore, the total profit paid to the country h 's households is given by

$$Pr_t^h = \sum_{k=1}^{k < h} Pr_{kt}^h + \sum_{k=1}^{k < h} \tilde{P}r_{kt}^h.$$

Technology. A typical firm in country h produces a differentiated good with the linear technology

represented by the product function:

$$Y_t^h(j) = A_t^h L_t^h(j)$$

where A_t^h is a country-specific stochastic productivity. The nominal marginal cost is common across domestic firms and given by

$$NMC_t^h = \frac{W_t^h}{A_t^h}.$$

Let $Y_t^h \equiv \left[\left(\frac{1}{n_h} \right)^{1/\phi} \int_{j \in \Omega_h} (Y_t^h(j))^{\frac{\phi-1}{\phi}} dj \right]^{\frac{\phi}{\phi-1}}$ define an index for aggregate domestic output, analogous to the one introduced for consumption.

Price-setting decisions. Prices follow a partial adjustment rule as in Calvo (1983). Producers of differentiated goods know the form of their individual demand functions and maximize profits taking overall market prices as given. In each period a fraction $\lambda \in [0; 1)$ of randomly chosen producers is not allowed to change the nominal price. The remaining fraction of firms, $1 - \lambda$, chooses prices optimally by maximizing the expected discounted value of profits. When a firm gets a chance to reset its price it does it for all its destination markets.²⁰ An individual domestic firm produces a unique variety ω .

Let me consider prices of export from country h to country i . A fraction $\Omega_h^{i,p}$ of exporters from h to i sets prices in the producer currency, a fraction $\Omega_h^{i,l}$ sets prices in the consumer currency, and a fraction $\Omega_h^{i,d}$ sets prices in the U.S dollars. Without loss of generality I assume that a firm that produces variety $\omega \in \Omega_h^{i,p} \equiv [\sum_{i=0}^{h-1} n_i, \sum_{i=0}^{h-1} n_i + \theta_h^{i,p}]$ sets price in the producer currency. I denote the price that households in i are charged for this variety as $P_{ht}^{i,p}(\omega)$ (*expressed in h-currency*). Next a firm that produces variety $\omega \in \Omega_h^{i,l} \equiv [\sum_{i=0}^{h-1} n_i + \theta_h^{i,p}, \sum_{i=0}^{h-1} n_i + \theta_h^{i,p} + \theta_h^{i,l}]$ sets the price in the local currency $P_{ht}^{i,l}(\omega)$ (*expressed in i-currency*). Finally, a firm that produces variety $\omega \in \Omega_h^{i,d} \equiv [\sum_{i=0}^h n_i - \theta_h^{i,d}, \sum_{i=0}^h n_i]$ sets the price in the dominant currency $P_{ht}^{i,d}(\omega)$ (*expressed in base-currency*). Then the price index of goods imported from country h to country i can be written as

$$\begin{aligned} (P_{ht}^i)^{1-\phi} &= \frac{1}{n_h} \int_{\omega \in \Omega_h} (P_{ht}^i(\omega))^{1-\phi} d\omega \\ &= \frac{1}{n_h} \int_{\omega \in \Omega_h^{i,p}} (P_{ht}^{i,p}(\omega)/\mathcal{E}_{ht}^i)^{1-\phi} d\omega + \frac{1}{n_h} \int_{\omega \in \Omega_h^{i,l}} (P_{ht}^{i,l}(\omega))^{1-\phi} d\omega + \frac{1}{n_h} \int_{\omega \in \Omega_h^{i,d}} (P_{ht}^{i,d}(\omega)/\mathcal{E}_{1t}^i)^{1-\phi} d\omega. \end{aligned}$$

It is convenient to study the evolution of $P_{ht}^{i,p}(\omega)$, $P_{ht}^{i,l}(\omega)$, and $P_{ht}^{i,d}(\omega)$ separately.

Producer currency pricing. A firm, that is allowed to reset prices in period t , chooses the set of prices $\{\bar{P}_{ht}^{i,p}\}_{i=1}^N$ that maximizes the current market value of the profits generated while that remains effective. Prices are set in the producer currency and the price that the household in country i is charged at time $t+k$ is $\bar{P}_{ht}^{i,p}/\mathcal{E}_{ht+k}^i$. Formally, the firm solves the problem of maximization of discounted sum of profits,

²⁰There are two possible specifications. First, there is a unique firm that produces the variety j in country h and sells it at home and in all foreign countries. Second, there is a single firm that produces a variety j in country h and sells it either to domestic household or to one of the foreign countries. As the nominal marginal costs do not depend on Y_t^h , both specifications are the same.

summarized by

$$\max_{\{\bar{P}_{it}^{h,p}\}_{i=1}^N} \sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h \left(\sum_{i=1}^N \bar{P}_{ht}^{i,p} Y_{ht+k|t}^{i,p} - \Psi \left(\sum_{i=1}^N Y_{ht+k|t}^{i,p} \right) \right) \right\}$$

where $\Psi \left(\sum_{i=1}^N Y_{ht+k|t}^{i,p} \right) = \frac{W_{t+k}^h}{e^{a_{t+k}^h}} \left(\sum_{i=1}^N Y_{ht+k|t}^{i,p} \right)$ is the total cost function, subject to the sequence of demand constraints

$$Y_{ht+k|t}^{h,p} = (1 - a^h) \left(\frac{\bar{P}_{ht}^{h,p}}{P_{ht+k}^h} \right)^{-\phi} \left(\frac{P_{ht+k}^h}{P_{t+k}^h} \right)^{-\theta} C_{t+k}^h,$$

$$Y_{ht+k|t}^{i,p} = a^i a_h^i \left(\frac{\bar{P}_{ht}^{i,p}}{\mathcal{E}_{ht+k}^i P_{ht+k}^i} \right)^{-\phi} \left(\frac{P_{ht+k}^i}{P_{Ft+k}^i} \right)^{-\gamma} \left(\frac{P_{Ft+k}^i}{P_{t+k}^i} \right)^{-\theta} C_{t+k}^i.$$

for $k = 0, 1, 2, \dots$, where $q_{t,t+k}^h \equiv \beta^k (C_{t+k}^h / C_t^h)^{-\sigma} (P_t^h / P_{t+k}^h)$ is the stochastic discount factor for nominal payoffs, and $Y_{it+k|t}^{h,p}$ denotes output sold in country i in period $t+k$ for a firm that last reset its price in period t in the domestic currency.

The first-order condition associated with the problem above takes the form

$$\sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h Y_{ht+k|t}^{i,p} (\bar{P}_{it}^{h,p} - \mathcal{M} \psi_{t+k|t}) \right\} = 0 \text{ for all } i \in \{1, \dots, N\},$$

where $\psi_{t+k|t} \equiv \Psi'_{t+k}(Y_{it+k|t}^{h,p}) = W_{t+k}^h / e^{a_{t+k}^h}$ denotes the (nominal) marginal cost in period $t+k$ for a firm which last reset its price in period t and $\mathcal{M} = \frac{\phi}{\phi-1}$.

Local currency pricing. Next, let me consider a problem of firms that set prices in the consumer currency. A firm, that is allowed to re-set prices in period t , chooses the set of prices $\{\bar{P}_{ht}^{i,l}\}_{i=1}^N$ (expressed in currency i) that maximizes the current market value of profits:

$$\max_{\{\bar{P}_{ht}^{i,l}\}_{i=1}^N} \sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h \left(\sum_{i=1}^N \mathcal{E}_{ht+k}^i \bar{P}_{ht}^{i,l} Y_{ht+k|t}^{i,l} - \Psi \left(\sum_{i=1}^N Y_{ht+k|t}^{i,l} \right) \right) \right\}$$

subject to the demand constraints

$$Y_{ht+k|t}^{i,l} = a_h^i a^i \left(\frac{\bar{P}_{ht}^{i,l}}{P_{ht+k}^i} \right)^{-\phi} \left(\frac{P_{ht+k}^i}{P_{Ft+k}^i} \right)^{-\gamma} \left(\frac{P_{Ft+k}^i}{P_{t+k}^i} \right)^{-\theta} C_{t+k}^i,$$

$$Y_{ht+k|t}^{h,l} = (1 - a^h) \left(\frac{\bar{P}_{ht}^{h,l}}{P_{ht+k}^h} \right)^{-\phi} \left(\frac{P_{ht+k}^h}{P_{t+k}^h} \right)^{-\theta} C_{t+k}^h.$$

The first order condition with respect to $\bar{P}_{ht}^{i,l}$ is

$$\sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h Y_{ht+k|t}^{i,l} (\mathcal{E}_{h,t+k}^i \bar{P}_{ht}^{i,l} - \mathcal{M} \psi_{t+k|t}) \right\} = 0.$$

Dominant currency pricing. Lastly, I consider a problem of a firm that set prices in the U.S. dollars. The firm sets prices in U.S. dollars for foreign household and in the domestic currency for domestic households. Therefore, the producer in country h chooses an optimal price for domestic consumers $\bar{P}_{ht}^{h,d}$, and the set of prices set in base currency for all foreign destinations $\{\bar{P}_{ht}^{i,d}\}_{i=1, i \neq h}^N$

$$\max_{\{\bar{P}_{ht}^{i,d}\}_{i=1, i \neq h}^N, \bar{P}_{ht}^{h,d}} \sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h \left(\mathcal{E}_{ht+k}^1 \sum_{\substack{i=1, \\ i \neq h}}^N \bar{P}_{ht}^{i,d} Y_{ht+k|t}^{i,d} + \bar{P}_{ht}^{h,d} Y_{ht+k|t}^{h,d} - \Psi \left(\sum_{i=1}^N Y_{ht+k|t}^{i,d} \right) \right) \right\}$$

subject to demand constraint

$$Y_{ht+k|t}^{i,d} = a_h^i a^i \left(\frac{\bar{P}_{ht}^{i,d}}{P_{ht+k}^i \mathcal{E}_{1t+k}^i} \right)^{-\phi} \left(\frac{P_{ht+k}^i}{P_{Ft+k}^i} \right)^{-\gamma} \left(\frac{P_{Ft+k}^i}{P_{t+k}^i} \right)^{-\theta} C_{t+k}^i$$

$$Y_{ht+k|t}^{h,d} = (1 - a^h) \left(\frac{\bar{P}_{ht}^{h,d}}{P_{ht+k}^h} \right)^{-\phi} \left(\frac{P_{ht+k}^h}{P_{t+k}^h} \right)^{-\theta} C_{t+k}^h.$$

The first order condition with respect to $\bar{P}_{ht}^{i,d}$ is

$$\sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h \sum_{\substack{i=1, \\ i \neq h}}^N Y_{ht+k|t}^{i,d} (\mathcal{E}_{h,t+k}^1 \bar{P}_{ht}^{i,d} - \mathcal{M}\psi_{t+k|t}) \right\} = 0.$$

Government. The government budget constraint is

$$R_{t-1}^h \hat{B}_t^h + n_h T_t^h = \hat{B}_{t+1}^h,$$

\hat{B}_t^h is government supply of bonds denominated in the local currency.

Monetary Policy. The monetary authority sets the nominal interest rate according to the following Taylor rule, written in the log-linearized form

$$i_t^h = \rho_i^h i_{t-1}^h + (1 - \rho_i^h) (\alpha_\pi^h \mathbb{E}_t \{\pi_{t+1}^h\} + \alpha_{\Delta y}^h (y_t^h - y_{t-1}^h) + \alpha_{\Delta \pi}^h (\pi_t^h - \pi_{t-1}^h)) + \epsilon_{m,t}^h,$$

where $i_t^h = \log(R_t^h) - \log(R)$, ρ_i^h , α_π , $\alpha_{\Delta \pi}$, and $\alpha_{\Delta y}$, are the monetary policy parameters, π_t^h is the CPI inflation rate, $y_t^h - y_{t-1}^h$ is the country h 's output growth and $\epsilon_{m,t}^h$ is a country-specific monetary policy shock.

Equilibrium conditions. The model is closed by the market equilibrium conditions. *Bond market equilibrium conditions* are

$$n_h B_{t+1}^h + \sum_{k>h}^N Q_{ht+1}^k - \sum_{k=1}^{k<h} \frac{Q_{kt+1}^h}{\mathcal{E}_{kt}^h} + \sum_{k>h}^N Z_{ht+1}^k - \sum_{k=1}^{k<h} \frac{Z_{kt+1}^h}{\mathcal{E}_{kt}^h} = \hat{B}_{t+1}^h.$$

Consumption goods market equilibrium conditions are:

$$\begin{aligned}
Y_t^h(j) &= \sum_{i=1}^N n_i C_{ht}^i(j) \\
&= (1 - \alpha^h) \left(\frac{P_{ht}^h(j)}{P_{ht}^h} \right)^{-\phi} \left(\frac{P_{ht}^h}{P_t^h} \right)^{-\theta} C_t^h + \sum_{i=1, i \neq h}^N \alpha^i \alpha_h^i \frac{n_i}{n_h} \left(\frac{P_{ht}^i(j)}{P_{ht}^i} \right)^{-\phi} \left(\frac{P_{ht}^i}{P_{Ft}^i} \right)^{-\gamma} \left(\frac{P_{Ft}^i}{P_t^i} \right)^{-\theta} C_t^i \\
&= \left(\frac{P_{ht}^h(j)}{P_{ht}^h} \right)^{-\phi} C_{ht}^h + \sum_{i=1, i \neq h}^N \frac{n_i}{n_h} \left(\frac{P_{ht}^i(j)}{P_{ht}^i} \right)^{-\phi} C_{ht}^i
\end{aligned}$$

The labor market equilibrium condition is

$$\int_{j \in \Omega_h} L^h(j) dj = n^h L^h$$

The country h budget constraint for $h \in \{2, \dots, N\}$ is

$$(n_h B_{t+1}^h - \hat{B}_{t+1}^h) - (n_h B_t^h - \hat{B}_t^h) R_{t-1}^h = \sum_{\substack{i=1 \\ i \neq h}}^N n_i \mathcal{E}_{ht}^i P_{ht}^i C_{ht}^i - \sum_{\substack{i=1 \\ i \neq h}}^N n_h P_{it}^h C_{it}^h + Pr_t^h.$$

A.5.2 The steady state

I consider the zero inflation, zero trade imbalances steady state, where $\mathcal{E}_h^i = 1$, $B^h = 0$, $N_j^i = 0$, $n_h C^h = Y^h$, for any $h, i, j \in \{1, \dots, N\}$. Each firm sets the same price for all destinations, $P_h^h(j) = P_h^i(j) = \frac{\phi}{\phi-1} \frac{W^h}{A^h}$ for any $j \in \Omega_h$ and $i \in \{1, \dots, N\}$. More over, I assume that PPP holds at the steady state, so $P_h^i(j) = P_h^h(j) = P_h^h = P_h^i = P_F^h = P^h = P$ for any $i, h \in \{1, \dots, N\}$. Therefore, it should be that $W^h/A^h = W/A$ for all $h \in \{1, \dots, N\}$. Then $C_h^h = (1 - \alpha^h) C^h$, $C_F^h = \alpha^h C^h$, $C_i^h = \alpha_i^h C_F^h = \alpha^h \alpha_i^h C^h$, $C_h^h(j) = \frac{1}{n^h} C_h^h$, $C_i^h(j) = \frac{1}{n^i} C_i^h$. From the Euler equation it follows that $R = 1/\beta$ for all $h \in \{1, \dots, N\}$. All households supply the same amount of labor and all firms produce the same amount of goods, as $L^h = N^h$. The labor supply condition becomes

$$\kappa (C^h)^\sigma (N^h)^{1/\nu} = \frac{W^h}{P} = \frac{\phi-1}{\phi} A^h$$

As all firms set the same prices at the steady state $Y^h(j) = \sum_{i=1}^N \frac{n_i}{n_h} C_h^i$ for all $j \in \Omega_h$ and $Y^h = n^h Y^h(j)$. From the production function it follows that $L^h = Y^h(j)/A^h = Y^h/A^h n^h$, plugging this into the labor supply equation yields

$$\kappa (C^h)^\sigma (Y^h/n^h)^{1/\nu} = \frac{\phi-1}{\phi} (A^h)^{1+\nu}.$$

It is assumed that $(A^h)^{1+1/\nu} = A^{1+1/\nu}$, so the difference in countries' economic sizes comes from different size of population. The steady state trade imbalances are equal to zero

$$\sum_{i \neq h} \alpha^h \alpha_i^h n_h P C^h = \sum_{i \neq h} \alpha^i \alpha_h^i n_i P C^i.$$

A.5.3 The log-linearize equilibrium

- Total consumption index:

$$c_t^h = (1 - \alpha^h) c_{ht}^h + \alpha^h c_{Ft}^h.$$

- The index of imported goods:

$$c_{Ft}^h = \sum_{i=1, i \neq h}^N \alpha_i^h c_{it}^h.$$

- The demand function:

$$c_{it}^h = -\gamma(p_{it}^h - p_{Ft}^h) + c_{Ft}^h.$$

- The imported goods price index:

$$p_{Ft}^h = \sum_{\substack{i=n, \\ i \neq h}}^n \alpha_i^h p_{it}^h.$$

- The demand functions:

$$c_{ht}^h = -\theta(p_{ht}^h - p_t^h) + c_t^h, \quad c_{Ft}^h = -\theta(p_{Ft}^h - p_t^h) + c_t^h.$$

- The total price index:

$$p_t^h = (1 - \alpha^h) p_{ht}^h + \alpha^h p_{Ft}^h.$$

- The Euler equation:

$$i_t^h = \sigma(c_{t+1}^h - c_t^h) + (p_{t+1}^h - p_t^h).$$

- The optimal labor supply equation:

$$\sigma c_t^h + 1/\nu n_t^h = w_t^h - p_t^h.$$

- The demand for bonds:

$$-q_{it+1}^j = \frac{1}{\Gamma^h} \left[\mathbb{E}_t \{ e_{it+1}^j \} - e_{it}^j + i_t^j - i_t^i \right].$$

where $q_{it+1}^j = Q_{it+1}^j$, $i_t^j - i_t^i = \log(R_t^i) - \log(R_t^j)$, and $e_{it}^j = e_{1t}^j - e_{it}^i$.

- The country h budget constraint combined with the bond market equilibrium conditions yields

$$\begin{aligned}
& \left(\sum_{k=1}^{k < h} \frac{Q_{kt+1}^h}{\mathcal{E}_{kt}^h} - \sum_{k > h}^N Q_{ht+1}^k + \sum_{k=1}^{k < h} \frac{N_{kt}^h}{\mathcal{E}_{kt}^h} - \sum_{k > h}^N N_{ht}^k \right) \\
& - \left(\sum_{k=1}^{k < h} \frac{Q_{kt}^h}{\mathcal{E}_{kt-1}^h} - \sum_{k > h}^N Q_{ht}^k + \sum_{k=1}^{k < h} \frac{N_{kt-1}^h}{\mathcal{E}_{kt-1}^h} - \sum_{k > h}^N N_{ht-1}^k \right) R_{t-1}^h \\
& = \sum_{\substack{i=1 \\ i \neq h}}^N n_i \mathcal{E}_{ht}^i P_{ht}^i C_{ht}^i - \sum_{\substack{i=1 \\ i \neq h}}^N n_h P_{it}^h C_{it}^h + Pr_t^h.
\end{aligned}$$

This condition is written in terms of h currency. It can be written in terms of dominant currency (by multiplying the whole equation by \mathcal{E}_{1t}^h), such that it is easier to solve for dominant currency exchange rates as

$$\begin{aligned}
& \left(\sum_{k=1}^{k < h} Q_{kt+1}^h \mathcal{E}_{1t}^k - \sum_{k > h}^N Q_{ht+1}^k \mathcal{E}_{1t}^h + \sum_{k=1}^{k < h} N_{kt}^h \mathcal{E}_{1t}^k - \sum_{k > h}^N N_{ht}^k \mathcal{E}_{1t}^h \right) \\
& - \left(\sum_{k=1}^{k < h} Q_{kt}^h \mathcal{E}_{1t-1}^k - \sum_{k > h}^N Q_{ht}^k \mathcal{E}_{1t-1}^h + \sum_{k=1}^{k < h} N_{kt-1}^h \mathcal{E}_{1t-1}^k - \sum_{k > h}^N N_{ht-1}^k \mathcal{E}_{1t-1}^h \right) R_{t-1}^h \\
& = \sum_{\substack{i=1 \\ i \neq h}}^N n_i \mathcal{E}_{1t}^i P_{ht}^i C_{ht}^i - \sum_{\substack{i=1 \\ i \neq h}}^N n_h \mathcal{E}_{1t}^h P_{it}^h C_{it}^h + Pr_t^h \mathcal{E}_{1t}^h.
\end{aligned}$$

The log-linearized equation is

$$\begin{aligned}
& \sum_{\substack{i=1, \\ i \neq h}}^N \alpha^h \alpha_i^h \mathcal{E}^h P_i^h n_h C^h [e_{1t}^h + p_{it}^h + c_{it}^h] - \sum_{\substack{i=1, \\ i \neq h}}^N \alpha^i \alpha_h^i \mathcal{E}^i P_h^i n_i C^i [e_{1t}^i + p_{ht}^i + c_{ht}^i] \\
& = \sum_{k=1}^{k < h} \left(q_{kt+1}^h - \frac{1}{\beta} q_{kt}^h \right) - \sum_{k > h}^N \left(q_{ht+1}^k - \frac{1}{\beta} q_{ht}^k \right) + \sum_{k=1}^{k < h} \left(n_{kt}^h - \frac{1}{\beta} n_{kt-1}^h \right) - \sum_{k > h}^N \left(n_{ht}^k - \frac{1}{\beta} n_{ht-1}^k \right)
\end{aligned}$$

where $n_{kt}^h = N_{kt}^h$. Assuming that $n_{kt}^h = 1/\beta n_{kt-1}^h + z_{kt}^h$ one obtains

$$\begin{aligned}
& \sum_{\substack{i=1, \\ i \neq h}}^N \alpha^h \alpha_i^h \mathcal{E}^h P^h Y_h [e_{1t}^h + p_{it}^h + c_{it}^h] - \sum_{\substack{i=1, \\ i \neq h}}^N \alpha^i \alpha_h^i \mathcal{E}^i P^i Y^i [e_{1t}^i + p_{ht}^i + c_{ht}^i] \\
& = \sum_{k=1}^{k < h} \left(q_{kt+1}^h - \frac{1}{\beta} q_{kt}^h \right) - \sum_{k > h}^N \left(q_{ht+1}^k - \frac{1}{\beta} q_{ht}^k \right) + \sum_{k=1}^{k < h} z_{kt}^h - \sum_{k > h}^N z_{ht}^k
\end{aligned}$$

where, the fact that $n_h C^h = Y^h$ was used. Next, the equation is divided by the nominal GDP

expressed in USD of N th country that represents the rest of the world:

$$\begin{aligned} & \sum_{\substack{i=1, \\ i \neq h}}^N \alpha^h \alpha^i \underbrace{\frac{\mathcal{E}_1^h P^h Y^h}{\mathcal{E}_1^N P^N Y^N}}_{=n_h/n_N} [e_{1t}^h + p_{it}^h + c_{it}^h] - \sum_{\substack{i=1, \\ i \neq h}}^N \alpha^i \alpha_h^i \underbrace{\frac{\mathcal{E}^i P_h^i Y^i}{\mathcal{E}^N P^N Y^N}}_{=n_i/n_N} [e_{1t}^i + p_{ht}^i + c_{ht}^i] \\ &= \sum_{k=1}^{k < h} \left(\tilde{q}_{kt+1}^h - \frac{1}{\beta} \tilde{q}_{kt}^h \right) - \sum_{k > h}^N \left(\tilde{q}_{ht+1}^k - \frac{1}{\beta} \tilde{q}_{ht}^k \right) + \sum_{k=1}^{k < h} \tilde{z}_{kt}^h - \sum_{k > h}^N \tilde{z}_{ht}^k \end{aligned}$$

where $\tilde{q}_{kt}^h \equiv q_{kt}^h / (\mathcal{E}_1^N P^N Y^N)$, $\tilde{z}_{kt}^h \equiv z_{kt}^h / (\mathcal{E}_1^N P^N Y^N)$.

- Individual variety production function:

$$y_t^h(j) = a_t^h + l_t^h(j)$$

- The total labor supply is equal to the total labor demand:

$$n_t^h = -a_t^h + \frac{1}{n_h} \int_{j \in \Omega_h} y_t^h(j) = -a_t^h + y_t^h.$$

- The total output:

$$y_t^h = \frac{1}{n_h} \int_{j \in \Omega_h} y_t^h(j) dj.$$

- The consumption good equilibrium condition:

$$y_t^h(j) = \sum_{i=1}^N \frac{n^i C_h^i}{n^h Y^h(j)} [-\phi(p_{ht}^i(j) - p_{ht}^i) + c_{ht}^i].$$

- The price index is:

$$p_{ht}^i = \frac{1}{n_h} \int_{\Omega_h} p_{ht}^i(j) dj.$$

Then

$$y_t^h = \frac{1}{n_h} \int_{\Omega_h} y_t^h(j) dj = \sum_{i=1}^N \frac{n_i C_h^i}{n^h Y^h(j)} c_{ht}^i = \sum_{i=1}^N \frac{n_i C_h^i}{Y^h} c_{ht}^i.$$

Notice that with PCP $Y_t^h = \sum_{i=1}^N n_i C_{ht}^i$. The equation above can be rewritten as

$$\begin{aligned} y_t^h &= \sum_{i=1}^N \frac{n_i C_h^i}{Y^h} c_{ht}^i \\ &= \sum_{\substack{i=1, \\ i \neq h}}^N \underbrace{\frac{\mathcal{E}^i P_h^i n_i C_h^i}{\mathcal{E}^i P_F^i n_i C_F^i}}_{=\alpha_h^i} \underbrace{\frac{\mathcal{E}^i P_F^i n_i C_F^i}{\mathcal{E}^i P^i n_i C^i}}_{=\alpha^i} \underbrace{\frac{\mathcal{E}^i P^i Y^i}{\mathcal{E}^{rw} P^{rw} Y^{rw}}}_{=n_i/n_N} \underbrace{\frac{\mathcal{E}^{rw} P^{rw} Y^{rw}}{\mathcal{E}^h P^h Y^h}}_{=1/n_h/n_N} c_{ht}^i + \underbrace{\frac{\mathcal{E}^h P_h^h n_h C_h^h}{\mathcal{E}^h P^h n_h C^h}}_{1-\alpha^h} c_{ht}^h \\ &= \sum_{\substack{i=1 \\ i \neq h}}^N \alpha_h^i \alpha^i \frac{n_i}{n_h} c_{ht}^i + (1 - \alpha^h) c_{ht}^h. \end{aligned}$$

- Price dynamics can be summarized by the definition of the price index of goods exported from h to i

$$\begin{aligned} (P_{ht}^i)^{1-\phi} &= \frac{1}{n_h} \int_{j \in \Omega_h} (P_{ht}^i(j))^{1-\phi} dj \\ &= \frac{1}{n_h} \int_{j \in \Omega_h^{i,p}} (P_{ht}^{i,p}(j)/\mathcal{E}_{ht}^i)^{1-\phi} dj + \frac{1}{n_h} \int_{j \in \Omega_h^{i,l}} (P_{ht}^{i,l}(j))^{1-\phi} dj + \frac{1}{n_h} \int_{j \in \Omega_h^{i,d}} (P_{ht}^{i,d}(j)/\mathcal{E}_{1t}^i)^{1-\phi} dj \end{aligned}$$

where three price indexes are defined as

$$\begin{aligned} P_{ht}^{i,p} &\equiv \left(\frac{1}{n_h} \int_{j \in \Omega_h^{i,p}} (P_{ht}^{i,p}(j))^{1-\phi} dj \right)^{\frac{1}{1-\phi}}, & P_{ht}^{i,l} &\equiv \left(\frac{1}{n_h} \int_{j \in \Omega_h^{i,l}} (P_{ht}^{i,l}(j))^{1-\phi} dj \right)^{\frac{1}{1-\phi}}, \\ P_{ht}^{i,d} &\equiv \left(\frac{1}{n_h} \int_{j \in \Omega_h^{i,d}} (P_{ht}^{i,d}(j))^{1-\phi} dj \right)^{\frac{1}{1-\phi}}. \end{aligned}$$

From the definition of price indexes it follows that

$$(P_{ht}^i)^{1-\phi} = \left(P_{ht}^{i,p}/\mathcal{E}_{ht}^i \right)^{1-\phi} + \left(P_{ht}^{i,l} \right)^{1-\phi} + \left(P_{ht}^{i,d}/\mathcal{E}_{1t}^i \right)^{1-\phi}.$$

At the steady state, $P_h^{ip}(j) = P_h^{il}(j) = P_h^{id}(j) = P_h^i = P$, $P_h^{ip} = (\theta_h^{ip}/n_h)^{\frac{1}{1-\phi}} P$, $P_h^{il} = (\theta_h^{il}/n_h)^{\frac{1}{1-\phi}} P$, $P_h^{id} = (\theta_h^{id}/n_h)^{\frac{1}{1-\phi}} P$. The log-linearized version of the equation above is

$$p_{ht}^i = \frac{\theta_h^{ip}}{n_h} (p_{ht}^{ip} - e_{ht}^i) + \frac{\theta_h^{il}}{n_h} p_{ht}^{il} + \frac{\theta_h^{id}}{n_h} (p_{ht}^{id} - e_{1t}^i).$$

- It is shown below that evolution of p_{ht}^{ip} , p_{ht}^{il} , and p_{ht}^{id} is given by

$$\begin{aligned} \pi_{ht}^{ip} &= \beta \mathbb{E}_t \{ \pi_{ht+1}^{ip} \} + \frac{(1-\beta\lambda)(1-\lambda)}{\lambda} mc_{ht}^{ip}, \\ \pi_{ht}^{il} &= \beta \mathbb{E}_t \{ \pi_{ht+1}^{il} \} + \frac{(1-\beta\lambda)(1-\lambda)}{\lambda} (mc_{ht}^{il} - e_{ht}^i), \\ \pi_{ht}^{id} &= \beta \mathbb{E}_t \{ \pi_{ht+1}^{id} \} + \frac{(1-\beta\lambda)(1-\lambda)}{\lambda} (mc_{ht}^{id} - e_{1t}^i). \end{aligned}$$

where $mc_{ht}^{ip} = w_t^h - a_t^h - p_{ht}^{i,p}$, $mc_{ht}^{il} = w_t^h - a_t^h - p_{ht}^{i,l}$, $mc_{ht}^{id} = w_t^h - a_t^h - p_{ht}^{i,d}$, $\pi_{ht}^{ip} = p_{ht}^{ip} - p_{ht-1}^{ip}$, $\pi_{ht}^{il} = p_{ht}^{il} - p_{ht-1}^{il}$, and $\pi_{ht}^{id} = p_{ht}^{id} - p_{ht-1}^{id}$.

A.6 Appendix. Evolution of prices.

Next section represents the log-linearization of firms' optimal pricing equations.

Producer currency pricing. The optimal price-setting condition is linearized around the zero inflation steady state. In particular, dividing by $P_{ht-1}^{i,p}$ and letting $\Pi_{ht,t+k}^{i,p} = P_{ht+k}^{i,p}/P_{ht}^{i,p}$ - being the relative

prices/inflation of goods exported from h to i , equation above can be written as

$$\sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h Y_{ht+k|t}^{i,p} \left(\frac{\bar{P}_{ht}^{i,p}}{P_{ht-1}^{i,p}} - \mathcal{M} \frac{\psi_{t+k|t}}{P_{ht+k}^{i,p}} \Pi_{ht-1,t+k}^{i,p} \right) \right\} = 0 \text{ for all } i \in \{1, \dots, n\}.$$

In the zero inflation steady state, $\bar{P}_{ht}^{i,p}/P_{ht+k}^{i,p} = 1$, and $\Pi_{ht-1,t+k}^{i,p} = 1$ for any $i \in \{1, \dots, n\}$. Furthermore, constancy of price level implies that $\bar{P}_{ht}^{i,p} = P_{ht+k}^{i,p}$ in that steady state, from which it follows that $Y_{ht+k|t}^i = Y_h^i$ and $MC_{ht+k|t}^{i,p} = \frac{\psi_{t+k|t}}{P_{ht+k}^{i,p}} = MC_h^{i,p}$, because all firms will be producing the same quantity of output. In addition, $q_{t,t+k}^h = \beta^k$ must hold in the steady state. Accordingly, $MC_h^{i,p} = 1/\mathcal{M}$. A first-order Taylor expansion of the equation around the zero inflation steady state yields

$$\bar{p}_{ht}^{i,p} - p_{ht-1}^{i,p} = (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ mc_{ht+k|t}^{i,p} + (p_{ht+k}^{i,p} - p_{ht-1}^{i,p}) \} \text{ for all } i \in \{1, \dots, n\}.$$

where $mc = -\mu$ and where $\mu = \log \mathcal{M}$. The equations above can be rewritten as²¹

$$\bar{p}_{ht}^{i,p} = \mu + (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ mc_{ht+k|t}^{i,p} + p_{ht+k}^{i,p} \} \text{ for all } i \in \{1, \dots, n\}.$$

From this equation it follows that the firm changes the same price in all destinations. Therefore, now I could determine the law of motion for $p_{ht}^{h,p}$ and after use the law of one price.

As real marginal costs does not depend on real output $y_{ht+k|t}^{i,p}$ (because of linear production function), we have that $mc_{ht+k|t}^{i,p} = mc_{ht+k}^{i,p}$. Next, we have

$$\begin{aligned} \bar{p}_{ht}^{i,p} - p_{ht-1}^{i,p} &= (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ mc_{ht+k}^{i,p} + (p_{ht+k}^{i,p} - p_{ht-1}^{i,p}) \} \\ &= (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ mc_{ht+k}^{i,p} + \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ \pi_{ht+k}^{i,p} \} \} \end{aligned}$$

Notice that the above discounted sum can be rewritten more compactly as the difference equation

$$\bar{p}_{ht}^{i,p} - p_{ht-1}^{i,p} = \beta\lambda \mathbb{E}_t \{ \bar{p}_{ht+1}^{i,p} - p_{ht}^{i,p} \} + (1 - \beta\lambda) mc_{ht}^{i,p} + \pi_{ht}^{i,p} \text{ for all } i \in \{1, \dots, n\}. \quad (15)$$

For the rest see the derivations below.

Local currency pricing

²¹This equation shows that under optimal pricing the law of one price is satisfied if firms set prices in domestic currency. More precisely, the equation can be rewritten as

$$\bar{p}_{ht}^{i,p} = (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ w_{t+k}^h - a_{t+k}^h \} \text{ for all } i \in \{1, \dots, n\}$$

therefore, $\bar{p}_{ht}^{i,p}$ does not depend on i .

Again, we divide the equation above by $P_{ht-1}^{i,l}$

$$\sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h Y_{ht+k|t}^{i,l} \left(\mathcal{E}_{h,t+k}^i \frac{\bar{P}_{ht}^{i,l}}{P_{ht-1}^{i,l}} - \mathcal{M} \frac{\phi_{t+k|t}}{P_{ht+k}^{i,l}} \Pi_{ht+k,t-1}^{i,l} \right) \right\} = 0.$$

where again $MC_{ht+k|t}^{i,l} = \psi_{t+k|t} / P_{ht+k}^{i,l}$, $\Pi_{ht+k}^{i,l} = P_{ht+k,t-1}^{i,l} / P_{ht-1}^{i,l}$.

In the zero inflation steady state, $\bar{P}_{ht}^{i,l} / P_{ht+k}^{i,l} = 1$ and $\Pi_{ht-1,t+k}^{i,l} = 1$. Furthermore, constancy of price level implies that $\bar{P}_{ht}^{i,l} = P_{ht+k}^{i,l}$ in that steady state, from which it follows that $Y_{ht+k|t}^{i,l} = Y_h^{i,l}$ and $MC_{ht+k|t}^{i,l} = MC_h^{i,l}$, because all firms will be producing the same quantity of output. In addition, $q_{t,t+k}^h = \beta^k$ must hold in the steady state, Accordingly, $MC_h^{i,l} = 1/\mathcal{M}$. A first-order Taylor expansion of the equation around the zero inflation steady state yields

$$\begin{aligned} \bar{p}_{ht}^{i,l} - p_{ht-1}^{i,l} &= (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ (mc_{ht+k|t}^i - e_{ht+k}^i) + (p_{ht+k}^{i,l} - p_{ht-1}^{i,l}) \} \\ &= (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ (mc_{ht+k|t}^i - e_{ht+k}^i) + \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ \pi_{ht+k}^{i,l} \} \}, \end{aligned}$$

where $e_{ht+k}^i = \log \mathcal{E}_{ht+k}^i = \mathcal{E}_{ht+k}^i - 1$.

Notice that the above discounted sum can be rewritten more compactly as the difference equation

$$\bar{p}_{ht}^{i,l} - p_{ht-1}^{i,l} = \beta\lambda \mathbb{E}_t \{ \bar{p}_{ht+1}^{i,l} - p_{ht}^{i,l} \} + (1 - \beta\lambda)(mc_{ht}^{i,l} - e_{ht}^i) + \pi_{ht}^{i,l}.$$

For the rest see the derivations below.

Dominant currency pricing. I assume that local firms can set prices only in dominant currency for foreign household and in local currency for local household. Therefore, the producer in country h chooses an optimal price for domestic consumers $\bar{P}_{ht}^{h,d}$, and the set of prices set in dollars for all foreign destinations $\{\bar{P}_{ht}^{i,d}\}_{i=1, i \neq h}^n$

$$\max_{\{\bar{P}_{ht}^{i,d}\}_{i=1, i \neq h}^n, \bar{P}_{ht}^{h,d}} \sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h \left(\mathcal{E}_{ht+k}^1 \sum_{\substack{i=1, \\ i \neq h}}^n \bar{P}_{ht}^{i,d} Y_{ht+k|t}^{i,d} + \bar{P}_{ht}^{h,d} Y_{ht+k|t}^{h,d} - \Psi \left(\sum_{i=1}^n Y_{ht+k|t}^{i,d} \right) \right) \right\}$$

subject to demand constraint

$$\begin{aligned} Y_{ht+k|t}^{i,d} &= \alpha_h^i \alpha^i \left(\frac{\bar{P}_{ht}^{i,d}}{P_{ht+k}^i \mathcal{E}_{1t+k}^i} \right)^{-\phi} \left(\frac{P_{ht+k}^i}{P_{Ft+k}^i} \right)^{-\gamma} \left(\frac{P_{Ft+k}^i}{P_{t+k}^i} \right)^{-\theta} C_{t+k}^i \text{ for } i \neq h, \\ Y_{ht+k|t}^{h,d} &= (1 - \alpha^h) \left(\frac{\bar{P}_{ht}^{h,d}}{P_{ht+k}^h} \right)^{-\phi} \left(\frac{P_{ht+k}^h}{P_{t+k}^h} \right)^{-\theta} C_{t+k}^h. \end{aligned}$$

The first order condition with respect to $\bar{P}_{ht}^{i,d}$ is

$$\sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h \sum_{\substack{i=1, \\ i \neq h}}^n Y_{ht+k|t}^{i,d} (\mathcal{E}_{h,t+k}^1 \bar{P}_{ht}^{i,d} - \mathcal{M} \psi_{t+k|t}) \right\} = 0.$$

Again, we divide the equation above by $P_{ht-1}^{i,d}$:

$$\sum_{k=0}^{\infty} \lambda^k \mathbb{E}_t \left\{ q_{t,t+k}^h Y_{ht+k|t}^{i,d} \left(\mathcal{E}_{h,t+k}^1 \frac{\bar{P}_{ht}^{i,d}}{P_{ht-1}^{i,d}} - \mathcal{M} MC_{ht+k|t}^{i,d} \Pi_{ht+k,t-1}^{i,d} \right) \right\} = 0.$$

where again $MC_{ht+k|t}^{i,d} = \psi_{t+k|t} / P_{ht+k}^{i,d}$, $\Pi_{ht+k}^{i,d} = P_{ht+k,t-1}^{i,d} / P_{ht-1}^{i,d}$.

In the zero inflation steady state, $\bar{P}_{ht}^{i,d} / P_{ht+k}^{i,d} = 1$ and $\Pi_{ht-1,t+k}^{i,d} = 1$. Furthermore, constancy of price level implies that $\bar{P}_{ht}^{i,d} = P_{ht+k}^{i,d}$ in that steady state, from which it follows that $Y_{ht+k|t}^{i,d} = Y_h^{i,d}$ and $MC_{ht+k|t}^{i,d} = MC_h^{i,d}$, because all firms will be producing the same quantity of output. In addition, $q_{t,t+k}^h = \beta^k$ must hold in the steady state, Accordingly, $MC_h^{i,d} = 1/\mathcal{M}$. A first-order Taylor expansion of the equation around the zero inflation steady state yields

$$\begin{aligned} \bar{p}_{ht}^{i,d} - p_{ht-1}^{i,d} &= (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ (mc_{ht+k|t}^{i,d} - e_{ht+k}^1) + (p_{ht+k}^{i,d} - p_{ht-1}^{i,d}) \} \\ &= (1 - \beta\lambda) \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ (mc_{ht+k|t}^{i,d} - e_{ht+k}^1) + \sum_{k=0}^{\infty} (\beta\lambda)^k \mathbb{E}_t \{ \pi_{ht+k}^{i,d} \} \}, \end{aligned}$$

where $e_{ht+k}^1 = \log \mathcal{E}_{ht+k}^1 = \mathcal{E}_{ht+k}^1 - 1$.

Notice that the above discounted sum can be rewritten more compactly as the difference equation

$$\bar{p}_{ht}^{i,d} - p_{ht-1}^{i,d} = \beta\lambda \mathbb{E}_t \{ \bar{p}_{ht+1}^{i,d} - p_{ht}^{i,d} \} + (1 - \beta\lambda)(mc_{ht}^{i,d} - e_{ht}^1) + \pi_{ht}^{i,d}.$$

For the rest see the derivations below.

Overall price level P_{ht}^{ip} . The manipulation of the price index, P_{ht}^{ip} yields

$$\begin{aligned} (P_{ht}^{ip})^{1-\phi} &= \frac{1}{n_h} \int_{j \in \Omega_h^{ip}} (P_{ht}^{ip}(j))^{1-\phi} dj = \frac{1}{n_h} \int_{j \in \mathcal{S}_h^{ip}(t)} (P_{ht-1}^{ip}(j))^{1-\phi} dj + (1 - \lambda) \frac{\theta_h^{ip}}{n_h} (\bar{P}_{ht}^{ip})^{1-\phi} \\ &= \lambda (P_{ht-1}^{ip})^{1-\phi} + (1 - \lambda) \frac{\theta_h^{ip}}{n_h} (\bar{P}_{ht}^{ip})^{1-\phi} \end{aligned}$$

or

$$\left(\frac{P_{ht}^{ip}}{P_{ht-1}^{ip}} \right)^{1-\phi} = \lambda + (1 - \lambda) \frac{\theta_h^{ip}}{n_h} \left(\frac{\bar{P}_{ht}^{ip}}{P_{ht-1}^{ip}} \right)^{1-\phi}.$$

The log-linearized version of the equation above is

$$p_{ht}^{ip} - p_{ht-1}^{ip} = (1 - \lambda)(\bar{p}_{ht}^{ip} - p_{ht-1}^{ip}). \quad (16)$$

Similarly, it can be shown that

$$\begin{aligned} p_{ht}^{il} - p_{ht-1}^{il} &= (1 - \lambda)(\bar{p}_{ht}^{il} - p_{ht-1}^{il}), \\ p_{ht}^{id} - p_{ht-1}^{id} &= (1 - \lambda)(\bar{p}_{ht}^{id} - p_{ht-1}^{id}). \end{aligned} \tag{17}$$

As it was shown above

$$\begin{aligned} \bar{p}_{ht}^{ip} - p_{ht-1}^{ip} &= \beta\lambda\mathbb{E}_t\{\bar{p}_{ht+1}^{ip} - p_{ht}^{ip}\} + (1 - \beta\lambda)mc_{ht}^{ip} + \pi_{ht}^{ip}, \\ \bar{p}_{ht}^{il} - p_{ht-1}^{il} &= \beta\lambda\mathbb{E}_t\{\bar{p}_{ht+1}^{il} - p_{ht}^{il}\} + (1 - \beta\lambda)(mc_{ht}^{il} - e_{ht}^i) + \pi_{ht}^{il}, \\ \bar{p}_{ht}^{id} - p_{ht-1}^{id} &= \beta\lambda\mathbb{E}_t\{\bar{p}_{ht+1}^{id} - p_{ht}^{id}\} + (1 - \beta\lambda)(mc_{ht}^{id} - e_{ht}^1) + \pi_{ht}^{id}, \end{aligned} \tag{18}$$

Combining (16), (), () yields

$$\begin{aligned} \pi_{ht}^{ip} &= \beta\mathbb{E}_t\{\pi_{ht+1}^{ip}\} + \frac{(1 - \beta\lambda)(1 - \lambda)}{\lambda}mc_{ht}^{ip}, \\ \pi_{ht}^{il} &= \beta\mathbb{E}_t\{\pi_{ht+1}^{il}\} + \frac{(1 - \beta\lambda)(1 - \lambda)}{\lambda}(mc_{ht}^{il} - e_{ht}^i), \\ \pi_{ht}^{id} &= \beta\mathbb{E}_t\{\pi_{ht+1}^{id}\} + \frac{(1 - \beta\lambda)(1 - \lambda)}{\lambda}(mc_{ht}^{id} - e_{ht}^1). \end{aligned}$$