Managing Inequality over the Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks*

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Abstract

We present a projection theory on the space of idiosyncratic histories for heterogeneous-agents models. This allows solving for optimal Ramsey policies in heterogeneous-agent models with aggregate shocks, using a Lagrangian approach. In addition, it allows improving current simulation methods using perturbation techniques, by using more steady-state information. We apply this to study the optimal level distorting tax on labor and unemployment insurance over the business cycle in a production economy. In the quantitative exercise, the average optimal replacement rate is 10% higher than the one implies by a sufficient-statistics approach, due to saving distortions. Moreover, the optimal replacement rate is countercyclical.

Keywords: Incomplete markets, optimal policies, heterogeneous agent models.

JEL codes: E21, E44, D91, D31.

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1 Introduction

Incomplete insurance-market economies provide a useful framework for examining many relevant aspects of inequalities and individual risks. In these models, infinitely-lived agents face incomplete insurance markets and borrowing limits that prevent them from perfectly hedging their idiosyncratic risk, in line with the Bewley-Huggett-Aiyagari literature (Bewley 1983, Imrohoroğlu 1989, Huggett 1993, Aiyagari 1994, Krusell and Smith 1998). These frameworks are now widely used, since they fill a gap between micro- and macroeconomics, and enable the inclusion of aggregate shocks and a number of additional frictions on both the goods and labor markets. However, little is known about optimal policies in these environments, due to the difficulties generated by the large and time-varying heterogeneity across agents. This is unfortunate, since a vast literature, reviewed below, suggests that the interaction between wealth inequalities and capital accumulation has first-order implications for the optimal design of time-varying policies.

This paper presents a methodological contribution that offers a general and tractable representation of incomplete insurance-market economies. This representation first improves on current simulation methods, such as Reiter (2009), which rely on perturbation techniques. Second and more importantly, it allows us to easily solve Ramsey problems in economies with both capital and aggregate shocks. We apply our framework to the design of optimal time-varying unemployment insurance (UI) over the business cycle. In the vast literature studying this issue, our contribution is to identify general equilibrium effects on the optimal UI, due to both the endogeneity of the real wage, and of the real interest rate, and due to the distortions on saving incentives. For instance, compared to a partial-equilibrium analysis, typically performed in the so-called *sufficient statistics* approach, we find the optimal UI to be 10% higher, because of distortions due to precautionary savings.

In incomplete insurance market economies, heterogeneity increases with time because agents differ according to the full history of their idiosyncratic risk realizations. Huggett (1993) and Aiyagari (1994), using the results of Hopenhayn and Prescott (1992), have shown that economies without aggregate risk have a recursive structure when the distribution of wealth is introduced as a state variable. Unfortunately, the distribution of wealth has infinite support, which is at the root of many analytical difficulties. Our basic idea is to go back in the sequential representation to consider the set of idiosyncratic histories at each period, in three steps. First, one can use a time-invariant partition of these histories $\mathcal{H}$ such that each agent, at each period, belongs to one and exactly one element of this partition $h \in \mathcal{H}$. Second, we show that there is a simple way to aggregate heterogeneity within each element h, such that one can construct exact aggregated budget constraints and Euler equations over the finite number of "agents" $h \in \mathcal{H}$, instead of the whole distribution. Third, using the steady-state outcome of the model (without aggregate shocks, but with idiosyncratic shocks), one can deduce the steady-state relevant distributions within each elements $h \in \mathcal{H}$. Our modeling assumption (which can be tested) is that changes in the distributions within elements $h \in \mathcal{H}$ has a second-order effect on welfare compared to changes of distribution across $h \in \mathcal{H}$. As a consequence, one can assume that the distributions within each $h \in \mathcal{H}$ is time-invariant, and only consider the dynamics of a "simple" projected model.

What is the proper choice of the partition? First, some explicit partitions can be constructed, based on a truncation of idiosyncratic histories. Each agent having the same history of the idiosyncratic shock for the last N periods are in the same element of $\mathcal{H}$ (for a given length N).[1] Although intuitive, these partitions can be inefficient as the number of elements in $\mathcal{H}$ grows exponentially with N. Thus, informed by the recursive representation of incomplete market economies, we use the steady-state distribution of wealth to construct efficient partitions for a large state-space.

Finally, for any projected economy, as we have relevant Euler equations and budget constraints over $\mathcal{H}$, one can use tools developed in dynamic contracts, sometimes called the Lagrangian approach and developed by Marcet and Marimon (2011), applied to elements of the partition $h \in \mathcal{H}$, to derive first-order conditions for the planner. These conditions are then easy to simulate with aggregate shocks. We provide algorithm to find the optimal Ramsey policies, and check that it is does not depend on the chosen partition.

We apply this methodology to the optimal UI over the business cycle. We consider the economy with both idiosyncratic productivity and unemployment risks of Krueger, Mittman, and Perri (2018), which generates a realistic wealth distribution and as well as realistic unemployment and income risks. In this economy, we introduce a labor supply choice and an UI scheme financed by a distorting tax on labor. The Ramsey problem to optimally choose UI is based on the standard trade-off between insurance and incentives, but in an environment where prices are endogenous due to the evolution of the capital stock. We derive a formula for the optimal replacement rate which highlights the role of saving incentives on the optimal value of UI. Using this formula, we can quantify the difference with previous analysis that do not consider

[1] In a previous version of this paper, LeGrand and Ragot (2017), we studied in depth the properties of partition based on truncations.

this margin. First, we find that the optimal UI is 10% higher, when one considers distortions due to precautionary savings, compared to the case where prices are considered exogenous. There is indeed capital over-accumulation in our calibrated economy.[2] Second, we find that UI is countercyclical after a productivity shock, and contribute to increase labor supply when productivity is high.

This paper is related to three strands of the literature. The first one is the computation of incomplete insurance markets with aggregate shocks. After the seminal paper of Krusell and Smith (1998), incomplete insurance market models with aggregate shocks have first been solved using a fixed point on simple expectation rules. Since the work of Reiter (2009) the literature has moved toward the use of perturbation method to simulate these models (see also the recent contribution of Boppart, Krusell, and Mitman (2018) to derive a linear approximation of the dynamics). These techniques are now used in various setups, to solve discrete-time models, as inWinberry (2016) or models first written in continuous time as in Ahn, Kaplan, Moll, Winberry, and Wolf (2017). Compared to Reiter (2009), our method has two main differences. First, we use more steady-information to simulate the model, namely the distribution of wealth within elements of $\mathcal{H}$. Second, and more importantly, our projection strategy construct relevant Euler equations and budget constraints to follow the wealth dynamics. This difference allows us to derive optimal policies with aggregate shocks.

Second, this paper is related to the literature on optimal (Ramsey) policies in general heterogeneous agent models. This literature is thin and very recent. First, Açikgöz, Hagedorn, Holter, and Wang (2018) provide an algorithm to solve for the steady-state allocation of the Ramsey program, based on assumptions on functional form. Nuño and Moll (2017) use a continuous-time approach without aggregate shock and rely on projection methods to determine the steady-state allocation. Bhandari, Evans, Golosov, and Sargent (2016) present a method using a "primal approach", which relies on perturbation methods around time-varying allocations. This strategy requires that credit constraints are not occasionally binding in the dynamics. Compared to these methods, our representation can be used in general cases with aggregate shocks.

Third, this paper contributes to the literature on optimal unemployment insurance. This literature is huge and a big part of it is based on a new approach to bring theory to the data, known as the sufficient-statistics approach (see the surveys of Chetty (2009), Chetty and Finkelstein (2013) and Kolsrud, Landais, Nilsson, and Spinnewijn (2018) for recent developments).

[2] It is known that incomplete-insurance markets can generate either over-accumulation or under-accumulation, see Dávila, Hong, Krusell, and Ríos-Rull (2012) and Aiyagari (1995) for two different cases. In our setup with both employment and productivity risk, we observe an over-accumulation of capital.

This literature provides an estimation strategy for a wide class of model in partial equilibrium. We instead rely on a general equilibrium model to assess both the partial and the general equilibrium effect of UI within the model. Some papers consider some general equilibrium effects, such as Mitman and Rabinovich (2015) or Landais, Michaillat, and Saez (2018), but focusing on externalities on the labor market and not considering the difficult question of saving distortions. To our knowledge, the only paper analyzing optimal UI in general equilibrium with saving is Krusell, Mukoyama, and Sahin (2010). To simplify the quantitative exercise, the authors performs welfare analysis comparing different steady-states with different level of UI. Instead, we solve for the general Ramsey problem, in both the steady-state and with aggregate shocks.

The rest of the paper is organized as follows. Section 2 presents the simple environment, on which our methodology will be applied. Section 3 presents the projection in the space of idiosyncratic histories in the general case. Section 5 presents solution techniques to derive optimal policies. Section 6 provides two numerical examples, a first one without optimal policies to benchmark our method with other ones presented in the literature. The second one computes optimal time-varying fiscal policy.

2 The economy

We consider a discrete-time setup. The economy features a single good and is populated by a population of size 1 of agents distributed on a segment I according to a measure $\ell(\cdot)$. We assume that the law of large number holds.

2.1 Preferences

Agents derive utility in each period from two goods: private consumption c and labor l. The period utility function is denoted $U(c,l)$. As standard in this literature, the utility function U is assumed to be Greenwood-Hercowitz-Huffman (GHH) utility function, exhibiting no wealth effect for the labor supply:

$$U(c,l) = u\left(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}\right),$$

where $\varphi > 0$ is the Frisch elasticity of labor supply, $\chi > 0$ scales labor disutility, and $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously derivable, increasing, and concave, with $u'(0) = \infty$. Our results do not critically depend on this specific functional form and we could use a more general utility function U. However, the algebra is simplified, especially in the Ramsey program, because of the absence of wealth effect for the labor supply.

Agents have standard additive intertemporal preferences, with a constant discount factor $\beta > 0$. They therefore rank consumption and labor streams, denoted respectively by $(c_t)_{t\geq 0}$ and $(l_t)_{t\geq 0}$, using the intertemporal utility criterion $\sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$.

2.2 Risks

We consider a general setup where agents face both aggregate risk, time-varying unemployment risk, and idiosyncratic productivity risk, as modeled by Krueger, Mittman, and Perri (2018). As will be clear in the quantitative analysis below, this general setup allows us to match the wealth distribution and a realistic dynamics of the labor market.

Aggregate risk. The aggregate risk will affect both aggregate productivity and the unemployment risk. Formally, the aggregate risk is represented by a probability space $(\mathcal{Z}^{\infty}, \mathcal{F}, \mathbb{P})$. At a given date t, the aggregate state is denoted z_t and takes values in the state space $\mathcal{Z} \subset \mathbb{R}^+$. We assume the aggregate risk to be a Markov process.[3] The history of aggregate shocks up to time t is denoted $z^t = \{z_0, \ldots, z_t\} \in \mathcal{Z}^{t+1}$. Finally, the period-0 probability density function of any history z^t is denoted $m^t(z^t)$.

For the sake of clarity, for any random variable $X_t : \mathcal{Z}^t \to \mathbb{R}$, we will denote X_t, instead of $X_t(z^t)$, its realization in state z^t,

Employment risk. At the beginning of each period, agents face an uninsurable idiosyncratic employment risk, denoted e_t at date t. The employment status e_t can take two values, e and u, corresponding to employment and unemployment respectively. We denote by $\mathcal{E} = \{e, u\}$ the set of possible employment status. Employed agents with $e_t = e$ can freely choose their labor supply l_t, and they earn a before-tax real wage, denoted $w_t l_t$ at date t. Unemployed agents with $e_t = u$ cannot work and will receive unemployment benefits financed by social contributions, and will suffer from a fixed disutility reflecting a level of domestic production. The two last aspects are further described below. A history of idiosyncratic shocks up to date t is denoted $e^t = \{e_0, \ldots, e_t\} \in \{0, 1\}^{t+1}$.

The employment status $(e_t)_{t\geq 0}$ follows a discrete Markov process with transition matrix $M_t(z^t) \in [0, 1]^{2\times 2}$ that is assumed to depend on the history of aggregate shocks up to date t. The job separation rate between periods $t-1$ and t is denoted $\Pi_{ue}(z^t) = 1 - \Pi_{uu}(z^t)$, while

[3] In the quantitative part, we will assume that it more specifically follows an AR(1) process.

$\Pi_{ee}(z^t) = 1 - \Pi_{eu}(z^t)$ is the job finding rate between $t-1$ and t. The time-varying transition matrix across employment status is therefore:

$$M_t(z^t) = \begin{bmatrix} \Pi_{uu}(z^t) & \Pi_{ue}(z^t) \\ \Pi_{eu}(z^t) & \Pi_{ee}(z^t) \end{bmatrix}. \tag{1}$$

As in Krusell and Smith (1998) and Krueger, Mittman, and Perri (2018), we assume that the share of the population that unemployed only depends on the current *aggregate* state, and that transition probabilities g and f actually only on the current and past aggregate states. We denote by $S_{u,t}(z_t)$ and $S_{e,t}(z_t)$ the populations of unemployed and employed agents respectively – where $S_{u,t}(z_t) + S_{e,t}(z_t) = 1$ at all dates.

Productivity risk. The individual productivity of agents is stochastic. At any date t, the individual productivity status is denoted y_t and takes values in a finite set $\mathcal{Y} \subset \mathbb{R}_+$. The cardinality of the set $\mathcal{Y}$ is denoted $Card\ \mathcal{Y}$ and is thus the number of different idiosyncratic productivity levels. Large values of y_t correspond to highly productive agents. The before-tax wage earned by an employed agent will be the product of an aggregate wage denoted w_t, depending on aggregate shock, of labor effort l_t, and of the individual productivity y_t. The total before-tax wage will therefore amount to $y_t w_t l_t$. An unemployed agent will also carry an idiosyncratic productivity level that will affect her unemployment benefits and her disutility level ζ_y.

The history of productivity shocks of a given agent up to date t is denoted $y^t = \{y_0, \ldots, y_t\}$. The productivity status follows a first-order Markov process where the transition probability from state $y_{t-1} = y$ to $y_t = y'$ is constant and denoted $\pi_{yy'}$. In particular, it is independent of the employment status of the agent. We denote by η_y the share of agents endowed with individual productivity level y. This share is constant through time because of assumptions on transition probabilities $\pi_{yy'}$.

The individual status of any agent i is characterized by her employment status e and personal productivity level y. At any date t, we will denote by $\sigma_t^i = (e_t^i, y_t^i)$ the date-t individual status of any agent. The set of possible individual status is denoted $\mathcal{S} = \mathcal{E} \times \mathcal{Y}$. Finally, we denote as σ^t a history until period t: $\sigma^t = \{..., \sigma_{t-1,}\sigma_t\}$. From the transition probability over employment and productivity status, one can derive the measure $\mu_t : \mathcal{S}^t \to [0,1]$, such that $\mu_t(\sigma_t)$ is the measure of agents having history σ_t at period t.

2.3 Production

The good is produced by a unique profit-maximizing representative firm. This firm is endowed with a production technology that transforms, at date t, labor L_t and capital K_{t-1} into Y_t output units of the single good. The production function is a Cobb-Douglas function with parameter $\alpha \in (0,1)$ featuring constant returns-to-scale. The capital must be installed one period before production and the total productivity factor Z_t is stochastic. Denoting as $\delta > 0$ the constant capital depreciation, the output Y_t is formally defined as follows:

$$Y_t = F(K_{t-1}, L_t) = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}. \tag{2}$$

where L_t is the labor supply expressed in efficient units and F is the aggregate production function subsuming capital depreciation. We have:

$$L_t = \int_{i \in I} y_t^i l_t^i \ell(di),$$

where we recall that $\ell(\cdot)$ represents the distribution of agents. The total productivity factor is simply the exponential of the aggregate shock z_t:

$$Z_t = \exp(z_t). \tag{3}$$

The two factor prices at date t are the aggregate before-tax wage rate w_t and the capital return r_t. The profit maximization of the producing firm implies the following factor prices.

$$w_t = F_L(K_{t-1}, L_t) = (1-\alpha)\, Z_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha}, \tag{4}$$

$$r_t = F_K(K_{t-1}, L_t) = \alpha Z_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha-1} - \delta. \tag{5}$$

2.4 Social contributions

The government raises distorting social contributions that serve to finance unemployment benefits. This setup is a minimal extension of Krueger, Mittman, and Perri (2018) that allows for thinking about Ramsey policies, and in particular about optimal unemployment insurance along the business cycle. To allow for a non-trivial Ramsey program that does not feature full unemployment insurance, we indeed need to introduce a downside to unemployment benefit. We have made the choice to introduce a friction on the financing of unemployment benefits through a distorting tax on endogenous labor supply.

We now describe more formally the unemployment insurance setup. Unemployed agents receive at any date an unemployment benefit that is equal to a constant fraction of the wage the agent would earn if she were employed (with the same productivity level). The replacement rate being denoted ϕ_t, the unemployment benefit of an agent endowed with productivity y equals $\phi_t w_t y_t l_{t,e}$, where $l_{t,e}$ is the labor supply of a (fictive) employed agent with productivity y_t and aggregate wage rate w_t.

The unemployment benefits are solely financed by social contributions. These contributions are only paid by employed agents. Contributions amount to a constant proportion $\tau_t(z)$ of the wage and this proportion is identical for all employed agents, but depends on the current aggregate state z. The contribution τ_t is set such that the unemployment insurance (UI) scheme is balanced at any date t, since we rule out the possibility of social debt. The balance budget of the unemployment insurance scheme can be expressed as:

$$\phi_t w_t \int_{i \in \mathcal{U}} y_t^i l_{t,e}^i \ell(di) = \tau_t \phi_t w_t \int_{i \in \mathcal{E}} y_t^i l_t^i \ell(di), \tag{6}$$

where $\mathcal{U}$ represents the set of unemployed agents and $\mathcal{E}$ the set of employed agents.

2.5 Agents' program and resource constraints

2.5.1 Sequential formulation

We consider an agent i. She can save in a riskless asset that pays off the gross interest rate $1 + r_t$. She is prevented from holding too negative savings and the latter must remain greater than an exogenous threshold denoted $-\bar{a}$. At date 0, the agent chooses her consumption $(c_t^i)_{t\geq 0}$, her labor supply $(l_t^i)_{t\geq 0}$, and her saving plans $(a_t^i)_{t\geq 0}$ that maximize her intertemporal utility, subject to a budget constraint and the previous borrowing limit. Formally, her program can be expressed as follows:

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u \left(c_t^i - \chi^{-1} \frac{l_t^{i,1+1/\varphi}}{1+1/\varphi} \right) \tag{7}$$

$$c_t^i + a_t^i = (1 + r_t) a_{t-1}^i + \left((1 - \tau_t) 1_{e_t^i = e} + \phi_t 1_{e_t^i = u} \right) l_{t,e}^i y_t^i w_t, \tag{8}$$

$$a_t^i \geq -\bar{a}, \tag{9}$$

where $1_{e_t^i = e}$ is an indicator function equal to 1 if the agent is currently employed ($e_t^i = e$) and to 0 otherwise. The budget constraint (8) is very standard and the expression $\left((1 - \tau_t) 1_{e_t^i = e} + \phi_t 1_{e_t^i = u} \right) l_t^i y_t^i w_t$ is a compact formulation for the net (i.e., after taxes and after social contributions) wage of the agent i endowed with productivity y_t^i, depending on whether she is employed ($e_t^i = e$) or unemployed ($e_t^i = u$).

We denote by $\beta^t \nu_t^i$ the Lagrange multiplier of the credit constraint of agent i. The Lagrange multiplier is obviously null when the agent is not credit-constrained. Taking advantage of the GHH utility function, the first-order conditions of the agent' program (7)–(9) can written as follows:

$$u'(c_t^i - \chi^{-1} \frac{l_t^{i,1+1/\varphi}}{1+1/\varphi}) = \beta \mathbb{E}_t \left[(1+r_{t+1}) u'(c_{t+1}^i - \chi^{-1} \frac{l_{t+1}^{i,1+1/\varphi}}{1+1/\varphi}) \right] + \nu_t^i, \tag{10}$$

$$l_t^{i,1/\varphi} = \chi(1-\tau_t) y_t^i 1_{e_t^i = e}. \tag{11}$$

The GHH utility function implies a very simple expression for the labor supply. Unemployed agents do not supply any labor, but they earn unemployment benefits and suffer from disutility related to home production.

We now turn to the economy-wide constraints. First, the financial market clearing implies the following relationship:

$$\int_i a_t^i \ell(di) = K_t. \tag{12}$$

The clearing of goods market implies that the total consumption, made of private individual consumption, private firm consumption and public consumption equals total supply, made of output and past capital:

$$\int_i c_t^i \ell(di) + K_t = Y_t + K_{t-1}. \tag{13}$$

Since every employed agent endogenously supplies labor, while unemployed agents do not work, the labor L_t in efficient units is defined as:

$$L_t = \int_i y_t^i l_t^i \ell\,(di)\,, \tag{14}$$

since agents have different individual productivities.

Using the transition matrix M_t in equation (1), we deduce that the law of motion for the populations of employed and unemployed agents, denoted S_{et} and $S_{u,t}$ respectively, is:

$$S_{u,t} = 1 - S_{e,t} = \Pi_{eu,t} S_{e,t-1} + \Pi_{uu,t} S_{u,t-1}. \tag{15}$$

The share of agents S_y with productivity y is defined as follows:

$$S_y = \sum_{y \in \mathcal{Y}} S_{y'} \pi_{y'y},$$

which is constant through time since the transition matrix $(\pi_{y'y})_{y,y'}$ is constant.

We can finally formulate our equilibrium definition.

Definition 1 (Sequential equilibrium) *A sequential competitive equilibrium is a collection of individual allocations* $(c_t^i, l^i, a_t^i)_{t\geq 0, i\in I}$*, of aggregate quantities* $(K_t, L_t, Y_t)_{t\geq 0}$*, of price processes* $(w_t, r_t)_{t\geq 0}$*, and of social contributions* $(\tau_t)_{t\geq 0}$*, such that, for an initial wealth distribution* $(a_{-1}^i)_{i\in I}$*, and for initial values of capital stock* $K_{-1} = \int_{i\in I} a_{-1}^i \ell(di)$*, and of the initial aggregate shock* z_{-1}*, we have:*

1. *given prices, individual strategies* $(c_t^i, l^i, a_t^i)_{t\geq 0, i\in I}$ *solve the agents' optimization program in equations (7)–(9);*

2. *financial, labor, and good markets clear at all dates: for any* $t \geq 0$*, equations (12), (13) and (14) hold;*

3. *the UI scheme is balanced at all dates balance: equation (6) holds for all* $t \geq 0$*;*

4. *factor prices* $(w_t, r_t)_{t\geq 0}$ *are consistent with (4), and (5).*

3 Solving the model with history representation

We now provide our projection theory to obtain a finite-dimensional state-space representation. The basic idea is to group, in each period, agents according to their histories.

3.1 Partitions

At any date t, each agent i is uniquely characterized by her personal history of idiosyncratic risk realizations $\sigma^{i,t} = (e^{i,t}, y^{i,t})$, that include both employment and productivity risk histories. Obviously, the number of idiosyncratic risk histories exponentially grow over time and the steady-state is characterized by an unbounded (technically a countably finite) number of idiosyncratic histories. The core idea of our projection method is to group idiosyncratic histories in a finite number of "buckets", such that the agents' population, and henceforth the economy, is represented by this finite set of buckets. Agents can change buckets according to the realization of their idiosyncratic history.

Let us now proceed with a more formal presentation. We start with the partition of idiosyncratic histories. A partition $\mathcal{H}$ is a collection of sets of idiosyncratic histories $h \in \mathcal{H}$, such that any idiosyncratic history σ^t, for any date t, belongs to one and exactly one element of the partition $\mathcal{H}$. Formally, for any history σ^t, there exists a unique element $h \in \mathcal{H}$, such that $\sigma^t \in h$. In the remainder of the paper, with a slight abuse of notation, we will say that that an agent at date t belongs to $h \in \mathcal{H}$ if she is endowed with an idiosyncratic history σ^t belonging to h.

The idea of our theory is consider elements of the partition instead of the individual agents. We will indeed *project* individual programs and first-order conditions onto a given history partition $\mathcal{H}$. We will then simulate the projected model – that will consist of a finite number of representative partition elements – instead of the full individual model. A preliminary, though crucial remark, is that each partition element $h \in \mathcal{H}$ embeds some time-varying heterogeneity among agents, since different idiosyncratic histories are represented with the same partition element h. This heterogeneity within partition element will matter when thinking about agents' transition between buckets and when projecting the model onto the partition.

For the sake of concreteness, we now present two types of partitions, explicit partitions based on the truncation of idiosyncratic histories, and then implicit partition, using insights from the steady-state distribution of wealth.

3.1.1 Explicit partition

A first solution to construct a partition consists in relying on the truncation of idiosyncratic histories. More precisely, each idiosyncratic history σ^t is represented by the realizations of idiosyncratic status over the N consecutive previous periods, where N is a given history length. More precisely, each idiosyncratic history corresponds to a history with finite length $N \geq 1$, represented as a vector $\sigma^N = (\sigma_{-N+1}, \ldots, \sigma_0) \in \mathcal{S}^N$. The partition $\mathcal{H}$ is then the collection of N-length vectors $\sigma^N \in \mathcal{S}^N$ and is therefore the set $\mathcal{S}^N$ itself. All agents endowed with an identical idiosyncratic realization over the last N periods will belong to the same element of the partition $\mathcal{H}$.

Even though it has not been formalized in those terms, such partitions have already been used in the literature. First, Challe, Matheron, Ragot, and Rubio-Ramirez (2017) consider a sole idiosyncratic risk, which an unemployment risk, similar to the one in this paper – productivity is simply pinned down by employment status. In each period, agents can be either employed or unemployed. Authors consider a three-state partition of unemployment risk histories, which is $(\{e\}, \{eu\}, \{uu\})$. In each period, an agent can be represented by one and only one of these three elements: employed now, unemployed now and in the previous period, or unemployed now and employed before. This three-state partition is shown to be sufficient to capture time-varying precautionary savings. One advantage of this approach is reflected in the partition transition matrix, that is straightforward to derive from labor market transitions. For instance $\Pi_{\{e\},\{eu\},t} = \Pi_{eu,t}$, $\Pi_{\{eu\},\{e\},t} = \Pi_{\{uu\},\{e\},t} = \Pi_{ue,t}$, and finally $\Pi_{\{eu\},\{uu\},t} = \Pi_{uu,t}$, where we recall that $\Pi_{uu,t}$ and $\Pi_{eu,t}$ are the job-market transition probabilities – see equation (1).

Another application of explicit partitions can be found in LeGrand and Ragot (2017), who rely on a more general truncation. Again, they consider the full partition of unemployment risk history, denoted $\mathcal{E}^N$, where $N \geq 0$ represents as before history length. Formally, the partition contains all unemployment histories of length N, that are vectors $(e_{-N+1}, \ldots, e_0) \in \mathcal{E}^N$. As in the previous case, the transition matrix between partition elements can be easily derived from he job-market transition probabilities.

This construction is rather intuitive but has the drawback to generate a large number of partition elements, since the size of the partition, equal to $(Card\ \mathcal{S})^N$, grows exponentially with N. Given that the sizes of partition elements are not homogeneous, a relatively large N may be needed to properly capture the heterogeneity of certain idiosyncratic histories. The cost of this large N is a large partition size, that includes some elements of very small size, whose contribution to the global dynamics or welfare is limited.

Implicit partitions using the steady-state distribution of wealth. A second option we develop here is to define implicit partitions, based on the steady-state distribution of wealth. At the steady-state, in absence of aggregate shocks (i.e.. $Z_t = 1$), it is known that the equilibrium is characterized by an invariant steady-state distribution of beginning-of-period wealth $(a, \sigma) \mapsto \Gamma(a, \sigma)$ for $[-\bar{a}; +\infty[$ and $\sigma \in \mathcal{S}$, such that the size of the agent population currently in state σ and endowed with a wealth in the interval $[a_0, a_1]$ amounts to $\int_{a_0}^{a_1} d\Gamma(a, \sigma)$. Instead of follwing the wealth for any idiosyncratic history σ^t, namely $a(\sigma^t)$, we can measure the measure of agents having a wealth between $[a; a + da]$.

We can then define an implicit partition $\mathcal{H}$ of the agents' population based on a partition of the wealth space $[-\bar{a}, +\infty[$. Formally, the wealth partition is a collections of sets $b \in \mathcal{B}$, such that:

$$\begin{cases} [-\bar{a}, +\infty[= \cup_{b \in \mathcal{B}} b, \\ b \cap b' = \emptyset \text{ for all } b, b' \in \mathcal{B}, \text{ such that } b \neq b'. \end{cases}$$

Any interval of wealth $b \in \mathcal{B}$ defines buckets of history (σ, b) such that $h_{(\sigma,b)} \in \mathcal{H}$:

$$\sigma^t \in h_{(\sigma,b)} \iff a\left(\sigma^t\right) \in b \text{ and } \sigma_t = \sigma$$

In words, this partition puts in the same buckets the histories with the sae current idiosyncratic state σ, which generate a wealth level in given interval *at the steady state*. Note that histories with the same current state $\sigma = (y, e)$ are in the same bucket. As a consequence, we can write unambiguously, for any elemnts $h \in \mathcal{H}$ $y(h)$ or $e(h)$, to refer to the current productivity and employment status of agents in the bucket h.

After aggregate shocks, the wealth level of agents in the same bucket may fluctuate, an we will indeed follow the average wealth in the bucket. Note that Reiter (2009) also uses the steady-state distribution of wealth to follow agents. But as, will be clear below, Reiter's strategy is to identify agents according to their wealth level during the dynamics. Instead, we follow agents according to their histories. We will show below that it is a key difference, which allow to solve for optimal policies. Before, we have to present the projection properties of the model.

3.2 Projection of the model

We now consider a general partition $\mathcal{H}$, containing a finite set of histories $h \in \mathcal{H}$. We will now explain how to project the economic model onto this partition. This projection consists in following the dynamics of the average consumption and wealth of each bucket $h \in \mathcal{H}$, instead of whole distribution. We here show how to represent the exact dynamics of the model over $h \in \mathcal{H}$. We first introduce general concept and the we provide the numerical approximation to solve the model.

3.2.1 Projecting variables: The basics

The first step of the model projection consists in assigning for each variable of interest (such as consumption, asset holdings, labor supply, etc.) a value for all elements of the partition. Formally, this representative value is simply an average over the population of agents belonging to the partition element.

We begin with computing the population size of partition elements. For instance, for a partition element $h \in \mathcal{H}$, this size corresponds to the measure of agents with current idiosyncratic σ^t such that $\sigma^t \in h$. We denote this population size at date t as $S_{h,t}$, which is formally defined as:

$$S_{h,t} = \sum_{\sigma^t \in \mathcal{H}} \mu_t(\sigma^t),$$

As $\mathcal{H}$ is a partition, any agents being in an elements $\tilde{h} \in \mathcal{H}$ at any period $t-1$ is in an element $h \in \mathcal{H}$ at period t (possibily the same). Denote as $\Pi^S_{\tilde{h},h,t-1}$ i the fraction of agents going from elements $\tilde{h}$ at period $t-1$ to elements h at period t. At each period, the transition matrix $\left(\Pi^S_{\tilde{h},h,t}\right)_{(\tilde{h},h)\in\mathcal{H}\times\mathcal{H}}$, verifies $\Pi^S_{\tilde{h},h,t} \in [0,1]$ and $\sum_{h\in\mathcal{H}} \Pi^S_{\tilde{h},h,t} = 1$, such that the evolution of population sizes of partition elements is defined as follows:

$$S_{h,t} = \sum_{\tilde{h}\in\mathcal{H}} \Pi^S_{\tilde{h},h,t-1} S_{\tilde{h},t-1}. \tag{16}$$

We now turn to the projection of a generic individual choice variable onto partition elements. This generic choice variable, denoted X, can represent savings or consumption for instance. As any individual variable, this variable will depend at date t on the idiosyncratic risk histories σ^t and of aggregate risk history z^t. The variable and its dependence will therefore be denoted as $X_t\left(\sigma^t, z^t\right)$ at date t.

The projection on the partition elements consists in averaging the variable among all agents sharing the $\sigma^t, \in h$, where $h \in \mathcal{H}$. More formally, the variable value for partition element $h \in \mathcal{H}$, which is denoted as $X_{h,t}(z^t)$ or simply $X_{h,t}$, is defined as:

$$X_{h,t} = \mathbb{A}_{h,t}[X_t] \equiv \frac{\sum_{\sigma^t\in\mathcal{H}} X_t\left(\sigma^t, z^t\right)\mu_t(\sigma^t)}{S_{h,t}}, \tag{17}$$

where $\mathbb{A}_{h,t}[\cdot]$ denotes the projection operator onto the partition element h at date t.

3.2.2 Projecting variables: Other mechanisms

Equations (16) and (17) provide the basic mechanisms for projecting the model onto the partition. However, some other operations are also useful but are a bit more subtle.

First, the previous projection relies on the linear operator $\mathbb{A}_{h,t}$, which implies that the projection of transform of a variable is not the transform of the projected variable. In other words, $\mathbb{A}_{h,t}[f(X_t)]$ generally differs from $f(\mathbb{A}_{h,t}[X_t])$, unless particular cases (such as f affine, X_t having a degenerate distribution, etc.). We will write that:

$$\mathbb{A}_{h,t}[f(X_t)] = \xi^{f,X}_{h,t} f(X_{h,t}), \tag{18}$$

where the quantity $\xi^{f}_{h,t}$ embeds the non-linearity correction of f and the shape of the distribution within h. As a consequence, this correction depends on the function f and on the variable X.

Second, the projection of lagged variable also involves a twist, since the projecting of the lagged variable is not the lagged projected variables. It indeed involves transitions from past possible partition elements to the current one h. More formally, we can show that projecting the variable X_{t-1} onto the partition element h can be expressed as:

$$\mathbb{A}_{h,t}[X_{t-1}] = \frac{\sum_{\tilde{h}\in\mathcal{H}} \Pi^X_{\tilde{h},h,t} S_{\tilde{h},t-1} X_{\tilde{h},t-1}}{S_{h,t}}, \tag{19}$$

where $(\Pi^X_{\tilde{h},h,t})_{\tilde{h},,h\in(\mathcal{H}\times\mathcal{H})}$ is a probability transition matrix, for which we have $\Pi^X_{\tilde{h},h,t} \in [0,1]$ and $\sum_{h\in\mathcal{H}} \Pi^X_{\tilde{h},h,t} = 1$. In general, this matrix depends on the variable X under consideration. The interpretation of equation (19) is rather direct and similar to the one of equation (16). The beginning-of-period (value of the variable X_{t-1} on partition element h at t is a result of a pooling-like operation. At date $t-1$, the end-of-period variable X was pooled among all agents of the partition element to generate the quantity $S_{\tilde{h},t-1}X_{\tilde{h},t-1}$. The quantity was then transited to partition element h with probability $\Pi^X_{\tilde{h},h,t}$ and then equally shared among all agents of the partition element, such that each of them is endowed with the quantity $\mathbb{A}_{h,t}[X_{t-1}]$.

Finally, the projection of expectation also implies some correction. We will describe it when handling the Euler equation in Section 3.3. We will also explain how to handle the various corrections in the computational simulations.

3.3 The projected model

We consider the economic model presented in Section 2 and an implicit partition $\mathcal{H}$. The model is characterized by the following set of equations: (i) individual budget constraints, (ii) Euler equations, (iii) market clearing conditions, and (iv) balance of the unemployment scheme. We will project the model and obtain counterparts of these different equations for the partition $\mathcal{H}$ that we consider.

Budget constraints. The individual budget constraint (8) is rather direct to project and proceeds from the basic projection mechanism (17) and of the projection (19) for a lagged variable. Using the notation for variables of the partition, the budget constraint for any partition element $h \in \mathcal{H}$ can be expressed as:

$$\begin{aligned} c_{h,t} + a_{h,t} \leq (1+r_t) \sum_{\tilde{h}\in\mathcal{H}} \Pi^a_{\tilde{h},h,t} \frac{S_{\tilde{h}h,t-1}}{S_{h,t}} a_{\tilde{h},t-1} \\ + \left((1-\tau_t)1_{s=e} + \phi_t 1_{s=u}\right) l_{h,t} y_h w_t. \end{aligned} \tag{20}$$

The interpretation of the partition budget constraint (20) is rather immediate: resources made of labor income and saving payoffs are used to consume and save. The only actual subtlety is related to the projection of past savings, which can be interpreted as a pooling operation (see the discussion after equation (19)).

Market clearing conditions. The market clearing conditions are also rather straightforward to express. Equations (12)–(14) stating the clearing for markets for savings, goods and labor respectively, become in terms of partition element variables:

$$K_t = \sum_{h \in \mathcal{H}} S_{h,t} a_{h,t}, \tag{21}$$

$$\sum_{h \in \mathcal{H}} S_{h,t} c_{h,t} + K_t = Y_t + K_{t-1}, \tag{22}$$

$$L_t = \sum_{h \in \mathcal{H}} S_{h,t} y_h l_{h,t}. \tag{23}$$

Again the interpretation of partition clearing conditions (21)–(23) is rather direct and stems from the fact that partition elements can be represented by a unique agent with a specific size. More specifically, the labor clearing condition (23) uses the fact that unemployed agents do not work and therefore do not appear in the aggregation.

Unemployment insurance. The budget of the UI scheme has to be balanced at any date and no social debt is allowed. The condition of financial balance for the UI scheme is equation (6) and becomes for the projected model:

$$\phi_t \sum_{h \in \mathcal{H}} S_{h,t} y_h l_{h,t} = \tau_t \sum_{h \in \mathcal{H}} S_{h,t} y_h l_{h,t}, \tag{24}$$

where it should be remembered that unemployed agents earn an unemployment benefit that is proportional to ϕ_t and to the labor supply an employed with the same characteristics h would supply.

Euler equations. We now conclude the representation of the projected model by Euler equations for unconstrained agents. The Euler equation (11) for labor is straightforward to project, as it is linear. It implies that the labor supply for a partition element h can be expressed as:

$$l_{h,t} = \chi^{\varphi} (1 - \tau_t)^{\varphi} y_h^{\varphi} 1_{s=e}. \tag{25}$$

Only employed agents supply labor and their effort only depends on the labor tax (negatively) and the element productivity.

The projection of the Euler equation for consumption is more involved, because of two main reasons: (i) the non-linearity of marginal utilities, and (ii) the conditional expectation operator. We can show that the consumption Euler equation for agents can be expressed as:

$$\xi^u_{h,t} U_c\left(c_{h,t}, l_{h,t}\right) - \nu_{h,t} = \beta \mathbb{E}_t \Bigg[(1 + r_{t+1}) \sum_{\tilde{h} \in \mathcal{H}} \Pi^u_{h,\tilde{h},t+1} \xi^u_{\tilde{h},t+1} \times U_c\left(c_{\tilde{h},t+1}, l_{\tilde{h},,t+1}\right) \Bigg], \tag{26}$$

where $\nu_{h,t} = 0$ if all agents in h are not constrained. Let us comment equation (26). First, the quantities $\xi^u_{h,t}$ correct for the nonlinearity of U_c and are defined in (18). Second, the expectation operator $\mathbb{E}_t\left[\cdot\right]$ should be understood with respect to aggregate risk only, since individual risks are handled explicitly in a developed summation. Note that the terms $(\Pi^u_{h\tilde{h},t})$ are nonnegative and aggregate future transitions across buckets.

4 Simulating the projected model

The projected model consists of equations (20)–(26), together with the equations (2), (4), and (5) related to the production sector and that remain unchanged in the projected model.

The projection of the model provides a exact representation of the dynamics introducing times-varying parameters such as the transition matrices $\Pi^S_{h,\tilde{h},t}$, $\Pi^a_{h,\tilde{h},t}$ and $\Pi^u_{h,\tilde{h},t}$ and coefficients $\xi^u_{h,t}$ which capture the evolution of distributions within buckets. It should be clear that the dynamics of these coefficients is unknown. Our simplifying assumption is simple. We estimate these values at the steady-state, and we assume that the relevant variables are constant in the dynamics. As a consequence, although we consider heterogeneity in each bucket of wealth, we don't consider this heterogeneity as time-varying. We now provide the formal construction.

4.1 Steady-state economy

When there is no aggregate risk, the equations of projected model charaterizing the agents' problem are

$$\xi_h^u U_c\left(c_h, l_h\right) - \nu_h = \beta(1+r) \sum_{\tilde{h}\in\mathcal{H}} \Pi^u_{h,\tilde{h}} \xi^u_{\tilde{h}} U_c\left(c_{\tilde{h}}, l_{\tilde{h}}\right) \tag{27}$$

$$l_h = \chi^{\varphi}(1-\tau)^{\varphi} y_h^{\varphi} 1_{s=e} \tag{28}$$

$$c_h + a_h \leq (1+r) \sum_{\tilde{h}\in\mathcal{H}} \Pi^a_{\tilde{h},h} \frac{S_{\tilde{h}h}}{S_h} a_{\tilde{h}} + \left((1-\tau)1_{s=e} + \phi 1_{s=u}\right) l_h y_h w. \tag{29}$$

$$S_h = \sum_{\tilde{h}\in\mathcal{H}} \Pi^S_{\tilde{h},h} S_{\tilde{h}} \tag{30}$$

We state an obvious result as a Proposition

Proposition 1: *The variables* $\left(\xi_h^u, \Pi^u_{h,\tilde{h}}, \Pi^a_{\tilde{h},h}, \Pi^S_{\tilde{h},h}\right)$ $\tilde{h}, h \in \mathcal{H}$ *can be identified at the steady state.*

We skip the proof of this proposition. It is indeed obvious, as in steady state we can compute the stationary wealth distribution of the Bewley model, one can integrate policy rules and transition probabilities to determine the relevant variables. This can be done for both explicit and implicit partitions. We can now provide our simplified projected model.

4.2 Simulating the dynamics of the projected model

Definition : *The simplified projected model is defined by the equations*

$$\xi_h^u U_c\left(c_{h,t}, l_{h,t}\right) - \nu_h = \beta(1+r) \sum_{\tilde{h}\in\mathcal{H}} \Pi^u_{h,\tilde{h}} \xi^u_{\tilde{h}} U_c\left(c_{\tilde{h},t}, l_{\tilde{h},t}\right) \tag{31}$$

$$l_{h,t} = \chi^{\varphi}(1-\tau_t)^{\varphi} y_{h,t}^{\varphi} 1_{s=e} \tag{32}$$

$$c_{h,t} + a_{h,t} \leq (1+r_t) \sum_{\tilde{h}\in\mathcal{H}} \Pi^a_{\tilde{h},h} \frac{S_{\tilde{h}h}}{S_h} a_{\tilde{h},t} + \left((1-\tau_t)1_{s=e} + \phi_t 1_{s=u}\right) l_{h_t} y_h w_t. \tag{33}$$

where $\left(\xi_h^u, \Pi^u_{h,\tilde{h}}, \Pi^a_{\tilde{h},h}, \Pi^S_{\tilde{h},h}\right)$ *are set to their steady-state values, together with equations (2), (4), and (5).*

How can we think about this economy? A useful metaphor is the island-economy often used to simplify incomplete-market models. Each bucket should be thought as an island, and we impose constant flow of agents and resources across agents, set at their steady-state values. In the case of explicit partitions, LeGrand and Ragot (2017) show that one can provide a decentralization (in terms of limited risk-sharing) of such an island metaphor[4].

4.3 Comparison with Reiter's representation

It may be useful to compare the previous construction with the one of Reiter (2009). A first difference, is that the previous construction uses more steady-state information embedded in the variables $\left(\xi_h^u, \Pi_{h,\tilde{h}}^u, \Pi_{\tilde{h},h}^a, \Pi_{\tilde{h},h}^S\right)$. The difference is deeper, because our construction defines a dynamics relying on Euler equations. To see that, consider the case where labor market transition are constant to clarify the difference. Reiter proposes to follow the time-varying measure of agents within given brackets of wealth. As a consequence, after a TFP shock, the number of agents within any bracket of wealth is time-varying, as saving decisions change after a TFP shock. As a consequence, transition probabilities across buckets would be time-varying in Reiter's algorithm. In words, when idiosyncratic risk is not time-varying, in the Reiter's algorithm, the boundaries in terms of wealth defining brackets are constant, and the number of agents in any bracket is time-varying; In our setup, as labor market transition are constant, the transition matrices $\Pi_{h,h'}^{S,a,u}$ are not time-varying by construction. It is the wealth within each bucket which is time-varying for the projected Euler equations to be fulfilled. This difference will make it possible to solve for optimal policies.

5 Ramsey program

5.1 Formulation of the Ramsey program

We now derive optimal Ramsey policies in the Bewley model. Comparing the Ramsey allocations in our setup with those of a complete insurance-market economy will enable us to identify the specific role of redistribution and the lack of insurance. However, solving for Ramsey policies in the general case is difficult. Indeed, one has to introduce additional state variables, such as the distribution of Lagrange multipliers for the relevant individual constraint.[5] Solving for this joint distribution is particularly difficult.

The main idea of the current method is to solve for the Ramsey optimal policy for the

[4] To simplify the exposition, we present the case where labor-market transitions are constant. Time-varying transitions on the labor market only require correcting transition matrices for the exogenous change in transition across buckets.

[5] The relevant individual constraint depends on the way the Ramsey problem is written. As we discuss below, in the Lagrangian approach of Marcet and Marimon (2011), these relevant constraints are the individual Euler equations. Bhandari, Evans, Golosov, and Sargent (2016) use a primal approach and thus consider the individual Lagrange multiplier on the budget constraint.

projected model given a partition $\mathcal{H}$. We first explain the methodology to compute the Ramsey policies. We then describe our algorithm for computing Ramsey policies and we finally discuss the relationships with other methods.

The Ramsey problem consists in determining the unemployment insurance policy – here equivalently, unemployment benefit rate ϕ_t and labor tax rate τ_t – that corresponds to the "best" competitive equilibrium, according to an aggregate welfare criterion. In other words, the planner has to select UI tools and individual choices, subject to budget constraints, to UI scheme balance, and to Euler equations – that guarantee the optimality of individual choices. Formally, the Ramsey problem can be written as follows:

$$\max_{\left((a_{h,t},c_{h,t},l_{h,t})_{h\in\mathcal{H}},\phi_t,\tau_t\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\sum_{h\in\mathcal{H}}S_{h,t}\xi^U_{h,t}U(c_{h,t},l_{h,t})\right], \tag{34}$$

subject to: (i) the projected budget constraint (20), (ii) the labor Euler equation (25), (iii) the consumption Euler equation (26), (iv) the UI scheme budget balance (24), (v) the market clearing constraints (21) and (23), and finally (vi) the equations (5) and (4) determining factor prices r_t and w_t. Note that we also have to to take into account the constraint driving the evolution equation of the population shares of employed and unemployed agents (15). However, since the evolution is independent of the planner's choices, it has no impact on the Ramsey policies.

Our final remark involves the objective of the Ramsey program (34). It corresponds to the projection of an additive welfare criterion. So, due to the non-linearity of the function U, we have to introduce a correcting factor $\xi^U_{h,t}$. To compute it, we make the same assumption as for the other correcting factors and we assume that the steady-state value remains unchanged.

A reformulation of the Ramsey problem. We simplify the formulation of the Ramsey problem exposed in equation (34). We first denote by $\beta^t S_{h,t}\lambda_{h,t}$ the Lagrange multiplier of the Euler equation for partition element h at date t. We also define for all $\tilde{h}\in\mathcal{H}$:

$$\Lambda_{\tilde{h},t} = \frac{\sum_{h\in\mathcal{H}} S_{hh,t}\lambda_{h,t-1}\Pi_{h\tilde{h}}}{S_{\tilde{h},t}}, \tag{35}$$

which, for agents of partition element h, can be interpreted as the average of their previous period Lagrange multipliers for the Euler equation. Finally, note that $\lambda_{ht} = 0$ if $a_{h,t} = -\overline{a}$: the multiplier $\lambda_{h,t}$ is null when the credit constraint is binding. The product $\lambda_{h,t}\nu_{h,t}$ (for any t and any h) is thus always null. The following lemma summarizes our simplification of the Ramsey problem.

Lemma 1 (Simplified Ramsey problem) *The Ramsey problem in equations (34) can be simplified into:*

$$\max_{\left((a_{h,t},c_{h,t},l_{h,t})_{h\in\mathcal{H}},\phi_t,\tau_t\right)_{t\geq 0}} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} S_{(b,y,s),t} \left(\eta_{h,t} U(c_{h,t}, l_{h,t}) \right. \tag{36}$$

$$\left. - \left(\lambda_{h,t} - (1+r_t)\Lambda_{h,t}\right) \xi_{h,t} U_c(c_{h,t}, l_{h,t})\right)$$

$$\text{s.t. } \lambda_{h,t} = 0 \text{ if } a_{h,t} = -\overline{a}, \tag{37}$$

and subject to equations (20), (25), (24), (21), (23), and finally (5) and (4).

The proof is provided in Appendix. The simplification of the Ramsey problem, which eases the computation of the maximization problem, is based on a re-writing of the Lagrangian to introduce Lagrange multipliers into the objective, as in Marcet and Marimon (2011). It could also provide a recursive formulation of the Ramsey problem that we do not need, as the sequential representation allows us to derive first-order conditions, expressed in a way that aids interpretation.

5.2 Ramsey conditions

We can easily derive the first-order conditions of the Ramsey program (36)–(37). Indeed, using proper substitution, the program can be written as a maximization subject to two choice variables: the labor tax τ_t and the savings $(a_{h,t})$. Note that we discuss in Section 5.3 below issues related to second-order conditions.

To ease the interpretation of first-order conditions, we introduce the following notation:

$$\Psi_{h,t} = \xi^U_{h,t} U_{c,h,t} - \left(\lambda_{h,t} - (1+r_t)\Lambda_{h,t}\right) \xi^u_{h,t} U_{cc,h,t}, \tag{38}$$

which is the *marginal social valuation of liquidity* for agents in partition element h. Indeed, if an agent receives one additional unit of goods today, this additional unit will have a value proportional to $U_{c,h,t}$. This value only accounts for private valuation, but should also include the effect on the internalization cost of Euler equations. Indeed, this additional unit affects agent's incentive to save from period $t-1$ to period t and from period t to period $t+1$. This effect is captured by the second term at the right hand side, proportional to $U_{cc,h,t}$.

Using this notation, the two first-order conditions of the Ramsey program can be written as follows.

$$\Psi_{h,t} = \beta \sum_{\tilde{h},} \Pi^{a}_{h,\tilde{h}} \mathbb{E}_t \left[(1 + r_{t+1}) \Psi_{\tilde{h},t+1} \right] + \beta \frac{1-\alpha}{\varphi} \mathbb{E}_t \left[\frac{1}{L_{t+1}} \left(\frac{K_t}{L_{t+1}} \right)^{\alpha-1} \right. \tag{39}$$

$$\left. \times \sum_{\tilde{h}} \left((1 - \tau_{t+1}) 1_{\tilde{s}=e} + \frac{S_{e,t+1}}{S_{u,t+1}} \left(\tau_{t+1} \left(1 + \varphi\right) - 1 \right) 1_{\tilde{s}=u} \right) \Psi_{\tilde{h},,t+1} S_{\tilde{h},t+1} l_{\tilde{h},e,t+1} y_{\tilde{h}} \right]$$

and

$$\sum_{\tilde{h},} S_{\tilde{h},t} \left(\Lambda_{\tilde{h},t} \xi_{\tilde{h},,t} U_{c,\tilde{h},,t} + \Psi_{\tilde{h},t} \tilde{a}_{\tilde{h},t} \right) \tag{40}$$

$$= \frac{1}{\alpha\varphi} \frac{K_{t-1}}{L_t} \sum_{\tilde{h}} \left(-(1 - \tau_t) 1_{\tilde{s}=e} + \frac{S_{e,t}}{S_{u,t}} \left(1 + \alpha\varphi - (1 + \varphi)\tau_t \right) 1_{\tilde{s}=u} \right) S_{\tilde{h},,t} \Psi_{\tilde{h},t} l_{\tilde{h},t} y_{\tilde{h}}$$

The first-order condition (39) is the first-order condition on saving choices. The first part of the condition involves a Euler-like equation for the marginal liquidity valuation Ψ, stating that the planner aims at smoothing social liquidity benefits across time. The second part involves the impact of savings on labor supply through the price distortion involved by saving choices.

The first-order condition (40) relates to the labor tax. The left hand side concerns the distortions of the labor on savings through the modification of the interest rate. Savings can be affected directly by a change of the interest rate or indirectly through a modification of the consumption Euler equation. The left hand side concerns the distortions on labor supply. These distortions can be direct, through a higher tax burden for employed agents or a larger unemployment benefit for unemployed ones. There is also an indirect effect through a reduction in labor supply that affect both employed and unemployed agents – because the benefits of the latter are indexed on the labor supply of employed agents.

5.3 Remark on the convexity of the program

A traditional problem with Ramsey program is that the set of feasible allocations is not convex in general. This problem is quite general and also exists in a representative-agent economy. The non-convexity is precisely related to constraint associated to Euler equation – which is neither convex nor linear. Therefore, if first-order conditions are still necessary, they may be non-sufficient and generate three different types of problems:

1. the first order condition may characterize a local minimum;

2. the steady-state solution may not exist;

3. multiple equilibria may exist.

The first concern can be easily addressed, for instance by checking that small variations around the solution allocation do not yield a higher aggregate welfare. The second concern has been raised by Straub and Werning (2014), who show that in some cases the solution of the planner may not be an interior solution with constant real variables.[6] The possibility to solve the model with perturbation methods helps solve this issue. Indeed, studying the behavior of the model after perturbing the steady state with small aggregate shocks, provides insight regarding the subsequence convergence –or not– toward the interior solution. The last concern is more difficult to properly address. Up to our knowledge, the only imperfect solution consists in exploring the convergence for various initial values and checking that the local maximum is indeed a global one.

5.4 Comparison with other methods

To our knowledge, only three other papers provide general solution method to derive optimal Ramsey policies in incomplete insurance-market models.

First, Açikgöz (2015) provides an algorithm to solve for the steady-state allocation of the Ramsey program. He assumes some specific functional forms and show the convergence of the algorithm. This is a way to find the joint distribution over Lagrange multipliers and initial asset holdings. At this stage, we are not aware of any application of this algorithm to an economy with aggregate shocks.

Second, Nuño and Moll (2017) use a continuous-time approach and mean-field games to characterize optimal steady-state allocations. Their algorithm develops a projection method to characterize the relevant value functions and Lagrange multipliers. Our solution makes a more extensive use of the steady-state properties of the Bewley model, that enables us to properly distort the projection on a relevant grid. Although our model is expressed in discrete time, a methodology similar to ours can be applied to continuous-time models. An additional gain of our method in discrete time is that introducing aggregate shocks is straightforward, as we have seen above.

Third, Bhandari, Evans, Golosov, and Sargent (2016) present a solution method for models with aggregate shocks. Their solution relies on perturbation methods around time-varying

[6]Recent contributions such as Chari, Nicolini, and Teles (2016) show that the behavior of Lagrange multipliers depends on the set of instruments available to the planner. In addition, Chen, Chien, and Yang (2017) show theoretically that in an incomplete insurance-market model that the solution is interior.

allocations (and not around the steady-state). They solve the model by approximating the actual distribution by a very large number of agents. As we use more extensively the steady-state properties of the Bewley model, we can simulate the economies with a very small number of agents –see Section 6. As a consequence, our solution allows us to study Ramsey problems with a number of instruments.

6 Numerical examples

We first provide parameter values and then perform two quantitative exercises. In the first one in Section 6.2, we solve the model with constant replacement rate and exogenous labor supply, to compare our solution techniques with global solution methods, such as Krusell and Smith (1998). In the second exercise of Section 6.3, we derive the optimal unemployment benefits and study its dynamic properties.

6.1 Model parameters

The period is a quarter. The calibration for both aggregate risk and idiosyncratic earning risk is adapted from Krueger, Mittman, and Perri (2018), who provide a calibration strategy for both the employment, the productivity and the TFP risk.[7] The process for the aggregate risk is set to match the probability of severe recessions in the US postwar economy, over the period 1948Q1-2014Q3. A severe recession is defined as a period where the unemployment rate is above 9%. The frequency of severe recessions is 16.48% with an expected length of 22 quarters. The average unemployment rate during these recessions is 8.39%, whereas the average unemployment rate in good times is 5.33%. The average drop in GDP per capita is 7% compared to normal times. This process can be approximated by a two-state Markov process to solve the model with global methods, or by a continuous-support Markov process to solve the model with perturbation methods. We specify both solutions below.

Two-state Markov chain process. W consider two aggregate states (defined by the value of TFP), corresponding to a state of severe recession denoted by $Z_t = Z_l$ and to a normal state denoted with $Z_t = Z_h$. The Markov transition matrix for the aggregate states

[7] Den Haan (2010) uses a framework with only the employment risk to compare the simulations methods. Krueger, Mittman, and Perri (2018) consider both employment and productivity risks. This last calibration strategy is able to reproduce roughly the US wealth distribution, which is mostly the outcome of the productivity risk. We also compared our solutions method with other standard ones with unemployment risk only. The results are almost unchanged. For the sake of brevity, we provide these additional results in the Technical Appendix.

is straightforward to derive. Then, using the methodology of Shimer (2003), we can compute four employment-unemployment transition matrices between $t-1$ and t for $\{Z_{t-1}, Z_t\} \in \{\{Z_h Z_h\}, \{Z_l Z_h\}, \{Z_h Z_l\}, \{Z_l Z_l\}\}$. All these matrices are provided in Appendix, and will be used to solve the model with global methods.

Continuous state-space. The previous process is best approximated with the standard AR(1) process for TFP shocks,[8] $Z_t = \exp(z_t)$, with:

$$z_t = (1-\rho_z) + \rho_z z_{t-1} + \varepsilon_t^z, \tag{41}$$

where $\varepsilon_t^z \sim \mathcal{N}(0, \sigma_z^2)$, $\rho_z = 0.9464$ and $\sigma_z = 0.4\%$.

Then the job separation and job finding rates are functions of current and past values of the TFP level :

$$\begin{aligned} \Pi_{ue,t} &= \Pi_{ue}^{SS} + \rho_0^{ue} z_t + \rho_1^{ue} z_{t-1}, \\ \Pi_{eu,t} &= \Pi_{SS}^{eu} + \rho_0^{eu} z_t + \rho_1^{eu} z_{t-1}, \end{aligned} \tag{42}$$

where Π_{ue}^{SS} and Π_{SS}^{eu} are set to their postwar values. The two process generates the same first and second moments for $\Pi_{ue}^{SS} = 0.786$, $\Pi_{SS}^{eu} = 0.048$ and $(\rho_0^{ue}, \rho_1^{ue}) = (3.483, 0.436)$ and $(\rho_0^{eu}, \rho_1^{eu}) = (-0.613, 0.219)$. As expected, and as is consistent with the labor literature, the job separation rate is less volatile than the job finding rate and is countercyclical on impact.

Productivity risk. The income risk conditional on employment is estimated by Krueger, Mittman, and Perri (2018) using annual data. Translating these estimates into quarterly values (with the same autocorrelation and variance), we deduce the following process:

$$\log Inc_t = \rho^y \log Inc_{t-1} + \varepsilon_t^y,$$

with $\varepsilon_t \sim \mathcal{N}(0, \sigma_z^2)$. We find an autocorrelation equal to $\rho^y = 0.9923$ and a variance equal to $\sigma_y^2 = 0.0098$. The Rouwenhorst procedure is then used to discretize the process $\log Inc_t$ into a seven-state Markov process. When labor is exogenous, the seven idiosyncratic productivity levels are equal to these seven states, as in Krueger, Mittman, and Perri (2018). When labor is endogenous one can easily transform the productivity process to obtain the same process for

[8] Krueger, Mittman, and Perri (2018) estimate this process with a two-state Markov process, in order to use the Krusell and Smith (1998) algorithm. Our AR(1) representation is consistent with their estimation. In Appendix, we provide their estimated values for the sake of comparison.

Parameter	Description	Value
β	Discount Factor	0.99
α	Capital share	0.36
δ	Depreciation rate	0.025
ϕ	Replacement rate	0.5
Π_{ue}^{SS}	Average job finding rate	0.786
Π_{ue}^{SS}	Average job separation rate	0.048
ρ^z	Autocorrelation TFP	0.9464
σ_z	Standard deviation TFP shock	0.004
$(\rho_0^{ue}, \rho_1^{ue})$	Corr. job find. rate with TFP and TFP(-1)	$(3.483, 0.436)$
$(\rho_0^{eu}, \rho_1^{eu})$	Corr. job sep. rate with TFP and TFP(-1)	$(-0.613, 0.219)$
ρ^y	Autocorrelation idio. income	0.9923
σ^y	Standard dev. idio. income	0.0990
χ	Scaling param. labor supply	0.38
ξ	Frish elasticity labor supply	0.5

Table 1: Parameter values. See main text for descriptions and targets

total income (considering the Frish elasticity of the labor supply in the GHH specification). As agents can be either employed and unemployed, each agent can be in $14 = 7 \times 2$ idiosyncratic states.

Production function. The production function is $F(K, L) = ZK^{\alpha}L^{1-\alpha} - \delta K$, where the capital share is $\alpha = 0.36$ and the depreciation rate is $\delta = 0.025$. Z is the productivity process presented above.

Preferences. We assume a log period utility function ($\sigma = 1$) and a discount factor equal to $\beta = 0.99$. The Frish elasticity of labor supply is set to $\xi = 0.5$ following the estimates of Chetty, Guren, Manoli, and Weber (2011) and the scaling parameter is $\chi = 0.38$ to normalize aggregate labor supply at 1. Table 1 provides a summary of the model parameters.

6.2 Exogenous replacement rate and exogenous labor supply

To compare our solution method with global ones, we first study the model with an exogenous labor supply and a fixed replacement rate, set at $\phi = 50\%$, which is the standard value used in the quantitative literature. This model can be solved with global solution methods, such as Krusell and Smith (1998), and with our perturbation methods, to compare results.

First, to define an implicit partition we solve for the steady-state of the corresponding Bewley model, without aggregate shocks, where $\sigma^z = 0$, $\Pi_{ue,t} = \Pi_{ue}^{SS}$ and $\Pi_{eu,t} = \Pi_{eu}^{SS}$.

Second, we consider an implicit partition based on the distribution of wealth. We consider ten brackets of wealth. All constrained agents, independently of their type, belong to the first bracket of wealth, while and the remaining 9 brackets of wealth have the same size.[9] We then consider the distribution of the 14 different agents' types within these brackets. We remind that each agent can be in one of 14 different individual states. As a consequence, the partition has $140 = 7 \times 2 \times 10$ different elements.

Third, we compute the values for the ξ correction parameters that enable to exactly reproduce the distributions of wealth and of consumption on the projected partition. For ten brackets of wealth, the standard deviation across the different sets of histories is 1.62%.

Fourth, we simulate the mode using perturbation methods, with the exogenous process (41)-(42).

Steady-state results. We report the steady state distribution of wealth in the model and in the data, using the the Survey of Consumer Finance 2007, to avoid considering the temporary effects of the 2008 financial crisis. The distribution labeled "Model" is both the distribution of the Bewley model and the distribution of the projected model, as the brackets of wealth have been chosen accordingly. The Gini coefficient of wealth distribution generated by the model is high, and equals 0.72 and close to its empirical counterpart value of 0.78. Table 2 reports the distribution of wealth for the various quintiles and for the top decile of the wealth distribution. The distribution of wealth is close to its empirical counterpart. The model fails to reproduce the concentration of wealth at the top of the distribution, what is a well-known feature of these models in the literature. Heterogeneous discount factors or heterogeneity in the return to human capital (entrepreneurship) would help to reproduce this concentration. The average Gini

[9] Steady state capital stock is 34.70 and the wealth thresholds to define the brackets of wealths are (0.0100;0.2367; 0.3918; 1.7995; 5.2570; 12.3237; 24.0465; 50.0610; 93.6606; 814.5871). The last value is the highest amount of wealth held by any agent in the model.

	Q1	Q2	Q3	Q4	Q5	D10	Gini
Model	0.1	0.9	5.3	18.3	75.5	69.9	0.72
SCF 2007	−0.2	1.2	4.6	11.9	82.5	53.7	0.78

Table 2: Distribution of wealth

Moments	K-S	Model(sim)	Model(theory)	Description
$sd(Y_t)$	8.81	8.87	8.48	Standard deviation of output
$sd(C_t)$	5.43	5.57	5.33	Standard deviation of output
$sd(L_t)$	1.08	1.12	1.08	Standard deviation of output
$sd(w_t)$	3.32	3.09	2.94	Standard deviation of output
$sd(r_t)$	0.06	0.05	0.05	Standard deviation of output
$corr(Y_t, C_t)$	0.92	0.96	0.95	Correlation output and consumption
$corr(Y_t, L_t)$	0.99	0.99	0.99	Correlation output and labor

Table 3: Comparing second-order moments with different resolution techniques

coefficient in the model with aggregate shocks and in the Bewley model are indeed very close (0.726 and 0.725).

Dynamics results. We simulate the model with aggregate shocks to compute second-order moments. For the sake of comparison, we solve the model with three solution techniques. First, we simulate the model using the Krusell-Smith (K-S, henceforth) solution technique. More precisely, we use the Krueger, Mittman, and Perri (2018) algorithm, based on the computational strategy of **?** and **?**. This algorithm uses projection methods to solve for the optimal policies and simulation techniques to iterate on an aggregate law of motion in capital. Second, we solve for the dynamics of the projected model using DYNARE and simulate the model for 3500 periods to obtain second-order moments (a number of periods consistent with the simulation of the K-S model). Third, we also use DYNARE to find theoretical second-order moments. This last possibility is available because of the structure of our model, which has a high but finite number of equations to simulate. The difference between the last two economies allows us to identify sampling errors in our model. Results are reported in Table 3.

The moments generated by the three methods are very similar to each other. The simulated moments are very close between the K-S economy and the simulation of the projected model.

The theoretical moments are a little bit different from the simulated moments, due to sampling errors. The possibility to easily derive theoretical moments, avoiding sampling errors, is an advantage of our simulation technique.

6.3 Optimal unemployment benefits

We now consider the model with endogenous labor supply to determine the optimal replacement rate. To do so, we use the following algorithm.

1. Consider a partition of wealth $\mathcal{H}$, having N elements, with equal size (except the first one, which only include credit constrained agents). As there are 14 idiosyncratic states, there are thus $14 \times N$ different agents in the model.

2. Consider a replacement rate ϕ.

3. Solve the Bewley model.

4. Project the model on $\mathcal{H}$ at the steady state

5. Determine the optimal value of λ, Λ and Ψ in the model, using equation (39).

6. Iterate on ϕ until equation (40) is fulfilled.

7. Check that the optimal value of ϕ is not affected by the number of elements in the partition, N.

We now report the results. We consider a number of elements $N = 18$. The optimal replacement rate is $\phi = 63\%$ for $N = 18$. A summary of the model statistics is provided in Table 4. These statistics are the same asin the initial Bewley model and in the projected mode (no matter N) by construction. For the sake of comparison, we provide the same statistics for a smaller value of the replacement rate $\phi = 57\%$ (The choice for this value will be clear below). As expected, a lower replacement rate generates a higher precautionary saving and thus a higher capital stock and a higher GDP. The cost of this higher GDP is lower insurance and a lower relative consumption of unemployed workers compared to employed ones.

The first two lines of Table 5 report the convergence properties of correcting parameters of the model η and ξ^E, when the number of elements in the partition of wealth N increases. As expected, the variance of both $var(\eta)$ and $var(\xi^E)$ decreases monotonically when the number of partition elements N increases. The last line of Table 5 represents the optimal replacement

	r	w	K	L	Y	c_u/c_e	τ
$\phi = 63\%$	0.75%	2.474	42.35	0.990	3.828	79%	3.9%
$\phi = 57\%$	0.76%	2.474	42.41	0.990	3.834	78%	3.5%

Table 4: Model statistics for different replacement rate ϕ

	N	6	10	18	21
$\phi = 63\%$	$var(\eta)(\times 10^{-4})$	5.79	2.61	1.21	1.05
$\phi = 63\%$	$var(\xi^E)(\times 10^{-4})$	12.60	5.32	2.32	2.03
Optimal $\phi(\%)$		61	63	63	63

Table 5: Variance of aggregation coefficients

rate ϕ as a function of the number of partition elements N, to check that our results are not too sensitive to the size of the partition (item 7. of the previous algorithm). We find that the optimal replacement rate converges rapidly toward 63%.

6.4 Understanding the steady state replacement rate, comparison with the sufficient-statistics approach

The optimal social contribution rate we computed takes into account the saving distortions (captured by the Lagrange multiplier λ), and the effect on prices (captured by the concavity of production function α). It is interesting to assess how these two general equilibrium effects change the estimated optimal social-contribution rate. Indeed, the literature on the "sufficient statistics approach", described in the survey of Chetty (2009), typically assesses optimal social-insurance without considering these general equilibrium effects. This partial equilibrium rate is thus only determined by the tradeoff between providing insurance for unemployed, and reducing the labor supply of employed households, while keeping prices constant.

To evaluate the impact of general equilibrium effects, we construct the corresponding "sufficient statistics" optimal social-contribution rate by setting $\lambda = 0$ and $\alpha = 0$ in our optimal formula for τ. We find the expression:

$$\tau^{suff} = \frac{1}{1 + \varphi/\Delta_\Psi^{suff}},$$

with:

$$\Delta_\Psi^{suff} = 1 - \frac{U}{1-U} \frac{\sum_{\tilde{b},\tilde{y}} S_{(\tilde{b},\tilde{y},e),t} \eta_{(\tilde{b},\tilde{y},e)} U_c(\tilde{b},\tilde{y},e)\tilde{y} l_{\tilde{y}}}{\sum_{\tilde{b},\tilde{y}} S_{(\tilde{b},\tilde{y},u)} U_c(\tilde{b},\tilde{y},u)\tilde{y} l_{\tilde{y}}},$$

where U is the unemployment rate. We do not try to bring this last formula to the data, but we use it instead to assess the magnitude of general equilibrium effects within our model, as a relevant laboratory to assess the magnitude of general equilibrium effects.[10] We compute this statistics using the model outcomes simulated for the optimal replacement rate $\phi = 63\%$. We find $\tau^{suff} = 3.53\%$ and the implied replacement rate $\phi^{suff} = 57\%$. The model outcome for this replacement rate was provided in Table 4.

From this comparison, we observe that considering general equilibrium effect increases the optimal replacement rate by 10%. The reason is that in this economy the capital stock is too high due to the amount of precautionary saving. Increasing the replacement rate compared to the optimal partial equilibrium value helps reduce this over-accumulation.[11]

6.5 Optimal Unemployment benefits over the business cycle

We finally provide the dynamics of the optimal replacement rate over the business cycle. First, to identify the mechanisms at stake, we provide the dynamics for TFP shock only, shutting down the time-varying component of the employment risk. This experiment provides first insights on the optimal design of time-varying unemployment benefits. We thus use the process presented in equations (41)–(42) taking $(\rho_0^{eu}, \rho_1^{eu}, \rho_0^{ue}, \rho_1^{ue}) = (0, 0, 0, 0)$.

We first present the IRF after a positive TFP shock, plotted on Figure 1 for six relevant variables. The variable denoted ratio is the ratio of the average consumption of unemployed agents divided by the average consumption of employed agents. This provides n a simple measure of time-varying inequality. The replacement rate falls rapidly after a TFPshock. The implied social contribution falls, which contributes to increase labor supply, so as to benefit from the positive TFP shock. The ratio of the consumption of unemployed to employed households falls due to both the increase in wages and the decrease in the replacement rate.

Table 6 provides second-moment statistics for the economy with optimal time-varying replacement rate (first row) and for the economy where the replacement rate is constant, set at its optimal steady-state value (second row).

As explained above, the optimal replacement rate is countercyclical after a TFP shock. It decreases to induce an increase in labor supply. As a consequence, the unemployed households benefit less from a TFP shock when ϕ is time-varying (i.e., inequality decrease less when ϕ is time-varying and $corr(Y, \frac{c_u}{c_e})$ is less negative).

[10] Computing τ^{suff} using empirical moments would require the knowledge of many statistics, such as preference parameters and the joint distribution of consumption and labor supply

[11] It is known that incomplete insurance markets can generate either over-accumulation or under-accumulation of capital compared to the constrained efficient equilibrium, see Dávila, Hong, Krusell, and Ríos-Rull (2012) and Aiyagari (1995) for two different cases. In our setup, with both employment and productivity risk, we observe an over-accumulation of capital.

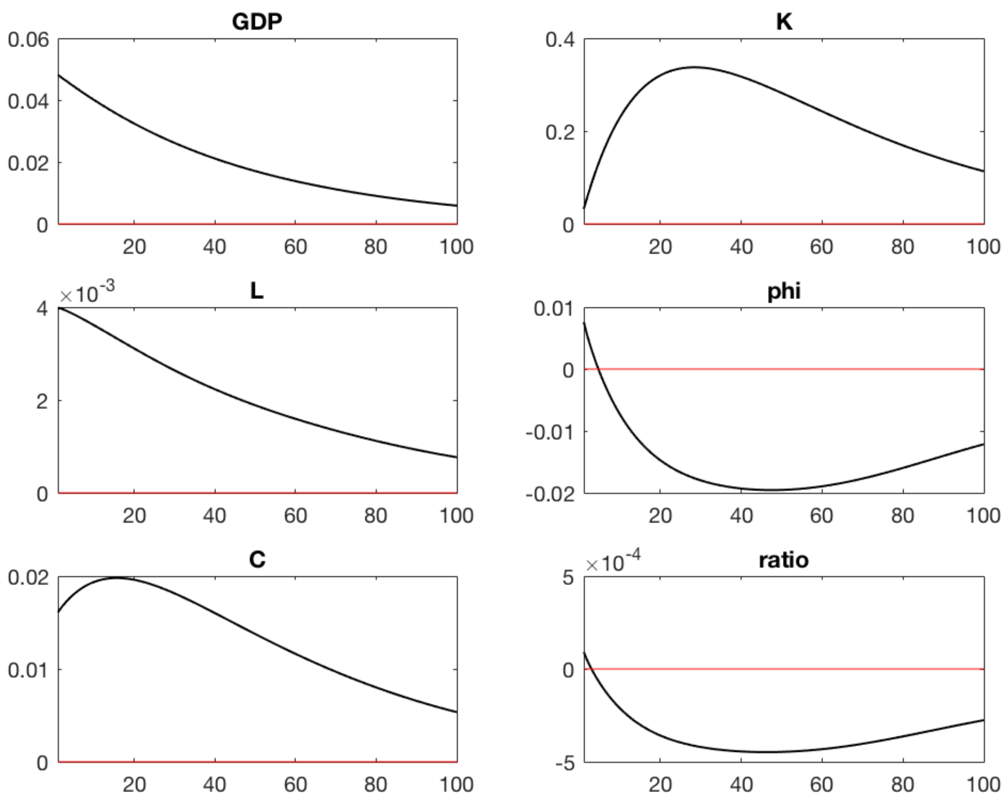

Figure 1: IRF after a TFP shock

		std(Y)	$corr(Y,\phi)$	$corr(Y,L)$	$corr(Y,C)$	$corr(Y,\frac{c_u}{c_e})$
TFP shock only	Optimal ϕ	0.25	-0.42	0.97	0.965	-0.44
	$\phi = \phi^{SS}$	0.24	0	1	0.966	-0.56
TFP and lab. mark. shock	Optimal ϕ	1.28	-0.9	0.99	0.98	-0.9

Table 6: Second moment statistics, for optimal ϕ (first line) and constant ϕ

We finally consider that both TFP and labor market risk are active, i.e., $\left(\rho_0^f, \rho_1^f, \rho_0^l, \rho_1^l\right)$ set to their estimated value. The replacement rate appears to fall more after a good shock, as unemployment decreases. This contributes to an additional increase in labor supply in booms. This last result is preliminary and we are currently working on a decomposition of the effect to understand it better.

7 Conclusion

This present a general projection theory of incomplete insurance market economies. The main idea is to use a partition in the space of idiodyncratic histories to obtain a relevant finite-dimensional state-space. We have performed comparisons with other simulating methods to check that our simulation strategy improves on current ones. Nevertheless, its main advantage is that it allows for the solution of Ramsey problem with aggregate shocks. This opens the possiblity to consider many relevant policy problems with heterogeneous agents and aggregate shocks.

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