The Optimal Maturity of Government Debt

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Abstract

We develop a novel class of perturbations to study the optimal composition of a government's debt portfolio. We derive a formula for the optimal portfolio and show that it can be expressed in terms of estimable "sufficient statistics". We use U.S. data to calculate the key moments required by our theory and show that they imply that the optimal portfolio is approximately geometrically declining in bonds of different maturities and requires little rebalancing in response to aggregate shocks. Our optimal portfolio differs from portfolios prescribed by existing models often used in the business cycle literature and also from those adopted by the U.S. Treasury. The key normative differences are driven by counterfactual asset pricing implications of standard models.

1 Introduction

In this paper we develop a framework to analyze how a government should manage its portfolio of debts and other financial assets. The government in our economy collects tax revenues, spends them on transfers as well as non-transfer expenditures, and uses a range of securities to smooth aggregate shocks to maximize its welfare. To break Ricardian equivalence, we assume that some agents cannot participate in the financial markets. We develop a novel class of pertubations that allow us to study the optimal composition of government portfolio in relatively general settings. Using these pertubations, we derive a formula for the optimal portfolio and show that it can be expressed in terms of (i) estimable "sufficient statistics" – such as covariances between returns and macro aggregates such as output or primary deficits, elasticities of bond prices and elasticity of tax revenues with respect to debt issuance, as well as (ii) parameters that determine government's attitude towards risk. In the benchmark case when the government and private agents share attitudes towards risk, all the preference parameters drop out. Then, applying our formula to data on bond returns and macro aggregates in the U.S., we find that the optimal portfolio involves issuing outstanding debt largely in form of a real 'consol' so that the amount of payments due in future periods decay exponentially with very little rebalancing over time.

The economic forces that drive the optimal composition are made transparent by decomposing the expression for the optimal portfolio in four intuitive terms: (i) *rollover risk*, which arises from fluctuations in the risk-free rate; (ii) *relative hedging concerns* that are captured by the difference between covariances of primary deficit and output with holding period returns on debts adjusted by the volatility of holding period returns; (iii) *price impact* of the government embedded in the elasticities of bond prices with respect to changes in bond supply, and (iv) a term that captures *fiscal externality*, i.e., the sensitivity of tax revenues with respect to changes in the portfolio composition of the debt.

The first term describes how the government can structure its portfolio to minimize risk from fluctuations in future interest rates. When all agents have the same attitude towards risk, uncertainty about future debt payments represents pure cost and, all other things being equal, the agents and the government would like to structure their portfolio to minimize it. The fluctuations in future rates, and in particular holding period returns, also provides hedging opportunity, which is captured by the second term. Here the key object is the difference in hedging opportunities that the fluctuation in these returns offer to the agents and the government. When all agents in the economy share the same attitudes towards risk, the first two terms, that is, the terms capturing rollover risk and hedging considerations, do not depend on the preference specification. Finally, the price impact and the fiscal externality terms in the formula captures the extent to which the government can use its monopoly power in issuing government debt for redistribution. If the changes in the composition of debts affects debts prices and tax revenues, the government can exploit this power to redistribute resources between investors in debt and transfer-recipients.

All the terms that appear in our formulas have direct empirical counterparts in the data. The existing empirical evidence from studies that looked at the quantitative easing programs of the federal reserve suggests that the price impact and fiscal externality terms are small. We then document in the data the magnitudes of first two terms: rollover risk and hedging, and study the implications for the debt management. We find that the rollover risk swamps hedging benefits and the optimal portfolio is primarily structured to minimize interest rate risk. This risk is minimized if the government issues pure discount debts of different maturities in such a way that the share of the market value of debt in a given maturity is exponentially declining in that maturity; alternatively this portfolio can be replicated by issuing a consol. We also find that short debt is a relatively better hedge for the government than for agents, and so the optimal portfolio has shorter maturity than the risk-minimizing portfolio, especially if the level of outstanding debt is small. Quantitatively, these hedging benefits are modest; optimal portfolio weights are stable over time and require almost no rebalancing in response to shocks.

Our findings contrasts to a large macro literature on optimal term structure of government debt that goes back to the seminal work of Angeletos (2002) and Buera and Nicolini (2004).[1] A typical finding in that literature is that in the canonical neoclassical growth model the government should issue long-term debt valued at tens or even hundreds times GDP while simultaneously taking an offsetting short position in short-term debt of a similar magnitudes. The optimal portfolio massively rebalances after aggregate shocks. Furthermore, the composition of an optimal portfolio is very sensitive to the menu of traded maturities. In contrast, we find moderate portfolios which are fairly stable over time. We show that the difference in findings is driven by counterfactual implications of the neoclassical growth model regarding the behavior of holding period returns on government debts. Standard parameterizations imply that such returns are very smooth and highly correlated with fluctuations in primary deficit, and as a result also with each other across maturities. This allows the government

[1] Other examples of such findings are in Farhi (2010); Faraglia et al. (2018); Lustig et al. (2008); Debortoli et al. (2017)

to hedge its shocks very well but it needs to take extreme debt positions to do so. Viewed through the lenses of our formula, such models imply that the relative hedging term is very large and time-varying. In contrast, we show that in the U.S. data it is very small and stable.

In addition to the macro Ramsey literature mentioned above, our work builds on two more strands of literature. The broad spirit of our approach, in that it derives optimal formulas in terms of empirically observable objects is closely related to the "sufficient statistics" literature used in public finance (Chetty (2006, 2009); Piketty and Saez (2013)). To derive interpretable formulas that literature often uses first order approximations of optimality conditions. This approach is not helpful for the optimal portfolio problems which can only be obtained by using second-, third- and even fourth- order approximations. Applying these approximations directly is impractical as they yield intractable high-order polynomial expressions. We instead pursue a different type of perturbation with respect to the amount of risk in the economy, that builds on the approaches developed by Samuelson (1970) and applied in other contexts by Guu and Judd (2001); Devereux and Sutherland (2011).

The second strand of literature we build on is portfolio problems studied in finance (e.g. Campbell and Viceira (1999); Viceira (2001); Campbell and Viceira (2001)). That literature typically thinks of an individual investor who trades at exogenously specified prices without taking a stance on who stands on the other side of those trades. This assumption is problematic for studying optimal government policies, since the preferences of the government, via social welfare functions, and the preferences of agents are closely linked. Incorporating this "general equilibrium" considerations can have a dramatic effect on the optimal prescriptions for managing portfolios. In particular, we show that one prediction of the "partial equilibrium" formulas that come from our analysis is that the government should issue only ultra-short debt (because it is the cheapest in the data), invest proceeds in a various risky assets to chase excess returns, and to rebalance its the portfolio frequently. All these implications disappear once the government takes into account the preferences of the agents it trades with.

The paper is organized as follows. In section 2 we develop our approach in the simplest settings, which we dub "a small government" case. The key assumption in this section is that the size of the pool of investor with whom the government trade is large, so that small changes in the composition of debts in government portfolio have no impact on debt prices. This assumption allows us to develop our main insights in the most transparent way. We then show that they extend to more general settings in section 3. In section 4, we use data on U.S. government debt, output, and primary deficit to document the key objects that appear in our formulas that characterize the optimal debt composition. In section 5, we calibrate a version of the neoclassical growth model which is extended so that it can match the key moments implied by the theory and evaluate quantitatively optimal the Ramsey policies. We also compare optimal portfolios to those held by U.S. Treasury. Section 6 concludes.

2 Theory: The small government case

In this section we use a simple model to illustrate our core insights. Time is discrete and lasts $\mathcal{T} \leq \infty$ periods. There are two sets of agents, "rich" (R) and "poor" (P), of measures $1-\lambda$ and λ respectively, and a government. Rich agents receive exogenous stochastic income Y_t in period t, pay taxes, and trade a set of $I+1 \leq \infty$ securities $\tilde{\boldsymbol{b}}_t = \left\{\tilde{b}_t^i\right\}_{i=0}^I$ at prices $\boldsymbol{q}_t = \{q_t^i\}_{i=0}^I$ that pay dividends $\boldsymbol{d}_{t+1} = \left\{d_{t+1}^i\right\}_{i=0}^I$. The dividends can be either deterministic at the time when the security was first issued (corresponding to real bonds of various maturities and coupon payments) or stochastic. Security $i=0$ is a one period discount bond.[2] Poor agents are hand-to-mouth and receive no income other than government transfers T_t. The government collects tax revenues, pays transfers to the poor as well as exogenous non-transfer expenditures, and trade securities with the rich agents.

It will be convenient to normalize all units in the government's budget constraint by the number of the poor agents. Let $\Upsilon_t(Y_t)$ and G_t be the amount of tax revenues and non-transfer expenditures per poor person, where Υ_t is an arbitrary differentiable function. Thus, the government budget constraint reads

$$T_t + \boldsymbol{q}_t \tilde{\boldsymbol{B}}_t = S_t + (\boldsymbol{d}_t + \boldsymbol{q}_t)\,\tilde{\boldsymbol{B}}_{t-1},$$

where $S_t \equiv \Upsilon_t(Y_t) - G_t$ and $\tilde{\boldsymbol{B}}_t$ is a vector of securities that the government holds, again per unit of poor agents. Negative values of $\tilde{\boldsymbol{B}}_t$ correspond to government issuing debt in that security. Since we mainly think of these securities as debts, they must be in zero net supply and satisfy feasibility

$$(1-\lambda)\,\tilde{\boldsymbol{b}}_t + \lambda \tilde{\boldsymbol{B}}_t = \boldsymbol{0}. \tag{1}$$

The taxes that a rich agent pays are given by $\frac{\lambda}{1-\lambda}\Upsilon_t(Y_t)$.

[2] A clarification might be helpful about the set of securities. The set of I securities includes all securities that can ever be traded. For exaple, in the economy in which agents can trade 2 period bonds, set I includes infinite many of such bonds indexed by the time when they are issued. It will be understood throughout that agents cannot trade such short-lived securities before they are issued or after they have matured. Keeping this convention about the set I allows us to avoid a cumbersome notation.

Agent $j \in \{P, R\}$ has preferences $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \eta_t^j u^j \left(c_t^j\right)$, where c_t^j is consumption of agent j in period t, $u^j(\cdot)$ is strictly concave, twice differentiable and satisfies the Inada conditions, and η_t^j is an exogenous stochastic process capturing preference shocks. Government objective is given by welfare function

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \eta_t^P u^P(T_t) + \mu \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \eta_t^R u^R\left(c_t^R\right)$$

for some Pareto weight $\mu \geq 0$ on the rich agent. We consider the problem of optimally managing portfolio $\tilde{\boldsymbol{B}}_t$ to maximize this welfare.

Our baseline case is when the fraction of poor agents to agents is small, $\lambda \to 0$. We refer to this case as a "small government", since government debt and taxes are the infinitesimal part of rich agents' income and their consumption satisfies $c_t^R = Y_t$. The main simplifying feature of this economy is that prices for government debts do not depend on government portfolio. This case not only illustrates most of the key insights from richer models but also naturally connects to both partial equilibrium portfolio choice problems studied in finance and open economy sovereign debt literature in international macro.

Before we proceed we want to make several points about this set up. Firstly, in this baseline model we treat both income Y_t and government taxes $\Upsilon_t(\cdot)$ as exogenous. We do that only for transparency. All the arguments in this section go through when income of the rich is endogenous and is determined by the optimal supply of labor. As for the choice of tax function, our approach to solving portfolio problem applies to any tax schedule, whether it is chosen optimally or not. Secondly, we assume that poor agents cannot access financial markets. There are two reasons why we make this assumption, one empirical and one theoretical. Empirically, most of the recipients of transfers appear to have little or no financial assets and exhibit very little consumption smoothing. The hand-to-mouth assumption appears to be a reasonably empirical description of their behavior (e.g. Mankiw (2000)). Theoretically, in order to study optimal management of government debt one needs to introduce some friction to break the Ricardian equivalence, and limited asset market participation appears to be an obvious one.[3] Finally, the assumption that there are only two types of agents simplifies the exposition without changing our main results.

[3] One common perception, based on the intuition from the representative agent economies, is that the introduction of distortionary taxes would also break the Ricardian equivalence. In heterogeneous agents economies this perception is generally false, as optimally set taxes are distortionary but the Ricardian equivalence holds (Werning (2007); Bhandari et al. (2017c)). In any case, as we show later in this paper our approach and insights extend to such settings.

Let $B_t \equiv \boldsymbol{q}_t \tilde{\boldsymbol{B}}_t$ be the market value of debt, $B_t^i \equiv q_t^i \tilde{B}_t^i$ be the market value of holdings of security i, $\omega_t^i \equiv B_t^i / B_t$ be the share of the portfolio allocated to security i, and $R_{t+1}^i \equiv \frac{d_{t+1}^i + q_{t+1}^i}{q_t^i}$ be the holding period return. In this new notation we can re-write government's budget constraint as

$$T_{t+1} + B_{t+1} = S_{t+1} + \left(R_{t+1}^0 + \sum_{i \geq 1} \omega_t^i \left(R_{t+1}^i - R_{t+1}^0 \right) \right) B_t. \tag{2}$$

Take any path of debts and taxes, and consider the following perturbation: in any period t decrease ω_t^0 by ε and increase ω_t^i by $\delta^i \varepsilon$ where $\sum_{i \geq 1} \delta^i = 1$ and then adjust transfers in period $t+1$ to absorb any change in income generated by this perturbation. This perturbation leaves the market value of debt B_t unchanged in all periods and only affects transfers in $t+1$. The reallocation generates stochastic return $\sum_{i \geq 1} \delta^i \omega_t^i \left(R_{t+1}^i - R_{t+1}^0 \right) B_t \varepsilon$ in period $t+1$. The welfare effect of this perturbation as $\varepsilon \to 0$ is then given by

$$\beta^t \mathbb{E}_t \eta_{t+1}^P u_c^P \left(T_{t+1} \right) \sum_{i \geq 1} \delta^i \omega_t^i r_{t+1}^i B_t,$$

where $r_{t+1}^i \equiv R_{t+1}^i - R_{t+1}^0$ are the excess return of security i over the one period bond.

If the portfolio in period t is chosen optimally, then there should be no gain from such perturbation for any $\boldsymbol{\delta}$. Therefore, in the optimum we must have

$$\mathbb{E}_t \eta_{t+1}^P u_c^P \left(T_{t+1} \right) r_{t+1}^i = 0 \text{ for all } i. \tag{3}$$

We use this expression to obtain the insights about the optimal structure of the government portfolio. To this end, we assume that conditional on information in time t exogenous shocks take the form

$$Y_{t+s} = \bar{Y}_{t+s} + \sigma \mathcal{E}_{Y,t+s}, \quad G_{t+s} = \bar{G}_{t+s} + \sigma \mathcal{E}_{G,t+s}, \quad \eta_{t+s}^j = \bar{\eta}_{t+s}^j + \sigma \mathcal{E}_{\eta,t+s}^j \text{ for } j \in \{R, P\},$$

where $\left\{ \bar{Y}_{t+s}, \bar{G}_{t+s}, \bar{\eta}_{t+s}^j \right\}_{s,j}$ are deterministic sequences, $\mathcal{E}_{t+s} \equiv \left\{ \mathcal{E}_{Y,t+s}, \mathcal{E}_{G,t+s}, \mathcal{E}_{\eta,t+s}^j \right\}_{s,j}$ are shocks that have bounded support and $\mathbb{E}_t \mathcal{E}_{t+s} = \mathbf{0}$ for all $s > 0$, but otherwise follow an arbitrary joint stochastic process, and σ is a positive scalar. We derive the approximations of our endogenous variables by considering Taylor expansions of various orders with respect to σ. Let $\partial_\sigma, \partial_{\sigma\sigma}, ...$ be first-, second-, and higher- order derivatives with respect to σ evaluated at $\sigma = 0$. We use bars to denote 0ths order approximations of random variables, so that a typical random variable X_{t+s} has an expansion $X_{t+s} = \bar{X}_{t+s} + \sigma\partial_\sigma X_{t+s} + \frac{1}{2}\sigma^2\partial_{\sigma\sigma}X_{t+s} + O(\sigma^3)$. Note that $\partial_\sigma X_{t+s}$ and $\partial_{\sigma\sigma}X_{t+s}$ are random variables that depend on the realization of shocks $\mathcal{E}_{t+1}, ..., \mathcal{E}_{t+s}$. Also note that $\mathbb{E}_t\partial_\sigma X_{t+s} = \mathbb{E}_t\partial_{\sigma\sigma}X_{t+s} = 0$ for all $s > 0$ when $X_{t+s} \in \{\bar{Y}_{t+s}, \bar{G}_{t+s}, \bar{\eta}^R_{t+s}, \bar{\eta}^P_{t+s}\}$.

Rich agents optimality condition satisfies a textbook asset pricing equation

$$\eta^R_t u^R_c(Y_t) = \beta\mathbb{E}_t\eta^R_{t+1}u^R_c(Y_{t+1})\frac{d^i_{t+1} + q^i_{t+1}}{q^i_t} \text{ for all } i, t, \tag{4}$$

where u^j_c denotes the derivative of u^j. Equation (4) can be re-written as $\mathbb{E}_t\eta^R_{t+1}u^R_c(Y_{t+1})r^i_{t+1} = 0$. Its zero- and first- order expansions imply that

$$\bar{r}^i_{t+1} = \mathbb{E}_t\partial_\sigma r^i_{t+1} = 0. \tag{5}$$

This equation shows a familiar insight that all securities to the first order approximation must pay the same expected return in equilibrium. Let $\frac{1}{2}a^i_t \equiv \mathbb{E}_t\partial_{\sigma\sigma}r^i_{t+1}$ be the risk premium, which at this point can be either positive or negative. Differentiate (3) twice and use (11) and (5) to obtain

$$\frac{\bar{T}_{t+1}}{\alpha^P_{t+1}}\left(a^i_t + \mathbb{E}_t\partial_\sigma r^i_{t+1}\partial_\sigma\eta^P_{t+1}\right) = \mathbb{E}_t\partial_\sigma r^i_{t+1}\partial_\sigma T_{t+1} \text{ for all } i, \tag{6}$$

where α^P_{t+1} is the coefficient of the relative risk aversion of the poor, $-u^P_{cc}(\bar{T}_{t+1})\bar{T}_{t+1}/u^P_c(\bar{T}_{t+1})$. Using the budget constraint to substitute for $\partial_\sigma T_{t+1}$ we obtain

$$\frac{\bar{T}_{t+1}}{\alpha^P_{t+1}}\left(\boldsymbol{a}_t + cov_t\left(\eta^P_{t+1}, \boldsymbol{r}_{t+1}\right)\right) - cov_t(S_{t+1}, \boldsymbol{r}_{t+1}) + cov_t(B_{t+1}, \boldsymbol{r}_{t+1}) - B_t\boldsymbol{\omega}_t var_t(\boldsymbol{r}_{t+1}) = \boldsymbol{0}. \tag{7}$$

This equation is the first substantive result of this section. If one is willing to take a stance on the form of the preferences of the poor agents, all other variables in this equation can be directly estimated using data on the returns of government bonds of different maturities, tax revenues, and the market value of debt. Using these estimates one can test whether it holds in the data empirically. A rejection of this equation in the data would then indicate that the government portfolio is suboptimal. Furthermore, one then can use this equation to see the direction in which the portfolio can be rebalanced to improve welfare. If the j^{th} row of the vector of the expression on the left hand side of (7) is positive then welfare is improved by increasing the share of government's portfolio in the j^{th} security and reducing it in the one period bond. The opposite rebalancing improves welfare if the j^{th} row is negative. The general spirit of this exercise would be similar to the sufficient statistics or tax reform approach in public finance (Chetty (2009); Golosov et al. (2014)), but unlike that literature, that focuses exclusively on the first order effects of policy changes, our approach allows us to consider higher-order effects that are central questions of the portfolio management. We show later in the paper that the effects of portfolio rebalancing on asset prices or tax revenues are easily incorporated into this formula.

One limitation of formula (7) is the it is not very transparent about the economic principles that govern the optimal management of government's portfolio. In particular, the covariance of returns $\boldsymbol{r}_{t+1}$ with future market values of debt B_{t+1} depends on the level of debt and on its portfolio composition, all of which are endogenous variables. To understand these economic principles better we now re-write these equations. In the text we assume that assume $\mathcal{T} = \infty$ and that $\left\{\bar{Y}_t, \bar{G}_t, \bar{\eta}_t^j\right\}_{t,j}$ do not depend on time t. As we show in the appendix these assumptions are not important for our results.

Sum the budget constraint (2) forward to get

$$\mathbb{E}_{t+1} \sum_{s=1}^{\infty} Q_{t+1,t+s} T_{t+s} = \mathbb{E}_{t+1} \sum_{s=1}^{\infty} Q_{t+1,t+s} S_{t+s} + \left[R_{t+1}^0 + \boldsymbol{\omega}_t \boldsymbol{r}_t\right] B_t, \tag{8}$$

where $Q_{t+1,t+s}$ is the inverse of the realized returns on the government's portfolio between periods $t+1$ and $t+s$. Taking a Taylor expansion of this equation we get

$$\mathbb{E}_{t+1} \sum_{s=1}^{\infty} \beta^{s-1} \left(\partial_\sigma T_{t+s} - \partial_\sigma S_{t+s}\right) = \left[\partial_\sigma R_{t+1}^0 + \boldsymbol{\omega}_t \partial_\sigma \boldsymbol{r}_t\right] B_t + (\bar{S} - \bar{T}) \mathbb{E}_{t+1} \sum_{s=1}^{\infty} \partial_\sigma Q_{t+1,t+s}. \tag{9}$$

The left hand side is the change in the present value of government's primary deficit. It is equal to change in the returns on government portfolio(the first term on the right hand side) plus the effect of change in the expectations about future interest rates (the second term on the right hand side). Since to the first-order expected excess returns are zero we have $\mathbb{E}_{t+1} \partial_\sigma Q_{t+1,t+s} = \mathbb{E}_{t+1} \partial_\sigma \left(q_{t+1,t+2}^0 \times ... \times q_{t+s-1,t+s}^0\right)$ so that the last sum in (9) is simply the sum of changes in long risk-free rates of all durations. The last term takes this form because we assumed that $\bar{S}_{t+s} - \bar{T}_{t+s}$ are independent of s; more generally this term is the infinite weighted sum with weights proportional to $\bar{S}_{t+s} - \bar{T}_{t+s}$ (see the appendix). Define $Q_{t+1}^{\infty} \equiv \mathbb{E}_{t+1} \sum_{s=1}^{\infty} Q_{t+1,t+s}$ and $PV_{t+1}(X) \equiv \mathbb{E}_{t+1} \sum_{s=1}^{\infty} \beta^{s-1} X_{t+s}$ for any random variable X.

A simple perturbational argument that lowers transfers in period t and increases them in period $t+s$ with adjustments to the holdings of the s maturity bond implies that the optimal allocation should also satisfy

$$\mathbb{E}_{t+1}\partial_\sigma T_{t+s} = \partial_\sigma T_{t+1} - \bar{T}\rho^P \left(\beta^{1-s}\mathbb{E}_{t+1}\partial_\sigma Q_{t+1,t+s} - \left(\mathbb{E}_{t+1}\partial_\sigma \eta^P_{t+s} - \partial_\sigma \eta^P_{t+1}\right)\right),$$

where ρ^P is the elasticity of the intertemporal substitution. With time separable preferences it coincides $1/\alpha^P$ but this restriction is not necessary for our analysis. Substitute this expression into (8) and rearrange to get

$$\frac{\partial_\sigma T_{t+1}}{1-\beta} = \partial_\sigma PV_{t+1}(S) + \left[\partial_\sigma R^0_{t+1} + \boldsymbol{\omega}_t \partial_\sigma \boldsymbol{r}_t\right] B_t + (\bar{S}-\bar{T})\partial_\sigma Q^\infty_{t+1} + \bar{T}\rho^P \left(Q^\infty_{t+1} - PV_t\left(\frac{\eta^P_{t+s}}{\eta^P_{t+1}}\right)\right).$$

Government's budget constraint implies that $\bar{T} - \bar{S} = \frac{1-\beta}{\beta}B_t$. Substitute this and previous equation into (6) and rearrange to get

$$B_t\boldsymbol{\omega}_t cov_t(\boldsymbol{r}_{t+1}, \boldsymbol{r}_{t+1}) = \boldsymbol{a}_t \frac{\bar{T}}{\alpha^P(1-\beta)} - cov_t\left(PV_{t+1}(S), \boldsymbol{r}_{t+1}\right) \tag{10}$$
$$+ \left[\frac{1}{\beta}B_t + \bar{T}\rho^P\right] cov_t(Q^\infty_{t+1}, \boldsymbol{r}_{t+1}) - \bar{T}\varrho^P cov_t\left(PV_t\left(\frac{\eta^P_{t+s}}{\eta^P_{t+1}}\right), \boldsymbol{r}_{t+1}\right).$$

This equation determines the optimal portfolio composition for the government. It has several elements that are familiar from individual investor portfolio analysis in finance, and some new ones. The first term on the right-hand-side captures agents' attitude towards financial risk. It is related to the textbook portfolio choice problem (see Samuelson (1970)) that abstracts from other sources of risk featured in this equation or from possible variation in risk or risk premium, and derives that the solution to investor's risk return trade-off is given by $\frac{\boldsymbol{a}}{\alpha^P}\left[cov(\boldsymbol{r}_{t+1}, \boldsymbol{r}_{t+1})\right]^{-1}$. Campbell and Viceira (1999) derived time-variant version of this term, observe that in the data the risk premium $\boldsymbol{a}_t$ exhibit a lot of predictability, and argued that the benefits of rebalancing portfolio and timing the markets are large. Since $\bar{T} = \bar{S} + \frac{1-\beta}{\beta}B_t$ one can also see the insight of Viceira (2001) that an agent who's revenues come mainly from non-financial assets should act in approximately risk-neutral way and invest a large fraction of her portfolio in a risky asset. The second term on the right hand side of (10) captures hedging benefits of financial assets with non-financial risk and generalizes results derived by Viceira (2001). The third term captures concerns about fluctuations in future risk-free rate. Campbell and Viceira (2001) derived a version of it and observed that as the risk version goes to infinity its importance grows relative to the first term, so that a very risk-averse investor with no outside income should structure her portfolio to minimize the risk of future interest rate fluctuations. The fourth term on the right hand side of (10) is new.

It is tempting to apply this partial equilibrium analysis to draw implications for government's portfolio management. Since most of transfers are financed out of tax revenues and interest income is relatively small this equation would suggest that the government should invest a lot in the assets with positive excess returns and rebalance frequently (for example, Lucas and Zeldes (2009) think about government portfolio allocations essentially from this point of view). Under such interpretation the Treasury should be running a giant hedge fund by issuing large quantities of short debt that are used to finance leveraged investments into various risky securities. In our view this approach would be misleading. It ignores the other side of the market, the investors who trade those assets with the government, and the fact that asset prices are shaped by their asset demand in general equilibrium.

The behavior of rich agents is characterized by their optimality conditions that can be shown to be

$$\boldsymbol{a}_t \frac{\bar{Y}}{\alpha^R} = cov_t \left(Y_{t+1}, \boldsymbol{r}_{t+1} \right)$$

and

$$\mathbb{E}_{t+1} \partial_\sigma Y_{t+s} = \partial_\sigma Y_{t+1} - \bar{Y} \rho^R \left(\beta^{1-s} \mathbb{E}_{t+1} \partial_\sigma Q_{t+1,t+s} - \left(\mathbb{E}_{t+1} \partial_\sigma \eta^R_{t+s} - \partial_\sigma \eta^R_{t+1} \right) \right).$$

These equations impose additional restrictions on behavior of returns on the one hand and preferences of rich and poor agents on the other. Although in principle one can come up with some preferences to make investors effectively irrational and irrelevant for government's portfolio choice we are skeptical about the validity of such suppositions. In our view, the most natural benchmark to consider is the one in which the rich and poor agents share the same preferences. To make our analysis simple we assume that

$$\eta^j_t u^j \left(c^j_t \right) = \eta_t \frac{\left(c^j_t \right)^{1-\alpha}}{1-\alpha} \text{ for } j \in \{R, P\}. \tag{11}$$

Under this assumption, the optimal portfolio expression becomes

$$B_t cov_t \left(\mathbf{r}_{t+1}, \mathbf{r}_{t+1} \right) \boldsymbol{\omega}_t = \frac{1-\beta}{\beta} B_t cov_t \left(Q^\infty_{t+1}, \boldsymbol{r}_{t+1} \right) + \left[\bar{T} cov_t \left(PV_{t+1} \left(\frac{Y}{\bar{Y}} \right), \boldsymbol{r}_{t+1} \right) - \bar{S} cov_t \left(PV_{t+1} \left(\frac{S}{\bar{S}} \right), \boldsymbol{r}_{t+1} \right) \right]. \tag{12}$$

This is the central equation of this section, and it is crucial to understanding our numerical results. the optimal portfolio is determined by two considerations. The first term on the top line captures the risk in future interest rates that we call the rollover risk. This risk is proportional to the level of debt B_t that the government needs to rollover. The second one, written in the square brackets on the right hand side on the bottom line, captures the *relative* hedging benefits of the risky securities for rich and poor households. Parameters capturing risk aversion or expected excess returns are absent from this formula because the only gains from trade comes from *differences* in hedging benefits that the bonds offer to different market participants. Divide both sides with B_t and let $\boldsymbol{\omega}$ be the vector of portfolio shares. The discussion above can be summarized with $\boldsymbol{\omega}_t = \boldsymbol{\omega}_t^R + \boldsymbol{\omega}_t^H$ and the two components given by

$$\boldsymbol{\omega}_t^R = \frac{1-\beta}{\beta} cov_t \left(\mathbf{r}_{t+1}, \mathbf{r}_{t+1}\right)^{-1} cov_t \left(Q_{t+1}^{\infty}, \boldsymbol{r}_{t+1}\right) \tag{13}$$

$$\boldsymbol{\omega}_t^H (B_t) = \frac{cov_t \left(\mathbf{r}_{t+1}, \mathbf{r}_{t+1}\right)^{-1}}{B_t} \left[\bar{T} cov_t \left(PV_{t+1}\left(\frac{Y}{\bar{Y}}\right), \boldsymbol{r}_{t+1}\right) - \bar{S} cov_t \left(PV_{t+1}\left(\frac{S}{\bar{S}}\right), \boldsymbol{r}_{t+1}\right)\right]. \tag{14}$$

One advantage of this formula is that all objects that appear in it are expressed in terms of statistics that can be estimated in the data directly. We do that in the next section. One useful and empirically relevant benchmark case for us will be the one in which the relative hedging benefits are small.

Proposition 1. *Suppose that $cov_t \left(PV_{t+1}(Y), \boldsymbol{r}_{t+1}\right) = cov_t \left(PV_{t+1}(S), \boldsymbol{r}_{t+1}\right) = 0$. If the government can trade pure discount bonds of any maturity, then the optimal share of the portfolio holding in bond that matures in j periods is given by $\omega_t^j = (1-\beta)\beta^j$ for all t.*

This proposition gives a remarkable simple prescription to the optimal public debt management. The government simply allocates geometrically declining share of its portfolio in debts of longer maturities. This shares never change, so the portfolio is never rebalanced. This portfolio ensures that the same amount of payments is due in each period equal to the difference $\bar{S} - \bar{T}$. Thus, this portfolio ensures that maturity of debt payments is perfectly matched with expected primary deficits, an effect that we refer to as the maturity matching.

Proposition 1 assumes that the government has access to infinitely many bonds. This assumption can easily be relaxed as the next two corollaries show

Corollary 1. *Suppose the assumptions are as in Proposition 1 but the government has access to only pure discount bonds for the first I periods. Then the optimal portfolio $\omega_t^j(I)$ satisfies*

$\omega_t^j(I) \to (1-\beta)\,\beta^j$ *as* $I \to \infty$.

and

Corollary 2. *Suppose the assumptions are as in Proposition 1 but the government has an access to a consol that pays one unit of consumption in each period. Then an optimal portfolio is to allocate all debt into consol and nothing to other securities.*

The first corollary comes from the fact that the importance of matching later maturities is declining with the rate β. So by even having access to a finite number of bonds the government can capture most of the gains by following the allocation rule of proposition 1 with finite but reasonably large I. The second corollary that just one security - a simple consol - allows to eliminate all the rollover risk. The intuition for this result is that such a consol is exactly equivalent to the portfolio of discount bonds described in Proposition 1. This result also gives us a natural benchmark to consider in a richer model where additional effects are present, where we can capture intuition for the additional effects by considering a portfolio of only two securities, a one period discount bond and a consol.

In data, the covariances mentioned in Proposition 1 are not zero but are slightly negative. To understand how results change relative to the implications of Proposition 1, it is useful first to assume that $G_t = 0$ and $\frac{S_t}{\bar{S}} = (1+x)\,\frac{Y_t}{\bar{Y}}$. The motivation behind this case is as follows. In the data, most of tax revenues are spent on transfers of various sorts, the share of revenues spent on pure public goods, such as military or infrastructure, is fairly small, especially at the federal level. This motivates us setting $G_t = 0$. Over the business cycle government primary deficit fluctuates much more than output, the effect driven by the progressivity in the tax code and various automatic stabilizers in the transfer system. We capture this progressivity by a parameter $x \geq 0$. Under these assumptions equation (12) becomes

$$B_t cov_t(\mathbf{r}_{t+1}, \mathbf{r}_{t+1})\boldsymbol{\omega}_t = \left[\left((1-\beta)\,B_t - x\frac{\bar{S}}{\bar{Y}}\right) cov_t\left(PV_{t+1}\left(Y\right), \boldsymbol{r}_{t+1}\right)\right] + \frac{1-\beta}{\beta} B_t cov_t\left(Q_{t+1}^{\infty}, \boldsymbol{r}_{t+1}\right).$$

In the data, covariances $cov_t\left(Y, r_{t+1}^j\right)$ are weakly negative. This has several implications. Firstly, issuing long debt has additional costs not captured by the assumptions of Proposition 1. Since the returns on the long debt are negatively correlated with output and government's income is more volatile than output, security with a payoff of the long debt offers a more valuable hedge for the government than for private agents. This implies that issuing long debt carries additional costs that need to be balanced against the rollover risk. Secondly, the optimal portfolio weights $\boldsymbol{\omega}_t$ are decreasing in B_t holding everything constant. Since negative values of B_t imply that the government is in debt, highly indebted governments should lengthen the maturity of its portfolio. All these insights become particularly transparent when we assume that the only long debt that the government has is consol. In which case equation (12) simple becomes

$$\omega_t^{consol} = 1 + \left((1-\beta) - x\frac{\bar{S}}{B_t \bar{Y}} \right) \frac{cov_t\left(PV_{t+1}\left(Y\right), r_{t+1}^{consol}\right)}{var_t\left(r_{t+1}^{consol}\right)}.$$

To the extent that the fluctuation in asset prices is much higher than fluctuation in output, it implies small departures from the portfolio rules highlighted in Proposition 1 and modest amount of rebalancing.

Although we mainly thought about our bonds as a riskless real securities, nothing in our derivations required that. The analysis equally applies if payout $\boldsymbol{d}_t$ is stochastic, either because the real asset itself is risky (for example, if it is stock market) or if the asset asset is riskless in nominal terms. Thus, it can be directly applied to broader questions about how the government should manage its portfolio of investments. One can ask, for example, whether it is beneficial to invest in the stock market. In the data stock market returns co-vary positively with output, so stock market is a bad hedge. When taxes are progressive, as in the example above, this implies that it is a worse hedge for the government than for private agents, so that the relative hedging in equation (12) is negative. Thus, investing in the stock market would unambiguously lower welfare if the government also can issue bonds of the type mentioned in Proposition 1 or its corollaries. It is optimal, if anything, to short the stock market. However, since $var_t\left(r_{t+1}^{stock\_mrk}\right)$ is much higher than $cov_t\left(\frac{Y_{t+1}}{\bar{Y}}, r_{t+1}^{stock\_mrk}\right)$ in the data, the benefits of shorting the stock market will be low and it is optimal to assign only a very small share of overall portfolio to short positions in stocks.

2.1 Connection Ramsey literature on debt management

The prescriptions for the optimal debt management that we derived in the previous section are seemingly at odds with Ramsey literature on optimal debt management. In his seminal work Angeletos (2002) showed that in a canonical neoclassical economy the government can construct a portfolio of uncontingent debts of various maturities to fully hedge its risks. Buera and Nicolini (2004) studies quantitative properties of such portfolio and found that it takes an extreme form. In their calibrations government's positions in in debts of different maturities are tens or even hundreds times greater than GDP, and the budget is balanced by buying equally huge debts in other maturities issued by private sector. Investments in debts of similar maturities are often vastly different but on balance government favors issuing very long debt and buying short debt of households. Small aggregate shocks trigger very large rebalancing of government portfolio. These findings have been confirmed in numerous calibrations and are now broadly accepted (Farhi (2010); Lustig et al. (2008); Debortoli et al. (2017); Faraglia et al. (2018)) but they contrast sharply with our results.

The key reason for this finding comes from the fact that neoclassical models have counterfactual implications for the prices of financial assets, such as government bonds. Those models predict that shocks to primary deficit are highly correlated with bond returns, and that the volatility of those returns is low. This implies that the government in the standard neoclassical economy can achieve complete market allocations by buying and selling bonds of different maturities but it needs to take extreme positions to leverage fluctuations in those returns. Since in the data returns show little correlation with deficits and are very volatile, these normative prescriptions of the neoclassical models are not robust.

A simple way to illustrate this using our baseline setup is to assume that output $\ln Y_t$ follows an AR(1) process with persistence ϱ and this is the only source of risk. It is easy to verify that $\partial_\sigma PV_{t+1}\left(\frac{Y}{\bar{Y}}\right) = \left[\frac{1}{1-\beta\varrho}\right]\partial_\sigma \log Y_{t+1}$ and $\partial_\sigma r^{consol}_{t+1} = \left[\frac{\alpha^R(1-\varrho)}{1-\beta\varrho}\right]\partial_\sigma \log Y_{t+1}$ when $Y_t = \bar{Y}$. Thus, the output and returns are closely correlated but volatility of returns is much smaller than volatility of output when shocks are persistent, $\varrho \to 1$. The hedging term in the optimal portfolio is proportional to $\frac{cov_t\left(PV_{t+1}\left(\frac{Y}{\bar{Y}}\right), r^{consol}_{t+1}\right)}{var_t\left(r^{consol}_{t+1}\right)} = \frac{1}{\alpha(1-\varrho)}$ which becomes arbitrary large for ϱ close to 1. If taxes are linear and the government needs to finance some exogenous expenditures $\bar{G} > 0$, equation (12) implies that

$$\omega_t^{consol} B_t = B_t - \frac{\bar{G} - (1-\beta)\,B_t}{\bar{Y}} \frac{1}{\alpha\,(1-\varrho)}.$$

When government is in debt, $B_t \leq 0$, the hedging term is negative. This captures the fact that poor agents face more risk than rich agents: tax revenues move one for one with output and consumption but they are used not only to pay transfers T_t but also non-transfer expenditures $\bar{G}$ and interest payments on debt $-(1-\beta)\,B_t$. This implies that $\omega_t^{consol} B_t \to -\infty$ as $\varrho \to 1$.[4] Thus, in line with the usual findings, in such an economy the government issues an arbitrarily large position in the long bond that is partially offset by buying a large quantity of one period debt issued by the rich. Similar logic implies that small shocks to Y_t

[4]When $B_t = 0$, portfolio share ω_t^{consol} is undefined but the level of investment in consol $\omega_t^{consol} B_t$ is define and given by the formula above.

or B_t get amplified triggering large rebalancing of the optimal portfolio.

3 Extensions

One advantage of the pertubational approach we developed in the previous section is that it is easy to apply to a variety of settings. In this section we extend our baseline economy in several different dimensions.

3.1 Preferences, risk, heterogeneity

In our baseline economy we assumed that preferences are time-separable but our derivations did not rely on this feature. In the appendix we follow essentially the same steps to extend our results to a version of Epstein-Zin preferences (Epstein and Zin (1989)), where utility at time t is define recursively as

$$V_{t-1} = \mathbb{E}_{t-1}\left[W_t^{1-\alpha}\right]^{\frac{1}{1-\alpha}},$$
$$W_t \equiv \left(\eta_t c_t^{1-\rho} + \beta V_t^{1-\rho}\right)^{\frac{1}{1-\rho}}.$$

As long as the preferences of the rich and poor agents are the same all our results from the previous section extend directly. Given the insights from the previous settings this is not suprising. When agents share the same preferences, their outlook on risk and rewards are the same. For this reason the optimal portfolio is structured as a simple trade-off between eliminating the common roll-over risk of the existing debt and the relative hedging benefits, emphasized in equation (12).

We chose our baseline specification of preferences with preference shocks η_t to follow Albuquerque et al. (2016). Their set up is a natural starting point for us as it is the only off-the-shelf model we are aware of that naturally delivers some of the key salient features of bond prices, such as the upward sloping real yield curve. Our analysis however is not restricted to preference shocks. Another popular explanation for variation in asset prices is time-varying volatility (Bansal and Yaron (2004); Bansal and Shaliastovich (2013)). Technically, it is more complicated to study the implications of time-varying volatility for portfolio problems as it requires taking fourth order expansions of equation (3). However, the basic steps of those expansions remain unchanged. In the appendix we show how this can be done in a three period economy, and show that a version of equation (12) also holds in such settings.

In our baseline economy we assumed that income is endogenous. Nothing changes in our analysis if modify our model to allow agents to choose labor supply given some stochastic exogenous wage rate w_t. The labor supply of agents is unaffected by our perturbation since all prices are unchanged. As long as the risk aversion and the intertemporal elasticity of substitution of the rich and the poor are the same (for example, if preferences between consumption and labor are separable and utility of consumption is given by (11)) the analysis goes unchanged. This further generalizes to a version of Epstein-Zin preferences with labor supply as in Karantounias (2018). It is also straightforward to extend this analysis to multiple agents, since the marginal utilities of all trading agents must satisfy (4).

3.2 The large government case

We now drop our assumption that the government is negligible relative to the size of population of agents it trades with and consider arbitrary $\lambda > 0$. To streamline the exposition we assume that one of the securities the government can trade is the consol; this assumption is not necessary for our results but significantly simplifies proofs.

Similarly to the arguments in section 2, take any equilibrium path of taxes, debts and bond prices, and consider the following perturbation. Suppose that in period t in some history h^t the government buys ϵ more of security j and sells the same amount of security 0; the government then increases its holdings in consol in period $t+1$ by $\frac{1}{1+q_{t+1}^{consol}}(R_{t+1}^j - R_{t+1}^0)\epsilon$ and transfers T_{t+1} by $\frac{q_{t+1}^{consol}}{1+q_{t+1}^{consol}}(R_{t+1}^j - R_{t+1}^0)\epsilon$.

This perturbation in general affects prices of all securities that the government can trade. Let $\xi_t^{i,j}$ be the marginal impact on price of security i in period t that results from this perturbation. By envelope theorem, the impact on utility of the rich agent from this perturbation is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \eta_t^R u_c^R \left(c_t^R\right) \left(\sum_{i \geq 0} \left(\tilde{b}_t^i - \tilde{b}_{t-1}^i \right) \xi_t^{i,j} \right).$$

Combining this with feasibility condition (1) we get that the welfare effect from this perturbation are given by

$$\begin{aligned} & \beta^{t+1} \Pr\left(h^t\right) \mathbb{E}_t \textstyle\sum_{s=1}^{\infty} \beta^{s-1} \left[\eta_{t+s}^P u_c^P \left(T_{t+s}\right) r_{t+1}^j\right] \\ + \quad & \mathbb{E}_0 \left[\textstyle\sum_{t=0}^{\infty} \beta^t \left(\sum_{i \geq 0} \xi_t^{i,j} \left(\tilde{B}_{t-1}^i - \tilde{B}_t^i \right) \right) \left(\eta_t^P u_c^P \left(T_t\right) - \frac{\lambda \mu}{1-\lambda} \eta_t^R u_c^R \left(c_t^R\right) \right) \right] \end{aligned} \quad (15)$$

The first term of this expression is equivalent to (3) in the previous section. The only difference is that we smoothed out consumption over time rather than consuming all the additional return from perturbation in period $t+1$. This change ensures that our price effects are zero to the first order, $\bar{\xi}_t^{i,j} = \partial_\sigma \xi_t^{i,j} = 0$. The second term captures the redistributory effect of this perturbation. This term depends on three parameter: on price elasticity $\xi_t^{i,j}$, the extent to which government rebalances its portfolio in the original economy $\tilde{B}_{t-1}^i - \tilde{B}_t^i$, and one how much more the government values a dollar in the hands of the poor, $\eta_t^P u_c^P(T_t)$, relative to the dollar in the hands of the rich, $\mu\eta_t^R u_c^R\left(c_t^R\right)$. Note that the difference in these two terms will be big only if the underlying tax system, which we took to be arbitrary so far, is inefficient. Thus, the optimal portfolios will deviate substantially from the ones in the small government benchmark only if the size of the investor pool is small *and* the government substantially rebalances its portfolio over time *and* the existing tax system is very inefficient.

We therefore expect price effects to be small in a reasonable calibration for two reasons. First, the magnitude of the price elasticity term $\xi_t^{i,j}$ depends on the size of the investor pool, and can be in principle estimated in the data. An example of this are the estimates of the effect of QE policies in the aftermath of the 2008 financial crises, when the U.S. Federal Reserve bank swapped one debt in its portfolio for another keeping the total outstanding debt unchanged. This swap is akin the perturbation considered in this section and, as we document in section 4, studies have found its effect to be small. Second, as we saw in section 2 the need for substantial rebalancing is unlikely to be satisfied for realistic behavior of returns.

In fact, it is possible to go one step further. We show in the appendix that a second order expansion with respect to σ yields the following proposition, the poof of which can be found in the appendix

Proposition 2. *There exists constants* c *and* d *such that the optimal portfolio at time* τ *solves*

$$\begin{aligned} cov_\tau(\mathbf{r}_{\tau+1}, \mathbf{r}_{\tau+1})\mathbf{B}_\tau = & \frac{1-\beta}{\beta}\bar{B}_\tau cov_\tau\left(\mathbf{r}_{\tau+1}, Q_{\tau+1}^\infty\right) \\ & + \left[\mathsf{c}\bar{T} cov_\tau\left(\mathbf{r}_{\tau+1}, PV_{\tau+1}(Y_{\tau+t}/\bar{Y})\right) - \mathsf{d}\bar{S} cov_0\left(\mathbf{r}_{\tau+1}, PV_{\tau+1}(S_t/\bar{S})\right)\right]. \end{aligned}$$

Constants c and d can be expressed in closed forms using the yield curve elasticities. One immediate takeaway from this proposition is that the government price effects only work through the relative hedging term, i.e. when there gains from trade between the government and the rich agents. This implies the following corollary

Corollary 3. *The conclusion of Proposition 1 and Corollary 2 holds for any* $\lambda > 0$*.*

The result of this corollary can most readily seen in the case where the government enters period 0 with all debt held in the consol. Absent price effects, the assumptions of Proposition 1 imply that the government will, to our order of approximation, wish to hold a constant maturity structure matching that of a consol. This portfolio structure would then imply $\tilde{B}^i_{t-1} - \tilde{B}^i_t = 0$ for all i and $t \geq 0$, which allows us to conclude that portfolio of Proposition 1 is in fact optimal.

3.3 Endogenous labor supply and distortionary transfers

To be written

3.4 Debt management without commitment

To be written

4 Data

Proposition 1 gives a simple prescription for the rollover risk, $\boldsymbol{\omega}^R_t$ – to issue debt such that the shares exponentially decline with maturity. The relative hedging component, $\boldsymbol{\omega}^H_t(\hat{B}_t)$ quantifies how much the optimal portfolio should deviate from this prescription. Our theory says that the magnitude depends on the level of outstanding debt $\hat{B}_t$ and the two ratios of conditional covariances to variances, namely, $cov_t\left(\mathbf{r}_{t+1}, \mathbf{r}_{t+1}\right)^{-1} cov_t\left(PV_{t+1}\left(Y\right), \boldsymbol{r}_{t+1}\right)$ and $cov_t\left(\mathbf{r}_{t+1}, \mathbf{r}_{t+1}\right)^{-1} cov_t\left(PV_{t+1}\left(S\right), \boldsymbol{r}_{t+1}\right)$. In this section, we will measure these moments, derive the optimal portfolio using our formula and compare the optimal debt profile to the current-in place debt profile for the US. We find that the hedging portfolio is small in magnitude, negative in sign so that it offsets the rollover risk component, and fairly stable over time. Comparing the optimal portfolio to the U.S. we find that the U.S. portfolios overweights shorter maturities.

Data construction Our primary data consists of US bond returns and macroaggregates. The sample period is 1960-2017, and the frequency is quarterly. To construct returns we use data from the Center for Research in Security Prices (CRSP). From the CRSP Fama Portfolio files, we obtain the holding period returns on 11 portfolios of nominal bonds that have outstanding maturities between 0-10 years. These include bonds with maturities below 6 months, 6-12m months, 12-18 months, 18-24 months, 24-30 months, 30-36months, 36-42 months, 42-48 months, 48-54 months, 54-60 months, and a final category with bonds that have maturities between 5-10 years.[5] We assume that the bondholders' labor income is proportional to GDP and use the growth rate of real GDP per capita from the national income and product accounts (NIPA) to measure Y_t. Finally, to compute S_t we use federal current tax receipts (including contributions to social insurance) less defense spending, also from the NIPA.

Our theory abstracts from inflation risk and requires us to construct holding period returns on real bonds. Conceptually, inflation-indexed bonds (TIPS) can be used to construct prices of real bonds. However, for the U.S., the data for TIPS is available only from 1997 and for a limited set of maturities. Instead, use the relationship between nominal and real bonds an assume a stochastic process of inflation to recover holding period returns on real bonds. Let $P_t^{\$}(n)$ and $P_t(n)$ be the prices of a zero-coupon bond at date t that pays of one unit of the numeraire and one unit of the consumption good, respectively at date $t+n$. Let $M_{t,t+n}$ and $\Pi_{t,t+n}$ be the real stochastic discount factor and the gross inflation rate between t and $t+n$. Absence of arbitrage implies that

$$P_t^{\$}(n) = \mathbb{E}_t \frac{M_{t,t+n}}{\Pi_{t,t+n}} \tag{16}$$

Using small cap letters for logs, and taking a log-linear approximation of (16), we get that

$$p_t(n) - p_t^{\$}(n) = \mathbb{E}_t \pi_{t,t+n}. \tag{17}$$

The log holding period returns on a real between t and $t+1$, hpr_t equals $p_{t+1}(n-1) - p_t(n)$, and similarly for the nominal bond $hpr_t^{\$} = p_{t+1}^{\$}(n-1) - p_t^{\$}(n)$. Equation (17) then implies

$$hpr_t(n) - hpr_t^{\$}(n) = \mathbb{E}_{t+1}\pi_{t+1,t+n} - \mathbb{E}_t \pi_{t,t+n}$$

We assume that $\{\pi_t\}$ follows an AR(1) process. We then use the Consumer Price Index to estimate the right-hand side, and back out $hpr_t(n)$ from $hpr_t^{\$}(n)$ for all n. Finally, we follow Beeler and Campbell (2012) and Schorfheide et al. (2018) to construct a measure of the ex-ante real rate $r_{f,t}$ by regressing the ex-post real rate (3 month T bill rate between t and $t+1$ minus the realized inflation) on a set of predictors that time t. We use this measure

[5] We also have "remaining" category for bonds of greater than 10 years that we exclude from our analysis. We only use the 10 plus years portfolio returns to compute the principal components in the factor analysis below.

Table I: Summary Statistics for Excess Holding Period Returns

	mean	std.	p25	p50	p75	corr $\Delta \log Y$	corr $\Delta \log S$
0-6m	0.08	0.52	-0.19	0.03	0.26	-0.33	-0.22
6-12m	0.15	1.21	-0.43	0.06	0.55	-0.34	-0.23
12-18m	0.22	1.72	-0.68	0.1	0.86	-0.34	-0.23
18-24m	0.24	2.05	-0.85	0.05	1.02	-0.34	-0.25
24-30m	0.28	2.38	-0.98	0.04	1.21	-0.33	-0.24
30-36m	0.32	2.62	-1.17	0.06	1.49	-0.33	-0.25
36-42m	0.36	2.81	-1.35	0.09	1.75	-0.32	-0.25
42-48m	0.37	3.02	-1.5	-0.02	1.83	-0.33	-0.23
48-54m	0.4	3.18	-1.5	-0.12	2.02	-0.33	-0.25
54-60m	0.33	3.46	-1.7	-0.2	2.09	-0.31	-0.24
60-120m	0.52	3.90	-1.75	0.15	2.60	-0.31	-0.22

Note: This table summarizes the time-series moments for holding period returns in excess of the risk-free rate across bonds of different maturities. Returns are quarterly and measured in percentage points. The sample is from 1960 Q1 to 2017 Q4.

of risk-free rate to construct excess holding period returns, $hprx_t(n) \equiv hpr_t(n) - rf_t$, which maps to object $\boldsymbol{r}$ that appears in our formulas.

In table I we summarize the descriptive statistics for the holding period returns in excess of the risk-free rate across maturities. All units are in percentage points and quarterly. In our sample, the unconditional mean for excess returns varies between 8 basis points for the below six month maturities to about 50 basis points for bonds with maturities between 5-10 years. The standard deviation of the excess holding period returns is increasing in the maturity and ranges between 0.52 percent for the below six month and 3.90 percent for the portfolio with maturities in the 5-10 years category. In the last two columns, we see that holding period returns are negatively correlated with output growth and growth rate of tax revenues.

Estimating inverse of the return covariance matrix A well-known concern in using inverses of covariance matrices in portfolio analysis is sampling uncertainty and how it manifests as extreme and unstable portfolio weights. For a detailed discussion, see Senneret et al. (2016), DeMiguel et al. (2007), Jagannathan and Ma (2003). These concerns apply equally to bond returns and to address them, we compute the hedging term using three approaches. The first is a "Naive approach", in which we use the inverse of the sample analog of the covariance matrix. The second is a " Diagonal approach," in which we zero all the off-diagonal elements of the sample covariance matrix and then take its inverse. The Diagonal approach ignores the correlation across returns and assigns portfolio weights as if there was only one security (in addition to the risk-free bond) available to hedge at a time. Finally, the third and our most preferred approach is what we call the "Factor approach." Here we assume that most of the variation in returns arises from a small set of common factors. We use the estimated factor loadings to compute the covariances and their inverses.

To implement the factor approach, we assume a single factor, and for all maturities n,

$$hprx_t(n) = A_{0,r}(n) + A_{1,r}(n) f_t + \sqrt{A_{\sigma,r}(n)}\epsilon_t^n, \tag{18}$$

$$f_t = \bar{f} + \sqrt{A_{\sigma,f}}\epsilon_t^f \tag{19}$$

as well as

$$\Delta \log Y_t = A_{0,y} + A_{1,y} f_t + \sqrt{A_{\sigma,y}}\epsilon_t^y, \tag{20}$$

$$\Delta \log S_t = A_{0,s} + A_{1,s} f_t + \sqrt{A_{\sigma,s}}\epsilon_t^s. \tag{21}$$

We set the common factor to be the first principal component of all the available returns.[6] This factor explains 93% of the returns variation. In table II, we list the OLS estimates for the factor loadings: $A_{1,r}(n)_n$, $A_{1,y}$, $A_{1,s}$ and loadings on the orthogonal components: $A_{\sigma,r}(n)_n$, $A_{\sigma,y}$, $A_{\sigma,s}$ and R^2. We find that the factor loadings are increasing in maturity n, and the R^2 are largest for maturities between 4 and 10 years. The loadings $A_{1,y}$ and $A_{1,s}$ are negative, reflecting the negative covariance of holding period returns and output growth. The point estimate of $A_{1,s}$ is roughly 4 times $A_{1,y}$ reflecting the progressivity of the U.S. tax code.

Exploiting the factor structure, the hedging term can be expressed as

$$\boldsymbol{\omega}^H(\hat{B}) = \frac{1}{\hat{B}}\left(\frac{A_{1,s}}{A_{1,y}} - 1 - \frac{1-\beta}{\beta}\hat{B}\right)\Pi(\boldsymbol{A}_{1,r}, \boldsymbol{A}_{\sigma,r}) A_{1,y} \boldsymbol{A}_{1,r}^\intercal A_f \tag{22}$$

with

$$\Pi(\boldsymbol{A}_{1,r}, \boldsymbol{A}_{\sigma,r}, A_f) = Diag(\boldsymbol{A}_{\sigma,r})^{-1} - \frac{Diag(\boldsymbol{A}_{\sigma,r})^{-1}\boldsymbol{A}_{1,r}\boldsymbol{A}_{1,r}^\intercal Diag(\boldsymbol{A}_{\sigma,r})^{-1}}{A_f^{-1} + \boldsymbol{A}_{1,r}^\intercal Diag(\boldsymbol{A}_{\sigma,r})^{-1}\boldsymbol{A}_{1,r}}.$$

We set $\hat{B} = 20$ so that it captures a debt to annual gdp of about 100% and a $\beta = 0.99$ so that steady steady-state annualized risk-free rate is 4%. We use the estimates of the factor

[6] We experimented with "CP" factor following the construction in Cochrane and Piazzesi (2005). However, for the returns we use, the R squares were too small.

Table II: Estimates from the Factor model

	A_0 estimate	std. err	A_1 estimate	std. err	A_σ	R^2
0-6m	0.10	0.02	0.04	0.002	0.10	0.67
6-12m	0.19	0.04	0.11	0.004	0.38	0.77
12-18m	0.27	0.05	0.16	0.005	0.56	0.83
18-24m	0.30	0.05	0.19	0.005	0.54	0.88
24-30m	0.35	0.05	0.23	0.005	0.54	0.91
30-36m	0.40	0.05	0.25	0.005	0.47	0.94
36-42m	0.43	0.04	0.27	0.004	0.37	0.96
42-48m	0.44	0.04	0.30	0.004	0.34	0.97
48-54m	0.48	0.04	0.31	0.004	0.33	0.97
54-60m	0.41	0.05	0.34	0.005	0.45	0.97
60-120m	0.56	0.04	0.39	0.004	0.33	0.98
$\Delta \log Y$	0.17	0.01	-0.0063	0.001	0.03	0.11
$\Delta \log S$	0.30	0.09	-0.0295	0.008	1.50	0.06

Note: This table summarizes the OLS estimates of the factor model in equations in equations (18) – (21).

loadings $\boldsymbol{A}_{1,r}, \boldsymbol{A}_{\sigma,r}$ along with the sample variance of the factor A_f to obtain the hedging portfolio. For the rollover risk term, we assume that the asset space is rich enough to span all maturities and this implies that the rollover risk component $\mathbf{1}^T\boldsymbol{\omega}^R \approx 100\%$. The portfolio that hedges the rollover risk is obtained by assigning shares that geometrically decline at the rate β^n and add up to one. The optimal portfolio in the risky assets $\boldsymbol{\omega}(\hat{B}) = \boldsymbol{\omega}^H(\hat{B}) + \boldsymbol{\omega}^R$ and the remaining $1 - \mathbf{1}^T\boldsymbol{\omega}(\hat{B})$ is issued via the one-period risk-free bond.

In table III, we compare the optimal portfolio computed using the Factor method with those obtained using the Naive and the Diagonal approaches. There are several observations. The optimal hedging portfolio spreads out across all maturities. The magnitudes are small, i.e., between -2% and -7% of the outstanding debt. The positions are negative, and hence *offset* those required by rollover risk. The longer maturities are more correlated with the common factor and hence dominate the hedging needs. In comparison to a situation where only rollover risk is present, the optimal maturity profile will decline at a faster rate as we take larger offsetting positions in longer maturities. In the table III, we also see that the hedging portfolios using the Naive approach are extremely large and fluctuate in signs. This is expected given the discussion about the fragility of portfolio weights coming from small sample issues highlighted by DeMiguel et al. (2007) and others.

Table III: Optimal Portfolio

	Hedging			Rollover	Optimal portfolio		
	Naive	Diagonal	Factor		Naive	Diagonal	Factor
0-6m	0.95	-1.97	-0.02	0.10	1.05	-1.87	0.08
6-12m	-2.06	-0.88	-0.02	0.10	-1.96	-0.78	0.08
12-18m	0.96	-0.62	-0.02	0.10	1.05	-0.52	0.08
18-24m	-0.12	-0.52	-0.02	0.10	-0.03	-0.42	0.08
24-30m	-0.16	-0.44	-0.02	0.09	-0.06	-0.34	0.07
30-36m	-0.34	-0.40	-0.03	0.09	-0.25	-0.30	0.06
36-42m	1.81	-0.36	-0.04	0.09	1.90	-0.27	0.05
42-48m	-0.94	-0.34	-0.05	0.09	-0.85	-0.25	0.04
48-54m	-0.97	-0.33	-0.05	0.09	-0.88	-0.24	0.03
54-60m	0.22	-0.28	-0.04	0.08	0.31	-0.20	0.04
60-120m	0.22	-0.24	-0.07	0.08	0.30	-0.17	0.01

Note: This table summarizes the optimal portfolio using the "Naive", "Diagonal", and "Factor" approaches.

Comparison to the US We next compare the optimal portfolio with the U.S. debt portfolio. We use CRSP to get the amount outstanding and Macaulay duration for all federally issued (marketable) debt.[7] Then for each date, we split the outstanding debt in bins indexed by maturities (at quarterly intervals). In the left panel of figure I, we plot the time-averaged maturity profile of the US debt. The area under the curve is 77%, with the rest of 23% issued in treasury bills. We see that the U.S. maturity profile weights heavily on the shorter maturities and portfolio weights declines sharply as we increase the maturities. In the right panel, we plot the time-series for the Macaulay duration of the U.S. debt. Over time, the Macaulay duration is stable and ranges within 2 and 4 years. Recall that we computed the optimal portfolio using a handful of maturities for which we had data on holding period returns. To be able to compare it to the US debt profile we fill in the optimal portfolio for intermediate maturities as well as longer maturities as follows. For the rollover risk component, Proposition 1 gives us a simple formula that can extend $\boldsymbol{\omega}^R$ for any set of maturities. For the relative hedging term, expression (22) shows that the maturity n appears in the factor loadings and the volatility of the orthogonal component $\{A_{1,r}(n)\}_n$ and $\{A_\sigma(n)\}_n$. For maturities between 0 to 30 quarters, we linearly interpolate the functions $\{A_{1,r}(n), A_\sigma(n)\}$, and for maturities

[7]For a few bonds where the duration is absent, we set duration equal to maturity date minus current quotation date. The CRSP database does not have outstanding amounts for bills. To address this, we supplement the CRSP data with data from Monthly Statements of Public Debt issued by the US Treasury and fill in the amount outstanding in bills.

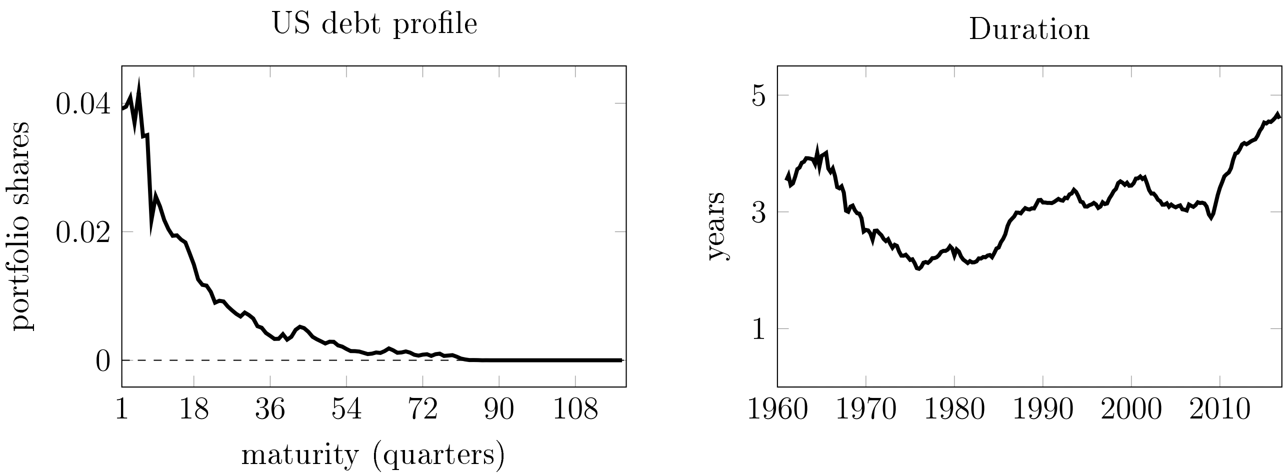

Figure I: The left panel plots the share of US debt by outstanding maturities averaged over the period 1960-2017. The right panel plots the time-series for Macaulay duration (years).

$m > 30$, we assume that the loadings flatten out, that is we set $A_{1,r}(m) = A_{1,r}(30)$ and $A_\sigma(m) = A_\sigma(30)$. Under these assumptions, we can obtain the optimal portfolio for all maturities that we see in the U.S. debt data.

In the right panel of figure II, we overlay the US debt profile with the optimal profile. The U.S. debt profile starts above the optimal and curves cross each other at around 25 quarters. We find that the US overweights short maturities relative to the optimal. In terms of Macaulay duration, the optimal portfolio has a duration of about 7.6 years which is much larger than the range 2-4 years that we found for the US debt profiles.

Time-varying covariances Finally, we want to investigate the question of rebalancing of the optimal portfolio. Our theory suggests that the rollover risk component requires no rebalancing and the relative hedging component should be rebalanced as $\hat{B}_t$, or as conditional covariances vary over time. In this section, we revisit the calculations of the optimal portfolios after allowing for time-variation in the variances and covariances of returns. We extend the factor model laid out in equations (18) – (21) so that the parameters $\left(\{A_{\sigma,r}(n)\}_n, A_{\sigma,y}, A_{\sigma,s}\right)$ and A_f are replaced by univariate GARCH processes:

$$A_{\sigma,x,t} = \bar{A}_{\sigma,x} + \rho_x^A A_{\sigma,x,t-1} + \theta_x^A \epsilon_t^x$$

where x is a generic place holder for the returns, output growth and tax revenues (net of non-discretionary spending). We impose that ϵ_t^x is Gaussian and each x, and estimate $\bar{A}_{\sigma,x}, \rho_x^A, \theta_x^A$

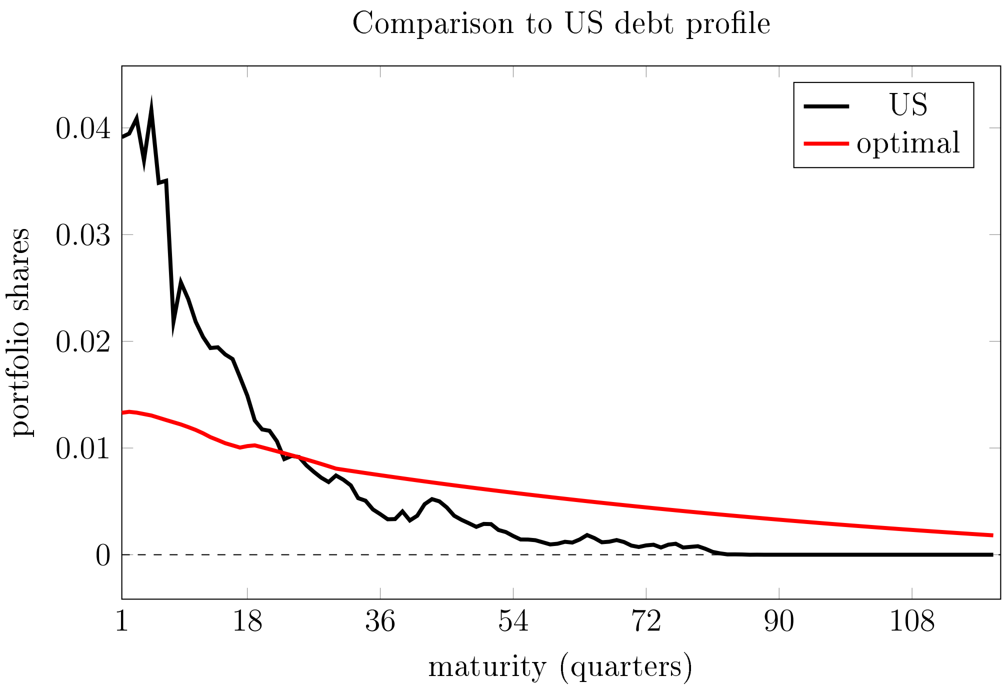

Figure II: The US debt profile is averaged for the period 1960-2017. The optimal debt profile is calculated using interpolated factor loadings.

using Maximum Likelihood.

In figure III, we plot estimated conditional volatilities for a subset of variables. There is a clear pattern. The volatilities for returns (including the factor) and macroaggregates are high in the early 80s and the great recession of 2008-2010 and quite stable in the intervening periods. Keeping everything else the same, periods when the factor is more volatile increases the covariance of returns, the covariance of returns with output growth (or tax revenues) as well as the variance of returns. Thus, its effect on the optimal portfolio is ambiguous. In table IV, we report the optimal portfolio separately for the high volatility and low volatility episodes. Quantitatively, we find that the optimal portfolios are stable, in spite of the fact that the volatities of the returns and factors are quite different in these sub periods.

Price impact Finally, we turn to the magnitude of the price impact term. We would ideally want to assess how a perturbation that changes the portfolio of the government keeping the total debt constant affects bond returns. In the fall of 2010, the Federal Reserve embarked on the second wave of quantitative easing (QE2) in which it decided to buy long-term Treasuries by issuing reserves and other short-term securities. Several studies such as

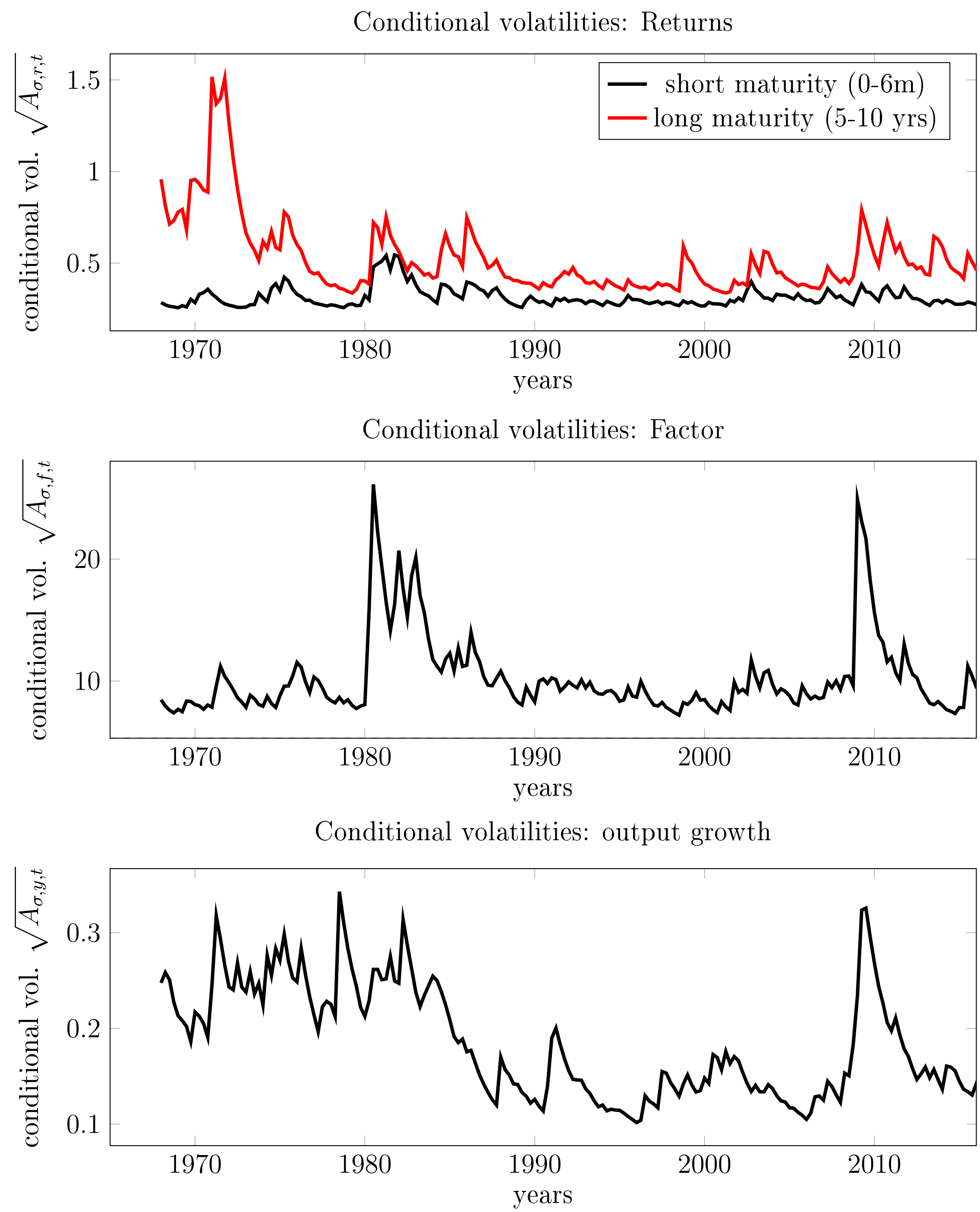

Figure III: Conditional volatilities of returns, factor, and output growth using the estimated GARH model

Table IV: Optimal Rebalacing

	Conditional Volatilties, $\sqrt{\boldsymbol{A}_\sigma}$					Optimal Portfolio, $\boldsymbol{\omega}$				
	1960-1979	1980-1984	1985-2007	2008-2010	2011-2016	1960-1979	1980-1984	1985-2007	2008-2010	2011-2016
0-6m	0.30	0.41	0.30	0.33	0.30	0.09	0.10	0.10	0.10	0.09
6-12m	0.59	0.66	0.61	0.64	0.62	0.08	0.09	0.09	0.09	0.09
12-18m	0.72	0.80	0.74	0.77	0.77	0.07	0.08	0.09	0.08	0.08
18-24m	0.72	0.78	0.72	0.75	0.79	0.06	0.08	0.08	0.08	0.07
24-30m	0.73	0.77	0.71	0.73	0.81	0.05	0.07	0.07	0.07	0.07
30-36m	0.70	0.76	0.60	0.68	0.90	0.04	0.05	0.05	0.05	0.07
36-42m	0.68	0.58	0.51	0.59	0.84	0.04	0.04	0.04	0.04	0.06
42-48m	0.71	0.55	0.44	0.53	0.77	0.01	0.00	0.02	0.03	0.02
48-54m	0.72	0.44	0.43	0.53	0.65	0.06	0.02	0.01	0.00	0.02
54-60m	1.05	0.50	0.39	0.44	0.57	0.02	0.02	0.01	0.02	-0.01
60-120m	0.71	0.54	0.43	0.56	0.51	0.08	0.08	0.08	0.08	0.08
factor	8.76	16.12	9.33	15.15	9.32					
g_y	0.25	0.25	0.14	0.23	0.15					
g_s	1.51	1.16	1.06	1.68	0.90					

Note: The table plots the average of the conditional volatilities and optimal portfolios for several subsamples. The periods $1980-85$ and $2008-2010$ are the high volatility years. For all the calculations $\hat{B}=20$, and $\beta=0.99$.

Krishnamurthy and Vissing-Jorgensen (2011), Joyce et al. (2012), Hamilton and Wu (2012), Chen et al. (2012) analyzed the effects of QE2 on the prices of bonds, thus providing us a natural experiment to quantify the price impact term. Using an event study approach, Krishnamurthy and Vissing-Jorgensen (2011) report that QE2 lowered the yields on 5-year bonds by 11 to 16 basis points and 10-year bonds by 7-10 basis points. However, they conclude that virtually none of this effect was through the portfolio rebalancing channel – an effect that arises because via QE2 the government changes the wealth of the marginal agent and thereby alters the equilibrium risk compensation all risky assets. Their analysis suggests that the estimated fall in yields is mostly an outcome of (i) a "signaling" effect, i.e., large asset purchases of long-term treasuries *signaled* a policy stance where short-rates will be low in the future, or (ii) the private sector updating its inflation expectations upwards. The price impact term in our model exclusively speaks to the rebalancing effect and our reading of the QE2 evidence is that it is economically small.

5 Quantitative Analysis

We now conduct a quantitative study of an optimal portfolio in a calibrated economy. Our strategy is to choose parameters values for preferences and shock distributions that allow the model economy with exogenous government policy to generate moments that match patterns of returns described in the previous section along with other U.S. business cycle facts. After estimating these parameters, we compute the Ramsey allocation, analyze its properties, and contrast our findings with the previous literature.

5.1 Calibration

As in section 2, the economy has two types of agents ("Poor", "Rich") with relative mass $(\lambda, 1-\lambda)$. We assume that the both types of private agents have the same preferences over consumption and perfectly correlated discount factor shocks. For $i \in \{R, P\}$ these preferences are given by

$$\mathbb{E}_0 \sum_t \eta_t \left(\frac{(c_t^i)^{1-\alpha}}{1-\alpha} \right).$$

The government's preferences are given by

$$(1-\mu)\mathbb{E}_0 \sum_t \eta_t \beta^t \left(\frac{\left(c_t^P\right)^{1-\alpha}}{1-\alpha} \right) + \mu \mathbb{E}_0 \sum_t \eta_t \beta^t \left(\frac{\left(c_t^R\right)^{1-\alpha}}{1-\alpha} \right)$$

where μ is the Pareto weight of the poor agents.

The stochastic processes for $\{Y_t, \eta_t, G_t\}$ are parameterized as

$$\log \frac{Y_t}{Y_{t-1}} = g_{y,t}, \quad \log \frac{\eta_t}{\eta_{t-1}} = g_{\eta,t}, \quad \log \frac{G_t}{Y_t} = \log \hat{G}_t$$

and

$$\begin{bmatrix} g_{y,t} \\ \log \hat{G}_t \\ g_{\eta,t} \end{bmatrix} = \mu + A \begin{bmatrix} g_{y,t-1} \\ \log \hat{G}_{t-1} \\ g_{\eta,t-1} \end{bmatrix} + \Sigma \begin{bmatrix} \epsilon_{y,t} \\ \epsilon_{G,t} \\ \epsilon_{\eta,t} \end{bmatrix}$$

A progressive tax function (as in Heathcote et al. (2017)) is used to raise revenues from the rich household. Their after-tax income is given by

$$Y_t - \Upsilon_t(Y_t) = Y_t - \tau_0 Y_t^{1-\tau_1} \Omega_t^{\tau_1}$$

$$\Omega_t = Y_{t-1}^{\theta} \Omega_{t-1}^{1-\theta}.$$

The parameter τ_0 pins down the level of taxes while the parameter τ_1 controls how sensitive tax revenues are to fluctuations in output in the short run. The last term Ω_t ensures that the tax function Υ is stationary under growth rate shocks. The parameter θ controls the speed with which the short-run elasticity $(1-\tau_1)$ approaches the long-run elasticity which necessarily needs to be one in order to keep the ergodic distribution of ratio of tax revenues to output stationary.

We construct a competitive equilibrium with two types of assets: a one period risk-free bond, and a consol. The government policy in the competitive equilibrium is given by

$$\frac{B_t}{Y_t} = \bar{B}, \qquad \frac{B_t^{consol}}{B_t} = \bar{\omega}.$$

We set $\alpha = 2$, and $\beta = 0.99$ for a quarterly calibration. The parameter λ is set such that the economy is consistent with stylized features of the distribution of tax revenues and bond holdings among U.S. households. About half of U.S. households directly or indirectly (i,e., through mutual funds, pension funds, and other institutional investors) participate in the bond markets, and more than 90% of the total tax revenue comes from the households with above median income. In light of these observations, we set $\lambda = 0.5$. The government policy parameters $\bar{B} = 60\%$ is calibrated to the average of federal debt to GDP over the period 1960-2015, and $\bar{\omega} = 20\%$ is calibrated such that the Macaulay duration of the government portfolio equals 5 years. See Hilscher et al. (2014) for estimates of duration for U.S. public debt.[8]

We simulate data from the the competitive equilibrium to calibrate μ, A, Σ, Υ. The matrix A is assumed to be diagonal with $A_y, A_{\hat{G}}$, and A_η calibrated to match the auto-correlations in output growth, non-transfer spending relative to output, and the risk-free rate. The matrix Σ is calibrated to match the variances and co-variances of growth rate of output, non-transfer spending, and risky returns. To compute these targets, risky returns are measured as the weighted-average of holding period returns across 5 maturity bins (0-1 year, 1-2years, 2-3years, 3-4years, 4-5years, 5-10years, 10+ years), output is measured as real per capita GDP, and non-transfer spending is measured as federal defense expenditures. Finally, the parameters for the tax function Υ, i.e., τ_0, τ_1 are calibrated to match a ratio of federal tax revenues to GDP of 17%, and an elasticity of tax revenues to output of 2.5. We set $\theta = 0.95$ and explore sensitivity of our results to its value. The calibrated parameters and the targeted moments are listed in Table V.

5.2 Ramsey Planning Problem

We compute the Ramsey allocation for the calibrated economy. Following standard steps as in Farhi (2010) and Bhandari et al. (2017b), we express the Ramsey planning problem using two Bellman equations, one for $t = 0$, and one for $t > 0$. We re scale variables by dividing by Y_t and denote the re-scaled variables using $\hat{\cdot}$. Define auxiliary variable $\hat{\mathcal{B}}_t$, $\hat{\mathrm{q}}_t$ as

$$\hat{\mathcal{B}}_t \equiv Y_t^{\alpha-1} c_t^{-\alpha} B_t,$$

$$\hat{\mathrm{q}}_t \equiv Y_t^{\alpha} c_t^{-\alpha} q_t^{consol}$$

[8]The value of $\bar{\omega}$ is obtained as a solution to

$$(1-\bar{\omega})\left(\frac{1}{4}\right) + \bar{\omega}\left(\frac{\sum_t t(1-\beta)\beta^t}{4}\right) = 5.$$

Table V: Baseline Parameters

Parameters		Target Moments	
Description	Value	Description	Value
λ	0.5	fraction of hh who own bonds	50%
A_y	0.32	auto-corr of real gdp growth	0.32
A_η	0.92	auto-corr of risk-free rate	0.92
$A_{\hat{g}}$	0.99	auto-corr of defense spending/gdp	0.99
Σ_{yy},Σ_{yr}	$0.002^2, -0.33$	std(gdp growth), cov(gdp growth, returns)	0.2%, -0.17
Σ_{GG},Σ_{Gr}	$0.02^2, 0.0$	std(defense exp/gdp), cov(defense exp/gdp, returns)	2%, -0.03
$\Sigma_{\eta\eta}$	0.002	std(returns)	2.73%
τ_0	.34	agg. tax revenues/ gdp	17%
τ_1	-1.5	cov(gdp growth,tax revenues growth)/var(gdp growth)	2.5
θ	0.95	–	–

Note: This table lists the parameter values for the baseline calibration and the targeted moments. Returns are measured as the weighted average of excess holding period returns across maturities where weights are the portfolio shares of the federal gross debt in those maturity bins.

where B_t is the market value of total debt and the q_t^{consol} is the price of the consol. The re-scaled tax revenues satisfy

$$\Upsilon_t(Y_t)/Y_t = \frac{\tau_0 Y_t^{1-\tau_1}\Omega_t^{\tau}}{Y_t} = \tau_0\left(\frac{\Omega_t}{Y_t}\right)^{\tau_1} = \tau_0 exp(\tau_1 \upsilon_t),$$

$$\upsilon_t = (1-\theta)\left(\upsilon_{t-1}\right) - g_{y,t}.$$

Using these auxiliary variables (and similarly $\hat{\mathcal{B}}_t^0$,$\hat{\mathcal{B}}_t^{consol}$ for the holdings in the two securities), the Euler equation for the rich agents with respect to the consol can be expressed as

$$\hat{\mathrm{q}}_{t-1} = \beta\mathbb{E}_{t-1}\left[\exp(g_{\eta,t} - \alpha g_{y,t})\left(\left(\hat{c}_t^R\right)^{-\alpha} + \hat{\mathrm{q}}_t\right)\right],$$

and their budget constraint as

$$\begin{aligned}\hat{\mathcal{B}}_t &= (\hat{\mathcal{B}}_{t-1} - \hat{\mathcal{B}}^{consol}_{t-1}) \frac{\exp\left(-g_{y,t}\right)\left(\hat{c}^R_t\right)^{-\alpha}}{\beta \mathbb{E}_{t-1}\left[\exp(g_{\eta,t} - \alpha g_{y,t})\left(\hat{c}^R_t\right)^{-\alpha}\right]} \\ &\quad + \hat{\mathcal{B}}^{consol}_{t-1} \frac{\exp(-g_{y,t})\left(\left(\hat{c}^R_t\right)^{-\alpha} + \hat{\mathrm{q}}_{\mathrm{t}}\right)}{\beta \mathbb{E}_{t-1}\left[\exp(g_{\eta,t} - \alpha g_y(s))\left(\left(\hat{c}^R_t\right)^{-\alpha} + \hat{\mathrm{q}}_{\mathrm{t}}\right)\right]} \\ &\quad - \left(\hat{c}^R_t\right)^{-\alpha}\left(\lambda \hat{T}_t + \hat{G}_t - (1-\lambda)\tau_0 \exp(\tau_1 \upsilon_t)\right)\end{aligned}$$

It helps to describe the $t > 0$ Ramsey plan recursively. Let $V(\mathcal{B}\_, \mathrm{q}\_, \upsilon\_, Y\_, g_y\_, g_\eta\_, \hat{G\_})$ be the value of the government entering with marginal-utility-adjusted debt, $\mathcal{B}\_ = c\_^{-\alpha} B\_$, and marginal-utility-adjusted price of consol, $\mathrm{q}\_ = c\_^{-\alpha} q^{consol}$, the state describing the rescaled tax revenues $\upsilon\_$, and the exogenous states $\left(Y\_, g_y\_, g_\eta\_, \hat{G\_}\right)$. Let $s\_ = (g_y\_, g_\eta\_, \hat{G\_})$, we can then define a stationary value function for the Ramsey Planner as

$$\begin{aligned}\hat{V}(\hat{\mathcal{B}}\_, \hat{\mathrm{q}}\_, \upsilon\_, s\_) &= \frac{V(\mathcal{B}\_, \mathrm{q}\_, Y\_, g_y\_, g_\eta\_, \hat{G\_})}{(Y\_)^{1-\alpha}\eta\_} \\ &= \mathbb{E}_{s\_}\left[\exp(g_\eta(s) + (1-\alpha) g_y(s))\left([1-\mu]\left(\frac{\hat{T}(s)^{1-\alpha}}{1-\alpha}\right) + \mu\left(\frac{\hat{c}^R(s)^{1-\alpha}}{1-\alpha}\right) + \beta \hat{V}(\hat{\mathcal{B}}(s), \hat{\mathrm{q}}(s), s)\right)\right]\end{aligned} \tag{23}$$

and the $t > 0$ Ramsey plan maximizes (23) by choosing $\left\{\hat{T}(s), \hat{\mathrm{q}}(s), \hat{c}(s), \hat{B}(s), \hat{B}\_^{consol}\right\}$ subject to implementability constraints

$$\hat{\mathrm{q}}\_ = \beta \mathbb{E}_{s\_}\left[\exp(g_{\eta,}(s) - \alpha g_y(s))\left(\hat{c}^R(s)^{-\alpha} + (1-\delta)\hat{\mathrm{q}}(s)\right)\right] \tag{24}$$

$$\begin{aligned}\hat{\mathcal{B}}(s) &= (\hat{\mathcal{B}}\_ - \hat{\mathcal{B}}\_^{consol}) \frac{\exp\left(-g_y(s)\right)\hat{c}^R(s)^{-\alpha}}{\beta \mathbb{E}_{s\_}\left[\exp(g_\eta(s) - \alpha g_y(s))\hat{c}^R(s)^{-\alpha}\right]} \\ &\quad + \hat{\mathcal{B}}\_^{1} \frac{\exp(-g_y(s))\left(\hat{c}^R(s)^{-\alpha} + \hat{\mathrm{q}}(s)\right)}{\beta \mathbb{E}_{s\_}\left[\exp(g_\eta(s) - \alpha g_y(s))\left(\hat{c}^R(s)^{-\alpha} + \hat{\mathrm{q}}(s)\right)\right]} \\ &\quad - \hat{c}^R(s)^{-\alpha}\left(\lambda \hat{T}(s) + \hat{G}(s) - (1-\lambda)\tau_0 \exp(\tau_1 \upsilon(s))\right)\end{aligned} \tag{25}$$

$$1 = (1-\lambda)\hat{c}^R(s) + \lambda \hat{T}(s) + \exp(g(s)) \tag{26}$$

$$\upsilon(s) = (1-\theta)\upsilon\_ - g_y(s) \tag{27}$$

Solving numerically the $t > 0$ Ramsey problem (24)-(27) is difficult with conventional numerical techniques because the state space consists of 3 endogenous variables, and 3 continuous shocks. To overcome this curse of dimensionality, we adopt numerical methods developed in Evans (2014) and Bhandari et al. (2017a). The details are relegated to the appendix. Given $\hat{V}$, we can recover the Ramsey plan by solving a standard optimization problem for the $t = 0$ planner who takes initial conditions on $\frac{B_{-}}{Y_{-}}, \nu_{-}, s_{-}$ and maximizes (23) subject to a time-0 budget and resource constraints.

5.3 Results

We present are two central findings: (1) the optimal government portfolio largely issues the debt via a consol; and (2) there is little rebalancing in response to shocks. Our findings are robust to the choice of Pareto weights, and so for our baseline we set the $\mu = 0$.

We start by reporting the optimal Ramsey portfolio $\frac{B^{consol}}{B}$ as a function of the initial debt B in Figure IV. We see that fraction of total debt issued in the form of the consol ranges between 75% and 95% as we vary the initial debt relative to (quarterly) output between 25% and 125%. The intuition for this optimal portfolio comes forces that work through formula (12). Returns induced by the optimal allocations retain the patterns of returns in the data, i.e., they are volatile and largely uncorrelated with shocks that drive primary deficits. This implies that the relative hedging term is in expression (12) is small. We match the negative covariance of holding period returns and output and this implies that the optimal hedging portfolio requires a long position of about 45% relative to output in the consol when initial debt is zero. As we decrease B_t, the incremental portfolio mostly holds a short position in the consol to hedge the rollover risk arising due to fluctuations in the short rate.

We can compare the U.S. maturity structure to that implied by the Ramsey portfolio. The position in the consol can be replicated using an alternative market structure where the government trades zero-coupon bonds all maturities. Proposition 1 makes a sharp prediction for the maturity profile of the Ramsey portfolio: the fraction of debt outstanding in the first period should be $(1 - \omega_t^{consol}) + (1 - \beta)\omega_t^{consol}$, and then the fractions outstanding in the rest of the periods are given by $\left\{(1 - \beta)\omega_t^{consol}\beta^j\right\}_{j=2}^{\infty}$. We compute the maturity profile for the U.S. debt as follows. For each month, we use bond-level data from CRSP to group the unexpired treasury securities in bins indexed by their Macaulay duration (in years) and compute the fraction in each bin. We then take the sample average over the period 1960-2015. In Figure V, we plot the Ramsey profile and U.S. alongside each other. We see that

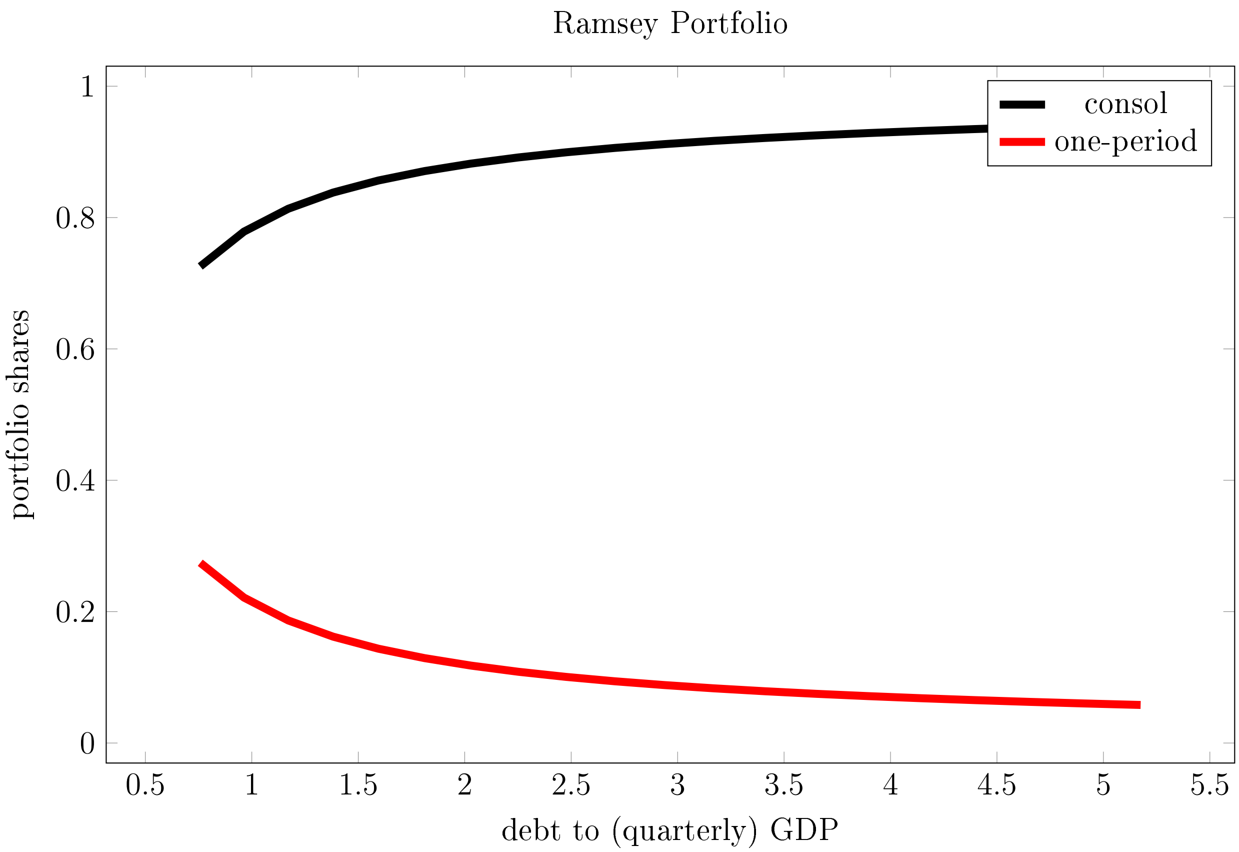

Figure IV: Percentage of total debt held in the consol and one period risk-free bond as a function of initial debt.

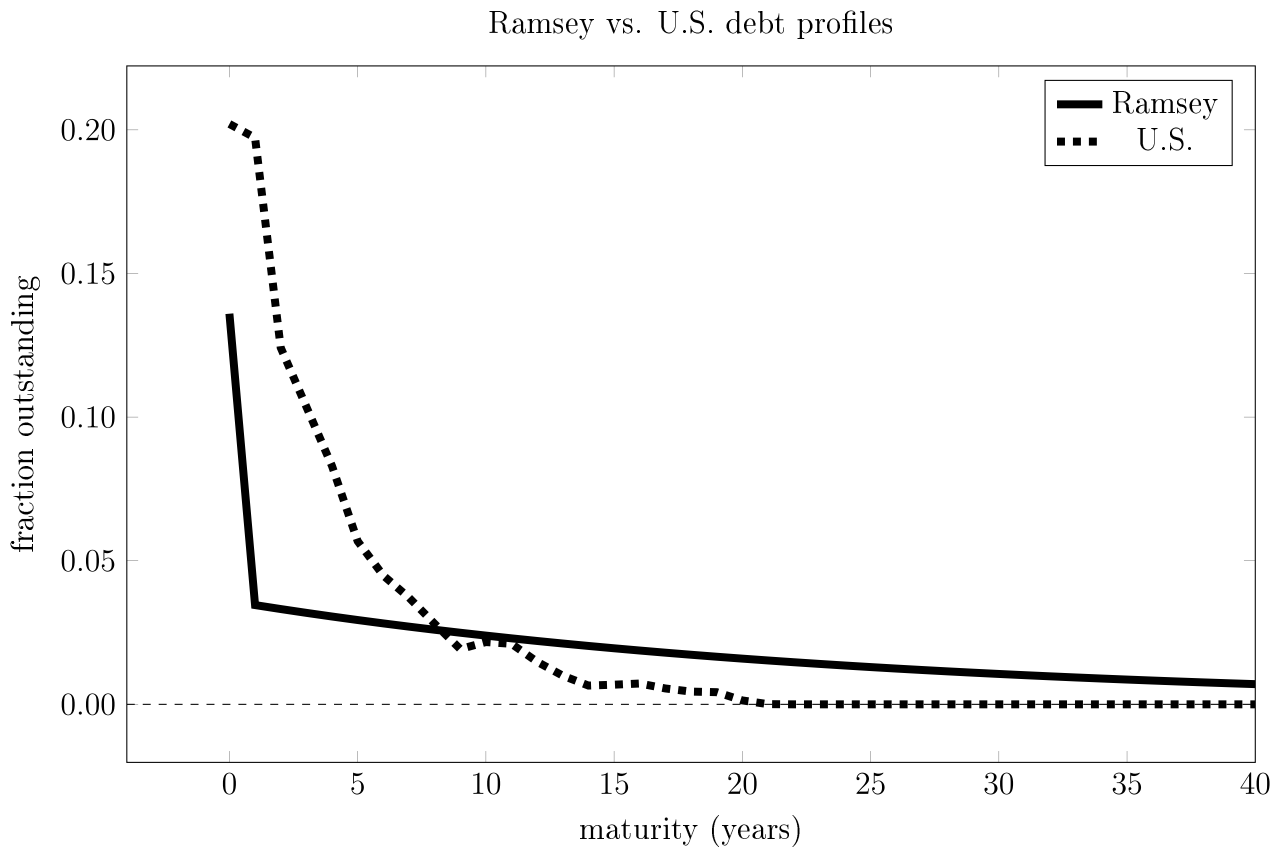

Figure V: Fraction of debt outstanding by maturity (years). For the U.S., we compute outstanding debt by bins of (Macaulay) duration and average for the sample period 1960-2015. The Ramsey debt profile is constructed using a replicating portfolio of zero-coupun bonds of all maturities.

the U.S. maturity profile declines at a much slower rate than optimal. For instance, the fraction of of debt outstanding in maturities 0-5 years is about 75%, and above 10 years is 13% in the U.S. data where as the Ramsey portfolio would have those fractions at 30% and 60%, respectively.

In Figure VI, we examine the accuracy of the formula in equation (12), by comparing the Ramsey portfolio to that predicted by the formula. More precisely, we simulate the Ramsey allocation to compute the ergodic covariances between returns, output, primary surplus, and then plug them in equation (12). The difference between the optimal Ramsey portfolio and that predicted by equation (12) is informative about the strength of the price effects. In our model economy, we find that the price effects are small and ignoring them would over estimate the holdings in the consol by 5%-10% of quarterly output. The Ramsey planner internalizes

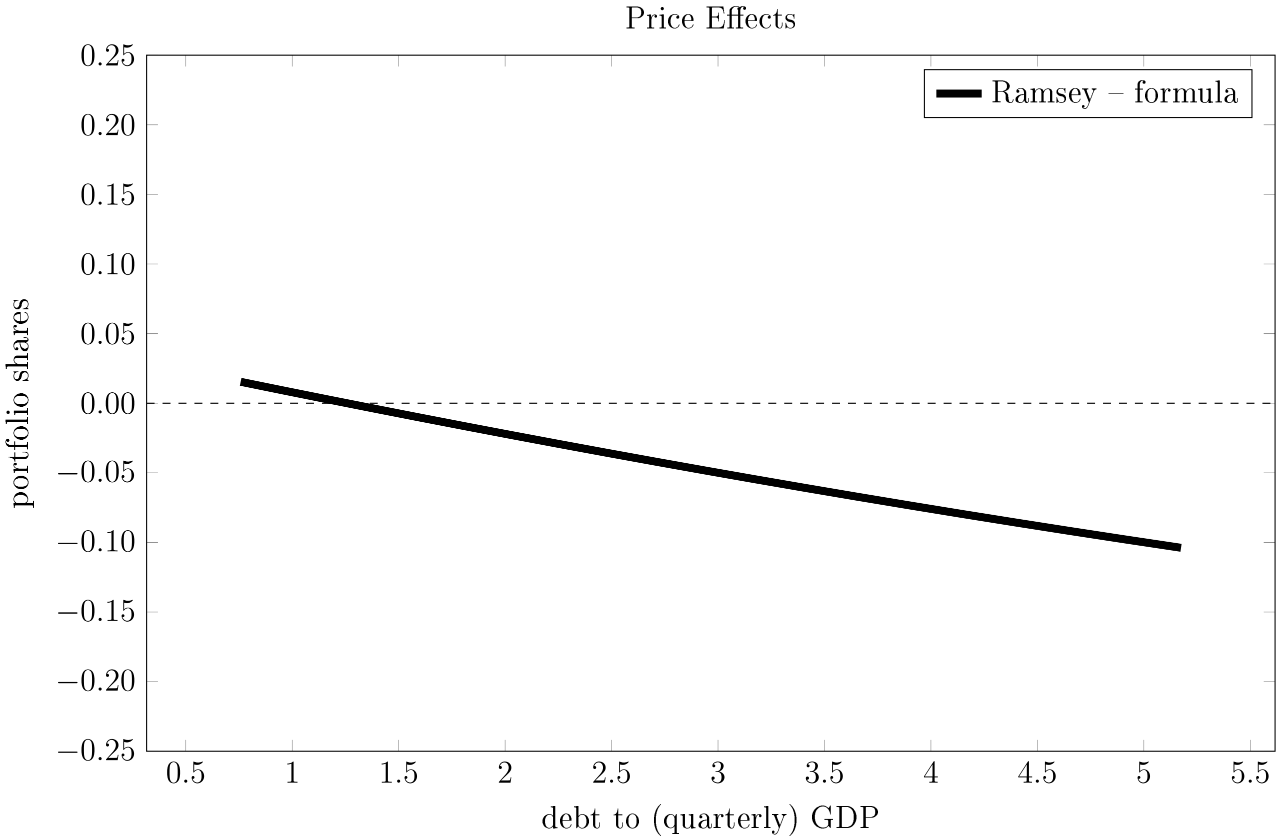

Figure VI: Price effects measured as the difference between Ramsey optimal and formula 12: amount of debt (relative to quarterly output) held in the consol.

that issuing more debt via a consol worsens the hedging avenues for the rich agents. Since these agents are also the marginal investors, they demand a higher a compensation for the incremental debt tempering the desire of the Ramsey planner to issue too much of its debt via a consol. However, these effects are small and consistent with the empirical papers that studied quantitative easing programs of the federal reserve (see section 4) which document small effects on the bond prices through the portfolio rebalancing channel.

Next we turn to how the Ramsey planner rebalances portfolios in response to shocks. In Figure VII, we plot the impulse response of total debt, the fraction of debt issued via the consol, and the transfers to a one standard deviation shock to ϵ_y, $\epsilon_G$and ϵ_η. The optimal portfolio shows small rebalancing in response to all of these shocks. For example, after a 0.2% permanent increase in output, the government reduces the debt-to-GDP ratio by slightly less than 1%, and the amount issued via the consol by 1.45%, keeping the maturity profile nearly unchanged. Transfers are smooth but not constant because with two assets only, the planner cannot implement the complete markets allocation.

Comparison to conventional RBC models We now study a special case of our model that is comparable to the Ramsey models of Angeletos (2002) and Buera and Nicolini (2004). These papers study optimal maturity in representative agent complete market settings with conventional RBC calibration. A crucial difference in our baseline is that asset returns are driven by discount factor shocks which are not too correlated to short run output and expenditure shocks. To recover their setting and outline the differences in a transparent way, we turn off the discount factor shocks as well as the growth rate shocks to output. We then recompute the optimal Ramsey portfolio and its rebalancing with respect to expenditure shocks.

In Figure VIII, we overlay the optimal holding in the consol relative to (quarterly) output for the RBC calibration on the graph for the baseline calibration. The optimal portfolio in the RBC calibration is about 10 times the baseline with large short positions in the consol and offsetting long positions in the one period risk-free asset. In Figure IX, we plot impulse responses for the RBC calibration to study the rebalacing after a one-standard deviation increase in non-transfer spending relative to output. We see that the responses of total debt and the tax rate are similar across the two parameter settings, but responses of portfolio holdings are about 25 times larger in the RBC calibration.

Robustness The optimal portfolio (and its rebalancing) are robust to assumptions on the Pareto weights and the relative mass of the poor agent. In Figure X, we plot the optimal holdings in the consol as a fraction for initial debt (fixed at 60% of annual output) for a range of μ, λ. In all the cases, we find that fraction of debt issued via a consol varies generally in the range of 80-100% of quarterly output. The logic for this is simple. Both these parameters affect the optimal portfolio mainly though the price effect term. As the mass of the poor agents increase, or the Pareto weight on the rich agents increase, the price effect term is smaller, and hence the planner increases the holdings in the risky consol. However, as discussed in section 3.2, these effects are bounded and affect the optimal portfolio only through second order terms.

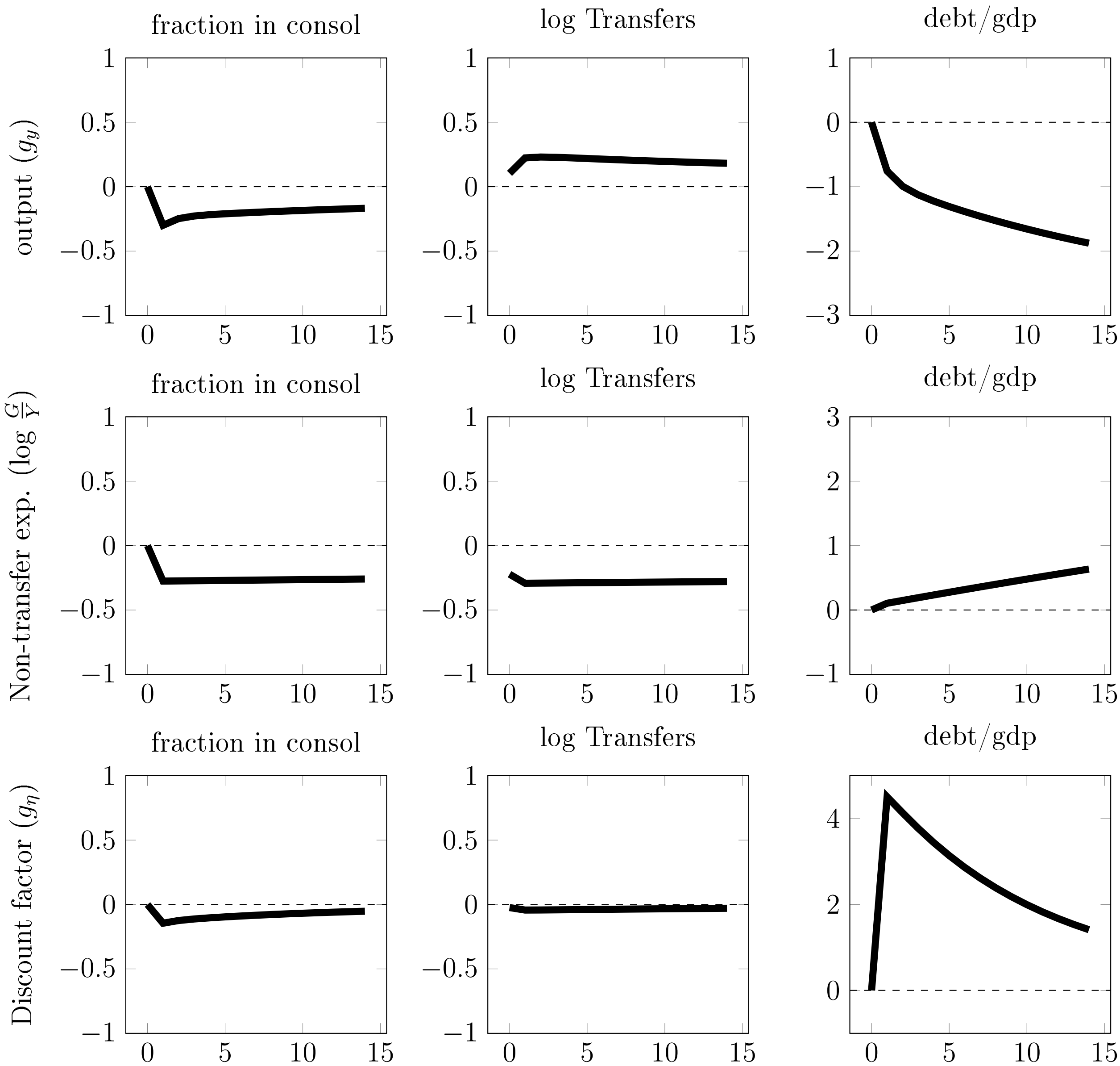

Figure VII: Impulse response functions under baseline calibration to a one standard deviation shock to $\epsilon_y, \epsilon_G, \epsilon_\eta$ respectively. Units on the y-axis are in percentage points.

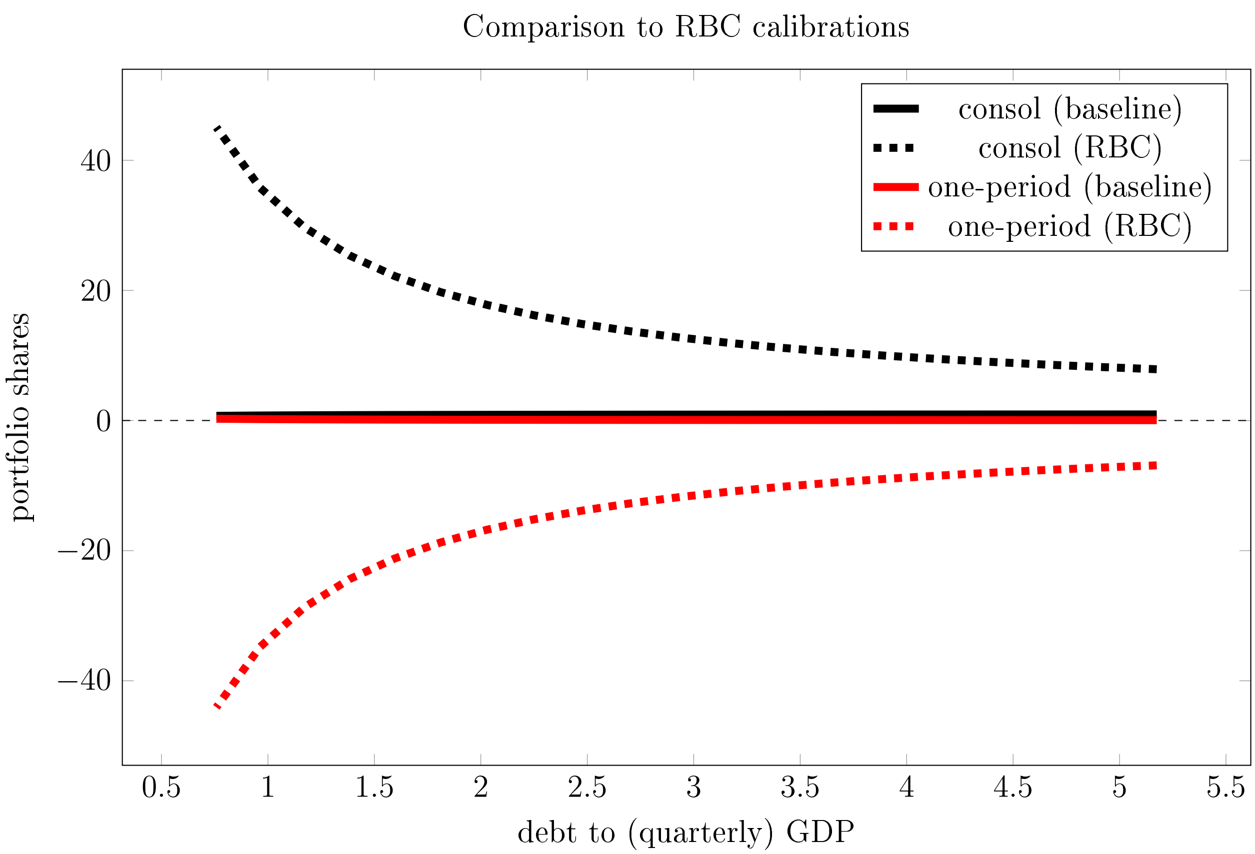

Figure VIII: Portfolio shares for the baseline calibration (solid lines) and the RBC calibration (dashed lines)

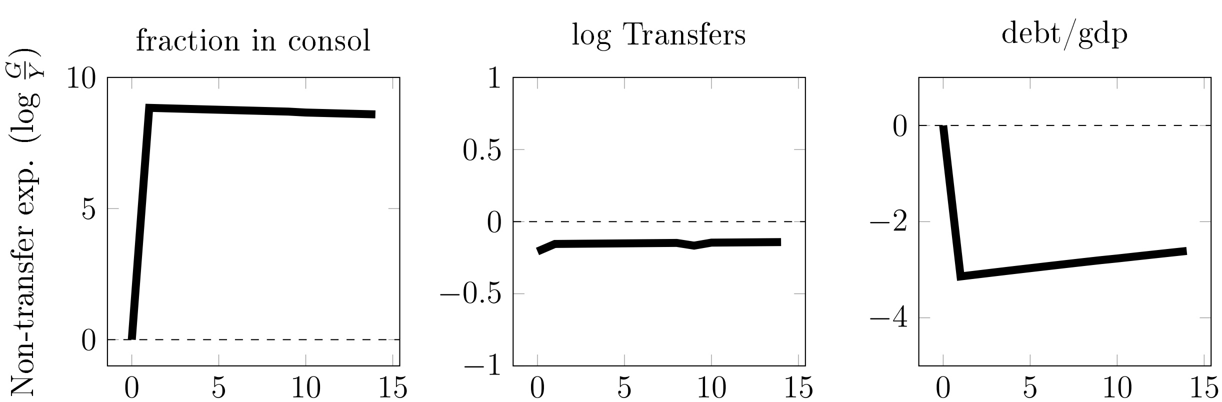

Figure IX: Impulse response functions under the RBC calibration to a one standard deviation shock to ϵ_G . Units on the y-axis are in percentage points.

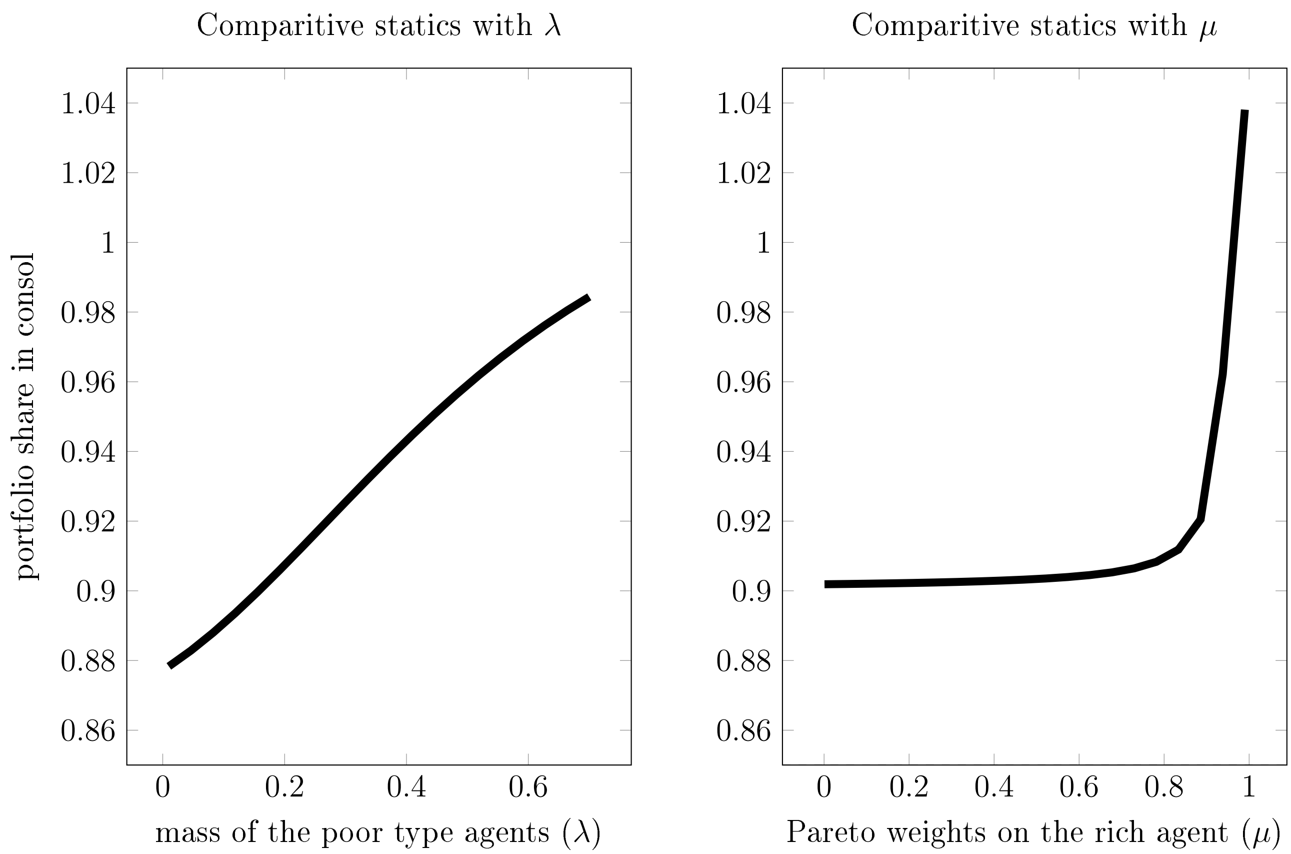

Figure X: Comparative statics with respect to the relative mass (λ) and the Pareto weights μ of the poor agents. The initial debt is fixed at 60% annual output as we vary λ, μ.

6 Conclusion

We develop a novel class of perturbations to study the optimal composition of a government's portfolio. We derive a formula for the optimal portfolio and show that it can be expressed in terms of estimable "sufficient statistics". We use U.S. data to calculate the key moments required by our theory and show that they imply that the optimal portfolio is approximately geometrically declining in bonds of different maturities and requires little rebalancing in response to aggregate shocks. Our optimal portfolio differs from portfolios prescribed by existing models often used in the business cycle literature and also from those adopted by the U.S. Treasury. The key normative differences are driven by counterfactual asset pricing implications of the standard models. A natural extension to our exercise is to allow for inflation and study joint portfolio and monetary policy in the direction of Lustig et al. (2008). We leave this for future work.

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