Speculative Bubbles, Heterogeneous Beliefs, and Learning *

Jan Werner

University of Minnesota

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Abstract: This paper develops a general theory of speculative bubbles and speculative trade in dynamic asset markets with short sales restrictions when agents have heterogeneous beliefs and are risk neutral. Speculative bubble arises when the price of an asset exceeds every trader’s valuation measured by her willingness to pay if obliged to hold the asset forever. Speculative bubble indicates speculative trade - whoever holds the asset intends to sell it at a later date. We identify a sufficient condition on agents’ heterogeneous beliefs for speculative bubbles in equilibrium. Our main focus is on heterogeneous beliefs arising from updating different prior beliefs in Bayesian model of learning. The sufficient condition for beliefs in Bayesian model is that no single prior dominates other agents’ priors in the sense of monotone likelihood ratio order. We study asymptotic properties of speculative bubbles in light of merging of conditional beliefs and consistency of priors.

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1. Introduction

If traders in asset markets have diverse (or heterogeneous) beliefs and short sales are restricted, asset prices will reflect the most optimistic beliefs. Pessimists who would want to short sell the asset, will be excluded from the market by the restriction on short selling. Harrison and Kreps (1978) pointed out that if heterogeneous beliefs are randomly changing over time so that different traders become the most optimistic at different times, asset prices may strictly exceed the most optimistic valuations because those traders anticipate to sell at a future date to new optimists. Trade becomes speculative as every trader who buys the asset intends to sell it at a future date, and hence she trades for short-term gain. Harrison and Kreps (1978) presented an example of a dynamic infinite-time market where agents are risk neutral, have heterogeneous beliefs about asset dividends, and short selling is prohibited. Because of risk neutrality, agents’ valuation of the asset which in general stands for the willingness to pay if obliged to hold the asset forever, is simply the discounted expected value of dividends under individual beliefs. Agents’ beliefs exhibit perpetual switching: there is no single agent who is more optimistic at all future dates and states than other agents about next period dividends of the asset. In equilibrium, the agent who has the most optimistic belief buys the asset and agents with less optimistic beliefs want to short-sell the asset but are restricted by the constraint. Asset prices persistently exceed all agents’ discounted expected values of future dividends.

Heterogeneity of beliefs and short sales restrictions are generally believed to be the primary reasons for the rapid rise and fall of stock prices during the dot.com bubble of 2000-2001. Ofek and Richardson (2003) provided compelling empirical evidence that traders beliefs about newly issued internet stocks were vastly diverse and that there were stringent short sales restrictions because of lockups. Hong, Scheinkman and Xiong (2006) developed a formal analysis in a model of asset markets with heterogeneous beliefs and short sales restrictions, and demonstrated that the model can account for price changes as in the dot.com bubble. Heterogeneity of beliefs in Hong, Scheinkman and Xiong (2006)\(^1\) model results from traders being too optimistic about information signals, that is, thinking that signals are more

\(^1\)See also Scheinkman and Xiong (2003).
accurate than they actually are.

Asset prices in Harrison and Kreps (1978) and Hong, Scheinkman and Xiong (2006) models strictly exceed the valuation of the most optimistic agents. The difference between the price and the highest valuation is termed *speculative bubble*. Speculative bubbles should not be confused with rational bubbles as the respective definitions are based on different notions of fundamental valuation. For speculative bubble, fundamental valuation is the willingness to pay for the asset if obliged to hold it forever. For rational bubble, fundamental valuation is the discounted expected value of future dividend under the risk-neutral pricing measure (or stochastic discount factor). While rational bubbles can arise in equilibrium under rather special conditions\(^2\), it is not so for speculative bubbles. Dynamic properties of rational and speculative bubbles are different, too. Speculative bubbles may “burst,” while rational bubbles have to persist indefinitely, with positive probability.

The assumption of heterogeneity of beliefs is often met with skepticism among economists for it is at odds with the common prior doctrine. Traders’ beliefs in the Harrison and Kreps (1978) example are dogmatic. They remain unchanged regardless of patterns of realized dividends. Heterogeneous beliefs in Scheinkman and Xiong (2003) are generated by agent’s overreacting to commonly observed signals. Belief updating for the agents deviates from Bayesian updating, and this gives rise to heterogeneous conditional beliefs. Agents’ updating rules remain unchanged regardless of observed signals. Morris (1996) introduced learning in the model of speculative trade. He considered an i.i.d dividend process parametrized by a single parameter of its distribution (probability of high dividend) that is unknown to the agents. Agents have heterogeneous prior beliefs about that parameter. Morris (1996) showed that, as the agents update their beliefs over time, their posterior beliefs will exhibit switching property that leads to speculative trade as long as the prior beliefs are not ranked in the maximum likelihood ratio order. Werner (2015) showed that speculative bubbles may arise with ambiguous beliefs that are common to all traders.

This paper develops a general theory of speculative bubbles and speculative bubbles arise only with low interest rates.\(^3\)
trade in dynamic asset markets with short sales restrictions when agents have heterogeneous beliefs and are risk neutral. There is a single asset with arbitrary dividend process over (discrete) infinite time-horizon. Heterogeneous beliefs may arise because of overconfidence in updating beliefs upon public information, Bayesian learning with heterogeneous priors, or simply be dogmatic beliefs. We show that a condition of valuation switching is sufficient for speculative bubble and speculative trade. Valuation switching holds if for every event at every date there does not exist an agent whose discounted expected value of future dividends exceeds all other agents’ discounted expected values from that date on forever. The condition of valuation switching is sufficient but not necessary for speculative bubbles. Interestingly, the example of Harrison and Kreps (1978) provides an illustration. One of the traders in that example is valuation dominant at every date, in every event.

Our main focus is on heterogeneous beliefs arising from updating different prior beliefs in Bayesian model of learning. We consider a general setting of priors on a parametric set of probability measures over arbitrary dividend sequences. Valuation dominance in the setting of Bayesian learning is closely related to the maximum likelihood ratio (MLR) order of priors. We show that dominance in the MLR order implies valuation dominance. For an i.i.d. binomial dividend process, valuation dominance is equivalent to MLR dominance (see Morris (1996)).

An important issue arising in settings with heterogeneous beliefs is whether or not difference in beliefs can persist in the long run as agents make observations. This is important for characterization of dynamic properties of speculative bubbles. The classical Blackwell and Dubins (1962) merging-of-opinions result states that if agents prior beliefs are absolutely continuous with respect to each other, then conditional beliefs for the future given the past converge over time. Slawski (2008) was the first to point out the relevance of merging of beliefs for the asymptotic behavior of speculative bubbles, see also Morris (1996). We show that if the true probability measure on dividends is absolutely continuous with respect to agents’ beliefs, then their valuations converge to the true valuation and, moreover, asset price converges to the true valuation. This makes speculative bubble vanish in the limit. The condition of absolute continuity in infinite time is a restrictive condition. In the setting of Bayesian learning with heterogeneous priors, a weaker
condition of consistency of priors with the true parameter combined with absolute
continuity of priors with respect to each other is shown to be sufficient for afore-
mentioned asymptotic properties of prices and valuations. Yet again, consistency
of priors with the true parameter is not an innocuous condition and may be easily
violated, for example, in infinite-dimensional parameter sets or misspecified priors.
We conclude that persistent (or non-vanishing) speculative bubbles are not at all
unlikely.

The paper is organized as follows. In Section 2 we present the model of dynamic
asset markets with heterogeneous beliefs and short sales restrictions. We prove the
main result about sufficiency of valuation switching for the existence of speculative
bubbles, and discuss two examples. In Sections 3 and 4 we discuss speculative
bubbles in settings with heterogeneous priors and Bayesian learning in general
and with i.i.d dividends. Section 5 is about asymptotic properties of speculative
bubbles in light of merging of conditional beliefs and consistency of priors.

2. Heterogeneous Beliefs and Speculative Trade.

Time is discrete with infinite horizon and begins at date 0. The set of possible
states at each date is a finite set $S$. The product set $S^\infty$ represents all sequences of
states. For a sequence (or path) of states $(s_0, \ldots, s_t, \ldots)$, we use $s^t$ the denote the
partial history $(s_0, \ldots, s_t)$ through date $t$. Partial histories are date-t events. The
set $S^\infty$ together with the $\sigma$-filed $\Sigma$ of products of subsets of $S$ is the measurable
space describing the uncertainty. There is a single asset with date-t dividend $x_t$.
Dividend $x_t$ is a random variable on $(S^\infty, \Sigma)$ assumed measurable with respect to $\mathcal{F}_t$, the $\sigma$-filed of date-$t$ events.

There are $I$ agents. Each agent $i$ is risk-neutral and discounts future consump-
tion by discount factor $\beta$, common to all agents. Agent’s $i$ beliefs are represented
by a probability measure $P^i$ on $(S^\infty, \Sigma)$. Agent’s $i$ utility function of consumption
plan $c = \{c_t\}_{t=0}^\infty$ adapted to $\mathcal{F}_t$ is

$$
\sum_{t=0}^\infty \beta^t E^i[c_t],
$$

where $E^i$ denotes the expectation under probability measure $P^i$. Endowments $e^i_t$
are measurable w.r. to \(\mathcal{F}_t\), positive, and bounded. Initial holdings of the asset are \(\hat{h}_0^t \geq 0\). The supply of the asset \(\hat{h}_0^t = \sum_i \hat{h}_0^i\) is strictly positive.

The agent faces the following budget and portfolio constraints

\[
\begin{align*}
    c(0) + p(0)h(0) &\leq c'(0) + p(0)\hat{h}_0^t, \quad (2) \\
    c(s^t) + p(s^t)h(s^t) &\leq c'(s^t) + [p(s^t) + x(s^t)]h(s^t), \quad \forall s^t, \quad (3) \\
    h(s^t) &\geq 0, \quad \forall s^t 
\end{align*}
\]

Condition (4) is the short-sales constraint.

An equilibrium consists of prices \(p\) and consumption-portfolio allocation \(\{c^t, h^t\}\) such that plans \((c^t, h^t)\) are optimal and markets clear. Market clearing is

\[
\sum_i c^t_i = \hat{e}_t x_t, \quad \text{and} \quad \sum_i h^t_i = \hat{h}_0,
\]

for every \(t\).

Because of the short-sales constraint, equilibrium asset price \(p_t\) at date \(t\) satisfies the relationship

\[
p_t(s^t) = \max_i \beta E^i[p_{t+1} + x_{t+1}|s^t]. \quad (5)
\]

The agent (or agents) whose one-period-ahead conditional belief \(P^i(\cdot|s^t)\) is the maximizing one on the right-hand side of (5) holds the asset in \(s^t\) while the other agents whose conditional beliefs give lower expectation have zero holding. We call the agent whose beliefs is the maximizing one the optimist (about next-period price plus dividend) at \(s^t\).

Market belief at \(s^t\) is the maximizing probability in (5), i.e., the optimist’s belief, and is denoted by \(\hat{P}(\cdot|s^t)\). Let \(\hat{P}\) be the probability measure on \(S^\infty\) derived from one-period-ahead probabilities \(\hat{P}(\cdot|s^t)\).\(^3\) It follows that \(\hat{P}\) is a risk-neutral pricing measure (or state-price process) for \(p\). Since the asset is in strictly positive supply and the discounted present value of the aggregate endowment \(\sum_{i=0}^{\infty} \beta^t E^i \hat{e}_t\) is finite, the no-bubble theorem (see Theorem 3.3 in Santos and Woodford (1997), or Theorem 32.1 in LeRoy and Werner (2014)) implies that equilibrium price of the asset is equal to the infinite sum of discounted expected dividends under the

\(^3\)The existence of probability measure \(\hat{P}\) on \(S^\infty\) follows from the Kolmogorov Extension Theorem, see Halmos (1974), Sec. 38.
market belief. That is,
\[ p_t(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-1} E_{\tau}^i[x_\tau|s^t], \] (6)

for every \( s^t \). The fundamental value of the asset under agent’s \( i \) belief is the discounted sum of expected dividends conditional on event \( s^t \), that is,
\[ V^i(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-1} E_t^i[x_\tau|s^t]. \] (7)

Because of risk-neutral utilities, agents’ fundamental values represent their willingness to pay for the asset if obliged to hold it forever. It follows from (5) that
\[ p_t(s^t) \geq V^i_t(s^t), \] (8)

for every \( i \), every \( s^t \). The following lemma will be often used in the analysis to follow.

**Lemma 1:** If \( p_t(s^t) > V^i_t(s^t) \) for agent \( i \) in some event \( s^t \), then \( p_\tau(s^\tau) > V^i_\tau(s^\tau) \) for every predecessor event \( s^\tau \) of \( s^t \), where \( \tau < t \).

**Proof:** We first prove that \( p_{t-1}(s^{t-1}) > V^i_{t-1}(s^{t-1}) \) for the immediate predecessor of \( s^t \). From (5) we have
\[ p_{t-1}(s^{t-1}) \geq \beta E_{t-1}^i[p_t + x_t|s^{t-1}] > \beta E_t^i[V^i_t + x_t|s^{t-1}] = V^i_{t-1}(s^{t-1}), \] (9)

where we used (8) and (for strict inequality) the assumption that \( p_t(s^t) > V^i_t(s^t) \).
The proof for non-immediate predecessor events is an iteration of the argument in (9). \( \square \).

We say that there is **speculative bubble** in event \( s^t \), if
\[ p_t(s^t) > \max_i V^i_t(s^t). \] (10)

If (10) holds, then the optimist who buys the asset at \( s^t \) pays the price exceeding her valuation of the asset if she were to hold the asset forever. This means, of course, that she intends to sell the asset at a later date. Thus, speculative bubble indicates speculative trade. It follows from Lemma 1 that if there is speculative
bubble in event $s^t$ at date $t$, then there is speculative bubble at every date $\tau < t$, in each predecessor event. Thus speculative bubble has to originate at date 0, or more generally at the time of initial offering, but it can cease to exist (or burst) at a later date, or be permanent.

Agent $i$ is (weakly) valuation dominant in event $s^t$ if

$$V^i(s^\tau) \geq \max_j V^j(s^\tau),$$

(11)

for every event $s^\tau$ which is a successor of $s^t$. If there is no valuation dominant agent in event $s^t$, then we say that agents’ beliefs exhibit valuation switching at $s^t$. There is perpetual valuation switching from $s^t$ on if beliefs exhibit valuation switching in every successor of $s^t$.

The main result of this section shows that valuation switching is sufficient for the existence of speculative bubble.

**Theorem 1:** If agents’ beliefs exhibit valuation switching in event $s^t$, then in equilibrium there is speculative bubble in $s^t$.

**Proof:** Suppose by contradiction that $p_t(s^t) = V^i_t(s^t)$ for some agent $i$. It follows from Lemma 1 that $p_\tau(s^\tau) = V^i_\tau(s^\tau)$ for every successor event $s^\tau$. Since agent $i$ is not valuation dominant, there exists $j$ and a successor event $s^\tau$ such that $V^j_\tau(s^\tau) > V^i_\tau(s^\tau) = p_\tau(s^\tau)$. But this contradicts (8). □.

If there is perpetual valuation switching from $s^t$ on, then, by Theorem 1, there is permanent speculative bubble in every successor event of $s^t$. The condition of valuation switching is sufficient for speculative bubble but it is not necessary. This is illustrated by the following example.

**Example 1, Harrison and Kreps (1978):** The dividend process $x_t$ is a Markov chain taking two values 0 and 1 for every $t \geq 1$. There are two agents whose beliefs are described by transition matrices $Q_1$ and $Q_2$ given by

$$Q_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

(12)

Note that agent 1 is more optimistic than agent 2 about next-period high dividend when current dividend is 1 and vice versa when the current dividend is 0. Discount factor is $\beta = 0.75$.  

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Fundamental values of the asset depend only on the current dividend and can found to be equal to

\[ V^1(0) = \frac{4}{3}, \quad V^1(1) = \frac{11}{9}, \quad (13) \]

\[ V^2(0) = \frac{16}{11}, \quad V^2(1) = \frac{21}{11}. \quad (14) \]

Thus agent 2 is valuation dominant.

In equilibrium, the agent who is more optimistic about next period dividend is the optimist (about price plus dividend) and holds the asset. Equilibrium prices can be found from equation (5). We have

\[ p(0) = \beta \left[ \frac{1}{2} p(0) + \frac{1}{2} (p(1) + 1) \right] \quad (15) \]

\[ p(1) = \beta \left[ \frac{1}{4} p(0) + \frac{3}{4} (p(1) + 1) \right] \quad (16) \]

It follows that

\[ p(0) = \frac{24}{13}, \quad p(1) = \frac{27}{13}. \quad (17) \]

One can easily verify that the right-hand sides of equations (15) and (16) are the respective maximal values among the two agents. We have

\[ p(0) > \max_i V^i(0) \quad \text{and} \quad p(1) > \max_i V^i(1), \]

implying that there is speculative bubble. □

We conclude this section with another example in which Theorem 1 is used to demonstrate that there is speculative bubble in equilibrium.

**Example 2, Overreaction to News.**

Suppose that dividends \( x_t \) are an i.i.d. sequence taking two values \( H \) or \( L \) such that \( L < H \). There are \( I \) agents. At each date \( t \geq 1 \), agents observe realization of a public signal \( y_t \). Signals are independent and with identical distribution conditional on next-period dividend. That is, conditional on \( x_{t+1} \) signals \( (y_1, \ldots, y_t) \) are i.i.d. Signals can take arbitrary positive or negative values. The distribution of \( y_\tau \) conditional on \( x_{t+1} \) for \( \tau \leq t \) is two-sided exponential with the following densities that can be different across agents:

\[ f_i(y|x) = \begin{cases} \frac{k_i a_i^y}{a_i}, & \text{if } y \geq 0, \\ \frac{k_i b_i^y}{b_i}, & \text{if } y < 0, \end{cases} \quad (18) \]
and
\[ f_i(y|L) = \begin{cases} k_i b^y_i & \text{if } y \geq 0, \\ k_i a^{y}_i & \text{if } y < 0, \end{cases} \] (19)
where \(0 < a_i < 1\) and \(0 < b_i < 1\), and \(k_i = 1/[(\frac{1}{\ln a_i} + \frac{1}{\ln b_i})].\)

The prior probability of high dividend is \(1/2\). The posterior probability of date-
\(t+1\) dividend equal to \(H\) after observing history of signals \(y^t = (y_1, \ldots, y_t)\) is
\[ \pi_i(H|y^t) = \frac{1}{1 + (\frac{b_i}{a_i})m} \] (20)
where we used Bayes rule and where \(m = y_1 + \ldots + y_t\). Thus the posterior probability
depends only on the cumulative value \(m\) of past and current signals. We assume
that \(\frac{b_i}{a_i} < 1\) for every \(i\), so that greater cumulative signals are considered favorable
to high dividend. The ratio \(\frac{b_i}{a_i}\) can be interpreted as the strength of reaction to
the signal. The higher the ratio, the higher are the probabilities assigned to high
dividend for positive cumulative values of the signal and to low dividends for
negative values of the cumulative signal.

Agent’s \(i\) fundamental valuation of the asset at date \(t\) depends only on the
cumulative signal and is given by
\[ V^i(m) = \frac{\beta}{1 - \beta}[L(1 - \pi_i(H|m) + H\pi_i(H|m))] \] (21)
for every \(m\). If agents differ in the strength of reaction to the signal, that is, if
there are \(i\) and \(j\) such that
\[ \frac{b_i}{a_i} \neq \frac{b_j}{a_j} \] (22)
then there is perpetual valuation switching. Indeed, if \(\frac{b_i}{a_i} > \frac{b_j}{a_j}\) so that agent \(i\)
over-react to the signal relative to \(j\), then \(\pi_i(H|m) > \pi_j(H|m)\) (and therefore
\(V^i(m) > V^j(m)\)) for large positive value of \(m\), and \(\pi_i(H|m) < \pi_j(H|m)\) for large
negative \(m\). Theorem 1 implies that there is speculative bubble. \(\Box\)

3. Speculative Trade and Bayesian Learning.

Bayesian learning in the setting of Section 2 is described as follows: There is
a family of probability measures \(P_{\theta}\) on \((S^\infty, \Sigma)\) parametrized by \(\theta\) in the set of
parameters \(\Theta\). The set \(\Theta\) can be finite or infinite. There is \(\sigma\)-filed \(G\) of subsets of
Θ and the mapping $\theta \rightarrow P_\theta(A)$ is measurable for every $A \in \Sigma$. An agent, who does not know the true probability measure on $(S^\infty, \Sigma)$, has a prior belief $\mu$ on $(\Theta, \mathcal{G})$. The prior $\mu$ induces a joint distribution of states and parameters $\Pi_\mu$ defined by

$$\Pi_\mu(A \times B) = \int_A P_\theta(B)\mu(d\theta),$$

for $A \in \mathcal{G}$ and $B \in \Sigma$. Conditional probability on $\mathcal{G} \times \Sigma$ upon observing date-$t$ history of states $s^t$ is $\Pi_\mu(\cdot|s^t)$ and it induces the posterior belief on $\Theta$ denoted by $\mu_t(\cdot|s^t)$ and conditional probability of the future given the past on $\Sigma$ denoted by $P_\mu(\cdot|s^t)$. For example, if $\mu$ is a Dirac point-mass measure at some $\theta$, then $\mu_t = \mu$ for every $t$ and $P_\mu(\cdot|s^t) = P_\theta(\cdot|s^t)$. This is “dogmatic” belief, as in Example 1, that is unaffected by learning.

Returning to the model of asset trading of Section 2, let agent’s $i$ prior belief be $\mu^i$ on $(\Theta, \mathcal{G})$. We use $E^i$ to denote the expectation under probability measure $P_{\mu^i}$ and $E^i[\cdot|s^t]$ (or simply $E^i_t$) for conditional expectation under conditional probability $P_{\mu^i}(\cdot|s^t)$. As pointed out by Morris (1996), the condition of valuation dominance is related to the monotone likelihood ratio order of priors. Suppose that $\Theta \subset R$ and that each prior $\mu^i$ has density function $f^i$ on $\Theta$. Recall that prior $\mu^i$ dominates $\mu^j$ in the monotone likelihood ratio (MLR) order if

$$\frac{f^i(\theta')}{f^i(\theta)} \geq \frac{f^j(\theta')}{f^j(\theta)} \quad \text{for every } \theta' \geq \theta. \quad (23)$$

**Proposition 2:** Suppose that $V^i_{\theta^t}(s^t) \geq V^j_{\theta^t}(s^t)$ for every $\theta^t \geq \theta$ and every $s^t \in S^\infty$. If agent’s $i$ prior $\mu^i$ MLR-dominates every other agents’ prior, then agent $i$ is valuation dominant.

**Proof:** It is well-known that if $\mu^i$ MLR-dominates $\mu^j$, then $\mu^i$ dominates $\mu^j$ in the sense of first-order stochastic dominance. Since $V^i_\theta(s^0)$ is increasing in $\theta$, it follows that

$$V^i(s^0) = \int V^i_\theta(s^0)f^i(\theta)d\theta \geq \int V^j_\theta(s^0)f^j(\theta)d\theta = V^j(s^0) \quad (24)$$

for every $j$. Further, if $\mu^i$ MLR-dominates $\mu^j$, then the posterior $\mu^i(\cdot|s^t)$ MLR-dominates the posterior $\mu^j(\cdot|s^t)$ for every $s^t$. As in (24), this implies $V^i(s^t) \geq V^j(s^t)$ for every $j$. □

In this section we consider an important special cases of the model of Section 3: i.i.d. dividends.

Suppose that there is a family of probability measures $\pi_{\theta}$ on the state space $S$. Let $P_{\theta}$ be the product measure $\pi_{\infty}$ making states independently and identically distributed with $\pi_{\theta}$ on $S$. Here, the mean of the posterior distribution $\mu_{t}^{i}(\cdot|s^{t})$ on $\Theta$ is the Bayes estimate of the unknown true parameter. Let $E_{\theta}[x]$ denote the expected value of the dividend under $\pi_{\theta}$.

Let $E_{i}[x_{t+1}|s^{t}]$ be the expected value of next-period dividend under the probability $P_{\mu_{t}^{i}}(\cdot|s^{t})$. Then

$$V^{i}(s^{t}) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t}E_{i}[x_{t+1}|s^{t}] = E_{i}[x_{t+1}|s^{t}] \frac{\beta}{1-\beta},$$

(25)

for every $s^{t}$. Note that $E_{i}[x_{t+1}|s^{t}] = \int_{\Theta} E_{\theta}[x] d\mu_{t}^{i} (\theta|s^{t})$. It follows that agent $i$ is valuation dominant in event $s^{t}$ if and only if her conditional expectation of the next period dividend (weakly) exceeds every other agent’s conditional expectation of the dividend in every successor event $s^{\tau}$ of $s^{t}$. Otherwise, there is valuation switching in $s^{t}$.

If each prior $\mu_{i}$ has density function $f^{i}$ on $\Theta$ and $\Theta \subset R$, then we have the following corollary to Proposition 2:

**Corollary 3:** In the i.i.d model, if agent’s $i$ prior $\mu_{i}$ MLR-dominates every other agents’ prior and $E_{\theta}[x]$ is a non-decreasing function of $\theta$, then agent $i$ is valuation dominant.

**Example 3, Morris(1996):** Suppose that $x_{t}$ can take two values, 0 or 1, and $\theta$ is the probability of dividend 1 where $\theta \in [0, 1] = \Theta$. Agent’s $i$ prior $\mu_{i}$ on $[0, 1]$ has density function $f^{i}$. States are identified with dividends so that date-$t$ event is denoted by $x^{t}$ for the history of dividends until date $t$. Expected value of next period dividend $E_{i}[x_{t+1}|x^{t}]$ conditional on $x^{t}$ is equal to the conditional probability of next-period dividend equal to one (or high). That probability depends only on the number of high dividends from date 0 through $t$, and is denoted by $\nu^{i}(t,k)$ for
\[ \nu^i(t, k) = \frac{\int_0^1 \theta^{k+1} (1 - \theta)^{t-k} f^i(\theta) d\theta}{\int_0^1 \theta^k (1 - \theta)^{t-k} f^i(\theta) d\theta} \]

(26)

If \( \mu^j \) the uniform prior on \([0, 1]\) with \( f^j(\theta) \equiv 1 \), then

\[ \nu^j(t, k) = \frac{(k + 1)}{(t + 2)}. \]

(27)

If \( \mu^j \) is the Jeffrey’s prior with \( f^j(\theta) \sim \frac{1}{\sqrt{\theta(1-\theta)}} \), then

\[ \nu^j(t, k) = \frac{(k + 1/2)}{(t + 1)}. \]

(28)

It can be easily seen that for every \( t \) and \( k \) such that \( \nu^i(t, k) > \nu^j(t, k) \), there exist \( \tau > t \) and \( k' > k \), such that \( \nu^i(\tau, k') > \nu^j(\tau, k') \), and vice versa. Since \( V^i(x^t) = \nu^i(t, k) \frac{\beta}{1-\tau} \), it follows that these two popular priors under ignorance give rise to perpetual valuation switching and, by Theorem 1, to permanent speculative bubble.

A general class of priors on the interval \([0, 1]\) which includes the uniform prior and the Jeffrey’s prior are beta priors with density functions of the form \( f^i(\theta) \sim \theta^\alpha - 1 (1 - \theta)^\beta - 1 \) for some \( \alpha^i > 0 \) and \( \beta^i > 0 \). The posterior probability of high dividend for beta prior is

\[ \nu^i(t, k) = \frac{(k + \alpha_i)}{(t + \alpha_i + \beta_i)} \]

(29)

(see Ghosh and Ramamoorthi (2003)). If \( \mu^j \) has beta distribution as well, with \( \alpha_j \) and \( \beta_j \), then \( \mu^j \) valuation dominates \( \mu^i \) at every \( (t, k) \) if and only if \( \alpha_j \geq \alpha_i \) and \( \beta_j \leq \beta_i \). Otherwise, there is perpetual valuation switching between \( \mu^i \) and \( \mu^j \). Note that prior \( \mu^j \) dominates \( \mu^i \) in the MLR-order if and only if \( \alpha_j \geq \alpha_i \) and \( \beta_j \leq \beta_i \). Thus, MLR-order dominance and valuation dominance are equivalent within the class of beta priors and iid binomial dividends (see Morris (1996)). \( \square \)

5. Merging of Beliefs and Speculative Bubbles.

In this section we discuss asymptotic properties of speculative bubbles. Slawski (2009) pointed out the relevance of the Blackwell and Dubins (1962) merging-of-opinions result for the asymptotics of bubbles. If conditional beliefs merge in
the sense of becoming close to each other in variational norm, then fundamental values converge to a common limiting value. Blackwell and Dubins theorem says that conditional beliefs merge if initial beliefs are absolutely continuous.

As in Section 2, suppose that the beliefs of agent $i$ are represented by a probability measure $P^i$ on $(S^\infty, \Sigma)$. Further, let $P^0$ be the true probability measure on $(S^\infty, \Sigma)$. Blackwell and Dubins theorem says that if $P^0$ is absolutely continuous with respect to $P^i$, then

$$\lim_{t \to \infty} \{ \sup_{A \in \Sigma} |P^i(A|s^t) - P^0(A|s^t)| \} = 0, \quad P^0 - a.e. \quad (30)$$

Condition (30) is called merging of conditional beliefs. If the merging condition holds, then $\lim_t [V^i_t(s^t) - V^0_t(s^t)] = 0$, $P^0$-a.e, where $V^0$ is the fundamental value of the asset under the true measure $P^0$.

Absolute continuity of $P^0$ with respect to $P^i$ says that $P^0(A) = 0$ for every $A \in \Sigma$ such that $P^i(A) = 0$. It is a strong condition. It does not follow from a rather innocuous condition that date-t marginal $P^0_t$ is absolutely continuous with respect to $P^i_t$ for all $t$. For example, if $P^0$ and $P^i$ are infinite products of measures on $S$ as in the case of iid true distribution and iid beliefs, then $P^0$ is absolutely continuous with respect to $P^i$ only if they are identical. The same holds for stationary Markov beliefs. The beliefs in Example 1 are, of course, not absolutely continuous with respect to each other.

By the same argument of the Blackwell-Dubins Theorem, if the true measure $P^0$ is absolutely continuous with respect to the market belief $\hat{P}$, then then equilibrium asset price $p$ converges to the true fundamental value $P^0$-a.e. We shall prove next that $P^0$ is absolutely continuous with respect to $\hat{P}$ if $P^0$ is absolutely continuous with respect to every agent’s belief $P^i$. We apply a criterion for absolutely continuity of measures on the product space $(S^\infty, \Sigma)$ due to Darwich (2009), which is a simplified version of the main result of a seminal paper by Kabanov, Liptser and Shiryaev (1985).

Probability measure $P^0$ is absolutely continuous with respect to another mea-

\footnote{Absolute continuity of probability measures on the product space $(S^\infty, \Sigma)$ is not only sufficient but also necessary for merging of conditional beliefs for any pair of measures whose date-t marginals are absolutely continuous for every $t$.}
sure $Q$ on $(S^\infty, \Sigma)$ if
\[
\sum_{t=0}^{\infty} E_Q[(1 - \frac{Q(s^{t+1}|s^t)}{P^0(s^{t+1}|s^t)})^2|s^t] < \infty, \quad P^0 - a.e., \quad (31)
\]
where the ratio of conditional probabilities is set to zero if the denominator is zero.

Recall from Section 3 that the market belief $\hat{P}$ is formed by selecting at each $s^t$ the one-period-ahead probability $P^i(\cdot|s^t)$ which maximizes (5). If the sum in (31) is finite for $Q = P^i$ for each $i$, then the sum for $Q = \hat{P}$ must be finite, as well. It follows that $P^0$ is absolutely continuous with respect to the market belief $\hat{P}$. We summarize our discussion in the following theorem

**Theorem 2:** Suppose that $P^0$ is absolutely continuous with respect to $P^i$ for every $i$. Then
\[
\lim_t [V^i_t(s') - V^0_t(s')] = 0, \quad P^0 - a.e. \quad (32)
\]
Moreover $P^0$ is absolutely continuous with respect to the market belief $\hat{P}$ and
\[
\lim_t [p_t(s') - V^0_t(s')] = 0, \quad P^0 - a.e. \quad (33)
\]
Consequently, if there is speculative bubble, it vanishes in the limit $P^0$-almost surely.

The analysis of asymptotic properties of speculative bubbles is somewhat different when beliefs arise from Bayesian learning with heterogeneous priors. As in Section 3, let $\Theta$ be the set of parameters with a $\sigma$-filed of subsets $\mathcal{G}$. Prior belief of agent $i$ is measure $\mu_i$ on $(\Theta, \mathcal{G})$. Let $\theta_0$ be the true parameter so that the true probability distribution on states is $P^0 = P_\theta$ for $\theta = \theta_0$. If $\Theta$ is a finite set, then the condition $\mu^i(\theta_0) > 0$ guarantees that the Dirac point-mass measure at $\theta_0$ is absolutely continuous with respect to $\mu_i$. This in turn implies that $P^0$ is absolutely continuous with respect to $P^i$, and by the Blackwell-Dubins Theorem, that conditionals $P^0(\cdot|s^t)$ and $P^i(\cdot|s^t)$ merge $P^0 - a.e$. If $\Theta$ is an infinite set, then the condition $\mu^i(\theta_0) > 0$ may be unnatural. In Example 3, there is no $\theta$ in the support of any of the priors that has strictly positive measure. Consistency of prior belief $\mu^i$ at $\theta_0$ becomes an important issue.

Recall that prior $\mu_i$ is consistent at $\theta_0$ if the posterior belief $\mu^i_t$ converges weakly
to the Dirac measure at $\theta_0$ in the weak-star topology that is
\[
\lim_{t \to \infty} \int_{\Theta} g d\mu^i_t(s^t) = g(\theta_0), \quad P^0 - a.e.
\] (34)
for every continuous and bounded function $g$ on $\Theta$. We have

Proposition 3: In the i.i.d model, if $E_\theta(x)$ is continuous in $\theta$, every prior $\mu^i$ is consistent at $\theta_0$, and $\mu^i$ are absolutely continuous with respect to each other, then the hypotheses (32) and (33) of Theorem 2 hold. If there is speculative bubble, it vanishes in the limit $P^0$-almost surely.

Proof: If $\mu^i$ is consistent at $\theta_0$, then $\mu^i_t$ converges weakly to the Dirac measure at $\theta_0$. This implies that $\lim_t [V^i_t(s^t) - V^0_t(s^t)] = 0$ $P^0$-a.e. Furthermore, if $\mu^i$ are absolutely continuous with respect to each other, then $P_{\mu^i}$ are absolutely continuous with respect to each other and, by the same argument as in Theorem 4, $P_{\mu^i}$ is absolutely continuous with respect to the market belief $\hat{P}$. This implies that $\lim_t [\nu^i_t(s^t) - V^0_t(s^t)] = 0$. $P^0$-a.e. $\square$

We illustrate Proposition 3 with the following example.

Example 4: Consider again the setting of Example 3. The uniform prior and the Jeffrey’s prior are absolutely continuous with respect to each other. Further, they are consistent at the true parameter $\theta_0$ for every $\theta_0 \in [0, 1]$. This follows from the general results which we discuss later, but it can also be seen from the results in Example 3. By the strong Law of Large Numbers, the frequency $k/t$ of high dividends converges to $\theta_0$ with $\pi_0$-probability 1. Means of the posteriors of $\mu^i$ and $\mu^j$ are $\nu^i(t, k)$ in (27) and $\nu^j(t, k)$ in (28), and they converge to $\theta_0$, as well. Variances of the posteriors converge to zero (see Ghosh and Ramamoorthi (2003)) implying consistency.

Fundamental valuations of the asset converge to $\frac{\beta}{1-\beta} \theta_0$ with $\pi_0$-probability 1. By Proposition 3, the price of the asset converges to the same value. There is permanent speculative trade and the speculative bubble vanishes in the limit. $\square$.

Conditions for consistency of a prior at true parameter have been extensively studied in Bayesian statistics (see, for example, Ghosh and Ramamoorthi (2003)). The classical Theorem of Doob (1948) for the i.i.d model states that if the mapping $\theta \to P^0_\theta$ is 1-to-1, then prior $\mu$ on $\Theta$ is consistent at $\mu$-almost every parameter $\theta$. The almost-every nature of the theorem is unsatisfactory, and there are stronger
results in the literature. With our maintained assumption of the finite set of states $S$, consistency holds for every parameter in the support of prior $\mu$. Support of $\mu$ is the smallest closed set of full measure. This follows from Freedman (1963), also from a general result of Schwartz (1965). If $S$ is infinite (countable or not), then the parameter set $\Theta$ may naturally be infinite dimensional, and consistency - beyond the Theorem of Doob - may not hold (see Diaconis and Freedman (1986)). A recent account of conditions for consistency for non-i.i.d. processes can be found in Shalizi (2009).


This paper is a contribution to theory of speculative bubbles and speculative trade in dynamic asset markets with short sales restrictions when agents have heterogeneous beliefs and are risk neutral. We demonstrated that the condition of valuation switching is sufficient for there being speculative bubbles in equilibrium. Our main focus has been on heterogeneous beliefs arising from updating different prior beliefs in Bayesian model of learning. The condition of valuation switching is closely related to agents’ priors not being ordered in the monotone likelihood ratio order. The take-away from the analysis of speculative bubbles for heterogeneous beliefs in Bayesian model of learning is that persistent speculative bubbles are not at all unlikely. We studied asymptotic properties of speculative bubbles which closely related to merging of conditional beliefs and consistency of priors.

Another case of heterogeneous beliefs that may give rise to speculative bubbles are misspecified priors. A prior is called misspecified if the true parameter lies outside of the support. Slawski (2009) provides an example in which the true parameter lies outside of the common support of priors and there is persistent speculative bubble. We leave the case of misspecified priors for future research.
References


Slawski, A. (2008), “The Dynamics of Speculative Bubbles under Learning,” Work-