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# TECHNOLOGY, SKILL AND THE WAGE STRUCTURE

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Working Paper 22176
http://www.nber.org/papers/w22176

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
April 2016

I thank Daron Acemoglu, Ufuk Akcigit, Treb Allen, Marios Angeletos, Enghin Atalay, Gadi Barlevy, Marco Bassetto, Jess Benhabib, Katarina Borovickova, Paco Buera, Ariel Burstein, Raquel Fernandez, Xavier Gabaix, Larry Jones, Boyan Jovanovic, Joe Kaboski, Rasmus Lentz, Jeremy Lise, Bob Lucas, Sydney Ludvigson, Jesse Perla, Ananth Seshadri, Jaume Ventura, Gianluca Violante, and seminar participants at UCL, the University of Wisconsin, the Federal Reserve Bank of Philadelphia, the Becker-Friedman Institute, the Stern School, the Federal Reserve Bank of Chicago, and CREi, for useful comments. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 22176
April 2016
JEL No. D50,E24,O33,O40

**ABSTRACT**

Technical change, even if it is limited in scope, can have employment, output, price and wage effects that ripple through the whole economy. This paper uses a flexible and tractable framework, with heterogeneous workers and technologies, and many tasks/goods, to analyze the general equilibrium effects of technical change for a limited set of tasks. Output increases and price falls for tasks that are directly affected. The effects on employment depend on the elasticity of substitution across tasks/goods. For high elasticities, employment expands to a group of more skilled workers. Hence for tasks farther up the technology ladder, employment falls, output declines, and prices and wages rise. For low elasticities, employment at affected tasks contracts among less skilled workers, as they shift to complementary tasks with unchanged technologies. In all cases, the output, price and wage changes are damped for more distant tasks, both above and below the affected group.

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## 1. INTRODUCTION

Technical change, even if it is limited in scope, can have employment, output, price and wage effects that ripple through the whole economy. This paper uses a flexible and tractable framework, with heterogeneous workers and technologies, and many tasks/goods, to analyze in detail the general equilibrium effects of technical change for a limited set of tasks. Technology and human capital are assumed to be complements in production, so the labor market, which is competitive, produces positively assortative matching between technologies and skills: tasks/goods with better technologies are produced by workers with more human capital. But the quantitative allocation of workers to technologies is endogenous, determined by demands for the tasks that are produced. Hence technical change for a limited set of tasks can produce substantial changes in employment, output levels, prices and wages, for tasks and workers not directly affected.

Why is a model of this type useful? Wage inequality in the U.S. and elsewhere has grown markedly over the last three decades, and technical change seems clearly involved in the trend. Empirical studies have documented two broad patterns. One is the polarization of labor markets, the growth in both employment and relative wages at the top and bottom of the job distribution, accompanied by declines in both in the middle. A second is that increases in between-firm wage inequality account for most or all of the overall change, with changes in within-firm inequality playing a distinctly minor role. The canonical model of skill-biased technical change, which features two types of labor and labor-augmenting technical change for each type, is inadequate to explain these trends.[1]

[1]See Autor, Katz and Kearney (2006), Machin and Van Reenen (2007), and Autor and Dorn (2013) for evidence on labor market polarization. See Song, et al. (2015) for evidence on the importance of the between-firm component in the U.S. See Acemoglu (2002) and Acemoglu and Autor (2016) for a detailed review of the canonical model and its limitations.

The model here has many tasks/goods, which are combined to produce a single final good. Tasks differ in terms of their technology level, so there is a one-dimensional technology ladder, and workers differ in their human capital, so there is also a one-dimensional skill ladder. All production functions display constant returns to scale, and all markets are perfectly competitive, so firms, as such, play no role. A competitive equilibrium consists of an allocation of skill types to tasks, and a supporting set of prices and wage rates. Complementarity between skill and technology implies that the equilibrium features positively assortative matching, as in Becker's (1973) classic model of partnership formation.

After an improvement in one technology, affecting a limited set of tasks, labor is reallocated across all tasks, and all prices and wage rates change. In the model here, those effects can be sharply characterized analytically and easily computed numerically.

The results are intuitively appealing. First, and unsurprisingly, output increases and price falls for tasks that are directly affected by the technical change. General equilibrium effects are never strong enough to offset the direct effect of the shock.

The effects on employment depend on the elasticity of substitution across tasks/goods. For elasticities that exceed unity, employment at the affected tasks expands to a group of more skilled workers. Hence employment falls at tasks farther up the technology ladder, so outputs decline, and prices and wages rise. The effects are stronger for tasks with technologies closer to the one enjoying the technical change, and damped for tasks farther up the ladder.

At the other end, employment at directly affected tasks can expand to less skilled workers, or it can contract. If it expands, employment falls at tasks farther down the ladder, so outputs decline, and prices and wages rise. If it contracts, outputs farther down the ladder increase. In either case the changes are damped for more distant tasks.

For elasticities of substitution across tasks below unity, these results are mirrored and reversed. At directly affected tasks, employment contracts among less skilled workers. Hence employment expands for tasks farther down the ladder, and outputs rise, with damped changes for more distant tasks. Prices and wages may fall for some tasks and workers closest to those affected by the technical change.

At the top end, employment at directly affected tasks can expand or contract. If it expands, employment falls at tasks farther up the ladder, so outputs decline, and prices and wages rise. If it contracts, employment and output increase for tasks farther up the ladder. In either case the changes are damped for more distant tasks.

In the cases where the sign of the effect is ambiguous, two model features determine which pattern occurs: the size of the elasticity of substitution across tasks and the range of skill levels employed at the tasks enjoying the technical change. For elasticities that are not too close to unity and narrow skill ranges, the effects are rather symmetric up and down the ladder from the tasks affected by the technical change. For elasticities that are close to unity, the signs are ambiguous, but the magnitude of the change is likely to be small.

As noted above, firms play no role in the analysis, and even the word is (mostly) avoided. Each worker chooses how to use his labor endowment, combining it with any of the available technologies. The worker's decision can be viewed as a choice about an occupation, with his output—called a task or a good—used in production of the single final good.

In some contexts the distinction between human capital and technology is blurred. Here, human capital is an asset that belongs to a single worker, who is the only one able to employ it in production. Technology is a nonrival input, used by all workers producing a particular good. Framed in terms of competitive firms, the technology for producing a particular good is available to all. However, as will be shown below, the equilibrium can readily be re-interpreted as one with monopolistically competitive firms, and the technology as intangible capital that is the property of the producer. In either case, the fact that it is a nonrival input distinguishes it from both human and physical capital.

The vast literature on vintage capital models suggests that the distinction between new technologies and new capital is also blurred. If a new technology requires new investment for its implementation, giving it one label or the other is largely a matter of taste. Here, physical capital is ignored, so implementing improved technologies requires no investment.

The rest of the paper is organized as follows. Section 2 discusses the related literature. Section 3 presents the basic model and characterizes the competitive equilibrium. It also looks at the effect of eliminating technology differences, showing how the wage distribution would change if all tasks had a common technology level.

The main results are contained in section 4, where the model is used to study the effect of technical change for one set of tasks. In particular, we ask how the labor allocation, output and price levels, and wage rates change, for all tasks and workers. A sufficient condition is provided for the conclusion that "a rising tide lifts all boats," that the improvement raises wages for all workers, even those paired with technologies that are unaffected.

Section 5 looks at a multi-sector version of the model. Here, tasks in each sector are used as inputs in producing a sector aggregate, and these sector aggregates are in turn used to produce final goods. The goals here are twofold. First, an assumption is provided under which the equilibrium in the multi-sector model is isomorphic to that in the simpler model. With this assumption, the conclusions in section 4 carry over unchanged. This fact is interesting because it allows a more flexible interpretation of the elasticities across tasks. Without that assumption, the multi-sector model is a framework to re-visit the "rising tide" question, and the answer is more nuanced. For example, technological improvement for higher-skill producers in a basically low-tech sector can reduce wages for the low-skill workers employed there.

Section 6 contains simulations that highlight some of the results, and section 7 concludes. Mathematical arguments and proofs are gathered in the Appendix.

## 2. RELATED LITERATURE

The model here is related to three literatures. One is the extensive theoretical literature on skill-biased technical change. A second is the literature that studies the role of search frictions in settings with heterogeneous firms and workers. The third is the literature examining recent trends in wage inequality in the U.S. and elsewhere.

The first models of skill-biased technical change had two types of workers, performing distinct and imperfectly substitutable tasks, with separate technology shocks for each type. Acemoglu (2002) provides an elegant treatment of this model, and studies its ability to account for the major trends in employment, wages, and skill premia in the U.S. Acemoglu and Autor (2016), who call it the 'canonical' model, provide a complementary treatment.

Subsequent contributions to this literature add physical capital as a third factor of production, and use the strong decline in capital (equipment) prices observed in the data as the technology shock. These models posit an aggregate production function that displays capital-skill complementarity, and the falling price of capital induces persistent capital deepening. Thus, even with realistic growth in the share of high-skill workers in the labor force, the skill premium increases. Papers in this group include Krusell, et. al (2000) and Autor and Dorn (2013).

A third strand looks at models where technical change can produce ex post heterogeneity among ex ante identical workers, because of the opportunities for investment in human capital or learning-by-doing that the new technologies offer. Examples here include Jovanovic (1998), Caselli (1999), and Violante (2002).

A fourth strand consists of models with heterogeneity in jobs as well as in labor.

Sattinger (1975) is the earliest in this group. Kurtzon (2015) analyzes a model in which jobs are distinguished by entry (skill acquisition) costs, and workers with higher ability have a comparative advantage in learning more complex tasks. The model is used to study the effects of immigration on the slope of the wage function.

The model here is closest to the one in Costinot and Vogel (2010), of which it is a special case. Their production function is assumed to be supermodular in its two inputs, while here a CES structure with a low substitution elasticity used. The additional structure is important because it allows a much more complete characterization of the effects of technology shifts. While the more general setup delivers straightforward results about the slope of the labor allocation and wage functions, and hence about inequality, it is difficult to obtain results about levels: who wins and who loses in absolute terms, as well as the effects on output levels and prices.

The model here is also related to those used to study the role of search frictions in labor markets with heterogeneous firms and workers. Examples include Sattinger (1995), Mortensen and Pissarides (1999), Postel-Vinay and Robin (2002), Menzio and Shi (2011), and Lise and Robin (2013).

The model here is closest to those in Bagger and Lentz (2015) and Lise, Meghir, and Robin (2016). Both use frameworks with heterogeneous workers and technologies, and a CES production function to combine the two inputs. Relative to those models, the one here drops search frictions, but endogenizes the prices of outputs across worker-technology pairs. Here there is a downward sloping demand curve for each task, and its position depends on final good production. This fact produces interactions between the wages of different workers employed at the same task and at different tasks. Closing the model in this way provides a micro-foundation for the match surplus function, a function that frictional search models take as exogenous. As a consequence, the model here produces a non-degenerate distribution of workers across technologies/tasks, even in the absence of search frictions. Thus, it offers a richer framework for asking how important frictions are in generating wage differentials across workers.

Finally, the model is related to work documenting two broad patterns in the recent increase in wage inequality. One pattern looks at wages across occupations, the other at wages across firms.

Autor and Dorn (2013) document what they call the polarization of the U.S. labor market. They rank occupations according their mean wage in 1980, and examine subsequent changes in employment and wages. Over the period 1980-2005, there was rapid employment growth at the top and bottom of the distribution, a large decline in the second quartile, and a small increase in third. At the same time, there was rapid wage growth at the top, substantial wage growth at the bottom, and slow growth in the middle. Their paper offers an explanation for the shift from jobs in the middle towards those at the bottom, but has little to say about growth at the top. The model here offers a potential explanation.

Other studies look at wage patterns across firms, in the U.S. and elsewhere. They find that the increase in between-firm wage inequality accounts for most or all of the change, with changes in within-firm inequality playing a distinctly minor role. For the U.S. over the period 1982-2012, across the whole economy, Song, et al. (2015) find that the between-firm component accounts for virtually all of the increase, while within-firm inequality is almost unchanged. In addition, they report that this conclusion is robust across industries, geographic regions, and firm size groups. Looking at U.K. data for a broad set of firms over the period 1984-2001, Faggio, Salvanes, and Van Reenen (2007) report a similar conclusion: virtually all of the increase in wage dispersion is between-firm rather than within-firm, and the vast majority of it is within industries. Helpman, et al. (2012) find the same pattern in the manufacturing sector in Brazil over the period 1986-98.

Two more studies find patterns that are similar, if more muted. For West Germany over the period 1985-2009, Card, Heining, and Kline (2012) find that an increase in the variation in wage premia across establishments explains about a quarter of the overall increase in wage inequality, with the rest coming from increases in labor heterogeneity and more strongly assortative matching. And for the Swedish manufacturing sector over the period 2001-2007, Akerman, et al. (2013) find that the between-firm component, while much smaller, does display a modest increase, while Häkanson, Lindqvist, and Vlachos (2015) find an increase in complementarities and more assortative matching.

In addition, several studies suggest that the increase in between-firm wage inequality is related to increased inequality in productivity. One of these includes a broad set of industries: for the U.K., Faggio, Salvanes, and Van Reenen (2007) find a substantial increase in the dispersion in labor productivity across firms. Other studies find similar patterns for more limited groups of firms or for specific technologies. For the U.S. manufacturing sector over the period 1975-92, Dunne, et al. (2004) find that wage and productivity dispersion increased, and that much of this increase was between plants. They also find that much of the between-plant change is within industries, and that a significant fraction is related to investment in computer equipment. For Norway, Akerman, Gaarder and Mogstad (2015) study a broadband internet rollout. Using geographic variation in the timing of the rollout, they find that better internet access improved the productivity and labor outcomes of skilled workers, and had the reverse effect on the unskilled.

Taken together, these studies suggests that technology differences across firms have increased over time and have played a major role in increasing in wage inequality.

## 3. ONE-SECTOR MODEL

In this section the technologies for final goods and tasks are described, and the competitive equilibrium is characterized.

## A. Final good technology

The final good is produced by competitive firms using tasks/differentiated goods as inputs. A task is characterized by its technology level $x_j > 0$. There are $J$ such levels, indexed by $j = 1, ..., J$, ordered so $0 < x_1 < x_2 < \ ... \ < x_J$. Let $\gamma_j$ be the share of tasks with technology level $x_j$. The total number (mass) of tasks is normalized to unity.

All inputs enter symmetrically into final good production, but demands for them differ if their prices differ. In equilibrium, price $p_j$ is the same for all tasks with technology level $x_j$. Hence demand is the same for such tasks. Let $y_j$ denote the (common) quantity for those tasks. The final good is produced with the CRS technology

$$y_F = \left( \sum_{j=1}^{J} \gamma_j y_j^{(\rho-1)/\rho} \right)^{\rho/(\rho-1)}, \tag{1}$$

where $\rho > 0$ is the substitution elasticity. For $\rho = 1$ the technology is Cobb-Douglas.

The final goods sector takes the prices $p_j$ as given. As usual, input demands are

$$y_j = \left( \frac{p_j}{p_F} \right)^{-\rho} y_F, \qquad \text{all } j, \tag{2}$$

and the price of the final good is

$$p_F = \left( \sum_{j=1}^{J} \gamma_j p_j^{1-\rho} \right)^{1/(1-\rho)}. \tag{3}$$

We will take the final good as numeraire throughout, indexing prices so $p_F = 1$. Input costs exhaust revenue, so there are no profits.

The analysis could be extended to include weights on tasks. Let $\{\omega_i\}_{i=1}^{I}$ be a set of values for the weights, and let $\sigma_{ji}$ be the share of tasks with the (technology, weight) pair $(x_j, \omega_i)$. Then output of the final good is

$$y_F = \left( \sum_{j=1}^{J} \sum_{i=1}^{I} \sigma_{ji} \omega_i^{1/\rho} \widetilde{y}_{ji}^{(\rho-1)/\rho} \right)^{\rho/(\rho-1)},$$

where $\widetilde{y}_{ji}$ is the input of a task with characteristics $(x_j, \omega_i)$. It is straightforward to show that in this setting prices $p_j$ do not depend on $i$, and demand for each task is

$$\widetilde{y}_{ji} = \omega_i y_j, \qquad \text{all } i, j,$$

where $\{p_j, y_j\}_{j=1}^{J}$ and the aggregates $y_F, p_F$ are as above, and

$$\gamma_j \equiv \sum_{i=1}^{I} \sigma_{ji} \omega_i, \qquad \text{all } j.$$

Output and employment vary with $\omega_i$ across tasks with the same technology $x_j$, but the wage structure in the economy depends only on the $\gamma_j$'s.

### B. Differentiated good technology

Tasks/differentiated goods are produced using heterogeneous labor, characterized by its skill level $h$, as the only inputs. Assume that $h$ has a continuous distribution. Let $G(h)$ with density $g(h)$ on $H \equiv (h_{\min}, h_{\max})$, with $0 < h_{\min} < h_{\max} \leq \infty$, denote the distribution of skill across workers. The total size (mass) of the workforce is normalized to unity, and labor supply is inelastic: each worker supplies one unit.

The total output of a task depends on the size and quality of the workforce producing it, as well as its technology level $x_j$. In particular, if a task with technology $x_j$ employs workers of various human capital levels, with $\ell_j(h) \geq 0$ denoting the number (density) of each type, then total output is

$$y_j = \int \ell_j(h) \phi(h, x_j) dh, \qquad \text{all } j,$$

where $\phi(h, x)$ is the CES function

$$\phi(h, x) \equiv \left[\omega h^{(\eta-1)/\eta} + (1-\omega)\, x^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)}, \qquad \eta, \omega \in (0,1). \tag{4}$$

The elasticity of substitution $\eta$ between technology and human capital is assumed to be less than unity, and $\omega$ is the relative weight on human capital.

### C. Equilibrium

An equilibrium consists of a final output level $y_F$, output levels and prices $\{y_j, p_j\}_{j=1}^{J}$ for all types of tasks/differentiated goods, a wage function $w(h)$, $h \in H$, and an allocation of labor across technologies, that satisfy the usual optimization and market clearing conditions.

The model allows two interpretations about market structure. One is that each task is produced by competitive firms, with each firm choosing to employ skill types that minimize unit cost. In this case competition insures that each worker is paid his marginal revenue product.[2] Alternatively, one can suppose that workers simply choose which task to produce, with each worker choosing a task—a job—that maximizes his income. In either interpretation, task prices are taken as given by the decision maker—the firm or the worker.

In principle, the allocation of labor could be quite complicated, with any technology level $x_j$ employing workers with skill $h$ in various disjoint intervals, and with workers of a given human capital level $h$ producing goods with different technologies $x_j$. This does not occur in equilibrium, and it is straightforward to see why not.

Since labor markets are competitive, the allocation of labor across technologies is efficient. And since the elasticity of the CES function $\phi$ is less than unity, it is log supermodular. Hence efficiency requires positively assortative matching: workers with higher skill $h$ work with higher technologies $x_j$ (Costinot, 2009). Consequently the equilibrium labor allocation is characterized by thresholds $h_{\min} = b_0 < b_1 < \ ... \ < b_{J-1} < b_J = h_{\max}$, where technology $x_j$ employs workers with skill $h \in (b_{j-1}, b_j)$. We will refer to the interval $(b_{j-1}, b_j)$ as skill bin $j$. An individual with human capital $h =$

[2]If $\rho > 1$, profit-making firms could be introduced by assuming that each task/differentiated good is supplied by a unique producer. Under this assumption, the allocation of labor, output of each task, and prices would be unchanged, but wages would reduced by the factor $(\rho - 1)/\rho$, with the residual revenue going to profits.

$b_j$ is indifferent between working with technologies $x_j$ and $x_{j+1}$. Since the distribution function $G$ is continuous, the set of such workers has measure zero, and they can be allocated to either bin.

Equilibrium also requires market clearing for goods and labor. Thus, the equilibrium conditions are:

a. income maximization by all types of labor,

$$w(h) \geq p_j \phi(h, x_j), \qquad \text{all } h, \;\; \text{w/ eq. if } h \in [b_{j-1}, b_j], \qquad \text{all } j; \tag{5}$$

b. market clearing for tasks: $\{y_j, p_j\}_{j=1}^{J}$ satisfy (2), with $y_F$ as in (1);

c. labor market clearing,

$$\int_{b_{j-1}}^{b_j} \phi(h, x_j) g(h) dh = \gamma_j y_j, \qquad \text{all } j. \tag{6}$$

The first condition implies that each task is priced at unit cost, and the last says that the total productive capacity of labor with skill $h \in (b_{j-1}, b_j)$ is sufficient for production of tasks with technology $x_j$.

The allocation of labor within any skill bin $(b_{j-1}, b_j)$ across tasks with technology $x_j$ is, to some extent, indeterminate. Equilibrium determines only the output level $y_j$, which is the same across tasks with technology level $x_j$. For concreteness we can suppose that each task is produced by skill types in the interval $(b_{j-1}, b_j)$ in proportion to their representation in the population, but this is not required.[3]

To characterize the thresholds $\{b_j\}_{j=1}^{J-1}$, note that (5) implies

$$\frac{w'(h)}{w(h)} = \frac{\phi_h(h, x_j)}{\phi(h, x_j)}, \qquad h \in (b_{j-1}, b_j), \;\; \text{all } j. \tag{7}$$

Hence the equilibrium wage function is piecewise continuously differentiable, with kinks at the points $\{b_j\}_{j=1}^{J-1}$.

[3]Since $\phi$ has constant returns to scale, the number of firms producing any task—if firm are introduced into the narrative—is indeterminate. Only the total (productivity-weighted) labor input and total output are determined in equilibrium.

Since workers with skill $b_j$ are indifferent between working with technologies $x_j$ and $x_{j+1}$, it follows immediately from (5) and (2) that

$$\frac{p_{j+1}}{p_j} = \frac{\phi(b_j, x_j)}{\phi(b_j, x_{j+1})}, \tag{8}$$

$$\frac{y_{j+1}}{y_j} = \left(\frac{\phi(b_j, x_{j+1})}{\phi(b_j, x_j)}\right)^{\rho}, \qquad j = 1, ..., J-1. \tag{9}$$

Unit cost and price are strictly decreasing in $j$, and output is strictly increasing: goods with better technologies have lower prices and higher sales. If $\rho > 1$ (if $\rho < 1$), total revenue is increasing in $j$ (decreasing in $j$).

To characterize equilibrium, combine (6) and (9) to find that $\{b_j\}_{j=1}^{J-1}$ satisfy

$$\int_{b_j}^{b_{j+1}} g(h)\phi(h, x_{j+1})dh = \frac{\gamma_{j+1}}{\gamma_j}\left(\frac{\phi(b_j, x_{j+1})}{\phi(b_j, x_j)}\right)^{\rho} \int_{b_{j-1}}^{b_j} g(h)\phi(h, x_j)dh, \tag{10}$$

$$j = 1, ..., J-1.$$

Since $\eta < 1$, the ratio $\phi(b_j, x_{j+1})/\phi(b_j, x_j)$ is strictly increasing in $b_j$. Therefore, since $b_0 = h_{\min}$ is given, for any conjectured $b_1$, the sequence $\{b_j\}_{j=2}^{J}$ defined recursively by using (10) is increasing in $b_1$. Equilibrium requires $b_J = h_{\max}$. Thus a solution exists and it is unique.

Define $\Psi_j$ to be 'total productivity' of labor in the $j^{\text{th}}$ skill bin,

$$\Psi_j \equiv \int_{b_{j-1}}^{b_j} \phi(h, x_j)g(h)dh, \qquad j = 1, ..., J. \tag{11}$$

Then use (6) to write output of each type of good as

$$y_j = \frac{1}{\gamma_j}\Psi_j, \qquad j = 1, ..., J, \tag{12}$$

and write (10) in the more symmetric form

$$\phi(b_j, x_{j+1})^{-\rho}\frac{1}{\gamma_{j+1}}\Psi_{j+1} = \phi(b_j, x_j)^{-\rho}\frac{1}{\gamma_j}\Psi_j, \qquad j = 1, ..., J-1. \tag{13}$$

## D. Skill allocation

To see more clearly how workers and technologies are matched, it is useful to look at potential wage functions, like those in Neal and Rosen (2000, Figure 3.1). Let $w^p(h, x_j)$ denote the wage a worker with skill $h$ would earn producing a task with technology $x_j$,

$$w^p(h, x_j) = p_j \phi(h, x_j), \qquad \text{all } h, \text{ all } j.$$

Figure 1 displays potential wages as a function of $h$, for $J = 4$ technology levels.

For fixed $x_j$, the potential wage $w^p(h, x_j)$ is strictly increasing in $h$, so each curve is upward sloping. As a function of $x_j$, there are two effects. First, the price $p_j$ is decreasing in $x_j$, so the intercept decreases with $x_j$. In addition, since $\phi$ is log supermodular, higher $x_j$ implies a steeper slope for $\phi$ as a function of $h$. Thus, plotted against $h$, for various $x_j$ values, the potential wage functions cross. A worker's actual wage is the maximum of his potential wages, as in (5). Hence the wage function $w(h)$ is defined by the upper envelop of the four curves, and the crossing points along the upper envelop are the thresholds $b_j$ that divide the skill range into bins.

The four small circles show the choices available to a worker with skill $h_i$. The potential wage for that worker increases moving from $x_1$ to $x_2$ and from $x_2$ to $x_3$. But it falls moving from $x_3$ to $x_4$, so that worker chooses $x_3$.

In a model with search frictions, these points would represent rungs on a job ladder for a worker with skill $h_i$. This worker's first job might come from an employer of any type. That firm would pay him his reservation wage, not his marginal revenue product, so his initial wage would lie below all of the displayed values. But subsequently, outside offers from other firms would raise his wage, for two reasons. If the outside firm was a better match, he would change jobs and receive a wage increase. But even if the outside firm was an equivalent (or possibly worse) match, his wage might be bid up by competition. In this case he would not change jobs.

### E. Homogeneous technologies

To assess the effect of technology heterogeneity on the wage distribution, we can compare the solution above with one in an economy with the same distribution of skill across workers, but a single technology level. To focus on wage inequality, we will choose the common technology level so that output of the final good is the same in both economies. Then the total wage bill is also the same, and only the distribution of wages across workers changes.

To keep output of the final good $y_F$ unchanged, output of each task in the homogeneous technology (HT) economy must be

$$y^{HT} = y_F.$$

To back out the technology level $x^{HT}$ in the HT economy, note that one competitive equilibrium allocation is for workers with a pro rata share of all skill levels, to produce each task. Hence $x^{HT}$ satisfies

$$y^{HT} = \int \phi(h, x^{HT}) g(h) dh.$$

It then follows from (2) and (5) that a worker with human capital $h \in (b_{j-1}, b_j)$, who is in skill bin $j$ if technologies are heterogeneous, gains or loses as

$$\begin{aligned} 1 \;&\lesseqgtr\; \frac{w^{HT}(h)}{w(h)} \\ &= \frac{\phi(h, x^{HT})}{\phi(h, x_j)} \left( \frac{y^{HT}}{y_j} \right)^{-1/\rho}, \qquad h \in (b_{j-1}, b_j). \end{aligned} \tag{14}$$

There are two forces, pushing in opposite directions.

With heterogeneous technologies, workers with lower skill are matched with worse technologies. Hence workers who would be in skill bin $j$, with $x_j < x^{HT}$, become individually more productive. This effect tends to raise their wages, the first term in (14). This own-productivity effect, which is reversed for workers in skill bins $j$ with $x_j > x^{HT}$, tends to compress the wage distribution.

But task prices also change. With heterogeneous technologies, output $y_j$ is increasing in $x_j$ and price $p_j$ is decreasing. Hence workers who would be in skill bin $j$, with output $y_j < y^{HT}$ and price $p_j > p^{HT}$, suffer a cut in the price of their product, the second term in (14). Output can rise because the technology improves, because the average skill of their co-workers rises, because employment rises, or any combination. This price effect, which is reversed for workers in skill bins $j$ with $y_j > y^{HT}$, tends to expand the wage distribution.

The relative strength of the two forces depends on the distributions for technology and skill. A second-order approximation allows a quantitative assessment. For each $j$, let $G_j \equiv G(b_j) - G(b_{j-1})$, denote the share of the workforce in skill bin $j$, and let $\overline{h}_j \in (b_{j-1}, b_j)$ denote the skill level that satisfies

$$y_j = \frac{G_j}{\gamma_j}\phi(\overline{h}_j, x_j), \qquad \text{all } j.$$

If all workers in skill bin $j$ have skill $\overline{h}_j$, then total output for that bin is unchanged. For any worker, the wage change from moving to the HT economy is the sum of the own-productivity and price effects in (14). Thus, for a worker with skill $\overline{h}_j$, it is

$$\Delta \ln w(\overline{h}_j) = \left[\ln \phi\left(\overline{h}_j, x^{HT}\right) - \ln \phi\left(\overline{h}_j, x_j\right)\right] - \frac{1}{\rho}\left[\ln y^{HT} - \ln y_j\right].$$

For a certain family of economies, the quadratic approximation to this expression takes a simple form. Suppose that $\rho > 1$ and that $x$ has a fine grid, with distribution function $F(x) \approx G(a_H x)$, all $x$, where

$$a_H = \left((\rho - 1)\frac{1-\omega}{\omega}\right)^{\eta/(\eta-1)}.$$

Then in equilibrium[4] $\overline{h}_j \approx a_H x_j$, $G_j/\gamma_j \approx 1$, and $y_j \approx x_j \phi(a_H, 1)$, all $j$. Define

$$\begin{aligned} \Delta_{xj} &\equiv \ln x^{HT} - \ln x_j, && \text{all } j, \\ \chi_0 &\equiv -\frac{1}{\rho}\left\{\ln y^{HT} - \ln\left[x^{HT}\phi(a_H, 1)\right]\right\}. \end{aligned}$$

[4]This solution is exactly correct if $x$ has a continuous distribution. The associated wage function is $w(h) = w_0 h^{1-1/\rho}$.

As shown in the Appendix, the wage change is approximately

$$\Delta \ln w(\overline{h}_j) \approx \chi_0 - \frac{1}{2}\frac{\rho-1}{\rho^2}\frac{1-\eta}{\eta}\Delta_{xj}^2. \tag{15}$$

For $\rho > 1$ the coefficient on the quadratic term in (15) is negative, and by construction the total wage bill unchanged, so $\mathrm{E}\big[\Delta \ln w(\overline{h}_j)\big] \approx 0$. Hence taking the expectation in (15) gives $\chi_0 > 0$, so the constant term is positive. Thus workers with skill near the mean enjoy a wage gain, and those that are sufficiently different from the mean experience losses.

## 4. TECHNICAL CHANGE

This section looks at the effects of technical change that improves one technology by a small increment, with all others unchanged. Specifically, it characterizes the effect on the labor allocation, described by the thresholds $\{b_j\}_{j=1}^{J-1}$, on the output levels and prices $\{y_j, p_j\}_{j=1}^{J}$ for all tasks, and on the wage function $w(h)$.

The main forces can be previewed in Figure 1. Suppose technology $x_k$ gets the improvement. The direct effect is to increase labor productivity for workers in skill bin $k$, raising $w^p(\cdot, x_k)$ and making it slightly steeper. But the higher labor productivity increases $y_k$, which depresses the price $p_k$, and tends to raise all other prices, $p_j$, $j \neq k$. These price changes lower $w^p(\cdot, x_k)$ partway back toward its original level and raise all the other curves, $w^p(\cdot, x_j)$, $j \neq k$. The thresholds defining the employment bins shift, changing employment patterns and wages for all workers.

The rest of this section analyzes these changes in detail. Throughout we will use 'hats' to denote proportionate changes induced by the perturbation, $\hat{z} \equiv z^{-1}\partial z/\partial \varepsilon$, for any variable $z$. All derivations and proofs are in the Appendix.

## A. Final output

Suppose that technical change increases technology $x_k$ by a small increment $\varepsilon > 0$, with all others unchanged. Note that the change in output of the final good $y_F$ is a weighted average of the output changes for tasks,

$$\hat{y}_F = \sum_{j=1}^{J} \nu_j \hat{y}_j, \tag{16}$$

where the weights

$$\nu_j \equiv \frac{1}{y_F} \gamma_j p_j y_j, \qquad \text{all } j, \tag{17}$$

with $\sum_{j=1}^{J} \nu_j = 1$, are their cost shares in producing the final good. With the price of the final good fixed at unity, the relative price changes for tasks are

$$\hat{p}_j = \frac{1}{\rho} \left( \hat{y}_F - \hat{y}_j \right), \qquad \text{all } j, \tag{18}$$

and the weighted average of the price changes is $\sum_{j=1}^{J} \nu_j \hat{p}_j = \hat{p}_F = 0$.

Consider first the short run effects, with labor immobile. Recall the definition of $\Psi_j$, all $j$, in (11), and let $\hat{\Psi}_k$ be the direct effect of the technology improvement on total labor productivity in skill bin $k$. Output increases for tasks produced with technology $x_k$,

$$\hat{y}_k^{SR} = \hat{\Psi}_k \equiv \frac{1}{\Psi_k} \frac{\partial \Psi_k}{\partial x_k} > 0, \tag{19}$$

and is unchanged for all other tasks. Hence the change in final output is

$$\hat{y}_F^{SR} = \nu_k \hat{\Psi}_k > 0.$$

In the longer run, with labor mobile, the changes in $\{y_j\}_{j=1}^{J}$ and $y_F$ must be augmented to account for the impact of changes in the skill bins, changes in the $b_j$'s. Let $\left\{ b_j^{(k)}(\varepsilon) \right\}_{j=1}^{J-1}$ denote the solution to (13) as a function of $\varepsilon$, where $b_0 = h_{\min}$ and $b_J = h_{\max}$ are fixed. Define the density-weighted changes in the thresholds

$$\beta_j^{(k)} \equiv g(b_j) b_j^{(k)\prime}(\varepsilon), \qquad j = 1, ..., J-1,$$

with $\beta_0^{(k)} = \beta_J^{(k)} = 0$. From (11) and (12), the long run changes in output levels for tasks are

$$\hat{y}_k = \frac{1}{\Psi_k}\left[\phi(b_k, x_k)\beta_k^{(k)} - \phi(b_{k-1}, x_k)\beta_{k-1}^{(k)}\right] + \hat{\Psi}_k, \tag{20}$$

$$\hat{y}_j = \frac{1}{\Psi_j}\left[\phi(b_j, x_j)\beta_j^{(k)} - \phi(b_{j-1}, x_j)\beta_{j-1}^{(k)}\right], \qquad \text{all } j \neq k.$$

The next proposition shows that, to a first-order approximation, the change in the labor allocation has no impact on output of the final good: the long-run increase is the same as the short-run increase.

PROPOSITION 1: In the long run, with labor mobile, the change in output of the final good is, to a first-order approximation, the same as in the short run, $\hat{y}_F = \hat{y}_F^{SR}$.

This result is not surprising. The potential additional effect in the long run arises only from the reallocation of labor, changes in the thresholds $\{b_j\}_{j=1}^{J-1}$ defining the skill bins. Since labor markets are competitive, the baseline allocation of labor maximizes $y_F$. Hence to a first order approximation, small changes in those thresholds have no effect on final output. An increase (decrease) in $b_j$ raises (lowers) the output of tasks with technology $x_j$, but the effect on final output is exactly offset by the decrease (increase) in the output of tasks with technology $x_{j+1}$.

## B. Labor allocation

The changes in the labor allocation do, however, affect task-level outputs and prices, as well as wages. The rest of this section describes these changes. To determine the effect on the labor allocation, differentiate (13) and use (11) to get a system of $J-1$ linear equations for the changes in the thresholds,

$$\underline{\beta}^{(k)} = M\underline{A}^{(k)}, \tag{21}$$

where the superscript denotes which technology has been perturbed, and for any $k$,

$$A_{k-1}^{(k)} = -\rho\hat{\phi}_x(b_{k-1}, x_k) + \hat{\Psi}_k, \tag{22}$$

$$
\begin{aligned}
A_k^{(k)} &= \rho\hat{\phi}_x(b_k, x_k) - \hat{\Psi}_k, \\
A_j^{(k)} &= 0, \qquad \text{otherwise.}
\end{aligned}
$$

Since $\underline{A}^{(k)}$ has at most two non-zero elements—and only one if $k = 1$ or $k = J$, for fixed $k$ the solution to (21) involves only $A_{k-1}^{(k)}$, $A_k^{(k)}$, and the columns $M_{\cdot k-1}$ and $M_{\cdot k}$. In particular,

$$
\beta_j^{(k)} = m_{j,k-1}A_{k-1}^{(k)} + m_{j,k}A_k^{(k)}, \qquad j = 1, ..., J-1, \tag{23}
$$

where the first term drops out if $k = 1$, and the second drops out if $k = J$.

$M$ is the inverse of a tridiagonal matrix, so it has a recursive structure. Lemma 2 shows that it has strictly positive elements, and that successive row elements above and below the diagonal have ratios that depend only on the row $j$.

Lemma 2: All elements of $M$ are positive, and the elements in each column $M_{\cdot n}$ satisfy

$$
\begin{aligned}
m_{j+1,n} &= q_{j+1}m_{j,n}, \qquad j \geq n, \\
m_{j-1,n} &= r_{j-1}m_{j,n}, \qquad j \leq n,
\end{aligned} \tag{24}
$$

where $\{q_{j+1}\}_{j=1}^{J-2}$ and $\{r_{j-1}\}_{j=2}^{J-1}$ are positive constants.

Lemma 2 can be used as follows. Fix $k$, and use the first line in (24) to compare successive rows $j+1 > j \geq k$ in (21). Similarly, use the second line in (24) to compare successive rows $j-1 < j \leq k-1$, concluding that

$$
\begin{aligned}
\beta_{j+1}^{(k)} &= q_{j+1}\beta_j^{(k)}, \qquad j \geq k, \\
\beta_{j-1}^{(k)} &= r_{j-1}\beta_j^{(k)}, \qquad j \leq k-1.
\end{aligned} \tag{25}
$$

Thus, all thresholds at and above the $k^{\text{th}}$ move in the same direction, and all those at and below the $(k-1)^{\text{th}}$ move in the same direction. It remains to determine the signs of $\beta_k^{(k)}$ and $\beta_{k-1}^{(k)}$. For this we need to characterize the two nonzero elements of $\underline{A}^{(k)}$.

LEMMA 3: For any $k$,

i. if $\rho = 1$, then $A_{k-1}^{(k)} > 0$ and $A_k^{(k)} > 0$;

ii. if $\rho > 1$, then $A_k^{(k)} > 0$ and $A_{k-1}^{(k)}$ can have either sign; and

iii. if $\rho < 1$, then $A_{k-1}^{(k)} > 0$ and $A_k^{(k)}$ can have either sign.

The intuition for Lemma 3 is straightforward from (22). The term $\hat{\phi}_x(h, x_k)$ is the proportionate change in labor productivity for a worker with skill $h$. Since $\eta < 1$, it is strictly increasing in $h$. The term $\hat{\Psi}_k$ is the average value of these changes in skill bin $k$. If $\rho \geq 1$, then for a worker with skill $b_k$, at the upper threshold of the bin, $\hat{\Psi}_k < \hat{\phi}_x(b_k, x_k) \leq \rho\hat{\phi}_x(b_k, x_k)$, so $A_k^{(k)} > 0$. If $\rho < 1$, the sign is ambiguous. Similarly, if $\rho \leq 1$, then for a worker with skill $b_{k-1}$, at the lower threshold of skill bin $k$, $\rho\hat{\phi}_x(b_{k-1}, x_k) \leq \hat{\phi}_x(b_{k-1}, x_k) < \hat{\Psi}_k$, so $A_{k-1}^{(k)} > 0$. If $\rho > 1$, the sign is ambiguous.

Can anything more be said about the terms with ambiguous signs? The answer depends, to a large extent, on how the technology levels are chosen/defined. If the technology grid is fine, then the skill bins are narrow, so $b_{k-1}$ is close to $b_k$, and $A_{k-1}^{(k)} \approx -A_k^{(k)}$. For $\rho = 1$, both are close to zero.

If $A_{k-1}^{(k)}$ and $A_k^{(k)}$ are both positive, then it follows immediately from (23) and Lemma 2 that all thresholds shift upward. But even if one term in (23) is negative, the sign of the sum can sometimes be determined. Proposition 4 characterize the signs of $\beta_k^{(k)}$ and $\beta_{k-1}^{(k)}$ to the extent that it is possible.

PROPOSITION 4: For any $k$, an increase in technology $x_k$ implies:

for $\rho = 1$,

$$\beta_j^{(k)} > 0, \qquad \text{all } j;$$

for $\rho > 1$,

$$\beta_j^{(k)} \;>\; 0, \qquad j \geq k,$$

$$\beta_j^{(k)} \lesseqqgtr 0, \qquad j < k-1, \qquad \text{as} \qquad \beta_{k-1}^{(k)} \lesseqqgtr 0;$$

and for $\rho < 1$,

$$\begin{aligned} \beta_j^{(k)} &> 0, \qquad j \leq k-1, \\ \beta_j^{(k)} &\lesseqqgtr 0, \qquad j > k, \qquad \text{as} \qquad \beta_k^{(k)} \lesseqqgtr 0. \end{aligned}$$

For $\rho = 1$, all thresholds shift upward. For $\rho > 1$, the thresholds at and above the $k^{\text{th}}$ shift upward, while those at and below the $(k-1)^{\text{th}}$ can shift either way. For $\rho < 1$, the thresholds at and below the $(k-1)^{\text{th}}$ shift upward, while those at and above the $k^{\text{th}}$ can shift either way.

### C. Task/good output levels

From (6) and (11), the change in output for a task of type $j \neq k$ depends on the sum of the productivity-weighted employment changes at the two thresholds,

$$\hat{y}_j^{(k)} = \frac{1}{\Psi_j} \left[ \phi(b_j, x_j) \beta_j^{(k)} - \phi(b_{j-1}, x_j) \beta_{j-1}^{(k)} \right], \qquad j \neq k, \tag{26}$$

where $\beta_0^{(k)} = \beta_J^{(k)} = 0$. For goods of type $k$, the direct effect of the productivity change must also be added, so

$$\hat{y}_k^{(k)} = \hat{\Psi}_k + \frac{1}{\Psi_k} \left[ \phi(b_k, x_k) \beta_k^{(k)} - \phi(b_{k-1}, x_k) \beta_{k-1}^{(k)} \right]. \tag{27}$$

Proposition 5 characterizes the changes in output.

PROPOSITION 5: For any $k$,

$$\begin{aligned} \hat{y}_k^{(k)} &> 0, \\ \hat{y}_j^{(k)} &\gtreqqless 0, \quad j > k, \qquad \text{as} \qquad \beta_k^{(k)} \lesseqqgtr 0, \\ \hat{y}_j^{(k)} &\gtreqqless 0, \quad j < k, \qquad \text{as} \qquad \beta_{k-1}^{(k)} \gtreqqless 0. \end{aligned}$$

Output rises for tasks of type $k$. The output change is in the same direction for all tasks of type $j > k$, rising if $\beta_k^{(k)} < 0$, so more labor is devoted to these tasks, and falling if $\beta_k^{(k)} > 0$. Similarly, the output change is in the same direction for all tasks of type $j < k$, falling if $\beta_{k-1}^{(k)} < 0$ and rising if $\beta_{k-1}^{(k)} > 0$. Thus, for $\rho \geq 1$, output falls for tasks of types $j > k$, and for $\rho \leq 1$, output rises for tasks of types $j < k$.

Proposition 6 shows that the size of the output changes above and below $k$ are damped—whatever their sign—for more distant technology types.

PROPOSITION 6: For any $k$,

$$\begin{aligned} \left|\hat{y}_1^{(k)}\right| &< \left|\hat{y}_2^{(k)}\right| < \ ... \ < \left|\hat{y}_{k-1}^{(k)}\right|, \\ \left|\hat{y}_{k+1}^{(k)}\right| &> \left|\hat{y}_{k+2}^{(k)}\right| > \ ... \ > \left|y_J^{(k)}\right|. \end{aligned}$$

## D. Prices and wages

Next consider prices and wages. The price of a task rises or falls as its output change is less than or greater than the output change for the final good. In particular, from (18) and Proposition 1,

$$\hat{p}_j^{(k)} = \frac{1}{\rho}\left(\nu_k \hat{\Psi}_k - \hat{y}_j^{(k)}\right), \qquad \text{all } j. \tag{28}$$

Proposition 7 describes price changes. For tasks of type $k$, price falls. For types $j \neq k$, price rises if output falls, and the size of the increase is damped for types more distant from $k$. The sign of the price change is ambiguous if output rises, but the price changes are nevertheless ordered, even if there is a sign change somewhere along the chain. Price decreases, if they occur, are clustered among types near $k$.

PROPOSITION 7: For any $k$, an increase in technology $x_k$ implies

$$\hat{p}_k^{(k)} < 0.$$

For $j < k$,

$$0 < \hat{p}_1^{(k)} < \hat{p}_2^{(k)} < \ ... \ < \hat{p}_{k-1}^{(k)}, \qquad \text{if } \beta_{k-1}^{(k)} < 0,$$

$$\hat{p}^{(k)}_{k-1} < \; ... \; < \hat{p}^{(k)}_{2} < \hat{p}^{(k)}_{1}, \qquad \text{if } \beta^{(k)}_{k-1} > 0,$$

and some or all of the latter price changes can be negative. For $j > k$,

$$0 < \hat{p}^{(k)}_{J} < \hat{p}^{(k)}_{J-1} < \; ... \; < \hat{p}^{(k)}_{k+1}, \qquad \text{if } \beta^{(k)}_{k} > 0,$$

$$\hat{p}^{(k)}_{k+1} < \; ... \; < \hat{p}^{(k)}_{J-1} < \hat{p}^{(k)}_{J}, \qquad \text{if } \beta^{(k)}_{k} < 0,$$

and some or all of the latter price changes can be negative.

Next consider wage changes. It follows immediately from (5) that

$$\begin{aligned} \hat{w}(h) &= \hat{p}^{(k)}_{k} + \hat{\phi}_x(h, x_k), \qquad h \in (b_{k-1}, b_k)\,; \\ \hat{w}(h) &= \hat{p}^{(k)}_{j}, \qquad h \in (b_{j-1}, b_j)\,, \qquad j \neq k. \end{aligned}$$

For workers in skill bins $j \neq k$, wages change only because the price of their output changes. Hence the direction and size of the wage change is the same as the price change, and is equal for all workers in a skill bin. Workers in skill bin $k$ also experience a direct productivity effect, which is increasing in the worker's own human capital $h$.

Proposition 8 describes the one case where a technology improvement necessarily raises all wages.

PROPOSITION 8: If $\rho > 1$, then for any $k$, $\beta^{(k)}_{k-1} < 0$, implies $\hat{w}(h) > 0$, all $h$.

If $\rho \leq 1$, then $\beta^{(k)}_{k-1} > 0$, leaving open the possibility that $p_{k-1}$ falls, so wages fall for skill bin $k-1$.

More generally, if $\beta^{(k)}_{k} > 0$, then workers in skill bins $j > k$ get wage increases, as do workers with human capital near the upper threshold of skill bin $k$. If $\beta^{(k)}_{k} < 0$, wages can fall for some workers at the top of skill bin $k$. In this case prices can fall for some or all tasks of type $j > k$, so that wages fall for workers in these skill bins. The wage declines are clustered near skill bin $k$, and are damped for more distant skill bins. Indeed, wages can rise for workers sufficiently far up the skill ladder.

If $\beta_{k-1}^{(k)} < 0$, then workers in skill bins $j < k$ get wage increases, as do workers with human capital near the lower threshold of skill bin $k$. If $\beta_{k-1}^{(k)} > 0$, wages can fall for some workers at the bottom of skill bin $k$. In this case prices fall for some or all tasks of type $j < k$, so that wages fall for workers in these skill bins as well. The wage declines are clustered near skill bin $k$, and are damped for more distant skill bins. Indeed, wages can rise for workers sufficiently far down the skill ladder. The Appendix provides an example where wages decline for some workers.

## 5. MULTI-SECTOR MODEL

In this section the model is extended to include multiple sectors, each producing an intermediate that is used in final goods production. There are two goals. The first is to show that the patterns for wage differentials and for technology-based changes in wage inequality in the one-sector model can carry over to a many-sector model. The second goal is to show that a multi-sector model provides more scope for technological improvements for a subset of tasks to reduce wages for workers at closely substitutable tasks that have unchanged technologies.

### A. Final good and intermediate technologies

The final good is produced by competitive firms using sector intermediates as inputs, and sector intermediates are produced by competitive firms using tasks/differentiated goods as inputs. Both technologies are CES, but the sector technologies have a higher (common) elasticity of substitution.

There are $S$ sectors. Output and price for the final good are

$$y_F = \left( \sum_{s=1}^{S} \theta_s Y_s^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}, \qquad (29)$$

$$p_F = \left( \sum_{s=1}^{S} \theta_s^\sigma P_s^{1-\sigma} \right)^{1/(1-\sigma)} = 1, \qquad \sigma > 0, \quad \sigma \neq 1,$$

where $\theta_s > 0$ with $\Sigma_s \theta_s = 1$, are sector weights, $\sigma$ is the elasticity of substitution, and prices are normalized as before. For $\sigma = 1$,

$$y_F = \prod_{s=1}^{S} Y_s^{\theta_s}, \qquad p_F = \left[ \prod_{s=1}^{S} \left( \frac{\theta_s}{P_s} \right)^{\theta_s} \right]^{-1} = 1.$$

In either case, demands for sector intermediates are

$$Y_s = y_F \left( \frac{\theta_s p_F}{P_s} \right)^\sigma, \qquad \text{all } s. \tag{30}$$

Within each sector, tasks/differentiated goods are produced, potentially, with any or all of the technologies levels $\{x_j\}_{j=1}^J$. But the number of tasks, and the distribution of technology levels across them, may differ. Thus, the technologies for sector intermediates, the demands for tasks, and the prices of sector intermediates are

$$Y_s = n_s \left( \sum_{j=1}^{J} \gamma_{sj} y_{sj}^{(\rho-1)/\rho} \right)^{\rho/(\rho-1)}, \qquad \text{all } s, \tag{31}$$

$$y_{sj} = Y_s \left( \frac{p_{sj}}{P_s} \right)^{-\rho}, \qquad \text{all } j, s, \tag{32}$$

$$P_s = n_s^{-1/\rho} \left( \sum_{j=1}^{J} \gamma_{sj} p_{sj}^{1-\rho} \right)^{1/(1-\rho)}, \qquad \text{all } s, \tag{33}$$

where $n_s$ is the share of firms in sector $s$, and $\{\gamma_{sj}\}_{j=1}^J$ are the shares for each technology within the sector. The substitution elasticity $\rho$ is common across sectors but $n_s$ and $\{\gamma_{sj}\}_{j=1}^J$ may differ. It will be assumed throughout that $\rho > \sigma$, so tasks/differentiated goods within a sector are more substitutable than sector intermediates.

As before, each task has the CES technology $\phi(h, x)$ in (4), where $\omega$ and $\eta$ are the same across sectors.

Notice from (32) that for any task in sector $s$, demand is proportional to $Y_s P_s^\rho$, so it is increasing in both its sector intermediate $Y_s$ and its sector price $P_s$. But $Y_s$ and $P_s$ are linked through demand by final goods producers, in (30), where $P_s$ is proportional to $Y_s^{-1/\sigma}$. Hence $Y_s P_s^\rho$ is proportional to $Y_s^{1-\rho/\sigma}$. Since $\rho > \sigma$, this fact implies that a change in sector output $Y_s$ has a stronger effect on demand for tasks in that sector through the price channel than through the direct channel. That is, an increase in $Y_s$ reduces price $P_s$ so sharply that demand falls for a task $y_{sj}$ with unchanged price $p_{sj}$.

### B. Equilibrium

The conditions for the equilibrium labor allocation and output levels for tasks are similar to those in the one-sector model. Let $G, g$, be as before. Assume labor is mobile across sectors and let $w(h)$ denote the wage function. The labor allocation is described, as before by thresholds $\{b_j\}_{j=0}^J$, that partition the skill distribution into bins. Workers in skill bin $j$, those with $h \in (b_{j-1}, b_j)$, work with technology $x_j$. Since labor is mobile across sectors, the thresholds do not depend on $s$, and price $p_j$ does not vary by sector. The wage function satisfies (5) and (7), as before, and (8) still holds, and needs no sector subscripts

$$\left(\frac{p_{j+1}}{p_j}\right)^{-\rho} = \left(\frac{\phi(b_j, x_{j+1})}{\phi(b_j, x_j)}\right)^{\rho}, \qquad \text{all } j, s. \tag{34}$$

The demands for tasks in (32) do vary by sector, however, so output levels can be written as

$$y_{sj} = A_s p_j^{-\rho} y_F, \qquad \text{all } j, s, \tag{35}$$

where from (30) and (32),

$$A_s = \theta_s^\rho \left(\frac{Y_s}{y_F}\right)^{1-\rho/\sigma}, \qquad \text{all } s. \tag{36}$$

The constants $\{A_s\}_{s=1}^S$ act as weights on the sectors.

Labor market clearing requires productivity-weighted labor supply in each skill bin to equal the sum of demand from all sectors, so (6) becomes

$$\begin{aligned} \Psi_j &= \sum_{s=1}^{S} n_s \gamma_{sj} y_{sj} \\ &= p_j^{-\rho} y_F \sum_{s=1}^{S} A_s n_s \gamma_{sj}, \qquad j = 1, ..., J, \end{aligned} \tag{37}$$

where $\{\Psi_j\}_{j=1}^{J}$ are defined as before in (11). A competitive equilibrium consists of thresholds $\{b_j\}_{j=1}^{J-1}$, prices $\{p_j\}_{j=1}^{J}$, weights $\{A_s\}_{s=1}^{S}$, and output levels $\{y_{sj}\}$ satisfying (34)-(37), where the aggregates $y_F, \{Y_s\}$ are given by (29) and (31).

To characterize the equilibrium, start with candidate values $\{A_s\}$, and define $\{\overline{\gamma}_j\}_{j=1}^{J}$ as the weighted averages

$$\overline{\gamma}_j \equiv \sum_{s=1}^{S} A_s n_s \gamma_{sj}, \qquad \text{all } j. \tag{38}$$

Use (37) and (38) in (34), to find that the thresholds $\{b_j\}_{j=1}^{J-1}$ satisfy

$$\frac{\phi(b_j, x_{j+1})^{-\rho}}{\overline{\gamma}_{j+1}} \Psi_{j+1} = \frac{\phi(b_j, x_j)^{-\rho}}{\overline{\gamma}_j} \Psi_j, \qquad \text{all } j. \tag{39}$$

By the same reasoning as for (13), a solution exists and is unique. The thresholds determine the $\Psi_j$'s, and using (37) and (38) in (35) gives the quantities

$$y_{sj} = A_s \frac{\Psi_j}{\overline{\gamma}_j}, \qquad \text{all } j, s.$$

Then use (31) to get $\{Y_s\}_{s=1}^{J}$, and (29) to get $y_F$. For an equilibrium, (36) must hold. For a computational method, use (36) to calculate updated weights $\{A_s'\}_{s=1}^{S}$ and iterate.

Two examples illustrate some of the possibilities.

### C. Example: identical technology distributions across sectors

If the technology distributions are the same in all sectors, then the equilibrium skill bins are exactly as in the one-sector model. Hence the effects of a technology change are also the same.

To see this, suppose $\gamma_{sj} = \gamma_j$, all $s, j$. Then $\overline{\gamma}_j = \overline{A}\gamma_j$, all $j$, where $\overline{A} \equiv \sum_{s=1}^{S} A_s n_s$. Hence (39) simplifies to (13), and the thresholds $\{b_j\}$ depend only on the $\gamma_j$'s. Then

$$\begin{aligned} y_{sj} &= \frac{A_s}{\overline{A}} \frac{\Psi_j}{\gamma_j}, \qquad \text{all } j, s; \\ Y_s &= n_s \frac{A_s}{\overline{A}} \overline{\Psi}, \qquad \text{all } s; \\ y_F &= \overline{\Psi} \left[ \sum_{s=1}^{S} \theta_s \left( n_s \frac{A_s}{\overline{A}} \right)^{(\sigma-1)/\sigma} \right]^{\sigma/(\sigma-1)}, \end{aligned} \tag{40}$$

where

$$\overline{\Psi} \equiv \left[ \sum_{j=1}^{J} \gamma_j^{1/\rho} \Psi_j^{(\rho-1)/\rho} \right]^{\rho/(\rho-1)},$$

For the weights $\{A_s\}$, use the second line in (40) in (36) to get

$$A_s = c_0 \theta_s^{\sigma} n_s^{(\sigma-\rho)/\rho}, \qquad \text{all } s,$$

where $c_0$ is a constant. Hence

$$\frac{A_s}{\overline{A}} = \frac{\theta_s^{\sigma} n_s^{(\sigma-\rho)/\rho}}{\sum_{s'=1}^{S} \theta_{s'}^{\sigma} n_{s'}^{(\sigma-\rho)/\rho}}, \qquad \text{all } s.$$

Consider an improvement in technology $x_k$. Clearly $\{b_j\}_{j=1}^{J-1}$, $\{\Psi_j\}_{j=1}^{J}$ and $\{p_j\}_{j=1}^{J}$ change exactly as in the one-sector model, with $\left\{A_s/\overline{A}\right\}_{s=1}^{S}$ unchanged. Hence wages for each skill type also change exactly as in the one-sector model. In every sector, output of tasks/goods of type $j$ changes in proportion to the change in $\Psi_j$. Hence all sector outputs $\{Y_s\}$ increase in proportion to the increase in $\overline{\Psi}$, as does output of the final good. From (30) or (33), sector prices $\{P_s\}$ are unchanged.

### D. Example: two sectors, low-tech and high-tech

At the other extreme, consider an example with two sectors, with very different technology distributions. For clarity, label the sectors $s = L, H$. Half of the firms are in each sector, $n_L = n_H = 1/2$, the upper-level elasticity is $\sigma = 1$, and the factor shares are $\theta_L \in (0,1)$ and $\theta_H = 1 - \theta_L$.

There are three technology levels, $x_1 < x_2 < x_3$. In the low-tech sector, $s = L$, the fraction $(1 - \hat{\gamma})$ have $x_1$, the fraction $\hat{\gamma} \in (0,1)$ have $x_2$, and none have $x_3$. In the high-tech sector, $s = H$, all firms have have technology $x_3$. For simplicity the skill distribution is also discrete, with three levels, $h_1 < h_2 < h_3$, and with total numbers $\ell_1 = (1 - \hat{\gamma})/2$, $\ell_2 = \hat{\gamma}/2$, $\ell_3 = 1/2$.

Suppose that in equilibrium all labor with skill $h_j$ produces with technology $x_j$, so

$$y_j = \phi(h_j, x_j), \qquad j = 1, 2, 3.$$

Then sector aggregates are

$$\begin{aligned} Y_L &= \frac{1}{2}\left[(1-\hat{\gamma})\, y_1^{(\rho-1)/\rho} + \hat{\gamma} y_2^{(\rho-1)/\rho}\right]^{\rho/(\rho-1)}, \\ Y_H &= \frac{1}{2} y_3, \end{aligned}$$

sector demands from (30) are

$$P_s Y_s = \theta_s y_F, \qquad s = L, H,$$

and demands for differentiated goods from (32) imply

$$\begin{aligned} \frac{p_j}{P_L} &= \left(\frac{y_j}{Y_L}\right)^{-1/\rho}, \qquad j = 1, 2, \\ \frac{p_3}{P_H} &= \left(\frac{y_3}{Y_H}\right)^{-1/\rho} = 1. \end{aligned}$$

Consider technical change that increases $x_2$. Suppose that employment patterns are unchanged. Then $0 < \hat{Y}_L < \hat{y}_2$, and $\hat{y}_F = \theta_L \hat{Y}_L$, with $Y_H$ unchanged. To calculate the effect on wages, note that

$$\hat{w}_j = \hat{p}_j + \hat{y}_j, \qquad j = 1, 2, 3.$$

For workers with skill $h_3$, since $Y_H = y_3$ is unchanged,

$$\hat{w}_3 = \hat{p}_3 = \hat{P}_H = \hat{Y}_F = \theta_L \hat{Y}_L > 0.$$

For workers with skill $h_2$,

$$\begin{aligned}
\hat{w}_2 &= \hat{p}_2 + \hat{y}_2 \\
&= \hat{P}_L - \frac{1}{\rho}\left(\hat{y}_2 - \hat{Y}_L\right) + \hat{y}_2 \\
&= \hat{y}_F - \hat{Y}_L + \frac{1}{\rho}\hat{Y}_L + \left(1 - \frac{1}{\rho}\right)\hat{y}_2 \\
&= \theta_L \hat{Y}_L + \frac{\rho - 1}{\rho}\left(\hat{y}_2 - \hat{Y}_L\right).
\end{aligned}$$

Since $\hat{Y}_L < \hat{y}_2$, and by assumption $\rho > \sigma = 1$, both terms in the last line are positive, and $\hat{w}_2 > 0$. For workers with skill $h_1$, since $y_1$ is unchanged,

$$\begin{aligned}
\hat{w}_1 &= \hat{p}_1 = \hat{P}_L + \frac{1}{\rho}\hat{Y}_L \\
&= \hat{y}_F - \hat{Y}_L + \frac{1}{\rho}\hat{Y}_L \\
&= \left(\theta_L - 1 + \frac{1}{\rho}\right)\hat{Y}_L.
\end{aligned}$$

Hence $\hat{w}_1 < 0$ if the within-sector elasticity is high and the factor share of the low-tech sector is small, if

$$\frac{1}{\rho} + \theta_L < 1.$$

The main idea in this example is that if the elasticity $\sigma$ across sectors is low, and the weight $\theta_H$ on the high-tech sector is large, then the technology shock leads to a large increase in $P_H$, so a substantial share of the increase in final output accrues to workers in the high-tech sector. For tasks in the low-tech sector, there are two effects: the price

$P_L$ falls and output $Y_L$ rises. If the elasticity $\rho$ is large, the price effect dominates. For workers who use the improved technology, the direct productivity effect offsets the sector price change. But for workers using the unchanged technology, the wage change is dominated by the negative price effect.

Equilibrium requires that no worker can increase his wage by changing jobs. As shown in the Appendix, the parameters can be chosen so this requirement is satisfied.

## 6. NUMERICAL EXAMPLES

This section displays results for a simulated example. The substitution elasticity[5] between technology and skill is set at $\eta = 0.5$. Since the results of a technology change are sensitive to the elasticity of substitution across tasks, four values are used, $\rho = 0.5$, 1.002, 2, and 6.

It is innocuous to set $\omega = 0.5$, since the input weights can be offset by choice of the relative means of the technology and skill distributions. In addition, the mean of one distribution can be normalized in any convenient way.

As suggested by Zipf's law for the firm size distribution, the probability vector $\gamma$ for technology types is a discrete approximation to a Pareto, with a shape parameter $\lambda_F = 1.0$. The location parameter is $x_{\min} = 1$, the maximum value truncates ten percent of the distribution, and there are $J = 50$ types. As suggested by the wage distribution, the distribution of human capital is lognormal. The mean and variance are unity, $\mu_h = \sigma^2 = 1$, and the range is symmetric, truncating about 2.1% of the distribution in total. There are $N_h = 20,000$ values, to mimic a continuous

[5]Bagger and Lentz (2015) report a value of about 0.31. Lise, Meghir and Robin (2016) find a value of 0.53 for college educated workers, and a value well above unity for workers with high school or less education, suggesting that complementarities are important only for better-educated workers.

distribution. In summary, the parameters are

$$\eta = 0.5, \quad \lambda_F = 1, \quad \mu_h = 1, \quad \rho = 0.5,\ 1.002,\ 2,\ 6,$$
$$\omega = 0.5, \quad x_{\min} = 1, \quad \sigma_h^2 = 1.$$

**A. Baseline economies**

Figure 2 displays the results for the baselines economy. The number of technology types is large enough so that the skill bins are quite narrow, except at the very low end. Figures 2a and 2b show the average skill and employment levels for each technology, for the four values of $\rho$. For the highest $\rho$ value, labor is concentrated on the better technologies: employment is strong increasing in $x$, and the average skill for each $x$ is lower. For the two lowest elasticities, employment is concentrated on the lower technologies. Figure 2c shows the wage functions for the four values of $\rho$. For lower skill levels, the wage is approximately the same for all four elasticities. For higher skill levels, the wage is higher for higher elasticities. Total output (which is also the total wage bill) is significantly higher for the higher elasticities.

**B. Homogeneous technologies**

Figure 3 displays the effects of eliminating heterogeneity in technologies, with the common technology $x^{HT}$ in each case chosen so that final output—and hence the total wage bill—are unchanged. Recall that the direct technology effect raises (lowers) individual productivity for workers in the lower (upper) end of the skill distribution. But the average productivity of co-workers moves in the same direction. In addition, employment is uniform across all tasks in the HT economy.

The net effect in all four economies is to depress wages at both ends of the skill distribution, and to raise wages for those in the middle, as shown in Figure 3. The loss function is approximately quadratic, as (15) suggests, even though here the distribution functions for skill and technology are very different from each other.

Interestingly, in every case the variance of log wages in the HT economy is slightly *higher* than in the baseline economy. This fact suggests that reported values in the empirical literature, which show positive contributions of technology inequality to wage inequality, are missing an important element—price effects.

### C. A limited technology improvement

Figure 4 shows the effects of a small improvement for a group of technologies in the middle of the distribution. There are $J = 50$ technology types, and types $k = 21, ..., 30$, are increased by 3%.

Panel (a) shows the resulting changes in employment. For the elasticities greater than unity, employment expands at the affected tasks and falls for all others. For $\rho = 1.002$, employment is almost unchanged, although it rises (falls) very slightly for tasks below (above) the affected group. For $\rho = 0.5$, it falls noticeably at the affected group, and rises very slightly (is unchanged) for tasks below (above) the affected group.

Panel (b) shows the output changes. Output always rises for the tasks that are directly affected, and the size of the increase is larger for the higher elasticities. Output always falls for tasks higher up the technology ladder, although the changes are very small for the low elasticities. For the higher elasticities, outputs also fall for tasks farther down the ladder. For the lower elasticities, there is a very slight increase.

Panel (c) shows the wage changes, plotted as a function of the worker's position in the skill distribution. In all four economies, wages rise for all workers, although the changes are small for skill levels below the group that is directly affected. For the high elasticity, $\rho = 6$, the group that is directly affected—and hence is getting a substantial wage increase—extends much farther down the skill ladder.

## 7. Conclusion

The analysis here has focussed on the effects of technology changes, but the framework could also be used to examine other questions, and it could be extended in a number of ways. For example, it could be used to study the effects of changes in the skill distribution resulting from a change in immigration policy, or of changes in the demand structure resulting from a change in trade policy. It could also be used to revisit the role of labor market frictions in generating unemployment and producing job ladders, as in Mortensen and Pissarides (1999) and Moscarini and Postel-Vinay (2015).

In addition, the multi-sector model in section 5 could be used study more targeted types of technical change. As illustrated by the examples there, change that hits some tasks/goods within a single sector may have quite different effects from change that hits uniformly across sectors. The two-sector example suggests that the multi-sector framework may be useful for studying labor market polarization.

In the framework here, individuals work in isolation to produce outputs, here called tasks. But most goods and services, whether for consumption or investment, are not produced by single individuals. Aggregating tasks into goods requires additional information about how goods and services are produced, about which tasks are involved. Tackling this question is important because it connects the job/occupation decisions of individual workers with the outputs of goods and services measured in so many data sources. The multi-sector model in section 5 may provide a starting point for thinking about this issue.

Defining the boundaries of firms seems even harder, since a firm may produce only one task or a wide variety of goods. And firms, whether large or small, also choose which tasks to produce in-house and which to purchase in the marketplace. Nevertheless, the patterns for recent wage changes suggest rather strongly that firms are important in determining how technical change gets translated into rising wages.

Wage inequality has displayed large and long-lived shifts over the last century, as described in Goldin and Margo (1992) and Goldin and Katz (2007), and many of these shifts are surely due to changes in technology. Large increase in wage inequality lead, understandably, to calls for policies to deal with it. But to guide policies targeted at reducing inequality, we need to better understand its sources.

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# APPENDIX

## A. Derivation of the quadratic loss

For economies where $F(x) \approx G(a_H x)$, all $x$, the wage change from eliminating heterogeneity in technologies is

$$\begin{aligned}
\Delta \ln w(\overline{h}_j) &\approx \chi_0 - \frac{1}{\rho}\left[\ln x^H - \ln x_j\right] + \left[\ln \phi\left(a_H, x^H/x_j\right) - \ln \phi\left(a_H, 1\right)\right], \\
&\approx \chi_0 - \frac{1}{\rho}\Delta_{xj} + \varepsilon_x \Delta_{xj} + \frac{1}{2}\varepsilon_{xx}\Delta_{xj}^2,
\end{aligned}$$

where

$$\varepsilon_x \equiv \frac{\partial \ln \phi\left(h, x\right)}{\partial \ln x}, \qquad \varepsilon_{xx} \equiv \frac{\partial}{\partial \ln x}\left(\frac{x\phi_x\left(h, x\right)}{\phi\left(h, x\right)}\right).$$

For the elasticities, note that

$$\begin{aligned}
\phi(h, x) &\equiv \left[\omega h^{(\eta-1)/\eta} + \left(1-\omega\right) x^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)} \\
\phi_x(h, x) &= \left(1-\omega\right) x^{-1/\eta} \phi(h, x)^{1/\eta}, \\
\phi_{xx}(h, x) &= \frac{1}{\eta}\left(1-\omega\right) x^{-1/\eta} \phi(h, x)^{1/\eta}\left(\frac{\phi_x}{\phi} - x^{-1}\right),
\end{aligned}$$

so

$$\begin{aligned}
\frac{x\phi_x}{\phi} &= \left(1-\omega\right)\phi(h/x, 1)^{-(\eta-1)/\eta}, \\
\frac{x^2\phi_{xx}}{\phi} &= \frac{1}{\eta}\left(1-\omega\right)\phi(h/x, 1)^{-(\eta-1)/\eta}\left(\frac{x\phi_x}{\phi} - 1\right).
\end{aligned}$$

Note, too, that

$$\phi\left(a_H, 1\right)^{-(\eta-1)/\eta} = \frac{1}{\rho\left(1-\omega\right)}.$$

Hence evaluating the elasticities at $(h, x) = (a_H, 1)$ gives

$$\varepsilon_x = \frac{x\phi_x}{\phi} = \left(1-\omega\right)\phi(a_H, 1)^{-(\eta-1)/\eta} = \frac{1}{\rho}$$

$$
\begin{aligned}
\varepsilon_{xx} &= x\left[\frac{\phi_x}{\phi} - x\left(\frac{\phi_x}{\phi}\right)^2 + \frac{x\phi_{xx}}{\phi}\right] \\
&= \frac{1}{\rho} - \frac{1}{\rho^2} + \frac{1}{\eta}(1-\omega)\,\phi(a_H,1)^{-(\eta-1)/\eta}\left(\frac{1}{\rho}-1\right) \\
&= \frac{\rho-1}{\rho^2}\frac{\eta-1}{\eta}.
\end{aligned}
$$

## B. Proof of Proposition 1

PROOF OF PROPOSITION 1: Use (20) in (16) to find that

$$
\hat{y}_F = \nu_k \hat{\Psi}_k + \sum_{j=1}^{J} \nu_j \frac{1}{\Psi_j}\left[\phi(b_j,x_j)\beta_j^{(k)} - \phi(b_{j-1},x_j)\beta_{j-1}^{(k)}\right].
$$

Hence it suffices to show that

$$
\begin{aligned}
0 &= \sum_{j=1}^{J-1}\left[\frac{\gamma_j p_j y_j}{\Psi_j}\phi(b_j,x_j) - \frac{\gamma_{j+1}p_{j+1}y_{j+1}}{\Psi_{j+1}}\phi(b_j,x_{j+1})\right]\beta_j^{(k)} \\
&= \frac{\rho}{\rho-1}\sum_{j=1}^{J-1}\left[\frac{\gamma_j y_j}{\Psi_j} - \frac{\gamma_{j+1}y_{j+1}}{\Psi_{j+1}}\right] w(b_j)\beta_j^{(k)} \\
&= \frac{\rho}{\rho-1}\sum_{j=1}^{J-1}\left[\frac{\gamma_j \phi(b_j,x_j)^\rho}{\Psi_j} - \frac{\gamma_{j+1}\phi(b_j,x_{j+1})^\rho}{\Psi_{j+1}}\right]\phi(b_j,x_j)^{-\rho} y_j w(b_j)\beta_j^{(k)},
\end{aligned}
$$

where the first line uses (17) and the fact that $\beta_0^{(k)} = \beta_J^{(k)} = 0$, the second uses (5), and the third uses (9). From (13), the term in brackets in the last line is zero, for all $j$. ■

## C. Matrix M and proofs of Results 2 - 6

Differentiate (13) and use (11) to get

$$
\begin{aligned}
A_j^{(k)} = &-\frac{1}{\Psi_j}\phi(b_{j-1},x_j)\beta_{j-1}^{(k)} \\
&+\left[\frac{1}{\Psi_j}\phi(b_j,x_j) + \frac{1}{\Psi_{j+1}}\phi(b_j,x_{j+1}) + \rho\delta_j\right]\beta_j^{(k)} \\
&-\frac{1}{\Psi_{j+1}}\phi(b_{j+1},x_{j+1})\beta_{j+1}^{(k)}, \qquad j = 1, ..., J-1.
\end{aligned}
$$

Write this in matrix form as

$$\underline{A}^{(k)} = T\underline{\beta}^{(k)},$$

where $\underline{A}^{(k)}$ is defined in (22), and $T$ is a tridiagonal matrix of dimension $(J-1)$, with rows $(0, ..., 0, c_j, a_j, d_{j+1}, 0, ..., 0)$, where

$$\begin{aligned} a_j &\equiv \rho\delta_j - (d_j + c_{j+1}) > 0, && j = 1, ..., J-1, \\ \delta_j &\equiv \left[\hat{\phi}_h(b_j, x_{j+1}) - \hat{\phi}_h(b_j, x_j)\right]/g(b_j) > 0, && j = 1, ..., J-1, \\ c_j &\equiv -\frac{1}{\Psi_j}\phi(b_{j-1}, x_j) < 0, && j = 2, ..., J, \\ d_j &\equiv -\frac{1}{\Psi_j}\phi(b_j, x_j) < 0, && j = 1, ..., J-1. \end{aligned} \tag{41}$$

The matrix in (21) is the inverse, $M = T^{-1}$.

To characterize $M$, define the constants $\{\theta_i\}_{i=0}^{J-1}, \{\psi_i\}_{i=1}^{J}$, by

$$\theta_0 \equiv 1, \qquad \theta_1 \equiv a_1, \tag{42}$$

$$\theta_i \equiv a_i\theta_{i-1} - c_i d_i \theta_{i-2}, \qquad i = 2, ..., J-1;$$

$$\psi_J \equiv 1, \qquad \psi_{J-1} \equiv a_{J-1}, \tag{43}$$

$$\psi_i \equiv a_i\psi_{i+1} - c_{i+1}d_{i+1}\psi_{i+2}, \qquad i = J-2, ..., 1.$$

Lemma A1 shows that these constants and certain sums are positive.

LEMMA A1: The constants satisfy $\theta_i > 0$, all $i$, and $\psi_i > 0$, all $i$, and in addition

$$\theta_{i-1} + c_i\theta_{i-2} > 0, \qquad i = 2, ..., J-1, \tag{44}$$

$$\psi_i + d_i\psi_{i+1} > 0, \qquad i = J-2, ..., 1. \tag{45}$$

PROOF OF LEMMA A1: Use (41) in (42) to find that

$$\theta_i + c_{i+1}\theta_{i-1} = \rho\delta_i\theta_{i-1} - d_i\left(\theta_{i-1} + c_i\theta_{i-2}\right), \qquad i = 2, ..., J-1.$$

Since $\rho\delta_i > 0$, $d_i < 0$, $c_{i+1} < 0$, all $i$, it follows that

$$\theta_{i-1} > 0 \quad \text{and} \quad \theta_{i-1} + c_i\theta_{i-2} > 0 \qquad \Longrightarrow \qquad \theta_i + c_{i+1}\theta_{i-1} > 0 \quad \text{and} \quad \theta_i > 0.$$

Since $\theta_1 = a_1 > 0$, and

$$\theta_1 + c_2\theta_0 = a_1 + c_2 > 0,$$

by induction (44) holds. Similarly, use (41) in (43) to find that

$$\psi_i + d_i\psi_{i+1} = \rho\delta_i\psi_{i+1} - c_{i+1}\left(\psi_{i+1} + d_{i+1}\psi_{i+2}\right), \qquad i = J-2, ..., 1,$$

so

$$\psi_{i+1} > 0 \quad \text{and} \quad \psi_{i+1} + d_{i+1}\psi_{i+2} > 0 \qquad \Longrightarrow \qquad \psi_i + d_i\psi_{i+1} > 0 \quad \text{and} \quad \psi_i > 0.$$

Since $\psi_{J-1} = a_{J-1} > 0$ and

$$\psi_{J-1} + d_{J-1}\psi_J = a_{J-1} + d_{J-1} > 0,$$

by induction (45) holds. ■

Proof of Lemma 2: The matrix $M$ has elements (see Huang and McColl, 1997)

$$\begin{aligned} m_{nn} &= \frac{1}{\theta_{J-1}}\theta_{n-1}\psi_{n+1}, \qquad n = 1, ..., J-1, && (46) \\ m_{j+1,n} &= -c_{j+1}\frac{\psi_{j+2}}{\psi_{j+1}}m_{j,n}, \qquad j = n, ..., J-2, \\ m_{j-1,n} &= -d_j\frac{\theta_{j-2}}{\theta_{j-1}}m_{j,n}, \qquad j = n, ..., 2. \end{aligned}$$

Since $\theta_i, \psi_i > 0$ and $d_i, c_i < 0$, all $i$, clearly $m_{jn} > 0$, all $j, n$. In addition, clearly the columns satisfy (24), where

$$\begin{aligned} q_{j+1} &\equiv -c_{j+1}\frac{\psi_{j+2}}{\psi_{j+1}}, \qquad j = 1, ..., J-2, && (47) \\ r_{j-1} &\equiv -d_j\frac{\theta_{j-2}}{\theta_{j-1}}, \qquad j = 2, ..., J. \quad ■ \end{aligned}$$

Proof of Lemma 3: From the definitions of $\hat{\phi}_x$ and $\hat{\Psi}_x$,

$$A_k^{(k)} = \rho \frac{\phi_x(b_k, x_k)}{\phi(b_k, x_k)} - \frac{\int_{b_{k-1}}^{b_k} \phi_x(h, x_k) g(h) dh}{\int_{b_{k-1}}^{b_k} \phi(h, x_k) g(h) dh},$$

and since $\phi$ is a CES function,

$$\phi_x(h, x) = (1 - \omega)\, x^{-1/\eta} \phi(h, x)^{1/\eta}.$$

Hence $A_k^{(k)} \gtreqqless 0$ as

$$\int_{b_{k-1}}^{b_k} \phi(h, x_k) \left[ \rho \phi(b_k, x_k)^{1/\eta - 1} - \phi(h, x_k)^{1/\eta - 1} \right] g(h) dh \gtreqqless 0. \tag{48}$$

An analogous argument (with careful attention to signs) establishes that $A_{k-1}^{(k)} \gtreqqless 0$ as

$$\int_{b_{k-1}}^{b_k} \phi(h, x_k) \left[ \rho \phi(b_{k-1}, x_k)^{1/\eta - 1} - \phi(h, x_k)^{1/\eta - 1} \right] g(h) dh \lesseqqgtr 0. \tag{49}$$

Recall that $\phi(\cdot, x_k)$ is increasing in its first argument, and $\eta < 1$. For $\rho \geq 1$, the term in square brackets in (48) is positive over the range of integration, so $A_k^{(k)} > 0$. For $\rho \leq 1$, the term in square brackets in (49) is negative, so $A_{k-1}^{(k)} > 0$. In other cases the signs are ambiguous. ■

Proof of Proposition 4: For $\rho = 1$, the claims are immediate from (23) and Lemmas 2 and 3. For $\rho \neq 1$, the same is true for $k = 1$ and $k = J$, since (23) has only one term.

For $\rho \neq 1$ and $k \neq 1, J$, use the first line of (24), with $j = n = k - 1$, in (23) to find that

$$\begin{aligned} \beta_k^{(k)} &= q_k m_{k-1,k-1} A_{k-1}^{(k)} + m_{k,k} A_k^{(k)} \\ &= \frac{\psi_{k+1}}{\theta_{J-1}} \left( -c_k \theta_{k-2} A_{k-1}^{(k)} + \theta_{k-1} A_k^{(k)} \right), \end{aligned} \tag{50}$$

where the second line uses (46) and (47). Similarly, use the second line of (24), with $j = n = k$, in (23) to find that

$$\begin{aligned} \beta_{k-1}^{(k)} &= m_{k-1,k-1} A_{k-1}^{(k)} + r_{k-1} m_{k,k} A_k^{(k)} \\ &= \frac{\theta_{k-2}}{\theta_{J-1}} \left( -\psi_k A_{k-1}^{(k)} - d_k \psi_{k+1} A_k^{(k)} \right). \end{aligned} \tag{51}$$

Suppose $\rho > 1$. Then $A_k^{(k)} > 0$, so the second term in (50) is positive. If in addition $A_{k-1}^{(k)} \geq 0$, then the first term is nonnegative, so $\beta_k^{(k)} > 0$. If $A_{k-1}^{(k)} < 0$, then

$$\begin{aligned} 0 &< \int_{b_{k-1}}^{b_k} \phi(h, x_k)^{1/\eta} g(h) dg < \rho \phi(b_{k-1}, x_k)^{1/\eta - 1} \int_{b_{k-1}}^{b_k} \phi(h, x_k) g(h) dh \\ &< \rho \phi(b_k, x_k)^{1/\eta - 1} \int_{b_{k-1}}^{b_k} \phi(h, x_k) g(h) dh, \end{aligned}$$

so $\left| A_{k-1}^{(k)} \right| < A_k^{(k)}$. Hence by Lemma A1 the sum in parenthesis in (50) is positive. In (51), the fact that $\left| A_{k-1}^{(k)} \right| < A_k^{(k)}$, does not help in applying Lemma A1, so the sign is ambiguous.

Similarly, suppose $\rho < 1$. Then $A_{k-1}^{(k)} > 0$, so the first term in (51) is positive. If in addition $A_k^{(k)} \geq 0$, then the second term is nonnegative, so $\beta_{k-1}^{(k)} > 0$. If $A_k^{(k)} < 0$, then

$$\begin{aligned} \int_{b_{k-1}}^{b_k} \phi(h, x_k)^{1/\eta} g(h) dg &> \rho \phi(b_k, x_k)^{1/\eta - 1} \int_{b_{k-1}}^{b_k} \phi(h, x_k) g(h) dh \\ &> \rho \phi(b_{k-1}, x_k)^{1/\eta - 1} \int_{b_{k-1}}^{b_k} \phi(h, x_k) g(h) dh > 0, \end{aligned}$$

so $\left| A_k^{(k)} \right| < A_{k-1}^{(k)}$. Hence by Lemma A1 the sum in parenthesis in (51) is positive. In (50), the fact that $\left| A_k^{(k)} \right| < A_{k-1}^{(k)}$, does not help in applying Lemma A1, so the sign is ambiguous. ■

Proof of Proposition 5: Recall from (41) that

$$\phi(b_j, x_j) c_j = \phi(b_{j-1}, x_j) d_j, \qquad \text{all } j. \tag{52}$$

For $j > k$, use the first line in (25) in (26) to find that

$$\begin{aligned}
\hat{y}_j^{(k)} &= \frac{1}{\Psi_j}\left[\phi(b_j, x_j)q_j - \phi(b_{j-1}, x_j)\right]\beta_{j-1}^{(k)} \\
&= \frac{1}{\Psi_j}\left[-c_j\frac{\psi_{j+1}}{\psi_j}\phi(b_j, x_j) - \phi(b_{j-1}, x_j)\right]\beta_{j-1}^{(k)} \\
&= -\frac{1}{\Psi_j}\left[d_j\frac{\psi_{j+1}}{\psi_j} + 1\right]\phi(b_{j-1}, x_j)\beta_{j-1}^{(k)} \\
&\gtreqqless 0 \quad \text{as} \quad \beta_k^{(k)} \lesseqqgtr 0,
\end{aligned} \tag{53}$$

where the second line uses the definition of $q_j$, the third uses (52), and the last uses Lemma A1 and Proposition 4. Similarly, for $j < k$, use the second line in (25) in (26), and the definition of $r_{j-1}$ to find that

$$\begin{aligned}
\hat{y}_j^{(k)} &= \frac{1}{\Psi_j}\left[\phi(b_j, x_j) - r_{j-1}\phi(b_{j-1}, x_j)\right]\beta_j^{(k)} \\
&= \frac{1}{\Psi_j}\left[\phi(b_j, x_j) + d_j\frac{\theta_{j-2}}{\theta_{j-1}}\phi(b_{j-1}, x_j)\right]\beta_j^{(k)} \\
&= \frac{1}{\Psi_j}\left[1 + c_j\frac{\theta_{j-2}}{\theta_{j-1}}\right]\phi(b_j, x_j)\beta_j^{(k)} \\
&\gtreqqless 0 \quad \text{as} \quad \beta_{k-1}^{(k)} \gtreqqless 0, \qquad j < k.
\end{aligned} \tag{54}$$

For $j = k$, the first term in (27) is clearly positive. If $\rho \geq 1$, then the second term is also positive. If in addition $\beta_{k-1}^{(k)} \leq 0$, then last term is nonnegative, and $\hat{y}_k^{(k)} > 0$. If $\beta_{k-1}^{(k)} > 0$, use the fact that equilibrium requires

$$\phi(b_{k-1}, x_{k-1})p_{k-1} = \phi(b_{k-1}, x_k)p_k,$$

before and after the shock. Hence

$$\hat{p}_{k-1} - \hat{p}_k = \left[\hat{\phi}_h(b_{k-1}, x_k) - \hat{\phi}_h(b_{k-1}, x_{k-1})\right]\frac{\beta_{k-1}^{(k)}}{g(b_{k-1})} + \hat{\phi}_x(b_{k-1}, x_k). \tag{55}$$

For $\beta_{k-1}^{(k)} > 0$, both terms on the right are positive, so $\hat{p}_k < \hat{p}_{k-1}$. Hence $\hat{y}_k > \hat{y}_{k-1}$, and as shown above, in this case $\hat{y}_{k-1}^{(k)} > 0$.

If $\rho < 1$, then the first and third terms in (27) are positive. If in addition $\beta_k^{(k)} \geq 0$, then second term is nonnegative, and $\hat{y}_k^{(k)} > 0$. If $\beta_k^{(k)} < 0$, use the fact that equilibrium requires

$$\phi(b_k, x_k)p_k = \phi(b_k, x_{k+1})p_{k+1},$$

before and after the shock. Hence

$$\hat{p}_k - \hat{p}_{k+1} = \left[\hat{\phi}_h(b_k, x_{k+1}) - \hat{\phi}_h(b_k, x_k)\right] \frac{\beta_k^{(k)}}{g(b_k)} - \hat{\phi}_x(b_k, x_{k+1}).$$

For $\beta_{k-1}^{(k)} < 0$, both terms on the right are negative, so $\hat{p}_k < \hat{p}_{k+1}$. Hence $\hat{y}_k > \hat{y}_{k+1}$, and as shown above, in this case $\hat{y}_{k+1}^{(k)} > 0$. ■

Proof of Proposition 6: For $j > k$, use (53), the fact that $\beta_j^{(k)}/\beta_{j-1}^{(k)} = q_j$ and the definition of $q_j$ to find that

$$\begin{aligned} \frac{\hat{y}_{j+1}^{(k)}}{\hat{y}_j^{(k)}} &= -\frac{c_{j+1}}{c_j} \frac{d_{j+1}\psi_{j+2}/\psi_{j+1} + 1}{d_j\psi_{j+1}/\psi_j + 1} \frac{\psi_{j+1}}{\psi_j} c_j \\ &= \frac{-c_{j+1}\left(d_{j+1}\psi_{j+2} + \psi_{j+1}\right)}{d_j\psi_{j+1} + a_j\psi_{j+1} - d_{j+1}c_{j+1}\psi_{j+2}} \\ &= \frac{-c_{j+1}\left(d_{j+1}\psi_{j+2} + \psi_{j+1}\right)}{\rho\delta_j\psi_{j+1} - c_{j+1}\left(\psi_{j+1} + d_{j+1}\psi_{j+2}\right)} < 1, \qquad j > k, \end{aligned}$$

where the second line uses the definitions of $\psi_j$, the third uses the definition of $a_j$, and the inequality follows from Lemma A1 and the fact that $c_{j+1} < 0$.

Similarly, for $j < k$, use (54), the fact that $\beta_{j-1}^{(k)}/\beta_j^{(k)} = r_{j-1}$ and the definitions of $r_{j-1}$, $\theta_{j-1}$, and $a_{j-1}$ to find that

$$\begin{aligned} \frac{\hat{y}_{j-1}^{(k)}}{\hat{y}_j^{(k)}} &= -\frac{d_{j-1}}{d_j} \frac{1 + c_{j-1}\theta_{j-3}/\theta_{j-2}}{1 + c_j\theta_{j-2}/\theta_{j-1}} \frac{\theta_{j-2}}{\theta_{j-1}} d_j \\ &= \frac{-d_{j-1}\left(\theta_{j-2} + c_{j-1}\theta_{j-3}\right)}{a_{j-1}\theta_{j-2} - d_{j-1}c_{j-1}\theta_{j-3} + c_j\theta_{j-2}} \\ &= \frac{-d_{j-1}\left(\theta_{j-2} + c_{j-1}\theta_{j-3}\right)}{\rho\delta_{j-1}\theta_{j-2} - d_{j-1}\left(\theta_{j-2} + c_{j-1}\theta_{j-3}\right)} < 1, \qquad j < k. \end{aligned}$$ ■

Proof of Proposition 7: For $j \neq k$, the claims are immediate from (28) and Propositions 5 and 6.

For $j = k$ there are two cases. If $\beta_{k-1}^{(k)} > 0$, then $\hat{y}_{k-1}^{(k)} > 0$ and $\hat{p}_{k-1}^{(k)} < 0$. Since both terms on the right in (55) are positive, it follows that $\hat{p}_k^{(k)} < \hat{p}_{k-1}^{(k)} < 0$. This argument always holds if $\rho \leq 1$, and holds for $\rho > 1$ if $\beta_{k-1}^{(k)} > 0$.

If $\rho > 1$ and $\beta_{k-1}^{(k)} < 0$, then $\hat{y}_j^{(k)} < 0$ and $\hat{p}_j^{(k)} > 0$, all $j < k$. In addition, since $\beta_k^{(k)} > 0$, in this case $\hat{y}_j^{(k)} < 0$ and $\hat{p}_j^{(k)} > 0$, all $j > k$. Since $\Sigma_{j=1}^{J} \nu_j \hat{p}_j^{(k)} = 0$, it follows that $\hat{p}_k^{(k)} < 0$. ■

Proof of Proposition 8: For $h \notin (b_{k-1}, b_k)$, the claim is immediate from Propositions 4 and 7. For skill bin $k$, note that $\hat{w}(b_{k-1}) = \hat{p}_{k-1} > 0$, and $\hat{w}(h)$ is increasing in $h$ for $h \in (b_{k-1}, b_k)$. ■

## D. One-sector example with wage declines

For a one-sector example where the wage falls for some workers, let $J = 3$ and $k = 2$, and let the skill distribution be discrete, also with three types. Let $h_i, \ell_j$, $j = 1, 2, 3$, be the skill types and the number of workers of each type. The parameters are

$$
\begin{array}{llll}
x_3 = 10,000, & x_2 = 4, & x_1 = 1, & x_2' = 1.01 x_2, \\
h_3 = 10,000, & h_2 = 4, & h_1 = 0.95, & \\
\gamma_3 = 0.99, & \gamma_2 = 0.0090, & \gamma_1 = 0.0010, & \\
\ell_3 = 0.988912, & \ell_2 = 0.007991 & \ell_1 = 0.003097 & \\
\eta = 0.22, & \omega = 0.5, & \rho = 1.2. &
\end{array}
$$

The vast majority of firms have technology $x_3$, and the vast majority of the workforce has skill $h_3 = x_3$, and these levels are much higher than the others. Hence the increase in technology $x_2$ leaves final output virtually unchanged, and the price change at $x_1$ firms depends almost entirely on their own output change. In the initial equilibrium all workers with skill $h_3$ are employed at firms with technology $x_3$, and all with skill $h_2$ are matched with technology $x_2$. Workers with skill $h_1$ are divided between firms with technologies $x_1$ and $x_2$. The increase in $x_2$ reallocates some additional $h_1$ workers to $x_1$ firms, and $p_1$ falls. Workers with skill $h_1$ take a wage cut equal to decline in $p_1$.

### E. Two-sector example with wage declines

Equilibrium in the example in section 4.D requires that no worker can increase his wage by changing jobs, or

$$p_k \phi(h_j, x_k) \le p_j \phi(h_j, x_j), \qquad j, k = 1, 2, 3.$$

It suffices to show that this condition holds for $j, k = 1, 2$, and for $j, k = 2, 3$.

Suppose $h_j = x_j$, so $\phi(h_j, x_j) = x_i$, $j = 1, 2, 3$. Since $j = 1, 2$, are in the same sector, $p_2/p_1 = (x_2/x_1)^{-1/\rho}$. Hence for $j, k = 1, 2$, we need

$$\frac{\phi(x_2, x_1)}{x_2} \le \left(\frac{x_2}{x_1}\right)^{-1/\rho} \le \frac{x_1}{\phi(x_1, x_2)},$$

Let $\alpha = x_1/x_2$, and write this condition as

$$\phi(1, \alpha) \le \alpha^{1/\rho} \le \frac{1}{\phi(1, \alpha^{-1})}.$$

Since $j = 2, 3$ are in different sectors, and $Y_H = y_3$,

$$\begin{aligned}
\frac{p_3}{p_2} &= \frac{P_H}{P_L}\left(\frac{y_2}{Y_L}\right)^{1/\rho} = \frac{1-\theta}{\theta}\frac{Y_L}{x_3}\left(\frac{x_2}{Y_L}\right)^{1/\rho} \\
&= \frac{1-\theta}{\theta}\frac{(1-\hat{\gamma})\, x_1^{(\rho-1)/\rho} + \hat{\gamma} x_2^{(\rho-1)/\rho}}{x_3^{(\rho-1)/\rho}}\left(\frac{x_2}{x_3}\right)^{1/\rho} \\
&= \frac{1-\theta}{\theta}\left[(1-\hat{\gamma})(\alpha\beta)^{(\rho-1)/\rho} + \hat{\gamma}\beta^{(\rho-1)/\rho}\right]\beta^{1/\rho} \\
&= \frac{1-\theta}{\theta}\beta^{-1}\left[(1-\hat{\gamma})\,\alpha^{(\rho-1)/\rho} + \hat{\gamma}\right].
\end{aligned}$$

where $\beta = x_2/x_3 < 1$. Then the required condition is

$$\phi(1, \beta) \le \frac{1-\theta}{\theta}\beta^{-1}\left[(1-\hat{\gamma})\,\alpha^{-(\rho-1)/\rho} + \hat{\gamma}\right]^{-1} \le \frac{1}{\phi(1, \beta^{-1})}.$$

For $\omega = 0.5$, $\eta = 0.3$, $\rho = 4.0$, $\theta = 0.65$, and $\hat{\gamma} = 0.75$, the required conditions hold around $\alpha = 0.15$ and $\beta = 0.70$, and for these parameters $\theta + 1/\rho < 1$.

Figure 1: potential wages

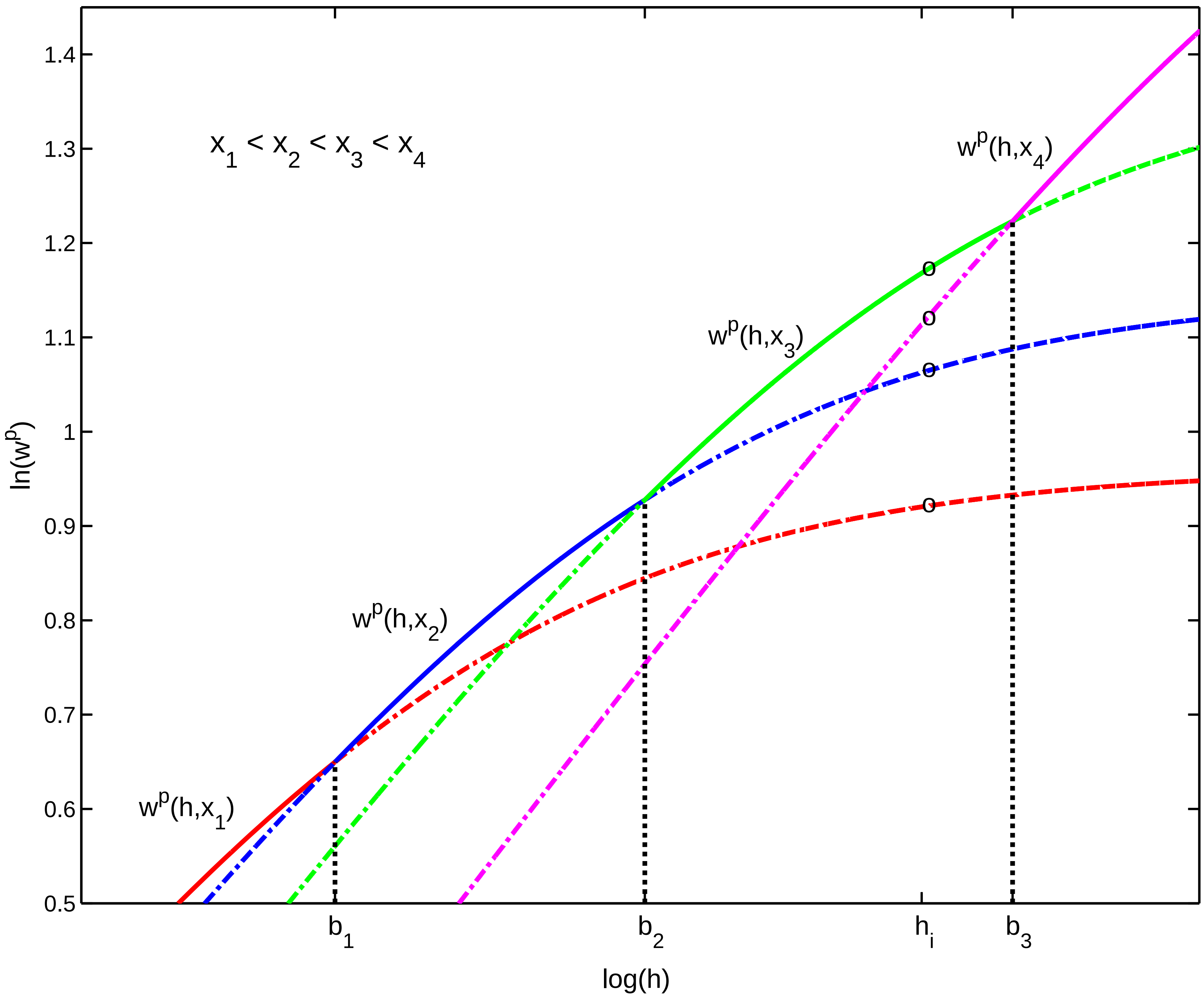

Figure 2a: average skill

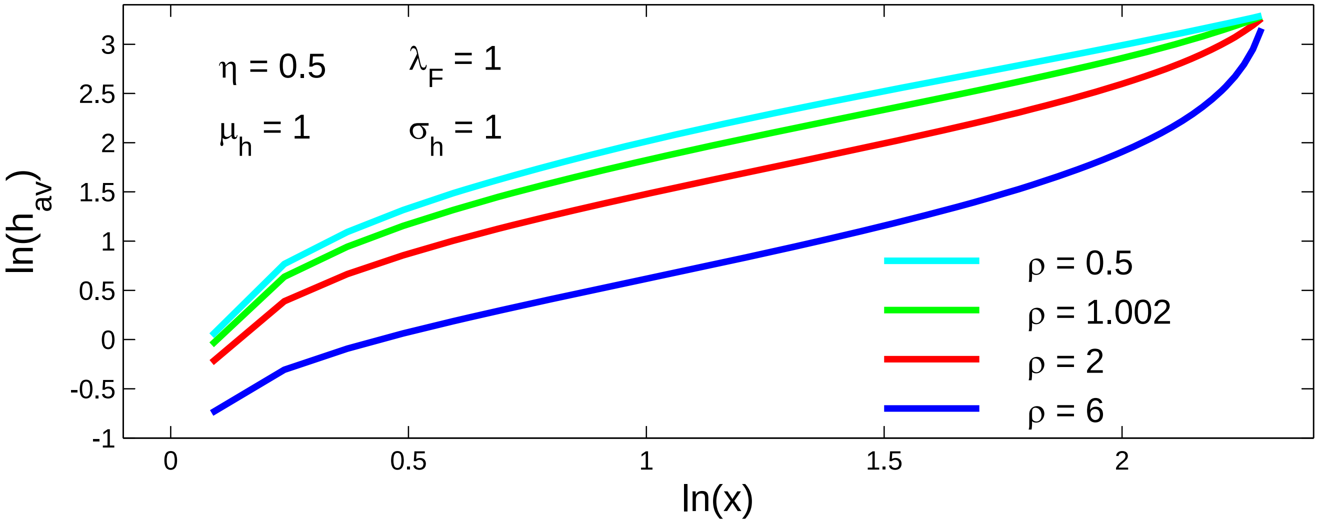


Figure 2b: employment

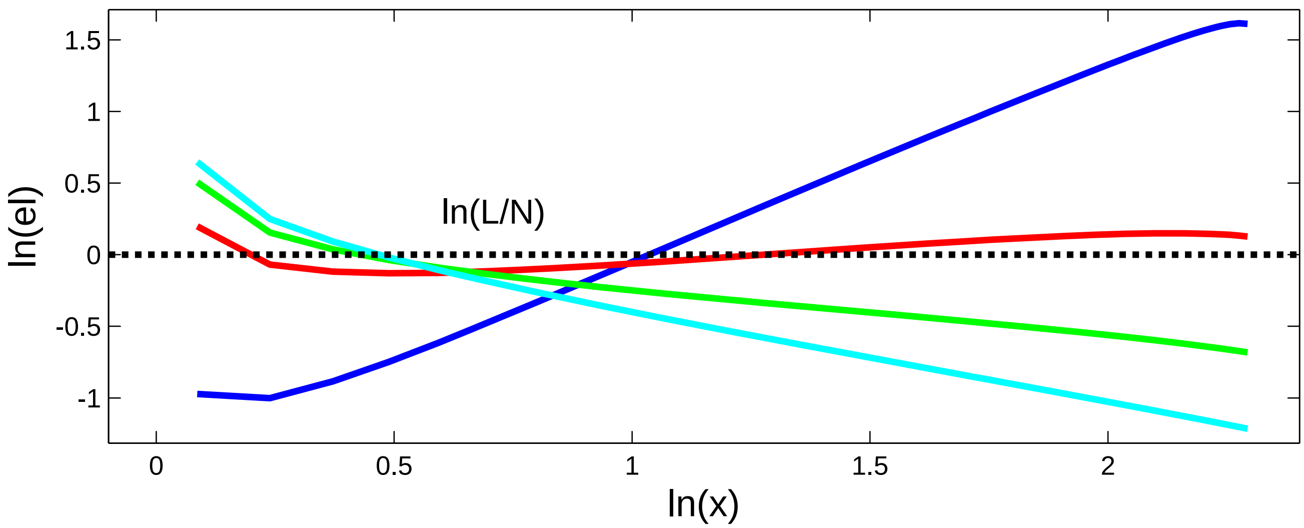


Figure 2c: wage function

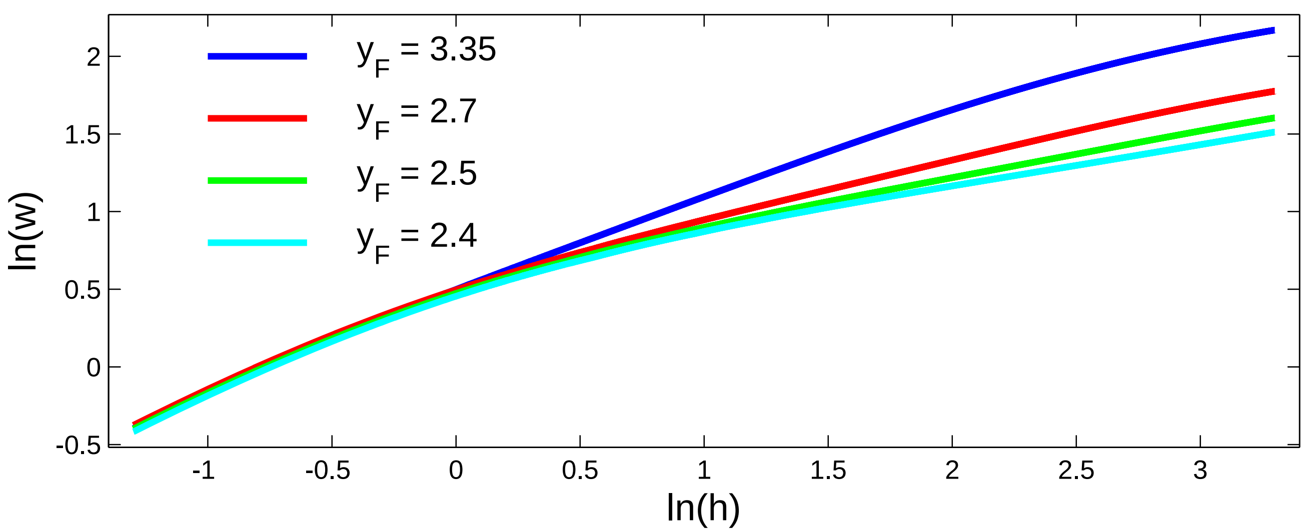

Figure 3c: wage change

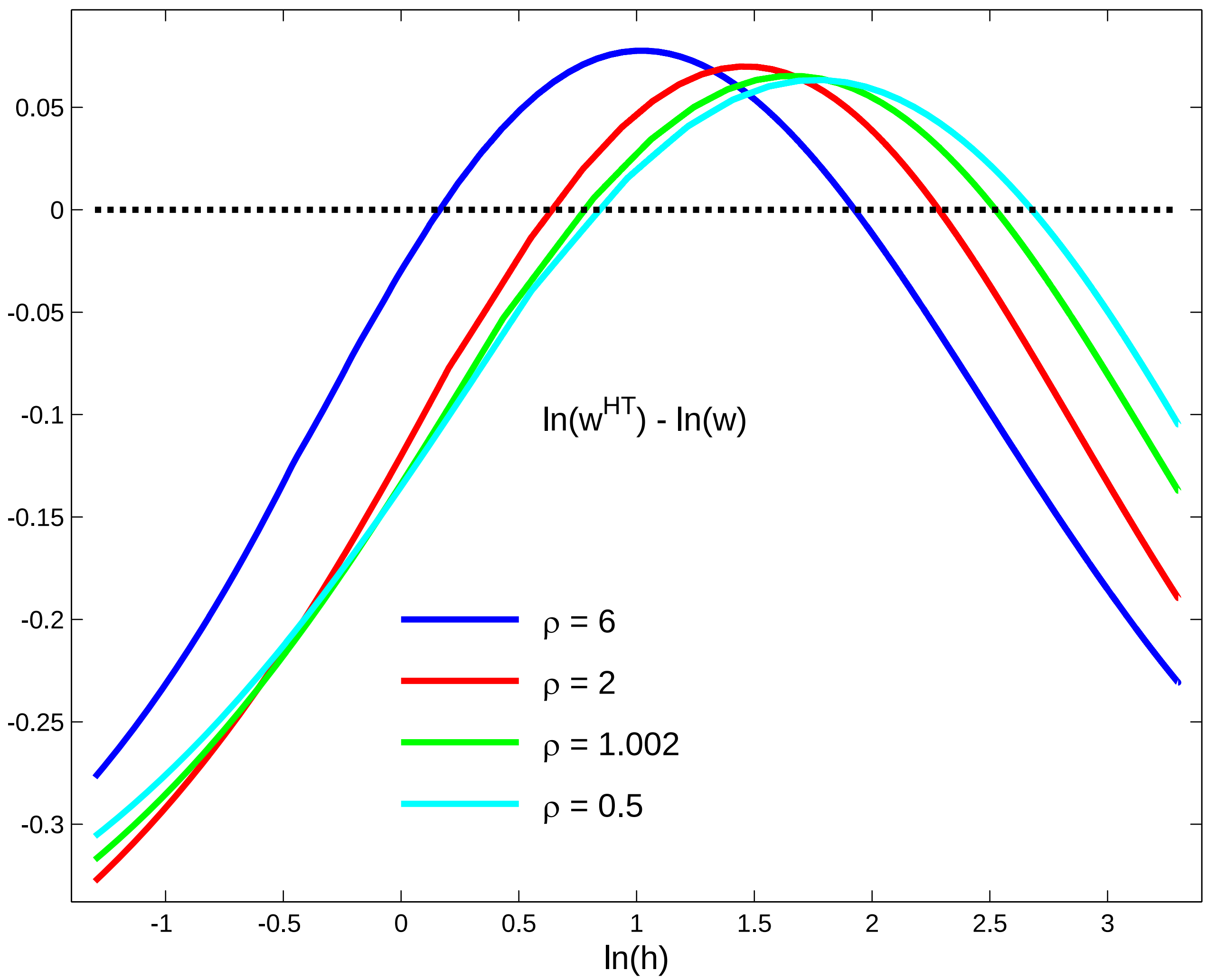

Figure 4a: change in employment

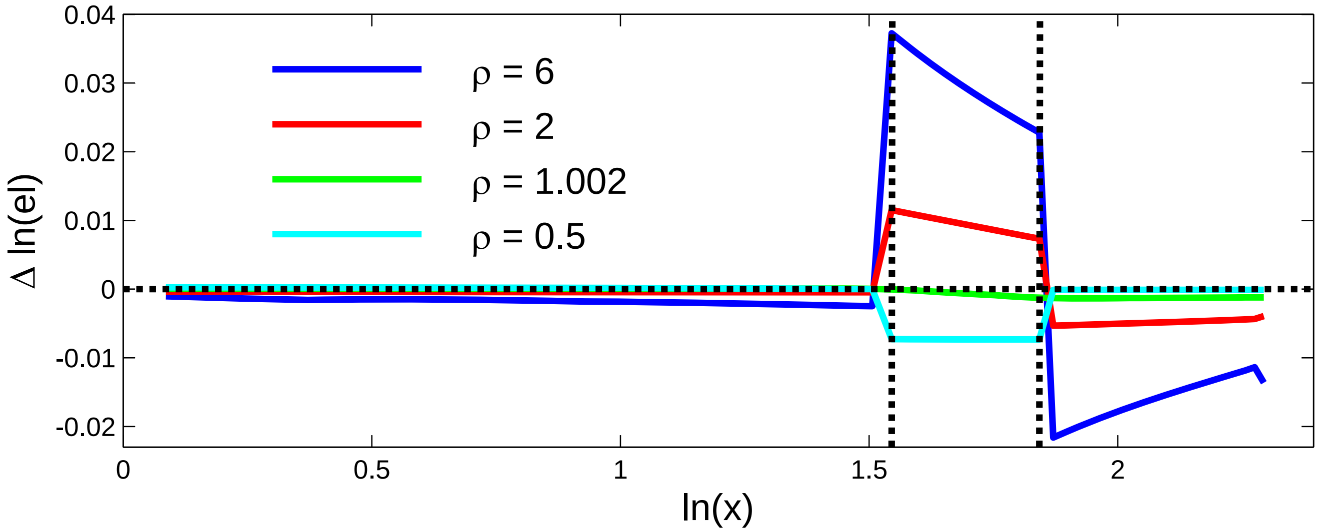


Figure 4b: change in output

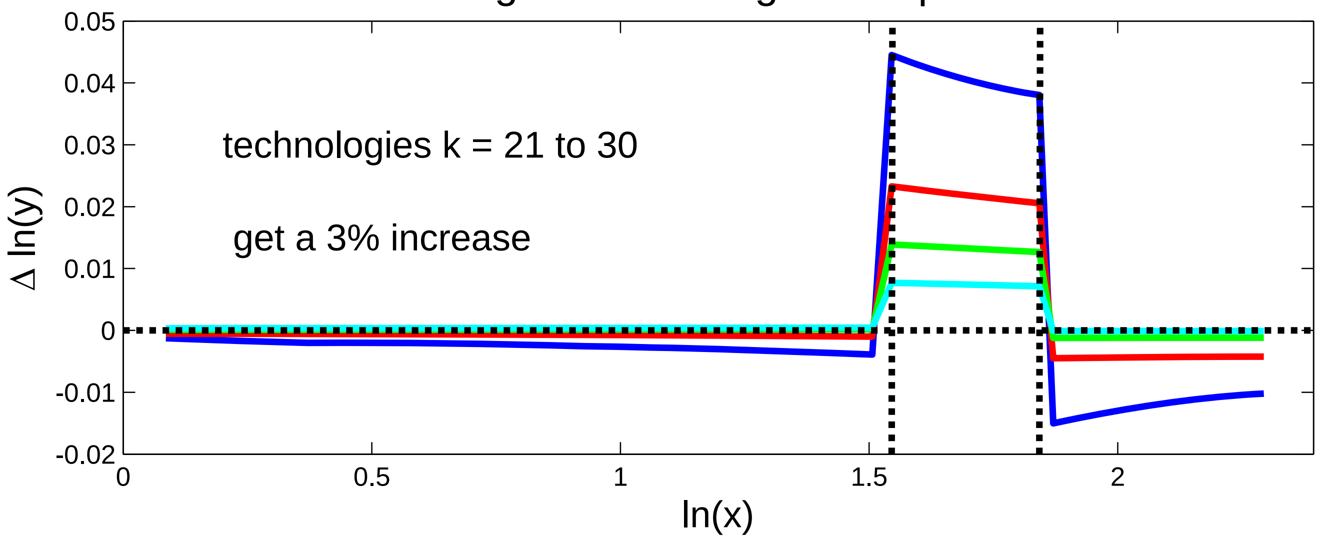


Figure 4c: wage changes