Anticipated Banking Panics*

Mark Gertler, Nobuhiro Kiyotaki and Andrea Prestipino

NYU, Princeton and Federal Reserve Board

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1 Introduction

As argued by Bernanke (2012), a distinctive feature of the recent crisis was "run-like" behavior on the major financial institutions in the shadow banking sector. Early on there were "slow" runs where depositors began a steady stream of withdrawals. The panic then culminated with a series of "fast runs" in September 2008, leading to the nearly instantaneous collapse of the entire investment banking sector. The resulting disruption of financial intermediation, Bernanke argues, was likely the major factor that led the downturn to devolve into the Great Recession.

In Gertler and Kiyotaki (2015; hereafter GK), two of the authors of this paper develop a simple macroeconomic model with banking panics to analyze the simultaneous feedback between real economic activity and banking instability. A corollary result of the paper is that allowing for anticipations of the possibility of a fast run can induce slow run-like behavior. As a result the model can capture the type of movement from slow to fast runs that was a feature of the Great Recession. As the market probability of a run increases, depositors withdraw some but not all of their funds, a pattern similar to the steady drain of credit from the shadow banking system that occurred prior to the outright collapse. Further, by pushing credit spreads up and asset prices down, the anticipation of a run can potentially have harmful effects on the economy even if the run itself does not occur ex post.

Critical to the analysis is how beliefs about the probability of a run are modeled. As in traditional models of runs (e.g. Diamond and Dybvig (1983)) a run in GK is a "sunspot" coordination failure. One important difference, though, is that whether a sunspot equilibrium exists depends on banks' financial exposure to systemic risk as measured by the depositor recovery rate in the event

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of failure. For tractability, GK posit that the run probability is a decreasing function of the recovery rate, the key fundamental that determines whether a run equilibrium exists. The run remains a sunspot but the probability of the sunspot is endogenous. The parameters of the belief function, however, are arbitrary.

In this paper we propose a simple alternative for forming beliefs about run probabilities.[1] Our approach will lead to bank run probabilities that vary countercyclically for purely endogenous reasons. In particular, we decompose the run probability into the product of two factors: first the probability that a bank run equilibrium exists; and second the probability that a sunspot run materializes conditional on the existence of the run equilibrium. To avoid building in arbitrary cyclicality we suppose that the latter is a fixed constant. On the other hand, the probability that a run equilibrium exists in the following period is endogenously determined by fundamentals: It is the probability that the recovery rate will be in the range where banking panics are self fulfilling. It remains the case that a run is not uniquely determined by fundamentals. However, as in the global games approach, the run probability is tied concretely to the rational forecast of the relevant fundamentals. A forecast of deteriorating fundamentals, for example, raises the run probability in a way that does not rely on arbitrariness in the belief function.

Section 2 briefly summarizes the GK macroeconomic model of banking panics and describes in detail our approach to modeling run probabilities. Section 3 presents some simulations of the model. Section 4 concludes.

2 The Basic Model

The framework is based on the infinite horizon macroeconomic model of banking instability developed in Gertler and Kiyotaki (2015). There are two types of agents - households and bankers - with a continuum of measure unity of each type. Banks have expertise in making loans and thus intermediate funds between households and productive assets. Households may also invest in productive assets directly, but are less efficient in doing so than are banks.

Households and bankers each get utility from consuming a perishable nondurable good. There is a durable asset, "capital", which yields a dividend stream of the nondurable good $\bar{Z}_{t+1}$ per unit at each time t and which is fixed in aggregate supply. The dividend process is given by

$$(Z_{t+1} - 1) = \rho(\bar{Z}_{t+1} - 1) + \varepsilon_{t+1} \tag{1}$$

where the random disturbance ε_{t+1} is i.i.d. with mean zero and is uniformly distributed over the closed support $[-\bar{\varepsilon}, \bar{\varepsilon}]$. In addition to the dividend stream generated by capital, both households and bankers also receive endowments of the nondurable good as we describe later.

[1] We would like to thank Michihiro Kandori for sugesting this formulation.

We assume capital does not depreciate and we normalize the total stock at unity. Claims on the capital may be either held by banks or directly by households. Let K_t^b be capital holdings by banks and K_t^h holdings by households. Given that total holdings must equal total supply:

$$K_t^b + K_t^h = 1 \tag{2}$$

Claims on capital may be traded in competitive markets as we discuss below. Let Q_t be the market price of a claim on a unit of capital. Then the gross rate of return on capital intermediated by banks, R_{t+1}^b, is given by

$$R_{t+1}^b = \frac{Z_{t+1} + Q_{t+1}}{Q_t} \tag{3}$$

To capture that households are less efficient than banks at holding capital we assume that they must pay a management cost each period that is increasing and convex in the size of their respective portfolios. In particular, to hold K_t^h units of capital that earns payoffs at $t+1$ a household must pay a management fee $f(K_t^h)$ at t, with $f'(K_t^h) > 0$; $f''(K_t^h) > 0$. The management fee captures the household's relative disadvantage in evaluating and monitoring direct capital holdings. The convex cost, further, is meant to capture limits on the capacity of households to manage a capital portfolio. Given the management cost, the household's return on capital R_{t+1}^h is given by

$$R_{t+1}^h = \frac{Z_{t+1} + Q_{t+1}}{Q_t + f'(K_t^h)} \tag{4}$$

Given $R_{t+1}^b > R_{t+1}^h$, absent financial frictions banks will intermediate the entire capital stock. Households in turn will save entirely in the form deposits. However, when banks are limited in their ability to obtain deposits, households will directly hold some of the capital. As the constraints tighten in a recession, as will happen in our model, the share of capital held by households will expand, forcing asset prices down. In the event of a run, which will become more likely in a recession, the household share will temporarily rise to unity as banks liquidate all their holdings, pushing asset prices down to firesale levels.

2.1 Households

Each household consumes and saves either by holding banks deposits or by holding claims on capital directly. In addition to returns on asset holdings, each household receives an endowment of the consumption good $\bar{Z}_{t+1}W^h$ that varies proportionately with the aggregate productivity shock $\bar{Z}_{t+1}$.

Intermediary deposits at t are one period bonds that promise to pay a non-contingent gross rate of return $\overline{R}_{t+1}$ in the absence of a run. In the event of a run at $t+1$, depositors receive the fraction x_{t+1} of the promised return, where the recovery rate x_{t+1} is the total liquidation value of bank assets per unit of promised deposit obligations. As we will discuss, bank runs are possible if and only if this ratio is strictly below unity. Let $p_t \epsilon\ [0,1]$ be the probability of run in $t+1$. Then we can express the gross rate of return on the deposit contract R_{t+1} as

$$R_{t+1} = \begin{cases} \bar{R}_{t+1} with probability 1 - p_t \\ x_{t+1}\overline{R}_{t+1} with probability p_t \end{cases}$$

A run in our model corresponds to a panic failure of households to roll over deposits as opposed to early withdrawal of demand deposits, as in the classic Diamond and Dybvig (1983) model.[2] For this reason we do not need to impose a "sequential service constraint" which is necessary to generate runs in Diamond/Dybvig. Instead we make the weaker assumption that all households receive the same pro rata share of output in a run. Later we describe the conditions that lead to the existence of an equilibrium where a "failure to rollover" run is possible.

Each period households choose consumption C_t^h, bank deposits D_t, and direct capital holding K_t^h to maximize an expected discounted stream of utility from consumption subject to a period budget constraint that equates consumption and saving to current asset and endowment income. Period utility is logarithmic in consumption and $\beta\ \epsilon\ (0,1)$ is the subjective discount factor. The first order condition for deposits is then given by

$$1 = [(1-p_t)E_t \{\Lambda_{t,t+1} | NoRun\} + p_t E_t \{\Lambda_{t,t+1} x_{t+1} | Run\}] \cdot \overline{R}_{t+1} \qquad (5)$$

Observe that the promised deposit rate $\overline{R}_{t+1}$ that satisfies equation (5) depends on the run probability p_t as well as x_{t+1}.

The first order condition for direct capital holdings is given in turn by

$$1 = E_t\{\Lambda_{t,t+1} R_{t+1}^h\} \qquad (6)$$

So long as households have some direct capital holdings, the first order condition given by (6) will be key in determining the market price of capital (see also equation (4)). Further, the market price of capital will tend to be decreasing in the share of capital since the marginal management cost $f'(K_t^h)$is increasing. As will become clear, in a panic run banks will sell all their securities to households, leading to a sharp contraction in asset prices.[3] The severity of the drop will depend on the quantity of sales and the convexity of the management cost function.

2.2 Bankers

Bankers manage financial intermediaries. They fund capital investments $Q_t k_t^b$ by issuing deposits d_t to households and also by using their own equity, or net

[2] Our modeling of runs as rollover crises follows the Cole and Kehoe (2000) model of self-fulfilling sovereign debt crises.

[3] In practice, the runs during the crisis occurred in wholesale funding markets where banks lend to one another, as opposed to retail markets where households lend to banks. Gertler, Kiyotaki and Prestipino, forthcoming, extend the GK model to allow for runs in wholesale markets.

worth n_t:

$$Q_t k_t^b = d_t + n_t \tag{7}$$

Due to financial market frictions bankers may be constrained in their ability to obtain deposits from households.

Each banker has an i.i.d. probability of surviving until the next period and a probability $1-\sigma$ of exiting. The expected lifetime is then $1/(1-\sigma)$. We introduce finite expected lifetimes for bankers to keep them from accumulating retained earnings to the point where they can fully self finance their investments. Each period $1-\sigma$ new bankers enter which keeps the total population constant.

Bankers consume their net worth upon exit. We assume each banker's utility is linear in terminal consumption (which is the same as their terminal net worth). Accordingly, we can express the expected utility of a surviving banker at t, V_t, which we refer to as the bank franchise value, as

$$V_t = E_t\{\beta\Omega_{t,t+1}n_{t+1}\}$$

where the bank uses the stochastic discount factor $\beta\Omega_{t,t+1}$ to value net worth realized in $t+1$,and $\Omega_{t,t+1}$ is the banker's shadow value of a unit of net worth at $t+1$, averaged across the likelihood of exit and the likelihood of survival, given by

$$\Omega_{t,t+1} = 1 - \sigma + \sigma\frac{V_{t+1}}{n_{t+1}} \tag{8}$$

With probability $1-\sigma$ the banker exits, implying a unit of net worth equals unity (the number of consumption goods it can purchase). With probability σ the banker survives implying the marginal value of n_t is $\frac{V_{t+1}}{n_{t+1}}$, the franchise value of the bank per unit of net worth. As will become clear, to the extent that an additional unit of net worth relaxes the financial market friction, $\frac{V_{t+1}}{n_{t+1}}$ in general will exceed unity.

We assume that surviving banks accumulate net worth through retained earnings. Conditional on the realization of $\bar{Z}_{t+1}$, n_t for surviving bankers is given by

$$n_t = R_t^b Q_{t-1}k_{t-1}^b - R_t d_{t-1}. \tag{9}$$

We suppose that for each new banker, n_t equals simply a "startup" endowment w^b, received only in the first period of business.

Absent a run at t (i.e. a failure of depositors to roll over), the bank pays its creditors the promised rate $\overline{R}_t$. In the event of a run, however, it liquidates its assets (by selling to households) and uses the proceeds to pay its creditors. Let Q_t^* be the liquidation price of bank assets conditional on a run. Then we can express the recovery rate on banks deposits

$$x_t = \min[1, \frac{(Q_t^* + \bar{Z}_{t+1})k_{t-1}^b}{\overline{R}_t d_{t-1}}] \tag{10}$$

Note that in this instance the bank's net worth goes to zero in the event of a run.

To motivate a limit on the bank's ability to issue deposits (which is also critical for open the possibility of a bank run equilibrium), we introduce the following moral hazard problem: After accepting deposits at the beginning of t and purchasing assets, but still during the period, the banker has the option of diverting a fraction θ of assets for personal use. The banker can do so by secretly selling the assets on the secondary market. Because this process takes time (in order to remain undetected), the banker must decide whether to divert at time t prior to the realization of uncertainty at $t+1$. The cost to the banker of siphoning funds is that the depositors can force the bank into liquidation at the beginning of next period.

The banker's decision over whether to divert funds at t reduces to comparing the franchise value of the bank, V_t, which measure the benefit in discounted profits from operating honestly, with the gain from diverting funds, $\theta Q_t k_t^b$. Rational depositors will not agree to lend funds if the bank prefers to divert them. Accordingly, any financial arrangement between the bank and its depositors must satisfy the following condition, which eliminates the bank's incentive to divert:

$$\theta Q_t k_t^b \leq V_t \tag{11}$$

Given the linearity in the bank's portfolio decision problem we conjecture and then verify subsequently, that the bank's franchise value V_t is proportional to it's net worth n_t.We can then restate the objective as to maximize V_t/n_t. Let $\phi_t \equiv Q_t k_t^b/n_t$ be the ratio of bank assets to net worth, which we will refer to as the "leverage multiple". Combining equations (7) and (9) with the expression for V_t yields the following representation of the bank's objective

$$\frac{V_t}{n_t} = \max_{\phi_t}(1-p_t)E_t\{\beta\Omega_{t,t+1}(R_{t+1}^b - \overline{R}_{t+1})\phi_t + \overline{R}_{t+1} \mid no\ run\} \tag{12}$$

subject to the incentive constraint (obtained from equation (11)):

$$\theta\phi_t \leq \frac{V_t}{n_t} \tag{13}$$

and the deposit rate constraint (obtained from equations (5) and (10)):

$$\overline{R}_{t+1} = \left[(1-p_t)E_t\left\{\Lambda_{t,t+1} \mid no\ run\right\} + p_t E_t\left\{\Lambda_{t,t+1}\frac{\phi_t}{\phi_t - 1}\frac{R_{t+1}^{b*}}{\overline{R}_{t+1}} \mid run\right\}\right]^{-1} \tag{14}$$

where $R_{t+1}^{b*} \equiv [Z_{t+1} + Q_{t+1}^*]/Q_t$ is the gross return on bank asset conditional on a run and where $\phi_t R_{t+1}^{b*}/(\phi_t - 1)\overline{R}_{t+1} = x_{t+1}$.[4]

[4] Notice that banks are not internalizing how changes in their leverage affect the probability they will experience individual runs. This does not alter their first order conditions however because an infinitesimal reduction in leverage allows banks to survive a run at the thresold level $\bar{Z}_{t+1}$, see below. But at this level thier networth is zero. See the Appendix for a detailed description of the bank's problem.

In what follows we restrict our attention to a symmetric equilibrium in which all banks choose the same leverage multiple ϕ_t and all depositors coordinate on the same rollover decision. It follows that all banks will default in the event of a run and will survive without a run.[5] Given this, p_t will be common across banks and we can proceed to characterize the representative bank's optimal choice of the leverage multiple ϕ_t. Let μ_t be the expected discounted excess return on banks assets relative to deposit costs and let ν_t be the expected discounted return from reducing deposits a unit:

$$\begin{aligned} \mu_t &= (1-p_t)E_t\{\beta\Omega_{t,t+1}[(R^b_{t+1}-\overline{R}_{t+1})\,|no\ run\} \\ \nu_t &= (1-p_t)E_t\{\beta\Omega_{t,t+1}\overline{R}_{t+1}\,|no\ run\} \end{aligned} \tag{15}$$

Next define μ^r_t as the expected discounted *marginal excess return to bank assets*

$$\mu^r_t = \mu_t - \nu_t \frac{(\phi_t - 1)}{\overline{R}_{t+1}} \frac{d\overline{R}_{t+1}(\phi_t)}{d\phi_t} < \mu_t \tag{16}$$

The second term on the right of equation (16) reflects the effect of the increase in $\overline{R}_{t+1}$ that arises as the bank increases ϕ_t. An increase in ϕ_t reduces the recovery rate, forcing $\overline{R}_{t+1}$ up to compensate depositors, as equation (14) suggests. The term $\nu_t(\phi_t - 1)$ then reflects the reduction in the bank franchise value that results from each percentage increase in $\overline{R}_{t+1}$. Due to the marginal effect on $\overline{R}_{t+1}$ from expanding ϕ_t, the marginal excess return μ^r_t is below the average excess return μ_t.

The solution for ϕ_t depends on whether or not the incentive constraint (13) is binding. In the case where it binds, ϕ_t is given by

$$\phi_t = \frac{\nu_t}{\theta - \mu_t}; \quad \text{with } \mu^r_t > 0 \tag{17}$$

Conversely, when the constraint is not binding,

$$\mu^r_t = 0; \quad \text{with } \phi_t < \frac{\nu_t}{\theta - \mu_t} \tag{18}$$

The constraint (17) limits the leverage multiple to the point where the bank's gain from diverting funds per unit of net worth is exactly balanced by the cost per unit of net worth of losing the franchise value, which is measured by $V_t/n_t = \mu_t\phi_t + \nu_t$. Note that the excess return μ_t tends to move countercyclically since the spread between $R^b_{t+1} - \overline{R}_{t+1}$ widens as the borrowing constraint tightens in recessions. As a results, ϕ_t tends to move countercyclically.

[5] It does not mean that there is no asymmetric equilibrium in which some banks survive in the event of a run. One simple environment in which banks behave symmetrically arises by asssuming that an individual bank survivng a systemic run would lose an additional fraction of the franchise value due to, for example, network externalities or reputation costs. The asymmetric equilibrium without such auxiliary assumption is a topic for further research. See Appendix for details.

If the constraint does not bind, banks asset positions may still be limited by its net worth, so long as there is a possibility that the incentive constraint could bind in the future. In this instance, as in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2015) banks have a precautionary motive for scaling back their respective leverage multiples.[6] The precautionary motive is captured by the presence of the discount factor $\Omega_{t,t+1}$ in the measure of the discounted excess return. The multiplier $\Omega_{t,t+1}$, which reflects the shadow value of net worth, tends to vary countercyclically given that borrowing constraints tighten in downturns. By reducing their leverage multiples, banks reduce the risk of taking losses when the shadow value of net worth is high.

In either case, as we conjectured, the franchise value of the bank V_t is proportionate to n_t by a factor that is independent of bank-specific factors. When the incentive constraint is binding, $V_t = \theta\phi_t \cdot n_t$. When is not currently binding, $V_t = \{[\nu_t \frac{(\phi_t - 1)}{\bar{R}_{t+1}} \frac{d\bar{R}_{t+1}(\phi_t)}{d\phi_t}]\phi_t + \nu_t\} \cdot n_t$ (since $\mu_t^r = 0$). An important corollary is that the bank cannot operate with zero net worth. In this instance V_t falls to zero, implying that the incentive constraint (11) would always be violated if the bank tried to issue deposits. As we show, a necessary condition for a bank run is that banks cannot operate with zero net worth.

2.3 Aggregation and equilibrium without bank runs

Given that individual bank portfolio decisions are homogenous in net worth, the leverage multiple ϕ_t is independent of bank-specific factors. Accordingly, we can aggregate across banks to obtain the following relation between aggregate bank asset holdings $Q_t K_t^b$ and the aggregate quantity of net worth in the banking sector:

$$\frac{Q_t K_t^b}{N_t} = \phi_t \tag{19}$$

Summing across both surviving and entering bankers yields the following expression for the evolution of net worth

$$N_t = \sigma[(R_t^b - R_t)\phi_{t-1} + R_t]N_{t-1} + W^b \tag{20}$$

where $W^b = (1-\sigma)w^b$ is the total endowment of entering bankers. The first term is the total net worth of bankers that operated at $t-1$ and survived until t.

Conversely, exiting bankers consume the fraction $1-\sigma$ of net earnings on assets:

$$C_t^b = (1-\sigma)[(R_t^b - R_t)\phi_{t-1} + R_t]N_{t-1}$$

Finally, output net management costs is consumed by bankers and households.

$$C_t^h + C_t^b = \bar{Z}_{t+1} + W^h + W^b - \frac{\alpha}{2}(K^h)^2$$

[6] One difference from these papers is that because default is possible (in the event of a run), the bank's decision over its leverage multiple also affects to promised deposit rate, which affects the cost of funds at the margin. This effect provides an additional motive for the bank to reduce its leverage multiple.

2.4 Condition for a bank run equilibrium

As in Diamond and Dybvig (1983), the runs we consider are runs on the entire banking system and not an individual bank. A run on an individual bank will not have aggregate effects as depositors simply shuffle their funds from one bank to another. We differ from Diamond and Dybvig though in that runs reflect a panic failure to roll over deposits as opposed to early withdrawal (similar to Cole and Kehoe 2000). In addition, runs are anticipated.

Consider the behavior of a household that acquired deposits at $t-1$. The household must then decide whether to roll over deposits at t. A self-fulfilling "run" equilibrium is possible if the household perceives that in the event all other depositors run, forcing the banking system into liquidation, the household will lose money if it rolls over its deposits. Note that this condition is satisfied if the liquidation makes the banking system insolvent, i.e. drives aggregate bank net worth to zero. Given the moral hazard problem, a household that deposits in a zero net worth bank will simply lose its money (as the bank runs away with it).

The condition for a bank run equilibrium at t, accordingly, is that in the event of liquidation following a run, bank net worth goes to zero. Recall that earlier we defined the depositor recovery rate, x_t, as the ratio of the value of bank assets in liquidation to promised obligations to depositors. Accordingly, the condition for a bank run equilibrium is simply that the recovery rate is below unity:

$$\begin{aligned} x_t &= \frac{(Q_t^* + \bar{Z}_{t+1})K_{t-1}^b}{\bar{R}_t D_{t-1}} < 1 \\ &= \frac{R_t^{b*}}{\bar{R}_t} \cdot \frac{1}{1 - 1/\phi_{t-1}} < 1 \end{aligned} \tag{21}$$

where as earlier R_t^{b*} is the return on bank assets conditional on liquidation.

2.5 Liquidation prices and recovery after a run

Key to the condition for a bank run equilibrium is the behavior of the liquidation price Q_t^*. A depositor run at t induces all banks to liquidate their assets by selling them to households. Accordingly in the wake of the run:

$$K_t^h = \overline{K} = 1 \tag{22}$$

The banking system then rebuilds itself over time as new banks enter. We suppose that new banks enter one period after the panic. The evolution of net worth following the run at t is given by

$$\begin{aligned} N_{t+1} &= W^b + \sigma W^b, \\ N_{t+i} &= \sigma[(Z_{t+i} + Q_{t+i})K_{t+i-1}^b - R_{t+i}D_{t+i-1}] + W^b, \text{ for all } i \geq 2. \end{aligned} \tag{23}$$

To obtain Q_t^*, we invert the household Euler equation to obtain:

$$Q_t^* = E_t\left[\sum_{i=1}^{\infty} \Lambda_{t,t+i}(Z_{t+i} - f'(K_{t+i}^h))\right] - .f'(1) \tag{24}$$

The liquidation price is thus equal to the expected discounted stream of dividends net marginal management costs. Since marginal management costs are at a maximum when K_t^h equal unity, Q_t^* is at a minimum, given the expected future path of K_t^h.[7] Further, the longer it takes the banking system to recover (so K_t^h to falls back to steady state) the lower will be Q_t^*. Finally, note that shocks to $\bar{Z}_{t+1}$ will cause Q_t^* to move procyclically.

2.6 The run probability

We next turn to the determination of the run probability. Let ξ_{t+1} be a binary variable that takes on a value of 1 with probability π and a value of 0 with probability $1-\pi$. In the event of 1, depositors coordinate on a run if a bank run equilibrium exists. Accordingly, a bank run arises at $t+1$iff (i) a bank run equilibrium exists at $t+1$ and (ii) $\xi_{t+1} = 1$. Let ω_t be the probability at t that a bank run equilibrium exists at $t+1$. Then the probability p_t that of run at $t+1$ is given by

$$p_t = \omega_t \cdot \pi \tag{25}$$

We find ω_t as follows. Define $\overline{Z}_{t+1}$ as the value of Z_{t+1} that makes the recovery rate x_{t+1} unity. That is

$$x(\overline{Z}_{t+1}) = \frac{(Q^*(\overline{Z}_{t+1})_{t+1} + \overline{Z}_{t+1})K_t^b}{\bar{R}_{t+1}D_t} = 1 \tag{26}$$

For values of Z_{t+1} below $\overline{Z}_{t+1}$, x_{t+1} is below unity and a bank run equilibrium exists. The probability of a bank run equilibrium existing is accordingly the probability that Z_{t+1} is below $\overline{Z}_{t+1}$:

$$\omega_t = prob\{Z_{t+1} < \overline{Z}_{t+1} \mid \bar{Z}_{t+1}\}$$

It follows that the probability of a run varies inversely with $E_t x_{t+1}$. The lower the forecast of the depositor recovery rate, the higher ω_t and thus the higher p_t. As discussed in Gertler and Kiyotaki (2015), negative shocks to banks returns will increase the probability of future runs through two channels: first by increasing banks' leverage they decrease expected recovery rates; second, as long as shocks are persistent, a negative shock to Z_t lowers the expected liquidation value of banks assets. In this way the model captures that an unexpected weakening of the banking system raises the likelihood of a run. As we show next, there is an interesting feedback: a rise in the run probability will weaken the banking system.

[7] Within our framework the management cost provides a simple way to motivate firesale prices being substantially below normal prices. For a more explicit modelling of this phenomenom, see Kurlat (forthcoming).

3 Numerical Examples

In this section we present several simulations to illustrate the dynamic interaction between the macroeconomy and banking panics. We focus in particular on how the model produces slow versus fast runs.

We begin with a description of the calibration and then turn to the simulations.

3.1 Parameter Choices

The numerical examples here are meant to be illustrative of the model's mechanisms that explain how banking fragility interacts with the real economy as opposed to any kind of serious attempt to explain the data. In this spirit Table 1 lists the parameter values we use in the numerical experiments. We set the discount factor β to its conventional value of .99. Households' endowment W^h, which is meant to capture employment income, equals three times average capital income. The rest of the parameters are calibrated by comparing moments from model simulations to their empirical counterparts as explained below.

We choose the value of α, the parameter controlling households managerial costs, in order for the average proportion of capital intermediated by households, $E\left\{K_t^h\right\}$, to equal one third of total capital. The parameters governing banker's survival probability, σ, and their seizure rate, θ, are set to obtain an average level of banks leverage , $E\left\{\phi_t\right\}$, of seven and an annual spread between the expected return on bank assets and the deposit rate of two hundred basis points.

The endowment of new bankers W^b is key in determining the dynamics of the economy after a run, as total banks net worth in the period right after a run has happened is given by $(1+\sigma)\,W^b$. Therefore, we set this parameter so that the increase in credit spreads upon a bank run matches the increase in the excess bond premium after the collapse of Lehmann in 2008.

We choose a value for the probability of observing a sunspot, π, in order for bank runs to occur once every twenty years on average. The standard deviation of productivity shocks is set to 2% in order to match the unconditional standard deviation of linearly detrended US consumption from 1983Q1 to 2014Q4. And finally, we choose a relatively low value for the serial correlation of $\bar{Z}_{t+1}$ in order to emphasize how transitory shocks to banks returns are endogenously propagated within our setup.

Parameter	*Description*	*Value*
β	*Discount factor*	.99
W^h	*Household endowment*	.0378
α	*Household Managerial Cost*	.007
σ	*Banker's Survival Probability*	.975
θ	*Banker's Seizure Rate*	.2
W^b	*New Bankers Endowment*	.001
π	*Probability of Sunspot*	.15
σ_z	*Std Deviation of Productivity Shocks*	.02
ρ	*Serial Correlation of Productivity*	.6

3.2 Impulse Responses: Recessions and Runs

We illustrate the workings of the model by showing the impulse response of the economy to a transitory shock to productivity $\bar{Z}_{t+1}$. We first solve the model nonlinearly, allowing for the incentive constraint to be only occasionally binding. We next define a steady state for economy as the (non-run) state where all variables remain constant as long as $\bar{Z}_{t+1}$ stays at its mean.

With the economy in steady state we then trace out the effect of a negative 4% shock to the aggregate dividends process $\bar{Z}_{t+1}$ assuming no other shocks to $\bar{Z}_{t+1}$ occur in the future. Figure 1 shows the result of the experiment. In order to capture the movement from slow to fast run the dotted line portrays the case in which a fast run occurs four periods after the shock while the solid line describes the case in which a run does not occur ex post.

Given our calibration, which is described in GKP (2015), the incentive constraint does not bind in steady state. However, the negative shock to $\bar{Z}_{t+1}$ leads to losses in returns on bank assets, causing bank net worth to fall 25% to the point where the incentive constraint binds. A symptom of the binding financial constraint is a sharp increase in the credit spread to nearly 300 basis points. The increase in the spread, in turn, raises the cost of capital, leading to a further drop in asset prices and a weakening of bank balance sheets. This is a common feature of financial accelerator and credit cycle models.

There is however an additional channel that opens up as the weakening of bank balance sheet increases market perceptions of the probability of a run p_t which increases from a steady state value of roughly 0.25% per quarter to 3.50% per quarter in response to the shock. The increase in the run probability places upward pressure on deposit spreads and downward pressure on asset prices, weakening bank's financial positions. This magnifies the financial accelerator. Further, the rise in the anticipation of a run intensifies the outflow of deposits from banks, which drop roughly 12% helping generate a slow run.

As shown by the dotted line, when the fast run is realized, there is a complete collapse of the banking system as depositors coordinate on a no rollover equilibrium. As a result, banks liquidate all their assets leading to a sharp drop in asset prices and rise in spreads. Asset prices drop 20% to their liquidation values while spreads increase to more than 3.5%. Output net of management costs drops to 8% below steady state, more than double the drop in $\bar{Z}_{t+1}$, reflecting the inefficiency from the complete loss of banking services.

Absent a government policy intervention, recovery from the run is quite slow. It takes time for banks to rebuild their balance sheets. Hindering the process is that the probability of a subsequent run stays high. High excess returns after the run permit banks to raise their leverage multiples. Doing so, however, raises the run probability which has a dampening effect by placing downward pressure on asset prices and upward pressure on spreads.

4 Conclusion

A salient feature of the Great Recession was a protracted period of turmoil in financial markets that started with the credit crunch in the summer of 2007 and culminated an year later with the collapse of the entire shadow banking system. The steady withdrawal of funds from major financial institutions that took place over the period was akin to a "slow run" that eventually turned into a "fast run" around the collapse of Lehman Brothers in the fall of 2008. The resulting disruption of financial intermediation was likely the major factor that led the downturn to devolve into the Great Recession.

In this paper we build on existing literature to develop a model that can endogenously generate this transition from a slow run to a fast run. Slow runs in the model arise as negative retruns on banks assets raise creditors concerns about financial stability, leading them to increase their assessment of the probability that a bank run will materialize in the future and hence withdraw deposits from banks. This in turn weakens banks balance sheet positions by forcing asset prices down and increasing the cost of banks borrowing, so that depositors worries are self confirming. When agents actually coordinate on a run equilibrium, a fast run ensues where the entire banking sector is forced to liquidate assets at firesale prices and the economy suffers a very long and deep recession.

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5 Appendix

In this appendix we provide a detailed description of the bank's problem. We also derive conditions under which the optimality conditions (17) and (18) characterize the optimal choice of banks.

Let the recovery rate on deposits held at an individual bank with leverage ϕ be given by

$$x_{t+1}(\phi) = \frac{Q_{t+1} + Z_{t+1}}{Q_t} \frac{\phi}{\phi - 1} \frac{1}{\bar{R}_{t+1}(\phi)}$$

where

$$\bar{R}(\phi) = \frac{1 - \Pr\{x_{t+1}(\phi) < 1\} E_t \left\{ \Lambda_{t+1} R^k_{t+1} \frac{\phi}{\phi-1} \,||x_{t+1}(\phi) < 1 \right\}}{\Pr\{x_{t+1}(\phi) \geq 1\} E_t \{\Lambda_{t+1} \,|x_{t+1}(\phi) \geq 1\}}$$

The problem of a bank is

$$\psi_t(\phi) = \max_{\phi \geq 0} \psi^k_t(\phi) I_{\{\phi \geq 1\}} + \psi^D_t(\phi) I_{\{\phi < 1\}}$$

Where $\psi^k_t(\phi)$ is the optimal value of a bank that borrows and invest in capital

$$\psi^k_t(\phi) = \max_{\phi \geq 1} v^k_t(\phi)\phi - v^w_t(\phi)(\phi - 1)\bar{R}_{t+1}(\phi)$$

$$s.t.$$

$$\theta\phi \leq \psi_t^k(\phi)$$

where

$$v^k(\phi) = \Pr\{x_{t+1}(\phi) \geq 1\} E\left\{\Omega_{t+1} R_{t+1}^k \,| x_{t+1}(\phi) \geq 1\right\}$$

$$v^W(\phi) = \Pr\{x_{t+1}(\phi) \geq 1\} E\{\Omega_{t+1} \,| x_{t+1}(\phi) \geq 1\}$$

and $\psi_t^D(\phi)$ is the optimal value of a bank that invest in other banks deposits as well as capital

$$\psi_t^D(\phi) = \max_{\phi \in [0,1]} v_t^k \phi - v_t^D(\phi - 1)$$

where

$$v_t^k = E_t\left\{\Omega_{t+1} R_{t+1}^k\right\}$$

$$v_t^D = E_t\{\Omega_{t+1} R_{t+1}\}$$

The following proposition shows that what is crucial in determining the shape of a bank's objective function is what we assume about its continuation value if they decrease leverage in order to survive a run. We will assume that bankers that decide to decrease their leverage in order to survive a run suffer a cost that is due to lost netowrk connections and reputation costs due to its refusal to accept deposits in the previous period. That is letting ψ_{t+1}^* be the marginal value of wealth of a bank if a run happens at time $t+1$, we have

$$\psi_{t+1}^* = (1-\rho)\left[1 - \sigma + \sigma\psi_{t+2}\right]$$

where $\rho \in (0,1)$ is our measure of reputation cost and

$$\psi_{t+2} = \psi\left((1+\sigma) W, Z_{t+2}, NoRun\right)$$

A deviation to $\phi = 1$ at time t yields

$$\psi(1) = (1-p_t) E_t\left\{\left[1 - \sigma + \sigma\psi_{t+1}\right] R_{t+1}^k \,| Run_{t+1} = 0\right\} + p_t E_t\left\{\left[1 - \sigma + \sigma\psi_{t+1}^*\right] R_{t+1}^{k*} \,| Run_{t+1} = 1\right\}$$

similarly a deviation to $\phi = 0$ at time t yields

$$\psi(0) = (1-p_t) E_t\left\{\left[1 - \sigma + \sigma\psi_{t+1}\right] \bar{R}_t \,| Run_{t+1} = 0\right\} + p_t E_t\left\{\left[1 - \sigma + \sigma\psi_{t+1}^*\right] R_{t+1}^{k*} \frac{\phi_t^*}{\phi_t^* - 1} \,| Run_{t+1} = 1\right\}$$

We now characterize the individual bank's optimization problem under a condition on the realtive sensitivity of housheolds and banks stochastic discount factor to the occurrence of a run. A corollary of the proposition is that as long as banks suffer big enough reputation and network costs when they individually survive a run, as reflected in a decrease in $\Omega_{t+1}^*(Z_{t+1})$, the optimality conditions (17) and (18) characterize the optimal choice of banks.

We introduce the folowing notation. Let $\phi_t^{ND,R}$ be the leverage at which a bank does not default even when there is a run

$$R_{t+1}^k\left(\min_t\{Z_{t+1}\} \frac{\phi_t^{ND,R}}{\phi_t^{ND,R} - 1}\right) = \bar{R}_t^{rf} = \frac{1}{E\left\{\tilde{\Lambda}_{t+1}\right\}}$$

and $\phi_t^{ND,NR}$ the level where no default occurs as long as there is no run

$$R_{t+1}^{k}\left(\min_{t}\left\{Z_{t+1}\right\}\frac{\phi_t^{ND,NR}}{\phi_t^{ND,NR}-1}\right)=\bar{R}\left(\phi_t^{ND,NR}\right)$$

For $\phi>\phi_t^{ND,R}$ and $\phi>\phi_t^{ND,NR}$ the associated thresholds for default are $Z_t^{D,R}(\phi)$ and $Z_t^{D,NR}(\phi)$

$$R_{t+1}^{k*}\left(Z_t^{D,R}(\phi)\right)\frac{\phi}{\phi-1}=\bar{R}(\phi)$$

$$R_{t+1}^{k}\left(Z_t^{D,NR}(\phi)\right)\frac{\phi}{\phi-1}=\bar{R}(\phi)$$

finally we let $\bar{\phi}_t$ be the maximum leverage allowed by the incentive constraint

$$\theta\bar{\phi}_t=\psi_t^{k}\left(\bar{\phi}_t\right)$$

.

We restrict attention to a symmetric strategy equilibrium in which all banks choose $\phi=\phi_t^*$ and restrict parameters so that in equilibrium banks default only when there is a run, $\phi_t^*<\phi_t^{ND,NR}$.

Proposition 1 *If the percentage increase in banks marginal value of wealth upon a run is smaller than households, i.e.*

$$\frac{\Omega_{t+1}^{*}\left(Z_{t+1}\right)}{E\left\{\tilde{\Omega}_{t+1}\right\}}<\frac{\Lambda_{t+1}^{*}\left(Z_{t+1}\right)}{E\left\{\tilde{\Lambda}_{t+1}\right\}}\ \forall Z_{t+1}\leq\bar{Z}_{t+1} \tag{27}$$

the objective of the banker satisfies

$$\begin{array}{cc}
\psi''(\phi)=0 & \phi\in\left(\phi_t^*,\phi_t^{ND,NR}\right)\\
\psi''(\phi)<0 & \phi\in\left(\phi^{ND,R},\phi_t^*\right)\ \textit{if}\ \phi_t^*>\phi^{ND,R}\\
\psi''(\phi)=0 & \phi\leq\phi^{ND,R}\ \phi\neq1\\
\psi'(\phi)<\psi'\left(\phi'\right) & \phi\in[0,1),\phi'\in\left(1,\phi^{ND,R}\right)
\end{array}$$

Morevoer in a symmetric strategy equilibrium with positive banks capital holdings we must have

$$\psi\left(\phi_t^*\right)>\psi(0)$$

and the following conditions characterizing the optimal leverage choice

$$\begin{array}{cc}
\phi_t^*=\phi^{IC}=\frac{\psi\left(\phi_t^*\right)}{\theta} & \textit{if}\ \psi'\left(\phi_t^*\right)>0\\
\psi'\left(\phi_t^*\right)=0 & \textit{if}\ \theta\phi_t^*<\psi_t\left(\phi_t^*\right)\quad\textit{and}\ \psi\left(\phi_t^*\right)>\psi\left(\phi^{IC}\right)
\end{array}$$

Proof. The general form of $\nu^k(\phi)$ and $\nu^w(\phi)$ is given by

$$v^k(\phi) = \left[\begin{array}{c} (1-\pi)\int_{Z_{t+1}>z^{D,NR}(\phi)} \Omega_{t+1} R^k_{t+1} dF_t(Z_{t+1}) + \\ \pi \int_{Z_{t+1}>\max\left\{\bar{Z}_{t+1}, z^{D,NR}(\phi)\right\}} \Omega_{t+1} R^k_{t+1} dF_t(Z_{t+1}) + \\ \pi \int_{z^{D,R}(\phi)<z'<Z^*_t} \Omega^*_{t+1}(z') R^{k*}(z') dF_t\left(z'\right) \end{array}\right]$$

$$v^W(\phi) = \left[\begin{array}{c} (1-\pi)\int_{Z_{t+1}>z^{D,NR}(\phi)} \Omega_{t+1} dF_t(Z_{t+1}) + \\ \pi \int_{Z_{t+1}>\max\left\{\bar{Z}_{t+1}, z^{D,NR}(\phi)\right\}} \Omega_{t+1} dF_t(Z_{t+1}) + \\ \pi \int_{z^{D,R}(\phi)<z'<\bar{Z}_t} \Omega^*_{t+1} dF_t(Z_{t+}) \end{array}\right]$$

In the region $\phi \in \left(1, \phi_t^{ND,NR}\right)$ the derivative of the objective w.r.t. ϕ is given by

$$\psi'(\phi) = v^k(\phi) - \frac{v^w(\phi)}{\Gamma(\phi)}\left(1 - \frac{(\phi-1)}{\phi}\delta(\phi)\right) \tag{28}$$

where

$$\begin{aligned} \Gamma(\phi) &= \Pr\{x_{t+1}(\phi) \geq 1\} E_t\{\Lambda_{t+1} | x_{t+1}(\phi) \geq 1\} \\ &= \left[(1-\pi)\int \Lambda_{t+1} dF_t(Z_{t+1}) + \pi \int_{Z_{t+1}>\bar{Z}_{t+1}} \Lambda_{t+1} dF_t(Z_{t+1}) + \pi \int_{z^{D,R}(\phi)<Z_{t+1}<Z^*_t} \Lambda^*_{t+1} dF_t(Z_{t+1})\right] \end{aligned}$$

$$\begin{aligned} \frac{(\phi-1)}{\phi}\delta(\phi) &= \Pr\{x_{t+1}(\phi) < 1\} E_t\left\{\Lambda_{t+1} R^k_{t+1} || x_{t+1}(\phi) < 1\right\} \\ &= \pi \int_{Z_{t+1}<z^{D,R}(\phi)} \Lambda^*_{t+1} R^{k*}_{t+1} dF_t(Z_{t+1}) \end{aligned}$$

and the second derivative when $\phi \in (1, \phi^*_t)$ is

$$\psi''(\phi) = \pi \Omega^*\left(z^{D,R}\right) f_t\left(z^{D,R}\right) \frac{dz^{D,R}}{d\phi} \frac{1}{\phi \Gamma(\phi)} \left[1 - \frac{\Lambda^*\left(z^{D,R}\right)}{\Gamma(\phi)} \frac{v^w(\phi)}{\Omega^*(z^{D,R})}\right] \tag{29}$$

while when $\phi \in \left(\phi^*_t, \phi^{ND,NR}\right)$

$$\psi''(\phi) = 0 \tag{30}$$

If (27)is satisfied then we have that multilying both sides by $\pi f(z_{t+1})$ and integrating over $z_{t+1} < Z^*_t$ we have

$$\frac{p_t E\left\{\Omega^*_{t+1} | Run_{t+1} = 1\right\}}{E_t\left\{\tilde{\Omega}_{t+1}\right\}} < \frac{p_t E\left\{\Lambda^*_{t+1} | Run_{t+1} = 1\right\}}{E_t\left\{\tilde{\Lambda}_{t+1}\right\}}$$

which implies

$$\frac{(1-p_t) E\{\Omega_{t+1} | Run_{t+1} = 0\}}{E_t\left\{\tilde{\Omega}_{t+1}\right\}} > \frac{(1-p_t) E\{\Lambda(z') | Run_{t+1} = 0\}}{E_t\left\{\tilde{\Lambda}_{t+1}\right\}} \tag{31}$$

So that

$$\lim_{\phi\uparrow\phi_t^*}\frac{\Omega^*\left(z^{D,R}(\phi)\right)}{v^w(\phi)} = \frac{\Omega^*(Z_t^*)}{(1-p_t)E\{\Omega_{t+1}|Run_{t+1}=0\}} \tag{32}$$

$$= \frac{\Omega^*(Z_t^*)}{E\left\{\tilde{\Omega}_{t+1}\right\}\frac{(1-p_t)E\{\Omega_{t+1}|Run_{t+1}=0\}}{E\left\{\tilde{\Omega}_{t+1}\right\}}} <$$

$$\frac{\Lambda^*(Z_t^*)}{E\left\{\tilde{\Lambda}_{t+1}\right\}\frac{(1-p_t)E\{\Lambda_{t+1}|Run_{t+1}=0\}}{E\left\{\tilde{\Lambda}_{t+1}\right\}}} =$$

$$\lim_{\phi\uparrow\phi_t^*}\frac{\Lambda^*\left(z^{D,R}(\phi)\right)}{\Gamma(\phi)}$$

Given (27) and (32) we show in Lemma (2) that for $\phi\in(\phi^{ND,R},\phi_t^*)$

$$\frac{\Omega^*\left(z^{D,R}(\phi)\right)}{v^w(\phi)} = \frac{\Omega^*\left(z^{D,R}(\phi)\right)}{(1-p_t)E\{\Omega_{t+1}|Run_{t+1}=0\}+\pi\int_{z^{D,R}(\phi)<z'<Z_t^*}\Omega_{t+1}^*dF_t(Z_{t+})} <$$

$$< \frac{\Lambda^*\left(z^{D,R}(\phi)\right)}{(1-p_t)E\{\Lambda_{t+1}|Run_{t+1}=0\}+\pi\int_{z^{D,R}(\phi)<z'<Z_t^*}\Lambda_{t+1}^*dF_t(Z_{t+})} =$$

$$= \frac{\Lambda^*\left(z^{D,R}(\phi)\right)}{\Gamma(\phi)}$$

so that by (29)

$$\psi''(\phi)<0 \quad \phi\in(\phi^{ND,R},\phi_t^*)$$

Notice that when $\phi\in(1,\phi^{ND,R}]$ the first derivative is constant since

$$\psi''(\phi)=0 \quad \phi\le\phi^{ND,R}\ \phi\neq 1$$

and it is equal to

$$\psi'\left(\phi^{ND,R}\right)=E_t\left\{\tilde{\Omega}_{t+1}\left(\tilde{R}_{t+1}^k-\frac{1}{E_t\left\{\tilde{\Lambda}_{t+1}\right\}}\right)\right\}$$

The derivative in the region $\phi\in[0,1)$ is given by

$$\psi'(\phi)=E_t\left\{\tilde{\Omega}_{t+1}\left(\tilde{R}_{t+1}^k-\tilde{R}_{t+1}\right)\right\}$$

One can show that under (27) we have $\frac{COV\left(\tilde{\Omega}_{t+1},\tilde{R}_{t+1}\right)}{E_t\tilde{\Omega}_{t+1}} > \frac{COV\left(\tilde{\Lambda}_{t+1},\tilde{R}_{t+1}\right)}{E_t\tilde{\Lambda}_{t+1}}$, See

Lemma 3, which implies that

$$\begin{aligned}\frac{E\left\{\tilde{\Omega}_{t+1}\tilde{R}_{t+1}\right\}}{E_t\tilde{\Omega}_{t+1}} &= E_t\tilde{R}_{t+1}+\frac{COV\left(\tilde{\Omega}_{t+1},\tilde{R}_{t+1}\right)}{E_t\tilde{\Omega}_{t+1}} > \\ &> E_t\tilde{R}_{t+1}+\frac{COV\left(\tilde{\Lambda}_{t+1},\tilde{R}_{t+1}\right)}{E_t\tilde{\Lambda}_{t+1}} \\ &= E_t\tilde{R}_{t+1}+\frac{E_t\left\{\tilde{\Lambda}_{t+1}\tilde{R}_{t+1}\right\}}{E_t\tilde{\Lambda}_{t+1}}-E_t\tilde{R}_{t+1} \\ &= \frac{1}{E_t\tilde{\Lambda}_{t+1}}\end{aligned}$$

This means that

$$\begin{aligned}\lim_{\phi\uparrow 1}\psi'(\phi) &= E_t\left\{\tilde{\Omega}_{t+1}\left(\tilde{R}^k_{t+1}-\tilde{R}_{t+1}\right)\right\} \\ &< E_t\left\{\tilde{\Omega}_{t+1}\left(\tilde{R}^k_{t+1}-\frac{1}{E_t\tilde{\Lambda}_{t+1}}\right)\right\} \\ &= \lim_{\phi\downarrow 1}\psi'(\phi)\end{aligned}$$

which implies

$$\psi'(\phi)<\psi'\left(\phi'\right)\quad \phi\in[0,1),\phi'>1$$

Given this last inequality in general one needs to check that either $E_t\left\{\Omega_{t+1}\left(R^k_{t+1}-R_{t+1}\right)\right\}>0$ or $\psi^k(\phi)>\psi(0)$. ■

Lemma 2 *If the inequality in* (27) *is satisfied, for* $\phi\in(\phi^{ND,R},\phi^*_t)$

$$\begin{aligned}\frac{\Omega^*\left(z^{D,R}(\phi)\right)}{v^w(\phi)} &= \frac{\Omega^*\left(z^{D,R}(\phi)\right)}{(1-p_t)E\left\{\Omega_{t+1}\left|Run_{t+1}=0\right.\right\}+\pi\int_{z^{D,R}(\phi)<z'<\bar{Z}_{t+1}}\Omega^*_{t+1}dF_t(Z_{t+})} < \\ &< \frac{\Lambda^*\left(z^{D,R}(\phi)\right)}{(1-p_t)E\left\{\Lambda_{t+1}\left|Run_{t+1}=0\right.\right\}+\pi\int_{z^{D,R}(\phi)<z'<\bar{Z}_{t+1}}\Lambda^*_{t+1}dF_t(Z_{t+})} = \\ &= \frac{\Lambda^*\left(z^{D,R}(\phi)\right)}{\Gamma(\phi)}\end{aligned}$$

Proof. Write

$$w(z)=\frac{\pi\Omega^*(z)\,pdf(z)}{(1-p_t)E\left\{\Omega_{t+1}\left|Run_{t+1}=0\right.\right\}}$$

$$l(z)=\frac{\pi\Lambda^*(z)\,pdf(z)}{(1-p_t)E\left\{\Lambda_{t+1}\left|Run_{t+1}=0\right.\right\}}$$

Notice that as we showed in (32), we have

$$w(z)<l(z)$$

and we can rewrite (27) as

$$\frac{w(z)}{1+\int\limits_{\min\{Z\}}^{\bar{Z}_{t+1}} w(s)\,ds} = \frac{\pi\Omega^*(z)\,pdf(z)}{E\left\{\tilde{\Omega}_{t+1}\right\}} < \frac{\pi\Lambda^*(z)\,pdf(z)}{E\left\{\tilde{\Lambda}_{t+1}\right\}} = \frac{l(z)}{1+\int\limits_{\min\{Z\}}^{\bar{Z}_{t+1}} l(s)\,ds}$$

the statement of the Lemma is equivalent to

$$\frac{l(z)}{w(z)}\frac{1+\int\limits_{z}^{\bar{Z}_{t+1}} w(s)\,ds}{1+\int\limits_{z}^{\bar{Z}_{t+1}} l(s)\,ds} > 1 \;\forall z \in \left(\min\{Z\}, \bar{Z}_{t+1}\right)$$

Fix a z and consider the function

$$D(x) = \frac{1+\int\limits_{x}^{\bar{Z}_{t+1}} w(s)\,ds}{1+\int\limits_{x}^{\bar{Z}_{t+1}} l(s)\,ds}$$

notice that

$$D(\min\{Z\}) = \frac{1+\frac{p_t E\left\{\Omega^*_{t+1}|Run_{t+1}=1\right\}}{(1-p_t)E\{\Omega_{t+1}|Run_{t+1}=0\}}}{1+\frac{p_t E\left\{\Lambda^*_{t+1}|Run_{t+1}=1\right\}}{(1-p_t)E\left\{\Lambda^*_{t+1}|Run_{t+1}=0\right\}}} < 1$$

$$D\left(\bar{Z}_{t+1}\right) = 1$$

$$\begin{aligned} D'(x) &= \frac{1+\int\limits_{x}^{\bar{Z}_{t+1}} w(s)\,ds}{1+\int\limits_{x}^{\bar{Z}_{t+1}} l(s)\,ds}\left[\frac{-w(x)}{1+\int\limits_{x}^{\bar{Z}_{t+1}} w(s)\,ds} + \frac{l(x)}{1+\int\limits_{x}^{\bar{Z}_{t+1}} l(s)\,ds}\right] = \\ &= D(x)\frac{w(x)}{1+\int\limits_{x}^{\bar{Z}_{t+1}} w(s)\,ds}\left[\frac{l(x)}{w(x)}D(x)-1\right] \end{aligned}$$

which implies that for each $z \in \left(\min Z, \bar{Z}_{t+1}\right)$

$$\frac{l(z)}{w(z)}D(z) > 1$$

To see this, assume not and let $\hat{Z} = \inf\left\{z \in \left(\min Z, \bar{Z}_{t+1}\right) \text{ such that} \frac{l(z)}{w(z)} D(z) \leq 1\right\}$, which implies that $\frac{l(z)}{w(z)} D(z) > 1$ for $z \in [\min Z, \hat{Z})$. But then

$$\begin{aligned}
\frac{l\left(\hat{Z}\right)}{w\left(\hat{Z}\right)} D\left(\hat{Z}\right) &= \frac{l\left(\hat{Z}\right)}{w\left(\hat{Z}\right)}\left[D(\min\{Z\}) + \int_{\min\{Z\}}^{\hat{Z}} D'(x)\,dx\right] = \\
&= \frac{l\left(\hat{Z}\right)}{w\left(\hat{Z}\right)}\left[1 + \int_{\min\{Z\}}^{\hat{Z}} D(x) \frac{w(x)}{1 + \int_{x}^{Z^{run}} w(s)\,ds}\left[\frac{l(x)}{w(x)} D(x) - 1\right] dx\right] > \\
&> \frac{l\left(\hat{Z}\right)}{w\left(\hat{Z}\right)} > 1
\end{aligned}$$

which is a contradiction. ■

Lemma 3 *If for* $Z_{t+1} \leq \bar{Z}_{t+1}$

$$\frac{\Omega^*(Z_{t+1})}{E\left\{\tilde{\Omega}_{t+1}\right\}} > \frac{\Lambda^*(Z_{t+1})}{E\left\{\tilde{\Lambda}_{t+1}\right\}} \tag{33}$$

then

$$\frac{COV\left(\tilde{\Omega}_{t+1}, \tilde{R}_{t+1}\right)}{E\left\{\tilde{\Omega}_{t+1}\right\}} < \frac{COV\left(\tilde{\Lambda}_{t+1}, \tilde{R}_{t+1}\right)}{E\left\{\tilde{\Lambda}_{t+1}\right\}}$$

Proof. Write

$$\mu^{\omega}_{t+1} = \frac{\tilde{\Omega}_{t+1}}{E\left\{\tilde{\Omega}_{t+1}\right\}} f(z_{t+1}) \;\; and \;\; \mu^{\lambda}_{t+1} = \frac{\tilde{\Lambda}_{t+1}}{E\left\{\tilde{\Lambda}_{t+1}\right\}} f(z_{t+1})$$

Then we have

$$
\begin{aligned}
\frac{COV\left(\tilde{\Omega}_{t+1},\tilde{R}_{t+1}\right)}{E\left\{\tilde{\Omega}_{t+1}\right\}} &= \bar{R}\left[\left[\frac{\left(1-p_t\right)E\left(\Omega_{t+1}\left|Run_{t+1}=0\right.\right)}{E\left\{\Omega_{t+1}\right\}}+\frac{p_t E\left(\Omega^*_{t+1}x_{t+1}\left|Run_{t+1}=1\right.\right)}{E\left\{\Omega_{t+1}\right\}}\right]-E\left\{x_{t+1}\right\}\right]= \\
&= \bar{R}\left[(1-\pi)\int\mu^{\omega}_{t+1}dz'+\pi\int_{z'>Z^*_t}\mu^{\omega}_{t+1}dz'+\pi\int_{z'<Z^*_t}\mu^{\omega}_{t+1}x_{t+1}\right]dz'-E\left\{x_{t+1}\right\} \\
&< \bar{R}\left[(1-\pi)\int\mu^{\lambda}_{t+1}dz'+\pi\int_{z'>Z^*_t}\mu^{\lambda}_{t+1}dz'+\pi\int_{z'<Z^*_t}\mu^{\lambda}_{t+1}x_{t+1}\right]dz'-E\left\{x_{t+1}\right\} \\
&= \bar{R}\left[\left[\frac{\left(1-p_t\right)E\left(\Lambda_{t+1}\left|Run_{t+1}=0\right.\right)}{E\left\{\tilde{\Lambda}_{t+1}\right\}}+\frac{p_t E\left(\Lambda^*_{t+1}x_{t+1}\left|Run_{t+1}=1\right.\right)}{E\left\{\tilde{\Lambda}_{t+1}\right\}}\right]-E\left\{x_{t+1}\right\}\right] \\
&= \frac{COV\left(\tilde{\Lambda}_{t+1},\tilde{R}_{t+1}\right)}{E\left\{\tilde{\Lambda}_{t+1}\right\}}
\end{aligned}
$$

where the inequality follows by noticing

$$
\int_{z'>Z^*_t}\left(\mu^{\omega}_{t+1}-\mu^{\lambda}_{t+1}\right)dz'=-\int_{z'<Z^*_t}\left(\mu^{\omega}_{t+1}-\mu^{\lambda}_{t+1}\right)dz'<-\int_{z'<Z^*_t}\left(\mu^{\omega}_{t+1}-\mu^{\lambda}_{t+1}\right)x_{t+1}dz'
$$

where the inequality follows from $x_{t+1}<1\forall z_{t+1}<Z^*_t$ and the equality from

$$
\int\mu^{\omega}_{t+1}dz=1=\int\mu^{\lambda}_{t+1}dz
$$

■