A Generalized Approach to Indeterminacy in Linear Rational Expectations Models*

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Abstract

We propose a novel approach to deal with the problem of indeterminacy in Linear Rational Expectations models. The method consists of augmenting the original model with a set of auxiliary exogenous equations that are used to provide the adequate number of explosive roots in presence of indeterminacy. The solution in this expanded state space, if it exists, is always determinate and is identical to the indeterminate solution of the original model. This approach accomodates both cases of determinacy and indeterminacy and can be implemented even when the boundaries of the determinacy region are unknown. As a result, the researcher can estimate the model by using standard packages without restricting the estimates to a certain area of the parameter space. Finally, this approach can easily accommodate different degrees of indeterminacy. We apply our method to simulated data from the New-Keynesian model for both regions of the parameter space. We show that our method successfully recovers the true parameter values independent of the initial values.

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1 Introduction

The possibility of sunspot and multiple equilibria has traditionally drawn a lot of attention in the macroeconomic literature since the work of Cass and Shell (1983), Farmer and Guo (1994) and Farmer and Guo (1995). Prominent examples in the monetary policy literature include the work of Clarida et al. (2000) and Kerr and King (1996), that study the possibility of multiple equilibria as a result of violations of the Taylor principle in New-Keynesian (NK) models. Applying the methods developed in Lubik and Schorfheide (2003) to the canonical NK model, Lubik and Schorfheide (2004) test for indeterminacy in U.S. monetary policy. More recently, Aruoba and Schorfheide (2015) study inflation dynamics at the Zero Lower Bound (ZLB) and during an exit from the ZLB. The authors show that at the time of the lift-off from the ZLB the equilibrium paths for inflation and real economic activity are not unique. On the methodological side, Farmer et al. (2015) propose a novel approach to solve and estimate linear rational expectations (LRE) models under indeterminacy which is equivalent to Lubik and Schorfheide (2003), while being implementable in standard software packages, such as Dynare and Sims' (2001) code Gensys.

The common practice to empirically test for determinacy by using Bayesian inference proceeds in two steps. First, the researcher estimates the same model in the regions of the parameter space associated with determinacy and indeterminacy. Then, she compares the two models to establish which one fits the data better when taking into account the number of parameters. The model that delivers the highest marginal data density is preferred and it indicates the prevalence of equilibrium (in)determinacy.

This procedure can sometimes be challenging to implement in practice. First, the procedure requires the researcher to solve the model differently depending on the area of the parameter space that is being studied. Second, the procedure requires to estimate the same model twice, first under determinacy, then under indeterminacy. This is the same procedure that would be followed if the researcher were comparing two *structurally different* models, while she is in fact estimating the *same structural* model in alternative regions of the parameter space. Finally, the estimation under indeterminacy is not generally implementable in standard estimation algorithms and requires a significant amount of coding work on the side of the researcher.

In this paper, we propose a novel approach to solving Linear Rational Expectation (LRE) models that easily accommodate both the case of determinacy and indeterminacy. As a result, the methodology can be used to estimate a LRE model that could potentially be characterized by multiplicity of equilibria over the entire parameter space. This approach is implementable even when the analytic condition for the region of determinacy or the degrees of indeterminacy are unknown and can be used in standard software packages, such as Dynare and Sims' (2001) code Gensys.

To understand how our approach works, it is useful to recall the conditions for determinacy as stated by Blanchard and Kahn (1980). Indeterminacy arises when the parameter values are such that the number of explosive roots is smaller than the number of non-predetermined variables. Our methodology augments the original model under study by appending additional autoregressive processes. The innovations of these exogenous processes are assumed to be linear combinations of a subset of the forecast errors associated with the expectational variables of the model and a newly defined vector of sunspot shocks. Whether the autoregressive processes are mean-reverting or explosive is central and the intuition follows. When a model is determinate, the roots of the additional autoregressive process are within the unit circle (i.e., the Blanchard-Kahn condition is satisfied) and we show that the law of motion for the endogenous variables is equivalent to the solution obtained using standard solution algorithms (King and Watson (1998), Uhlig (1999), Klein (2000), Sims (2001)). When the model is indeterminate, the appended autoregressive processes are explosive and the solution we obtain for the endogenous variables is equivalent to the one obtained with the methodology of Lubik and Schorfheide (2003) or, equivalently, Farmer et al. (2015). We prove the equivalence of the solutions under both determinacy and indeterminacy in Theorem 1 and in Section 4 we provide an analytic example.

Finally, we apply our theoretical results to the NK model of Lubik and Schorfheide (2004). We first generate two series of simulated data for parameter values which satisfy the condition for determinacy and indeterminacy, respectively. We then estimate the model by using the proposed augmented representation for both cases. In Section 5.1 we assume that the researcher knows the region of determinacy and we consider this case to provide an intuition for the relevance of our methodology when compared to the conventional approach of estimating the model over the two regions of the parameter space separately. In Section 5.2 the researcher is assumed to ignore the analytic condition defining the region of determinacy. With our methodology the model is estimated over the entire parameter space and the true parameter values are recovered, providing evidence in favor of determinacy or indeterminacy. Hence, our methodology can be used to test for indeterminacy in the wide class of medium- and large-scale models as well as small-scale models even when the region of determinacy cannot be derived analytically and is therefore *unknown* to the researcher.

Our methodology provides three main advantages. First, it accommodates both the case of determinacy and indeterminacy with the same system of equations and the model can therefore be solved by using standard solution algorithms. Instead, existing methods require to rewrite the model based on the existing degree of indeterminacy (Farmer et al. (2015)) or to construct the solution under indeterminacy ex-post following the seminal contribution of Lubik and Schorfheide (2003). Second, given that the method accommodates both the case of determinacy and indeterminacy, the researcher does not need to take a stance on which area of the parameter space she is interested in exploring. The methodology ensures that the Metropolis-Hastings algorithm used for estimation explores the entire parameter space, increasing the likelihood of finding a global maximum over the parameter space. This is particularly relevant when considering that the posterior mode is a crucial object used for Bayesian inference. Furthermore, the obtained posterior distributions of the model parameters could potentially lie in both regions and they result from the inference which can be drawn from the data about model determinacy. Finally, even when the region of determinacy is unknown, the methodology allows the researcher to estimate the model without imposing *a priori* assumptions about the uniqueness of the equilibrium, which can be equivalently thought of as restrictions on the parameter space over which Bayesian inference is conducted. Hence, information contained in the data indicates whether an estimated medium- or large-scale model is characterized by a unique solution or by multiplicity of equilibria.

The remainder of the paper is organized as follows. Section 2 builds the intuition by using an example borrowed from Lubik and Schorfheide (2004). Section 3 describes the methodology and provides the theorem which proves that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al.

(2015). An analytic example of the theoretical result is provided in Section 4. We show the empirical relevance of the proposed methodology with an estimation exercise in Section 5 and we then conclude with Section 6.

2 Building the intuition

Before presenting the theoretical results of the paper, this Section builds the intuition behind our approach by considering the univariate example proposed in Lubik and Schorfheide (2004). While Section 2.1 explains our approach from an analytical perspective, Section 2.2 addresses questions which could arise at the time of its practical implementation.

2.1 A useful example

To provide the intuition behind our methodology, we borrow the following example from Lubik and Schorfheide (2004). Consider the LRE model

$$y_t = \frac{1}{\theta} E_t(y_{t+1}) + \varepsilon_t, \tag{1}$$

where $\varepsilon_t \sim iidN(0, \sigma^2)$ and $\theta \in [0, 2]$. Defining $\xi_t \equiv E_t(y_{t+1})$, the one-period ahead forecast error is denoted by $\eta_t \equiv y_t - \xi_{t-1}$ and we rewrite the LRE model in (1) as

$$\xi_t = \theta \xi_{t-1} - \theta \varepsilon_t + \theta \eta_t. \tag{2}$$

Since the model in (1) has one expectational variable, the Blanchard-Kahn condition is satisfied when there is one unstable root. The equivalent model in (2) is thus determinate when $\theta > 1$ and to guarantee that the solution is stable it must hold that $\eta_t = \varepsilon_t$ and $\xi_t = 0$[1]. Hence, for $\theta > 1$, the solution for the endogenous variable is

$$y_t = \varepsilon_t. \tag{3}$$

[1] To be more precise, for the solution to be stable it is sufficient that $\xi_0 = 0$ combined with $\eta_t = \varepsilon_t$. However, when these two conditions hold, they imply $\xi_t = 0$ at any time t.

However, when $\theta \leq 1$, the Blanchard-Kahn condition is not satisfied, implying that the stability requirement imposes no restriction on the rational expectation forecast error η_t. Applying the methodology proposed by either Lubik and Schorfheide (2003) or equivalently Farmer et al. (2015), the solution takes the following expression

$$y_t = \theta y_{t-1} - \theta \varepsilon_t + \eta_t, \tag{4}$$

where the non-fundamental shock represented by the forecast error η_t can now affect the model dynamics.

The problem that a researcher faces when solving the LRE model in (2) using standard solution algorithms in both the determinacy and indeterminacy regions is the following. Under determinacy, the model already has a sufficient number of unstable roots to match the number of expectational variables. However, under indeterminacy, the model is missing one explosive root since it still has one expectational variables but no unstable root. Therefore, our approach proposes to augment the original model by appending an independent process which could be either stable or unstable. When the original model is determinate, the appended process must be stationary so that also the augmented representation satisfies the Blanchard-Kahn condition. The additional process should however be explosive under indeterminacy so that the Blanchard-Kahn condition is still satisfied for the augmented system, while not for the original model. In the following, we apply this intuition to the example considered in this Section.

Our methodology proposes to solve an augmented system of equations which can be solved using standard solution algorithms such as Sims (2001) under both determinacy and indeterminacy. Consider the following augmented system

$$\begin{aligned} \xi_t &= \theta \xi_{t-1} - \theta \varepsilon_t + \theta \eta_t, \\ \omega_t &= \alpha \omega_{t-1} - \nu_t + \eta_t, \end{aligned} \tag{5}$$

where ω_t is an independent autoregressive process, ν_t is a newly defined sunspot shock and $\alpha \in [0, 2]$. The following table summarizes the intuition behind our approach.

Table 1: Blanchard-Kahn condition in the augmented representation			
α	Unstable Roots	B-K condition in augmented model (5)	Solution
Determinacy $\theta > 1$ in original model (2)			
< 1	1	Satisfied	$\{y_t = \varepsilon_t,\, \omega_t = \alpha\omega_{t-1} - \nu_t + \varepsilon_t\}$
> 1	2	Not satisfied	-
Indeterminacy $\theta < 1$ in original model (2)			
< 1	0	Not satisfied	-
> 1	1	Satisfied	$\{y_t = \theta y_{t-1} - \theta\varepsilon_t + \eta_t,\, \omega_t = 0\}$

Note: The Table presents the regions of the parameter space for which the augmented representation in (5) satisfies the Blanchard-Kahn condition and shows the mapping to the original model in (2).

When the original LRE model in (2) is determinate, $\theta > 1$, the Blanchard-Kahn condition for the augmented representation in (5) is satisfied when $\alpha < 1$. Indeed, for $\theta > 1$ the original model has the same number of unstable roots as the number of expectational variables. Our methodology thus suggests to append a stable autoregressive process. Using the solution method of Sims (2001) to solve the augmented representation when $\theta > 1$ and $\alpha < 1$ delivers the same solution for the endogenous variable as in equation (3). Since the coefficient α is smaller than 1, the solution also includes the autoregressive process ω_t. Importantly, its dynamics do not impact the endogenous variable y_t.

Considering the case of indeterminacy (i.e. $\theta < 1$), the original model has one expectational variable but no unstable root, thus violating the Blanchard-Kahn condition. By appending an explosive autoregressive process, the augmented representation that we propose satisfies the Blanchard-Kahn condition and delivers the same solution as the one resulting from the methodology of Lubik and Schorfheide (2003) or Farmer et al. (2015) and described by equation (4). Moreover, stability imposes ω_t equal to zero, so that also in this case the solution for the endogenous variable does not depend on the appended autoregressive process.

Note that in both cases, the value of α is irrelevant for the law of motion of y_t. This makes clear that introducing the auxiliary processes does not affect the properties of the solution in the two cases. These processes only serve the purpose of providing the necessary explosive roots under indeterminacy and creating the mapping between the sunspot shocks and the expectation errors. As we will see in Section 3, this result can be generalized and applies to more complicated model with a potentially multiple degrees of indeterminacy.

2.2 Choosing α

A natural question that arises with the approach we propose is about how to choose α. We consider the following three cases: the researcher knows the analytic condition defining the region of determinacy; or she only has an relatively good idea of the parameter values for which the model changes region; or the region of determinacy is completely unknown to the researcher. We consider the three cases separately.

We first consider the case that the researcher is able to analytically derive the condition which defines when the model is determinate or indeterminate. For the example considered in this Section, this case corresponds to knowing that when $\theta < 1$ the model in (2) is indeterminate. We thus suggest to write the parameter α as a function of the parameter θ so that the augmented representation in (5) always satisfies the Blanchard-Kahn condition. This can be obtained by setting $\alpha \equiv 1/\theta$. When the original model is determinate ($\theta > 1$), the appended autoregressive process is stationary. Under indeterminacy ($\theta < 1$) of the original model, the parameter α is greater than 1 and the appended process is therefore explosive. Hence, when the region of determinacy is known, the researcher should carefully choose α such that the augmented representation always delivers a solution under both determinacy and indeterminacy. This would indeed increases the efficiency of the Metropolis-Hastings algorithm and allow for the data to provide information about model determinacy. Using the NK model in Lubik and Schorfheide (2004), we implement this suggestion in Section 5.1 where we estimate it assuming that the researcher knows the region of determinacy.

There are however instances when the researcher only has a rough idea about where the region of determinacy lies. Suppose that the researcher does not know the region but for values of the parameter θ slightly above 1 she can solve the original model under determinacy, while for values just below 1 the model is indeterminate. She thus has a relatively good idea that the region of determinacy is $\theta > 1$, even if she is not able to derive the analytical condition. In this case, she could set a prior distribution for the parameter α such that there is a higher probability of drawing values below 1 when the parameter θ is greater than 1 and vice versa. Similarly, the variance-covariance matrix used by the Metropolis-Hastings algorithm to make new draws should display a positive correlation between the values of θ and $1/\alpha$. This practice would increase the likelihood of obtaining a solution in the augmented representation and therefore the efficiency of the algorithm.

Finally, it could be the case that the region of determinacy is completely unknown to the researcher. For a given draw of the parameter θ, the researcher would like to make draws of α smaller or greater than 1 with equal probabilities. In this case, the researcher could use a uniform distribution over the interval $[0, 2]$ or any interval around 1 as a prior distribution. Also, note that the prior distribution does not necessarily have to be continuous. A discrete probability distribution that allows to make draws of α to be either equal 0.5 or 1.5 could also be specified as a prior. In this context, the efficiency of the algorithm would also be improved if it were to be designed as follows. If for a given draw of θ and α the augmented representation in (5) does not have a solution, the algorithm should be coded as to make a new draw of α' equal to the inverse of the earlier draw α. Also, in Section 5.2 we consider an application of this case for the NK model in Lubik and Schorfheide (2004).

3 Methodology

Given the general class of LRE models described in Sims (2001), this paper proposes an augmented representation which embeds the solution for the model under both determinacy and indeterminacy. In particular, the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) or equivalently Farmer et al. (2015).

In the following, we generalize the intuition built in the previous Section. Consider the following LRE model

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{6}$$

where X_t is the vector of endogenous variables, ε_t is a vector of exogenous shocks, η_t collects the one-step ahead forecast errors for the expectational variables of the system and θ is a vector of parameters. Consider the case in which the model is characterized by m degrees of indeterminacy for a region of the parameter space[2]. The proposed methodology appends to the original LRE model in (6) the following system of m equations

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \alpha_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \alpha_m \end{bmatrix} \tag{7}$$

where the vectors $\{\omega_t, \nu_t, \eta_{f,t}\}$ have dimension $m \times 1$, the vector $\eta_{f,t}$ is a subset of the endogenous shocks[3] and the elements α_i of the $m \times m$ diagonal matrix Φ belong to the interval $(-1, 1)$ when the model is determinate or they are outside the unit circle under indeterminacy. The equations in (7) are autoregressive processes whose innovations are linear combinations of a vector of sunspot shocks, ν_t , and a subset of forecast errors, $\eta_{f,t}$, where $E_{t-1}(\nu_t) = E_{t-1}(\eta_{f,t}) = 0$.

The proposed methodology works as in the example considered in the previous Section. Under indeterminacy the Blanchard-Kahn condition is not satisfied and, given that the system is characterized by m degrees of indeterminacy, then it is necessary to introduce m explosive roots to solve the model using standard solution algorithms. Equivalently, the (absolute value of the) diagonal elements of the matrix Φ are outside the unit circle.

[2]Denoting by n the minimum number of unstable roots of a LRE model, the maximum degrees of indeterminacy are defined as $m = p - n$. This implies that when the minimum number of unstable roots of a model is unknown, then m coincides with number of expectational variables p. This represents the largest possible numberof degrees of indeterminacy. However, in many cases the researcher knows that the degrees of indeterminacy are $m' < m$.

[3]To solve a LRE model under indeterminacy, Farmer et al. (2015) redefine a subset of endogenous shocks, $\eta_{f,t}$, as new fundamental disturbances. Importantly, they show that the choice of which forecast errors should be redefined as fundamental is irrelevant. Given the equivalence with our method proven in Theorem 1, it is without loss of generality that we consider a subset of forecast errors, $\eta_{f,t}$, to be included in (7).

On the other hand, under determinacy the (absolute value of the) diagonal elements of the matrix Φ are inside the unit circle, since the number of explosive roots of the original LRE model in (6) already equals the number of expectational variables in the model. Importantly, as shown in the analytic example, the block structure of the proposed methodology guarantees that the autoregressive processes, ω_t, do not affect the solution for the endogenous variables, X_t.

Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)'$ and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the system in (6) and (7) can be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t, \tag{8}$$

where

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1(\theta) & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n(\theta) & \Pi_f(\theta) \\ 0 & -\mathbf{I} \end{bmatrix},$$

and the matrix Π in (6) is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$ without loss of generality. The following theorem is proved.

Theorem 1 *Let ξ^{SIMS} be the equilibrium in model (6) under determinacy using Sims (2001) and ξ^{LS} the equilibrium in model (6) under indeterminacy using the methodology of Lubik and Schorfheide (2003). Let ξ^{BN} be the equilibrium in the augmented representation (8). Then, ξ^{BN} is equivalent to ξ^{SIMS} when model (6) is determinate and equivalent to ξ^{LS} when model (6) is indeterminate.*

Proof. See Section 7.1. ■

Given this result and the equivalence between the methodology of Lubik and Schorfheide (2003, 2004) with Farmer et al. (2015), the following corollary holds.

Corollary 2 *Let ξ^{FKN} be the equilibrium in model (6) under indeterminacy using Farmer et al. (2015). Then, ξ^{BN} is also equivalent to ξ^{FKN} under indeterminacy.*

Moreover, the following two considerations support Corollary 3 below, which describes a relevant result on the likelihood function of the augmented representation. First, as shown in the proof and in the analytic example described in Section 4, the reduced form of the augmented representation has a block structure which ensures that the endogenous variables, X_t, are not a function of the autoregressive processes, ω_t. Second, note that the appended autoregressive processes, ω_t, have no economic interpretation and therefore have no relation with the observable variables used in a measurement equation. These two considerations imply that the parameters of the matrix Φ introduced with the augmented representation are not identified[4]. Corollary 3 then follows[5].

Corollary 3 *The likelihood function associated with the newly defined vector of endogenous variables, $\hat{X}_t$, does not depend on the additional parameters included in the augmented representation, Φ, and is equivalent to the likelihood function associated with the endogenous variables, X_t.*

Theorem 1 proves that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). This theoretical result is crucial for the estimation exercise conducted in Section 5. The augmented representation guarantees that the Metropolis-Hastings algorithm explores the entire domain of the parameter space θ. Given the acceptance rule, the algorithm compares the posterior kernel obtained from draws in both regions and therefore Bayesian inference conducted using the proposed representation delivers posterior distributions which could lie over the entire parameter space.

[4]In particular, we refer to the definition of identification of structural parameters provided by Iskrev (2009).

[5]Notice that Corollary 3 holds when the augmented representation has a unique solution. This happens in two cases. First, values of the structural parameters θ which guarantee determinacy in the original LRE model should be combined with values for α_i in the matrix Φ whose absolute value lies within the unit circle. Second, values of the structural parameters θ for which the original model is indeterminate should be combined with (absolute) values of α_i outside the unit circle.

The logic of our approach is closely related to the rationale behind the design of the Metropolis-Hastings algorithm. In the latter, the choice of the scale parameter (which scales the inverse of the Hessian matrix evaluated at the posterior mode) represents a feature of the algorithm aimed at ensuring an efficient exploration of the posterior distribution around the posterior mode. Similarly, the acceptance rule of the algorithm allows for the possibility of accepting with a strictly positive probability a draw of parameters whose posterior kernel is lower than the one of the most recently accepted draw. Following the same spirit, our methodology ensures that the algorithm does not just visit either of the two regions of the parameter space separately. Rather, using the proposed augmented representation, the algorithm explores the entire parameter space.

4 Analytic example

This section considers the canonical NK model to provide an analytic example of the theoretical result in Theorem 1. Let

$$x_t = E_t(x_{t+1}) - \tau(R_t - E_t(x_{t+1})) \quad (9)$$

$$\pi_t = \beta E_{t-1}(\pi_{t+1}) + \kappa x_t \quad (10)$$

$$R_t = \psi \pi_t + \varepsilon_{R,t} \quad (11)$$

$$\eta_{1,t} = x_t - E_{t-1}(x_t) \quad (12)$$

$$\eta_{2,t} = \pi_t - E_{t-1}(\pi_t) \quad (13)$$

where equations (9)∼(11) represent the dynamic IS curve, the NK Phillips curve and a monetary policy reaction function, respectively. The variable x_t represents log deviations of GDP from a trend path and π_t and R_t are log deviations from the steady state level of inflation and the nominal interest rate. The one-step ahead forecast errors for the deviations of output from its trend and of inflation from its steady state are defined in (12) and (13). This model can be expressed in matrix form as

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \quad (14)$$

where $X_t = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}))'$, $\varepsilon_t = (\varepsilon_{R,t})$ and $\eta_t = (\eta_{1,t}, \eta_{2,t})'$.

It is well known that the region of determinacy is associated with an aggressive response of the monetary authority to changes in inflation, a condition satisfied when $|\psi| > 1$. Alternatively, the equilibrium is indeterminate when the monetary policy is "passive", that is $0 < |\psi| < 1$. In the latter case, there is one degree of indeterminacy ($m = 1$) since there are two forecast errors while the system is characterized by only one unstable root[6]. Given that $m = 1$, the proposed methodology consists in appending to the original LRE model in (14) the following equation[7]

$$\omega_t = \alpha\omega_{t-1} + \nu_t - \eta_{2,t}. \tag{15}$$

To provide the intuition, consider $\alpha \equiv 1/|\psi|$ without loss of generality. When the monetary authority is "passive", indeterminacy arises and the Blanchard-Kahn condition is not satisfied. Our representation augments the original system (14) with the explosive[8] autoregressive process in (15). The augmented representation not only mechanically satisfies the Blanchard-Kahn condition, but, as proven in Theorem 1, it describes all the set of equilibria which would be equivalently obtained using the methodology of Lubik and Schorfheide (2003, 2004) or Farmer et al. (2015). Alternatively, when the monetary policy adopts an "active" stance, the original system is determinate and the autoregressive process is stationary (i.e. $0 < \alpha < 1$), thus satisfying the Blanchard-Kahn condition also under determinacy. Importantly, as shown both in this example and more generally in the proof of Theorem 1, the block structure of the augmented representation ensures that the endogenous variables contained in the vector X_t are not a function of the process ω_t for both regions of the parameter space. While the remaining part of this section shows an analytic example of the results of Theorem 1, Section 5 generalizes the intuition and performs an estimation exercise for the most common case in which the analytic form of the region of determinacy is *unknown*.

We now show the equivalence result of Theorem 1 for the NK model described by (9) $\sim$ (13). As described in Section 3, the proposed methodology defines a new vector of en-

[6]As shown in Section 7.2, one of the roots of the system is always outside the unit circle. This implies that the maximum degree of indeterminacy is $m = 1$ and that we append only one auxiliary autoregressive process.

[7]Given the results of Farmer et al. (2015) highlighted in footnote 2, it is without loss of generality that we consider the case when $\eta_{f,t} = \eta_{2,t}$.

[8]Note that under indeterminacy $0 < |\psi| < 1$, which implies $\alpha > 1$.

dogenous variables $\hat{X}_t \equiv (X_t, \omega_t)' = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)'$ and a newly defined vector of exogenous shocks as $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)' = (\varepsilon_{R,t}, \nu_t)'$. The system in (14) and (15) can then be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{16}$$

The representation in (16) always delivers a solution of the following form under both determinacy and indeterminacy,

$$\hat{X}_t = \hat{T}\hat{X}_{t-1} + \hat{R}\hat{\varepsilon}_t. \tag{17}$$

In particular, under determinacy, the matrices $\hat{T}$ and $\hat{R}$ are

$$\hat{T} = \begin{pmatrix} \mathbf{0_{4\times 4}} & \mathbf{0_{4\times 1}} \\ \mathbf{0_{1\times 4}} & \alpha \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} -\frac{\tau}{1+\kappa\psi\tau} & \\ -\frac{\tau\kappa}{1+\kappa\psi\tau} & \mathbf{0_{4\times 1}} \\ 0 & \\ 0 & \\ \frac{\tau\kappa}{1+\kappa\psi\tau} & 1 \end{pmatrix}. \tag{18}$$

Therefore, the output gap and the deviations of inflation from its steady state level respond to the monetary policy shock only, and the sunspot shock does not affect the model dynamics as expected under a determinate equilibrium. Also, the matrix $\hat{T}$ reveals that, even if ω_t still follows a stationary autoregressive process, it does not affect the law of motion for the remaining endogenous variables.

On the other hand, under indeterminacy the matrices $\hat{T}$ and $\hat{R}$ are

$$\hat{T} = \begin{pmatrix} \mathbf{0_{4\times 3}} & G_{4\times 1} & \mathbf{0_{4\times 3}} \\ & & \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} H_{4\times 2} \\ \\ \mathbf{0_{1\times 2}} \end{pmatrix} \tag{19}$$

where

$$G_{4\times 1} \equiv \begin{pmatrix} -\frac{a_2}{2\kappa} \\ 1 \\ -\frac{a_1 a_2}{4\beta\kappa} \\ \frac{a_1}{2\beta} \end{pmatrix} \qquad H_{4\times 2} \equiv \begin{pmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \\ 0 & 1 \\ -\frac{\tau a_2}{a_3} & -\frac{a_2(1+\kappa\tau\psi)}{a_3\kappa} \\ \frac{2\kappa\tau}{a_3} & -\frac{2(1+\kappa\tau\psi)}{a_3} \end{pmatrix} \tag{20}$$

and $a_1 = (\beta - \phi + \kappa\tau + 1)$, $a_2 = (a_1 - 2)$, $a_3 = (a_1 + 2\phi)$ and $\phi = [(1 + \beta + \kappa\tau)^2 - 4\beta(1 + \kappa\tau\psi)]^{-1/2}$. From the matrices in (19) and recalling that $\hat{X}_t = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)$', two relevant comments can be made. First, under indeterminacy the endogenous variables are also affected by the sunspot shock. This is clear when looking at the form of $\hat{R}$. Second, comparing the form of the matrices under determinacy in (18) with those under indeterminacy in (19), it is evident that the propagation mechanism differs according to which region of the parameters is considered.

Section 4.1 and 4.2 explain the details behind the result for which the solutions in (18) are equivalent to those obtained under determinacy using the solution method of Sims (2001) as well as the solutions in (19) to those under indeterminacy when solving the model using the representation of Farmer et al. (2015)[9].

4.1 Determinacy

This section clarifies the details for the equivalence of the solutions which are obtained in the *determinacy* region of the parameter space when using the following two representations:

a) The matrix representation of the LRE model in (14) and reported here as equation (21)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\varepsilon_t + \Pi\eta_t \tag{21}$$

b) The proposed augmented representation in (16) and reported here as equation (22)

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{22}$$

Representations a) and b) deliver the equilibrium conditions reported in Table 2[10].

[9] While the methodologies of Lubik and Schorfheide (2003) and Farmer et al. (2015) are equivalent, the latter provides a more straightforward comparison with the method proposed here.

[10] Details on the derivation of the solutions are provided in the Section 7.2.

Table 2: Equivalence of solutions under determinacy

Sims (2001)	Bianchi-Nicoló
$E_t(x_{t+1}) = E_t(\pi_{t+1}) = 0$	$E_t(x_{t+1}) = E_t(\pi_{t+1}) = 0$
$\eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1\\ \kappa\end{bmatrix}\varepsilon_{R,t}$	$\eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 & 0\\ \kappa & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}$
$\begin{pmatrix}x_t\\ \pi_t\end{pmatrix} = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1\\ \kappa\end{bmatrix}\varepsilon_{R,t}$	$\begin{pmatrix}x_t\\ \pi_t\end{pmatrix} = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 & 0\\ \kappa & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}$
-	$\omega_t = \alpha\omega_{t-1} + \begin{bmatrix}\frac{\tau\kappa}{1+\kappa\tau\psi} & 1\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}$

where $\alpha \equiv 1/|\psi| < 1$. Comparing the obtained solutions, it is clear that they are equivalent. While our augmented representation potentially allows for the sunspot shock to affect the model dynamics, the coefficients which determine its impact on the endogenous variables equal zero. Moreover, the dynamics of the endogenous variables $X_t = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}))'$ are not affected by the autoregressive process ω_t, which therefore constitutes a separate block.

4.2 Indeterminacy

Under indeterminacy, the Blanchard-Kahn condition is not satisfied and to solve the model we use the solution method suggested by Farmer et al. (2015). Hence, the solutions which are compared in this section derive from the following two representations:

c) The matrix representation of the LRE model using the methodology of Farmer et al. (2015) when the forecast error for the deviations of inflation from its steady state,

$\eta_{2,t}$, is included as newly defined fundamental shock. Given the partition of the matrix Π in (14) as $\Pi = [\Pi_n \quad \Pi_f]$, then

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi_f \tilde{\varepsilon}_t + \Pi_n \eta_{1,t} \tag{23}$$

where $\tilde{\varepsilon}_t \equiv (\varepsilon_t, \eta_{2,t})'$ and $\Psi_f \equiv [\Psi \quad \Pi_f]$.

d) The proposed augmented representation, which is identical to the representation b) in Section 4.1

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t. \tag{24}$$

The equilibrium conditions obtained using representations c) and d) are reported in the following table[11].

[11] Details on the derivation of the solutions are provided in the Section 7.3.

Table 3: Equivalence of solutions under indeterminacy	
Farmer et al. (2015)	Bianchi-Nicoló
$E_t(x_{t+1}) = -\frac{a_2}{2\kappa}E_t(\pi_{t+1})$	$E_t(x_{t+1}) = -\frac{a_2}{2\kappa}E_t(\pi_{t+1})$
$\eta_{1,t}= \begin{bmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \end{bmatrix} \begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix}$	$\eta_t= \begin{bmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix}$
$\begin{pmatrix} x_t \\ \pi_t \\ E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = G_{4\times1}E_{t-1}(\pi_t) + H_{4\times2} \begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix}$	$\begin{pmatrix} x_t \\ \pi_t \\ E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = G_{4\times1}E_{t-1}(\pi_t) + H_{4\times2} \begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix}$
-	$\omega_t= 0$

Note: we show the equivalence with Farmer et al. (2015) since their representation is easier to compare with our results. However, their methodology is equivalent to Lubik and Schorfheide (2003) and therefore ours is too.

where the matrices $G_{4\times1}$ and $H_{4\times2}$ and the coefficients a_1, a_2 and a_3 are defined in (20). To understand the equivalence result, it is relevant to compare the linear restrictions imposed on the vector of forecast errors using the augmented representation. In particular, note that our methodology imposes the restriction $\eta_{2,t} = \nu_t$. Thus, the solution to the augmented representation sets restrictions on the forecast error, $\eta_{2,t}$, (which has been redefined as fundamental using the methodology of Farmer et al. (2015)) such that it corresponds to the sunspot shock, ν_t. Also, to guarantee a bounded solution, restrictions are imposed such that the autoregressive process ω_t equals zero at any time t. Therefore, the solutions for the two alternative representations are equivalent.

5 Application to Lubik-Schorfheide's model

While the previous section provides an analytic example clarifying the equivalence result proven in Theorem 1, this section highlights its importance for the estimation of LRE models using Bayesian inference. We consider the three-equation NK model of Lubik and Schorfheide (2004) and we conduct the following exercise. We run two simulations of the model for parameter values which lie in the region of the parameter space associated with determinacy and indeterminacy. Given the two simulations, Section 5.1 assumes that the region of determinacy is *known* and provides an intuition for the relevance of the proposed approach relative to the common practice of estimating the model for the two regions of the parameter space separately. In Section 5.2, we then assume that the region of determinacy is *unknown* and show that our methodology estimates a LRE model over the entire parameter space, thus using Bayesian inference to study whether the model is characterized by determinacy or indeterminacy.

We consider the NK model estimated by Lubik and Schorfheide (2004). The model is described by equations (25)$\sim$(30) and consists of a dynamic IS curve

$$x_t = E_t(x_{t+1}) - \tau(R_t - E_t(\pi_{t+1})) + g_t, \tag{25}$$

a NK Phillips curve

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa(x_t - z_t), \tag{26}$$

and a Taylor rule[12],

$$R_t = \rho_R R_{t-1} + (1-\rho_R)[\psi_1 \pi_t + \psi_2(x_t - z_t)] + \varepsilon_{R,t}. \tag{27}$$

The demand shock, g_t, and the supply shock, z_t, follow univariate AR(1) processes

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \tag{28}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{29}$$

where the standard deviations of the fundamental shocks $\varepsilon_{g,t}$, $\varepsilon_{z,t}$ and $\varepsilon_{R,t}$ are denoted by σ_g, σ_z and σ_R, respectively. We allow for the correlation between shocks (namely ρ_{gz}, ρ_{gR}

[12] The definition of the variables is the same as in Section 3.

and ρ_{zR}) to be nonzero. The rational expectation forecast errors are defined as

$$\eta_{1,t} = x_t - E_{t-1}\left[x_t\right], \qquad \eta_{2,t} = \pi_t - E_{t-1}\left[\pi_t\right]. \tag{30}$$

We define the vector of endogenous variables

$$X_t = \left(x_t, \pi_t, R_t, E_t\left(x_{t+1}\right), E_t\left(\pi_{t+1}\right), g_t, z_t\right)'$$

the vectors of fundamental shocks and non-fundamental errors,

$$\varepsilon_t = \left(\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}\right)', \qquad \eta_t = \left(\eta_{1,t}, \eta_{2,t}\right)'$$

and the vector of parameters

$$\theta = \left(\psi_1, \psi_2, \rho_R, \beta, \kappa, \tau, \rho_g, \rho_z, \sigma_g, \sigma_z, \sigma_R, \rho_{gz}, \rho_{gR}, \rho_{zR}\right)'.$$

This leads to the following representation of the model,

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t. \tag{31}$$

The LRE model in (31) is determinate when the following analytic condition is satisfied

$$|\psi^*| > 1, \tag{32}$$

where $\psi^* \equiv \psi_1 + \frac{(1-\beta)}{\kappa}\psi_2$. However, when the model is indeterminate, $0 < |\psi^*| < 1$, the system is characterized by one degree of indeterminacy ($m = 1$) since there are two expectational variables $\{E_t\left(x_{t+1}\right), E_t\left(\pi_{t+1}\right)\}$ and only one root outside the unit circle. The methodology we propose consists in augmenting the representation of the model in (31) with the autoregressive process

$$\omega_t = \alpha\omega_{t-1} + \nu_t - \eta_{2,t}. \tag{33}$$

Hence, we define a new vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)'$ and a newly defined vector of exogenous shocks as $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)' = (\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \nu_t)'$. The system in (31) and

(33) can then be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{34}$$

As in Lubik and Schorfheide (2004), the vector of observables, $\mathbf{y}_t = \{x_{obs,t}, \pi_{obs,t}, R_{obs,t}\}$, consists of

1. $x_{obs,t}$ the percentage deviations of (log) real GDP per capita from an HP-trend;
2. $\pi_{obs,t}$ the annualized percentage change in the Consumer Price Index for all Urban Consumers;
3. $R_{obs,t}$ the annualized percentage average Federal Funds Rate.

The measurement equations are described by

$$\mathbf{y}_t = \begin{bmatrix} 0 \\ \pi^* \\ \pi^* + r^* \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix} X_t.$$

where π^* and r^* are annualized steady-state inflation and real interest rates expressed in percentages. The discount factor, β is a function of the annualized real interest rate in steady-state r^* (i.e. $\beta = (1+r^*)^{-1/4}$). We then simulate the model under both determinacy and indeterminacy and Table 4 reports the parameter values used for the simulations.

Table 4: Parameter values for simulations

Parameter	Determinacy	Indeterminacy
ψ_1	2.1	0.7
ψ_2	0.17	0.17
ρ_R	0.60	0.60
π^*	4.28	4.28
r^*	1.13	1.13
κ	0.77	0.77
τ^{-1}	1.45	1.45
ρ_g	0.68	0.68
ρ_z	0.82	0.82
σ_R	0.23	0.23
σ_g	0.27	0.27
σ_z	1.13	1.13
ρ_{gz}	0.14	0.14
ρ_{gR}	0	0
ρ_{zR}	0	0
σ_ν	-	2.14
$\rho_{R\nu}$	-	-0.1
$\rho_{g\nu}$	-	0.06
$\rho_{z\nu}$	-	-0.87

The parameter values are those estimated by Lubik and Schorfheide (2004) for the Pre-Volcker period[13]. While under determinacy we set $\psi_1 = 2.1$ (thus guaranteeing $|\psi^*| > 1$), for the simulation under indeterminacy we impose $\psi_1 = 0.7$ for which $0 < |\psi^*| < 1$. Also, under indeterminacy we use the equivalence result in Farmer et al. (2015) to map the parameter estimates in Lubik and Schorfheide (2004) into values for the standard deviation of the sunspot shock and its correlation with the fundamental shocks.

Finally, Table 5 reports the prior distributions used for the estimation exercises in the

[13] Note that even if the estimates of Lubik and Schorfheide (2004) for the pre-Volcker period differ from those in the post-Volcker era, for the purpose of this paper changing the values assigned to the parameters which are *not* directly related to the analytic condition defining the region of determinacy is irrelevant.

following sections. The priors we consider are the same as in Lubik and Schorfheide (2004).

Table 5: Prior distribution for model parameters

Name	Range	Density	Mean	Std. Dev.	90% interval
ψ_1	R^+	*Gamma*	1.1	0.50	[0.43,2.03]
ψ_2	R^+	*Gamma*	0.25	0.15	[0.06,0.54]
ρ_R	$[0,1)$	*Beta*	0.50	0.20	[0.17,0.83]
π^*	R^+	*Gamma*	4.00	2.00	[1.35,7.75]
r^*	R^+	*Gamma*	2.00	1.00	[0.69,3.86]
κ	R^+	*Gamma*	0.50	0.35	[0.09,1.17]
τ^{-1}	R^+	*Gamma*	2.00	0.50	[1.25,2.88]
ρ_g	$[0,1)$	*Beta*	0.70	0.10	[0.52,0.85]
ρ_z	$[0,1)$	*Beta*	0.70	0.10	[0.52,0.85]
σ_R	R^+	*Inverse Gamma*	0.31	0.16	[0.14,0.60]
σ_g	R^+	*Inverse Gamma*	0.38	0.20	[0.17,0.74]
σ_z	R^+	*Inverse Gamma*	1.00	0.52	[0.47,1.95]
ρ_{gz}	[-1,1]	*Normal*	0.00	0.40	[-0.65,0.65]
ρ_{gR}	[-1,1]	*Normal*	0.00	0.40	[-0.65,0.65]
ρ_{zR}	[-1,1]	*Normal*	0.00	0.40	[-0.65,0.65]
σ_ν	R^+	*Inverse Gamma*	2	0.5	[1.33,2.92]
$\rho_{R\nu}$	[-1,1]	*Normal*	0.00	0.10	[-0.16,0.16]
$\rho_{g\nu}$	[-1,1]	*Normal*	0.00	0.10	[-0.16,0.16]
$\rho_{z\nu}$	[-1,1]	*Normal*	-0.70	0.10	[-0.86,0.54]

5.1 Known region of determinacy

In this section, we assume that the region of determinacy is *known* and provide an intuition for the relevance of the proposed approach relative to the common practice of estimating the model for the two regions of the parameter space separately. First, we show that our methodology is such that the optimization routine to compute the posterior mode is also performed over the entire parameter space, therefore increasing the likelihood of finding a global maximum over the parameter space. This is particularly relevant when considering that the posterior mode is a crucial object used for Bayesian inference. Second, we estimate the augmented representation under the assumption that the researcher knows the regions of determinacy and show that we successfully recover the true parameter values by estimating the model over the entire parameter space. The intuition behind this result also holds in Section 5.2, when the region of determinacy is assumed to be unknown.

In this section, we assume that the region of determinacy $|\psi^*| > 1$ is known and we set $\alpha \equiv 1/|\psi^*|$. This assumption implies that when the model is determinate, the autoregressive process is stable and the solution is equivalent to the solution of the original model (31). On the other hand, when the model is indeterminate (i.e. $0 < |\psi^*| < 1$), the autoregressive process is unstable, satisfying the Blanchard-Kahn condition. Given Theorem 1, the solution to the model is equivalent to the representation of the equilibrium in Lubik and Schorfheide (2004) or Farmer et al. (2015).

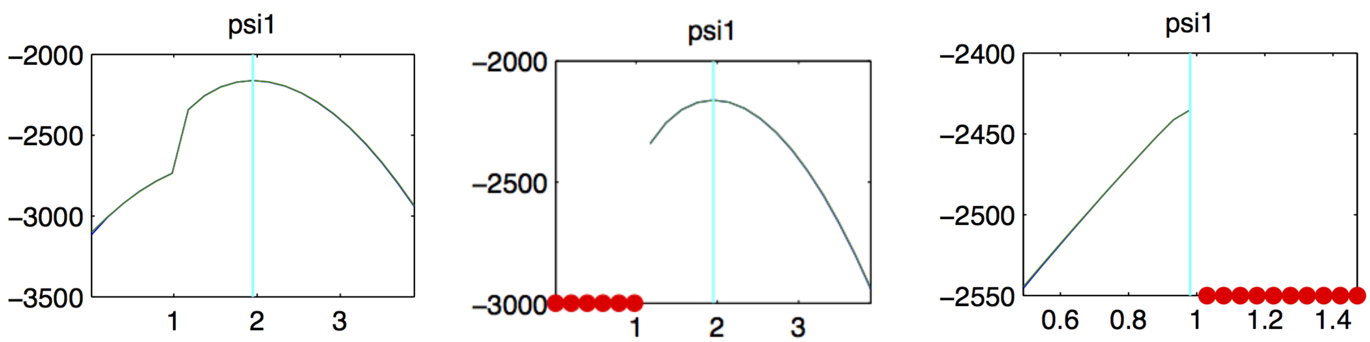
The assumption $\alpha \equiv 1/|\psi^*|$ enables to compute the posterior mode as an optimization routine over the *entire* parameter space. As indicated in An and Schorfheide (2007), the posterior mode is a crucial object used to compute the posterior distribution of the model parameters. Using Bayesian inference, the Metropolis-Hastings algorithm uses the posterior mode as a starting value for the algorithm as well as a scaled version of the inverse of the Hessian matrix evaluated at the posterior mode as the covariance matrix for the proposal distribution. Thus, "The algorithm constructs a Gaussian approximation around the posterior mode..." (An and Schorfheide (2007)), whose computation is pivotal to run the estimation procedure.

First, we consider the simulation of the model under determinacy and we compute the posterior mode of the model parameters using three different representations of the Lubik and Schorfheide's (2004) model. We consider the augmented representation proposed in this paper (34), the representation of the model under determinacy (31) and the representation of the model under indeterminacy using the methodology of Farmer et al. (2015)[14]. Figure 1 reports the posterior mode (vertical line) and how the posterior varies to changes

[14] As in section 4.2, we apply the methodology of Farmer et al. (2015) by redefining the forecast error for inflation, $\eta_{2,t}$, as fundamental shock, that is $\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi_f \tilde{\varepsilon}_t + \Pi_n \eta_{1,t}$, where $\tilde{\varepsilon}_t = (\varepsilon_{R,t}, \eta_{2,t})'$ and $\Pi = [\Pi_n \quad \Pi_f]$.

in the parameter ψ_1, while keeping the other structural parameters at their posterior mode estimates. While panel (a) considers the augmented representation, panel (b) and (c) report the plots for the representations under model determinacy and indeterminacy, respectively.

Figure 1: Posterior function and posterior modes (simulation under determinacy)



panel (a) panel (b) panel (c)

Note: Posterior function and posterior mode for the parameter ψ_1 for the augmented representation (panel (a)), the representation under determinacy (panel (b)) and under indeterminacy (panel (c)).

The red dots in panel b) and c) indicate parameter values for which, given the chosen model representation, the model could not be solved due to a violation of the Blanchard-Kahn condition. While in panel b) the model violates these conditions[15] for values of the parameter ψ_1 smaller than 1, panel c) shows that the representation of Farmer et al. (2015) does not allow to solve the model for values of ψ_1 greater than 1.

Figure 1 highlights the importance of computing the posterior mode over the entire parameter space (panel (a)) relative to some constrained regions. The optimization procedure of the posterior function delivers a substantially different posterior mode when the representation of the model under indeterminacy is used (panel (c)). Figure 1 also provides

[15]The violation of the Blanchard-Kahn conditions for values of ψ_1 close to 1 results from the values chosen for the simulation. Indeed, the term $\frac{(1-\beta)}{\kappa}\psi_2 \approx 0$, thus implying that the region of determinacy is approximated by the following condition $\psi^* \approx \psi_1 > 1$.

a graphical intuition of how Bayesian inference could be used in this context to extract information contained in the data. This same intuition also holds for the case when the analytic condition defining the region of determinacy is unknown (Section 5.2). The augmented representation guarantees that the Metropolis-Hastings algorithm explores the entire domain of the parameter space θ. Candidate parameter values are drawn from both the determinacy and the indeterminacy region, and the acceptance rule updates the proposal distribution from which subsequent draws are taken such that the posterior distribution of the model parameters could lie in either of the two regions.

We then estimate[16] the augmented representation using the data simulated under determinacy and Table 6 reports the mean and the 90% probability interval of the posterior distributions[17].

[16] Each estimation exercise performed in this paper use 500 observations from the simulated data. We then run two chains of 250,000 draws and we discard half of the draws. The acceptance ratio for all the estimations is between 25% and 33%.

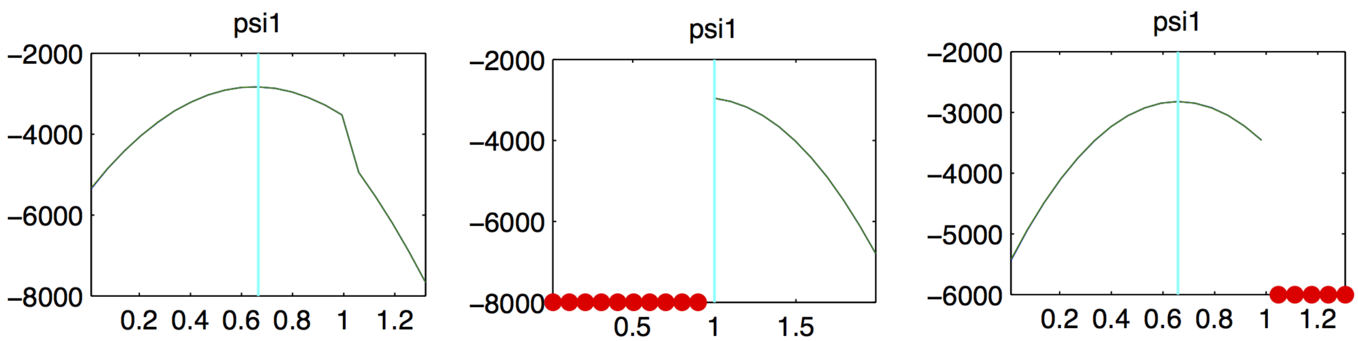
[17] Since the posterior estimates satisfy the analytic condition for determinacy, the endogenous variables, X_t, are not a function of the sunspot shock and we therefore do not report the estimates of the standard error of the sunspot shock and its correlation with the fundamental shocks.

Table 6: Posterior estimates, simulation under determinacy			
	True values	Posterior estimates	
		Mean	90% probability interval
ψ_1	2.1	1.92	[1.63,2.2]
ψ_2	0.17	0.38	[0.06,0.68]
ρ_R	0.60	0.61	[0.57,0.64]
π^*	4.28	4.40	[4.25,4.57]
r^*	1.13	1.29	[1.08,1.50]
κ	0.77	0.69	[0.40,0.99]
τ^{-1}	1.45	1.38	[1.11,1.66]
ρ_g	0.68	0.67	[0.62,0.72]
ρ_z	0.82	0.81	[0.76,0.85]
σ_R	0.23	0.22	[0.20,0.24]
σ_g	0.27	0.28	[0.23,0.32]
σ_z	1.13	1.12	[1.02,1.22]
ρ_{gz}	0.14	0.02	[-0.11,0.15]
ρ_{gR}	0	0.05	[-0.05,0.17]
ρ_{zR}	0	0.02	[-0.08,0.13]

The posterior estimates indicate that the true parameter values can be recovered under the augmented representation. All the parameter values used to simulate the model fall within the 90% probability intervals of the posterior distributions.

Secondly, we perform the same estimation exercise using the simulation of the model under indeterminacy. Figure 2 plots how the posterior varies with ψ_1 while the other parameters are constant at their posterior mode estimates. As before, the vertical line reports the corresponding posterior mode. Figure 2 provides similar evidence as in Figure 1. Panel (a), panel (b) and panel (c) refer to the augmented representation and the representation under determinacy and under indeterminacy, respectively.

Figure 2: Posterior function and posterior modes (simulation under indeterminacy)



panel (a) panel (b) panel (c)

Note: Posterior function and posterior mode for the parameter ψ_1 for the augmented representation (panel (a)), the representation under determinacy (panel (b)) and under indeterminacy (panel (c)).

Contrary to the alternative representations, the proposed augmented representation ensures to run the optimization routine to compute the posterior mode over the entire parameter space. Reasonably, the shape of the maximized functions in panel (a) of Figure 2 mirrors the plot of panel (a) in Figure 1.

As for the simulation under determinacy, we then estimate the model using the data simulated under indeterminacy and Table 7 reports the mean and 90% probability of the posterior distribution.

Table 7: Posterior estimates, simulation under indeterminacy

	True values	Posterior estimates	
		Mean	90% probability interval
ψ_1	0.7	0.64	[0.56,0.72]
ψ_2	0.17	0.24	[0.04,0.43]
ρ_R	0.60	0.61	[0.58,0.63]
π^*	4.28	4.40	[2.80,6.08]
r^*	1.13	1.14	[0.66,1.62]
κ	0.77	0.75	[0.52,1.00]
τ^{-1}	1.45	1.44	[1.16,1.76]
ρ_g	0.68	0.68	[0.56,0.81]
ρ_z	0.82	0.81	[0.77,0.85]
σ_R	0.23	0.22	[0.21,0.24]
σ_g	0.27	0.26	[0.16,0.36]
σ_z	1.13	1.14	[0.91,1.37]
ρ_{gz}	0.14	0.08	[-0.05,0.21]
ρ_{gR}	0	0.02	[-0.12,0.17]
ρ_{zR}	0	-0.03	[-0.13,0.07]
σ_ν	2.14	2.08	[1.97,2.18]
$\rho_{R\nu}$	-0.1	-0.04	[-0.16,0.07]
$\rho_{g\nu}$	0.06	0.11	[-0.01,0.25]
$\rho_{z\nu}$	-0.87	-0.88	[-0.94,-0.81]

Also in this case, we recover the true parameter values by estimating the proposed augmented representation. Since the posterior estimates indicate that the model is characterized by indeterminacy, we report the standard error of the sunspot shock, σ_ν, and its covariance with the fundamental shocks (i.e. $\rho_{\nu R}, \rho_{\nu g}, \rho_{\nu z}$).

5.2 Unknown region of determinacy

In this section, we assume that the region of determinacy, $|\psi^*| > 1$, is *unknown* and we show that the intuition provided in the previous section still holds, guaranteeing that the proposed methodology successfully recovers the true parameter values.

By considering this case, we show that our methodology applies to small-scale LRE models for which it is non-trivial to derive an analytic condition describing the region of determinacy as well as to medium- and large-scale LRE models that could potentially be characterized by indeterminacy. Our methodology allows the researcher to conduct Bayesian inference on the model parameters over the entire parameter space and to compute their posterior estimates which could potentially lie in both regions of determinacy and indeterminacy.

However, the assumption that the region of determinacy is unknown implies that it is no longer possible to impose $\alpha \equiv 1/|\psi^*|$. To ensure that the Metropolis-Hastings algorithm explores the entire parameter space, we assume a uniform distribution over the interval $(0, 2)$ as a prior distribution[18] for the parameter α. Equivalently, we assume that there is an equal probability of making draws of α from the interval $(0, 1]$ as well as from the interval $(1, 2)$. Draws of α from $(0, 1]$ combined with draws of the other parameters θ which satisfy the condition $|\psi^*| > 1$ ensure to solve the augmented representation under determinacy. Similarly, draws of α from $(1, 2)$ combined with draws of the other parameters of interest θ such that $0 < |\psi^*| < 1$ ensure to solve the proposed representation under indeterminacy.[19]

Importantly, the same intuition described in Section 5.1 still holds. The Metropolis-Hastings algorithm makes draws of α and θ which could solve the augmented representation under determinacy and indeterminacy and compare the posterior kernel obtained from draws in both regions. Given the acceptance rule specified for the algorithm, Bayesian inference conducted using the proposed representation delivers posterior estimates which could lie over the entire parameter space.

Having specified the prior for α, we estimate the augmented representation using the same two simulations of the data as in Section 5.1. We first estimate the augmented

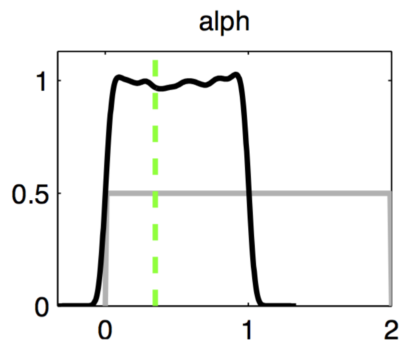
[18] The choice of the interval $(0, 2)$ is arbitrary. For any value $0 < a \leq 1$, it is sufficient to specify an interval $(-a + 1, a + 1)$ as the domain of the uniform distribution.

[19] The virtue of using a continuous distribution for α and treating it as any other parameter of the model is that the algorithm can be easily implemented in Dynare. However, the efficiency of the algorithm could be improved by using a discrete distribution for α given that the only thing that matters is if this parameter is inside or outside the unit circle. Furthermore, the MCMC algorithm could be modified to allow for the possibility that whenever the augmented model does not have a solution, the value of α is flipped. We consider this more efficient algorithm in the appendix.

representation of the model using the data simulated under determinacy and the same prior distributions reported in Table 5.

The posterior distribution for the parameter α is plotted in Figure 3. Two remarks should be made. First, more than 90% of the mass of the posterior distribution is distributed over the interval $(0, 1)$, thus providing evidence that the Metropolis-Hastings algorithm explores the entire parameter space and successfully recovers the information contained in the simulated data about model determinacy[20]. Second, the posterior distribution approximates a uniform distribution over the same interval. This result is in line with the non-identifiability of the parameter α on which Corollary 3 relies.

Figure 3: Posterior distribution of parameter α (simulation under determinacy)



Note: the grey line represents the prior distribution for the parameter α. The black line is the posterior distribution. The dashed green line is the posterior mode.

The posterior mean and 90% probability intervals of the parameters are reported in Table 8. Using the proposed augmented representation, the estimation procedure recovers the true parameter values and the (mean of the) posterior distributions indicate that the condition for determinacy is satisfied, that is $|\psi^*| > 1$. Also, as mentioned above, note that more than 90% of the mass of the posterior distribution of α lies in the interval $(0, 1)$.

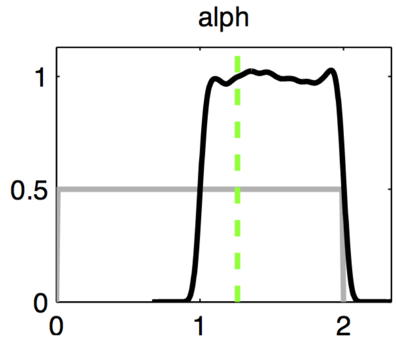
[20]Note that, while in Figure 3 it seems that a mass of the posterior distribution is below 0, we verified that the lowest value assumed by α is strictly positive, even if very close to 0.

Table 8: Posterior estimates, simulation under determinacy			
	True values	Posterior estimates	
		Mean	90% probability interval
α	-	0.5	[0.08,0.98]
ψ_1	2.1	1.92	[1.65,2.21]
ψ_2	0.17	0.38	[0.06,0.67]
ρ_R	0.60	0.61	[0.58,0.64]
π^*	4.28	4.40	[4.25,4.57]
r^*	1.13	1.29	[1.07,1.49]
κ	0.77	0.69	[0.40,0.97]
τ^{-1}	1.45	1.38	[1.11,1.65]
ρ_g	0.68	0.67	[0.62,0.73]
ρ_z	0.82	0.81	[0.77,0.85]
σ_R	0.23	0.22	[0.21,0.24]
σ_g	0.27	0.28	[0.23,0.33]
σ_z	1.13	1.12	[1.02,1.22]
ρ_{gz}	0.14	0.02	[-0.12,0.15]
ρ_{gR}	0	0.06	[-0.05,0.17]
ρ_{zR}	0	0.02	[-0.08,0.14]

The estimation of the augmented representation using simulated data under indeterminacy delivers a mirrored posterior distribution for the parameter α (Figure 4). In this case, more than 90% of the probability mass is distributed over the interval $(1, 2)$ and the posterior distribution of α closely resembles a uniform distribution over the same interval due to its non-identifiability[21].

[21] Similarly to Figure 3, it seems that a mass of the posterior distribution in Figure 4 is above 2. However, we verified that the largest value assumed by α is strictly below 2.

Figure 4: Posterior distribution of parameter α (simulation under indeterminacy)



Note: the grey line represents the prior distribution for the parameter α. The black line is the posterior distribution. The dashed green line is the posterior mode.

We report the mean and the 90% probability interval of the posterior distributions in Table 9.

Table 9: Posterior estimates, simulation under indeterminacy			
	True values	Posterior estimates	
		Mean	90% probability interval
α	-	1.50	[1.09,1.99]
ψ_1	0.7	0.65	[0.58,0.72]
ψ_2	0.17	0.24	[0.04,0.43]
ρ_R	0.60	0.62	[0.58,0.63]
π^*	4.28	4.33	[2.64,5.99]
r^*	1.13	1.17	[0.68,1.67]
κ	0.77	0.75	[0.53,0.96]
τ^{-1}	1.45	1.42	[1.12,1.70]
ρ_g	0.68	0.69	[0.57,0.81]
ρ_z	0.82	0.81	[0.77,0.85]
σ_R	0.23	0.22	[0.21,0.24]
σ_g	0.27	0.26	[0.15,0.36]
σ_z	1.13	1.14	[0.91,1.35]
ρ_{gz}	0.14	0.11	[-0.01,0.24]
ρ_{gR}	0	0.02	[-0.12,0.17]
ρ_{zR}	0	0.01	[-0.04,0.07]
σ_ν	2.14	2.08	[1.97,2.18]
$\rho_{R\nu}$	-0.1	0.00	[-0.17,0.16]
$\rho_{g\nu}$	0.06	0.00	[-0.17,0.16]
$\rho_{z\nu}$	-0.87	-0.87	[-0.94,-0.81]

Also in this case, we recover the true parameter values by estimating the augmented representation.

6 Conclusions

In this paper, we propose a generalized approach to solve and estimate LRE models over the entire parameter space. Our approach accommodates both cases of determinacy and indeterminacy and it does not require the researcher to know the analytic condition describing the region of determinacy or the degrees of indeterminacy.

When a LRE model is characterized by m degrees of indeterminacy, our approach augments it by appending m autoregressive processes whose innovations are linear combinations of a subset of endogenous shocks and a vector of newly defined sunspot shocks. The resulting augmented representation embeds both the solution which is obtained under determinacy using standard solution methods and those delivered by solving the model under indeterminacy using the approach of Lubik and Schorfheide (2003) and equivalently Farmer et al. (2015). We provide an analytical example for the theoretical result using a canonical NK model.

We finally apply our methodology to the NK model in Lubik and Schorfheide (2004). We simulate two series of data under the assumption of model determinacy and indeterminacy and we then estimate our augmented representation for both cases in which the region of determinacy is known or unknown to the researcher. The results of the estimation exercise indicate that our methodology successfully recovers the true parameter values.

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7 Appendix

7.1 Appendix A

7.1.1 Determinacy

This section shows the equivalence of the solutions obtained for a LRE model under determinacy using the proposed augmented representation and the standard solution algorithm in Sims (2001). Table 10 reports the equations which describe the solution for both methodologies and shows the equivalence.

Table 10: Equivalence of solutions under determinacy

Sims (2001)	Bianchi-Nicoló
$\xi_{2,t} \equiv Z_2 X_t = 0$	$\xi_{2,t} \equiv Z_2 X_t = 0$
$\eta_t = -\left(Q_2 \Pi\right)^{-1}\left(Q_2 \Psi\right)\varepsilon_t$	$\eta_t = -\left(Q_2 \Pi\right)^{-1}\left(Q_2 \Psi\right)\varepsilon_t$
$S_{11}\xi_{1,t} = T_{11}\xi_{1,t-1} + Q_1(\Psi\varepsilon_t + \Pi\eta_t)$	$S_{11}\xi_{1,t} = T_{11}\xi_{1,t-1} + Q_1(\Psi\varepsilon_t + \Pi\eta_t)$
-	$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}.$

In the following, we explicitly derive the equations reported in Table 10.

Canonical solution

Consider the LRE model in (6) and reported below as (35)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\varepsilon_t + \Pi\eta_t. \tag{35}$$

Under determinacy, Sims (2001) shows how to write the model in the form

$$SZ'X_t = TZ'X_{t-1} + Q\Psi\varepsilon_t + Q\Pi\eta_t,$$

where $\Gamma_0 = Q'SZ'$ and $\Gamma_1 = Q'TZ'$ is the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$ and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \qquad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad Z' = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}.$$

Consider the system of equations in the second block

$$S_{22}\xi_{2,t} = T_{22}\xi_{2,t-1} + Q_2(\Psi\varepsilon_t + \Pi\eta_t). \tag{36}$$

where $\xi_{2,t} \equiv Z_2 X_t$. Since the second block contains the explosive roots of the system, then the following conditions are imposed

$$\xi_{2,0} = 0 \tag{37}$$

$$Q_2(\Psi\varepsilon_t + \Pi\eta_t) = 0 \tag{38}$$

The number of explosive roots of the LRE model defines the number of equations in (36). Under determinacy, the number of explosive roots equals the number of forecast errors, η_t. Hence, equation (38) imposes the following linear restrictions on the forecast errors as a function of the fundamental shocks.

$$\eta_t = -\left(Q_2\Pi\right)^{-1}\left(Q_2\Psi\right)\varepsilon_t. \tag{39}$$

The solution for the endogenous variables, X_t, is obtained by combining these restrictions with the system of equations in the first block,

$$S_{11}\xi_{1,t} = T_{11}\xi_{1,t-1} + Q_1(\Psi\varepsilon_t + \Pi\eta_t). \tag{40}$$

Using the algorithm by Sims (2001), the solution of the LRE model in (2.1) is described by equations (37), (39) and (40).

Augmented representation

Consider the same LRE model as in (35). The proposed representation augments (35) with the following system of m equations (where m corresponds to the degree of indeterminacy of the model)

$$\omega_t = \Phi \omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \alpha_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \alpha_m \end{bmatrix}$$

where Φ is a $m \times m$ diagonal matrix whose elements α_i belong to the interval $[0,1]$ when the LRE model is determinate. Defining $\hat{X}_t \equiv (X_t, \omega_t)'$, the augmented representation takes the form

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t, \tag{41}$$

where $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$,

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix},$$

and the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$ without loss of generality. Since the Blanchard-Kahn condition is still satisfied, the solution algorithm by Sims (2001) can be used for the augmented representation and the QZ decomposition takes the form

$$\hat{S} \hat{Z}' \hat{X}_t = \hat{T} \hat{Z}' \hat{X}_{t-1} + \hat{Q} \hat{\Psi} \hat{\varepsilon}_t + \hat{Q} \hat{\Pi} \eta_t,$$

where $\hat{\Gamma}_0 = \hat{Q}' \hat{S} \hat{Z}'$ and $\hat{\Gamma}_1 = \hat{Q}' \hat{T} \hat{Z}'$ is the QZ decomposition of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and

$$\hat{S} = \begin{bmatrix} S_{11} & \mathbf{0} & S_{12} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_{22} \end{bmatrix}, \qquad \hat{T} = \begin{bmatrix} T_{11} & \mathbf{0} & T_{12} \\ \mathbf{0} & \Phi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T_{22} \end{bmatrix}, \qquad \hat{Z}' = \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Z_2 & \mathbf{0} \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Q_2 & \mathbf{0} \end{bmatrix}.$$

Note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}', \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z', Q\}$ obtained with the canonical solution.

Since under determinacy the diagonal elements of the matrix Φ are within the unit circle, then the explosive roots are grouped in the third block. Recalling the definition of the matrices $\hat{\Psi}$ and $\hat{\Pi}$, the system of equations in the third block is

$$S_{22}\xi_{2,t} = T_{22}\xi_{2,t-1} + Q_2(\Psi\varepsilon_t + \Pi\eta_t).$$

where $\xi_{2,t} \equiv Z_2 X_t$ as for the canonical solution. To eliminate the influence of the explosive roots of the system, then the following conditions are imposed

$$\xi_{2,0} = 0 \quad (42)$$

$$Q_2(\Psi\varepsilon_t + \Pi\eta_t) = 0 \quad (43)$$

Combining these restrictions with the system of equations in the first and second blocks, which are both characterized by stable dynamics,

$$S_{11}\xi_{1,t} = T_{11}\xi_{1,t-1} + Q_1(\Psi\varepsilon_t + \Pi\eta_t), \quad (44)$$

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}. \quad (45)$$

Finally, the solution in (42)$\sim$(45) obtained for the augmented representation of the LRE model is equivalent to the canonical solution described by equations (37), (39) and (40).

Two remarks should be made when comparing the two solutions. First, solving equation (43), the forecast errors are only a function of the exogenous shocks ε_t, and *not* of the sunspot shocks $\varepsilon_{\omega,t}$. Plugging the solution for the forecast errors in (44), it is therefore clear that the endogenous variables of the LRE model in (35), X_t, do not respond to sunspot shocks either, as expected under determinacy. Second, (44) and (45) indicate that under determinacy the appended system of equations constitutes a separate block, which does not affect the dynamics of the endogenous variables, X_t.

7.1.2 Indeterminacy

This section shows the equivalence of the solutions obtained for a LRE model under indeterminacy using the proposed augmented representation and the methodology of Lubik and Schorfheide (2003, 2004).

Lubik and Schorfheide (2003)

Consider the LRE model in (35) and reported below as (46)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \tag{46}$$

Applying the QZ decomposition used in Sims' algorithm, it results

$$S^* (Z^*)' X_t = T^* (Z^*)' X_{t-1} + Q^* \Psi \varepsilon_t + Q^* \Pi \eta_t,$$

where $\Gamma_0 = (Q^*)' S^* (Z^*)'$ and $\Gamma_1 = (Q^*)' S^* (Z^*)'$ is the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$ under indeterminacy and

$$S^* = \begin{bmatrix} S_{11}^* & S_{12}^* \\ 0 & S_{22}^* \end{bmatrix}, \qquad T^* = \begin{bmatrix} T_{11}^* & T_{12}^* \\ 0 & T_{22}^* \end{bmatrix}, \qquad (Z^*)' = \begin{bmatrix} Z_1^* \\ Z_2^* \end{bmatrix}, \qquad Q^* = \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix}.$$

Consider the system of equations in the second block

$$S_{22}^* \xi_{2,t}^* = T_{22}^* \xi_{2,t-1}^* + Q_2^* (\Psi \varepsilon_t + \Pi \eta_t),$$

where $\xi_{2,t}^* \equiv Z_2^* X_t$. The conditions imposed to eliminate the explosive dynamics of the model are[22]

$$\begin{aligned} \underset{n\times 1}{\xi_{2,0}^*} &= 0 \\ \underset{n\times \ell}{\tilde{\Psi}_2} \underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{\tilde{\Pi}_2} \underset{p\times 1}{\eta_t} &= 0, \end{aligned} \tag{47}$$

where $\tilde{\Psi}_2 \equiv Q_2^* \Psi$ and $\tilde{\Pi}_2 \equiv Q_2^* \Pi$. Hence, equation (47) imposes linear restrictions on the forecast errors, η_t, as a function of the fundamental shocks. However, under indeterminacy the number of forecast errors exceeds the number of explosive roots ($p > n$). To characterize the full set of solutions to the equation, Lubik and Schorfheide (2003) decompose the matrix $\tilde{\Pi}_2$ using the following singular value decomposition

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U} \left[\underset{n\times n}{D_{11}} \quad \underset{n\times m}{\mathbf{0}} \right] \underset{p\times p}{V^T}.$$

[22] In the following, it is useful to introduce the notation about the dimensionality of the matrices. Let n, l and p represent the number of explosive roots, fundamental shocks and forecast errors respectively.

Given the partition $\underset{p\times p}{V} \equiv \left[\underset{p\times n}{V_1} \quad \underset{p\times m}{V_2} \right]$, equation (47) can be written as

$$\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U^T} \underset{n\times \ell}{\tilde{\Psi}_2} \underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{V_1^T} \underset{p\times 1}{\eta_t} = 0. \tag{48}$$

Since the system is indeterminate, Lubik and Schorfheide append additional $m = p - n$ equations,

$$\underset{m\times \ell}{M_z} \underset{\ell\times 1}{\varepsilon_t} + \underset{m\times m}{M_\zeta} \underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2^T} \underset{p\times 1}{\eta_t}. \tag{49}$$

The $m \times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero and covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamentals, ε_t.

$$E[\zeta_t] = 0, \quad E\left[\zeta_t \varepsilon_t^T\right] = 0, \quad E\left[\zeta_t \zeta_t^T\right] = \Omega_{\zeta\zeta}.$$

The matrix M_z captures the correlation of the forecast errors, η_t, with fundamentals, ε_t, and Lubik and Schorfheide choose the normalization $M_\zeta = I_m$. Appending the additional equation (49) appended to equation (48) the expectational errors can be written as functions of the fundamentals, ε_t and the sunspot shocks, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(-\underset{p\times n}{V_1} \underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2} + \underset{p\times m}{V_2} \underset{m\times \ell}{M_z} \right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}.$$

More compactly

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1} \underset{n\times \ell}{N} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}, \tag{50}$$

where

$$\underset{n\times \ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2}.$$

is a function of the parameters of the model.

Augmented representation

The augmented representation takes the form

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t, \tag{51}$$

where $\hat{X}_t \equiv (X_t, \omega_t)'$, $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$ and

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix}. \tag{52}$$

where the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$ without loss of generality. Since under indeterminacy the diagonal elements of the matrix Φ are outside the unit circle, the Blanchard-Kahn condition is satisfied and the QZ decomposition takes the form

$$\hat{S}^* \left(\hat{Z}^*\right)' \hat{X}_t^* = \hat{T}^* \left(\hat{Z}^*\right)' \hat{X}_{t-1}^* + \hat{Q}^* \hat{\Psi} \hat{\varepsilon}_t + \hat{Q} \hat{\Pi} \eta_t,$$

where $\hat{\Gamma}_0 = \left(\hat{Q}^*\right)' \hat{S}^* \left(\hat{Z}^*\right)'$ and $\hat{\Gamma}_1 = \left(\hat{Q}^*\right)' \hat{T}^* \left(\hat{Z}^*\right)'$ is the QZ decomposition of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and

$$\hat{S}^* = \begin{bmatrix} S_{11}^* & S_{12}^* & \mathbf{0} \\ \mathbf{0} & S_{22}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{T}^* = \begin{bmatrix} T_{11}^* & T_{12}^* & \mathbf{0} \\ \mathbf{0} & T_{22}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi \end{bmatrix}, \qquad \left(\hat{Z}^*\right)' = \begin{bmatrix} Z_1^* & \mathbf{0} \\ Z_2^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{Q}^* = \begin{bmatrix} Q_1^* & \mathbf{0} \\ Q_2^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Note that the inner matrices of $\{\hat{S}^*, \hat{T}^*, \left(\hat{Z}^*\right)', \hat{Q}^*\}$ are the same as those which define the matrices $\{S^*, T^*, (Z^*)', Q^*\}$ using the methodology of Farmer et al. (2015). The explosive dynamics of the model are therefore grouped in the second and third block, which are reported below as (53)

$$\begin{bmatrix} S_{22}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \xi_{2,t-1}^* \\ \omega_{t-1} \end{bmatrix} = \begin{bmatrix} T_{22}^* & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix} \begin{bmatrix} \xi_{2,t-1}^* \\ \omega_{t-1} \end{bmatrix} + \begin{bmatrix} Q_2^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t\right). \tag{53}$$

The following restrictions are imposed to find a unique and bounded solution under indeterminacy

$$\xi_{2,0}^* = 0 \tag{54}$$

$$\omega_0 = 0 \tag{55}$$

$$\begin{bmatrix} Q_2^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t\right) = 0 \tag{56}$$

Recalling the definition of $\hat{\Psi}$ and $\hat{\Pi}$, equation (56) can be written as

$$\begin{bmatrix} Q_2^*\Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \hat{\varepsilon}_t + \begin{bmatrix} Q_2^*\Pi_n & Q_2^*\Pi_f \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \eta_t = 0.$$

Solving for η_t, the restrictions imposed on the forecast errors are described by the following system of equations[23]

$$\eta_t = - \begin{bmatrix} (Q_2^*\Pi_n)^{-1} Q_2^*\Psi & (Q_2^*\Pi_n)^{-1} Q_2^*\Pi_f \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \hat{\varepsilon}_t.$$

Equivalently,

$$\eta_t = C_1 \varepsilon_t + C_2 \nu_t, \tag{57}$$

where $C_1 \equiv - \begin{bmatrix} (Q_2^*\Pi_n)^{-1} Q_2^*\Psi \\ \mathbf{0} \end{bmatrix}$ and $C_2 \equiv - \begin{bmatrix} (Q_2^*\Pi_n)^{-1} Q_2^*\Pi_f \\ -\mathbf{I} \end{bmatrix}$.

Equivalence of Bianchi-Nicoló with Lubik-Schorfheide

Under indeterminacy, the equilibrium of Lubik and Schorfheide is parametrized by the following two sets of parameters: $\Theta = vec(\Gamma_0, \Gamma_1, \Psi, \Omega_{\varepsilon\varepsilon})^T$ and $\Theta^{LS} = vec(\Omega_{\zeta\zeta}, M_z)^T$.Similarly, the equilibrium of Bianchi and Nicoló is parametrized by the common set of parameters Θ and by the set $\Theta^{BN} = vec(\Omega_{\nu\nu}, \Omega_{\nu\varepsilon})^T$. To establish the equivalence between the two methodologies, we show that there exists a unique mapping from the parametrization of Lubik-Schorfheide to the one proposed here and vice-versa.

Consider equation (50) and (57) reported below as equation (58) and (59).

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1} \underset{n\times \ell}{N} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}, \tag{58}$$

$$\underset{p\times 1}{\eta_t} = \underset{p\times \ell}{C_1} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{C_2} \underset{m\times 1}{\nu_t} \tag{59}$$

[23] The blockwise inversion of a square matrix is $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(DD - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$.

Post-multiplying both equations by ε_t^T and taking conditional expectation,

$$\begin{aligned}
\underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times n}{V_1}\underset{n\times \ell}{N}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{V_2}\underset{m\times \ell}{M_z}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}}, \\
\underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times \ell}{C_1}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{C_2}\underset{m\times l}{\Omega_{\nu\varepsilon}}
\end{aligned}$$

Pre-multiplying both equations by V_2^T and combining them,

$$\underset{m\times \ell}{M_z}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} = \left(\underset{m\times p}{V_2^T}\underset{p\times \ell}{C_1} - \underset{m\times p}{V_2^T}\underset{p\times n}{V_1}\underset{n\times \ell}{N}\right)\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{m\times p}{V_2^T}\underset{p\times m}{C_2}\underset{m\times l}{\Omega_{\nu\varepsilon}}.$$

Using the properties of the vec operator, the following result holds

$$\underset{(m\times \ell)\times 1}{vec(M_z)} = \underset{(m\times \ell)\times(m\times \ell)}{(\Omega_{\varepsilon\varepsilon}\otimes I_m)^{-1}}\left[\underset{(m\times \ell)\times \ell^2}{\left[I_l\otimes\left(V_2^TC_1 - V_2^TV_1N\right)\right]}\underset{\ell^2\times 1}{vec\left(\Omega_{\varepsilon\varepsilon}\right)} + \underset{(m\times \ell)\times(m\times \ell)}{\left(I_l\otimes V_2^TC_2\right)}\underset{(m\times \ell)\times 1}{vec\left(\Omega_{\nu\varepsilon}\right)}\right]. \tag{60}$$

Also, considering again equation (58) and (59), post-multiplying by ζ_t^T and taking conditional expectation,

$$\begin{aligned}
\underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{V_2}\underset{m\times m}{\Omega_{\zeta\zeta}}, \\
\underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{C_2}\underset{m\times m}{\Omega_{\nu\zeta}}
\end{aligned}$$

Pre-multiplying both equations by V_2^T, combining them and taking the transpose

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \underset{m\times m}{\Omega_{\zeta\nu}}\underset{m\times m}{\left(V_2^TC_2\right)^T}. \tag{61}$$

Finally, to obtain an expression for $\Omega_{\zeta\nu}$, we post-multiply equation (58) and (59) by ν_t^T and take conditional expectations

$$\begin{aligned}
\underset{p\times m}{\Omega_{\eta\nu}} &= \left(\underset{p\times n}{V_1}\underset{n\times \ell}{N} + \underset{p\times m}{V_2}\underset{m\times \ell}{M_z}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{V_2}\underset{m\times m}{\Omega_{\zeta\nu}}, \\
\underset{p\times m}{\Omega_{\eta\nu}} &= \underset{p\times \ell}{C_1}\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{C_2}\underset{m\times m}{\Omega_{\nu\nu}}
\end{aligned}$$

Pre-multiplying both equations by V_2^T and solving for $\Omega_{\zeta\nu}$,

$$\underset{m\times m}{\Omega_{\zeta\nu}} = \left(\underset{m\times p}{V_2^T}\underset{p\times \ell}{C_1} - \underset{m\times p}{V_2^T}\underset{p\times n}{V_1}\underset{n\times \ell}{N} - \underset{m\times \ell}{M_z}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{m\times m}{\left(V_2^T C_2\right)}\underset{m\times m}{\Omega_{\nu\nu}}. \tag{62}$$

Post-multiplying (62) by $\underset{m\times m}{\left(V_2^T C_2\right)}^T$ and using (61), then

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \left(\underset{m\times p}{V_2^T}\underset{p\times \ell}{C_1} - \underset{m\times p}{V_2^T}\underset{p\times n}{V_1}\underset{n\times \ell}{N} - \underset{m\times \ell}{M_z}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}}\underset{m\times m}{\left(V_2^T C_2\right)}^T + \underset{m\times m}{\left(V_2^T C_2\right)}\underset{m\times m}{\Omega_{\nu\nu}}\underset{m\times m}{\left(V_2^T C_2\right)}^T. \tag{63}$$

Therefore, equation (60) and (63) define the one-to-one mapping between the parametrization in Lubik and Schorfheide $\{\Theta, \Theta^{LS}\}$ and the parametrization in Bianchi and Nicoló $\{\Theta, \Theta^{BN}\}$.

7.2 Appendix B

In this section, the derivations for the solutions under the two alternative representations discussed in Section 4.1 are provided.

a) Under determinacy, it is possible to use standard solution algorithms, such as Sims (2001).

Consider the three equations NK model in (9)∼(11) and reported below as equations (64)∼(68)

$$\begin{aligned} x_t &= E_t(x_{t+1}) - \tau(R_t - E_t(x_{t+1})) && (64)\\ \pi_t &= \beta E_{t-1}(\pi_{t+1}) + \kappa x_t && (65)\\ R_t &= \psi\pi_t + \varepsilon_{R,t} && (66)\\ \eta_{1,t} &= x_t - E_{t-1}(x_t) && (67)\\ \eta_{2,t} &= \pi_t - E_{t-1}(\pi_t) && (68) \end{aligned}$$

The LRE model can be written in the following matrix form

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi\eta_t, \qquad (69)$$

where $X_t = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}))'$, $\varepsilon_t = (\varepsilon_{R,t})$ and $\eta_t = (\eta_{1,t}, \eta_{2,t})'$.
The solution to (69) can be found following four steps. First, since the matrix Γ_0 is non-singular, the LRE model in (69) can be written as

$$X_t = \Gamma_1^* X_{t-1} + \Psi^*\varepsilon_t + \Pi^*\eta_t, \qquad (70)$$

where

$$\Gamma_1^* \equiv \Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} \mathbf{0_{2\times 2}} & \mathbf{I_{2\times 2}} \\ \mathbf{0_{2\times 2}} & A_{2\times 2} \end{bmatrix}, \qquad A \equiv \begin{bmatrix} 1 + \frac{\kappa\tau}{\beta} & \tau\left(\psi - \frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

$$\Psi^* \equiv \Gamma_0^{-1}\Psi = \begin{bmatrix} 0 \\ 0 \\ \tau \\ 0 \end{bmatrix}, \qquad \Pi^* \equiv \Gamma_0^{-1}\Pi = \begin{bmatrix} \mathbf{I_{2\times 2}} \\ \\ A_{2\times 2} \end{bmatrix}$$

Equivalently, the equations in (43) are

$$x_t = E_{t-1}(x_t) + \eta_{1,t} \tag{71}$$

$$\pi_t = E_{t-1}(\pi_t) + \eta_{2,t} \tag{72}$$

$$\xi_t = A\xi_{t-1} + \begin{bmatrix} \tau \\ 0 \end{bmatrix} \varepsilon_{R,t} + A\eta_t \tag{73}$$

where $\xi_t = (E_t(x_{t+1}), E_t(\pi_{t+1}))'$.

Second, in order to study the stability of the system, the matrix A is decomposed using the Jordan decomposition[24] and (73) can be written as

$$J^{-1}\xi_t = \Lambda J^{-1}\xi_{t-1} + J^{-1} \begin{bmatrix} \tau \\ 0 \end{bmatrix} \varepsilon_{R,t} + J^{-1}A\eta_t, \tag{74}$$

where

$$J^{-1} = \begin{bmatrix} -\frac{\kappa}{\phi} & -\frac{a_2}{2\phi} \\ \frac{\kappa}{\phi} & \frac{\beta+\phi+\kappa\tau-1}{2\phi} \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_{1,2} = \frac{(1+\beta+\kappa\tau) \pm \phi}{2\beta} \tag{75}$$

where $a_2 \equiv (\beta - \phi + \kappa\tau - 1)$, $\phi \equiv [(1+\beta+\kappa\tau)^2 - 4\beta(1+\kappa\tau\psi)]^{-1/2}$ and the diagonal elements of the matrix Λ are the roots of the system and under determinacy $|\lambda_{1,2}| > 1$.

Third, restrictions which eliminate the explosive dynamics of the system have to be imposed. Under determinacy both roots of (74) are unstable, which requires to impose the following conditions

$$\xi_t = \begin{pmatrix} E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = \mathbf{0_{2\times 1}} \tag{76}$$

[24]The Jordan decomposition of the matrix A is $A \equiv J\Lambda J^{-1}$, where the diagonal elements of the matrix Λ are the roots of the system.

$$\eta_t = -A^{-1}\begin{bmatrix}\tau\\0\end{bmatrix}\varepsilon_{R,t} = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1\\\kappa\end{bmatrix}\varepsilon_{R,t} \tag{77}$$

Fourth, the restrictions imposed on the endogenous variables and on the forecast errors are combined with the equations which define the remaining endogenous variables, that is (71) and (72). This implies

$$\begin{pmatrix}x_t\\\pi_t\end{pmatrix} = \eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1\\\kappa\end{bmatrix}\varepsilon_{R,t}. \tag{78}$$

b) The solution provided in Section 4.1 for the methodology proposed in this paper is derived.

The proposed methodology consists in appending to the original LRE model the following equation[25]

$$\omega_t = \alpha\omega_{t-1} + \nu_t - \eta_{2,t},$$

where without loss of generality $\alpha = (1/\psi) < 1$. Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)' = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)$' and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \varepsilon_{\zeta,t})' = (\varepsilon_{R,t}, \varepsilon_{\zeta,t})'$, the augmented representation of the LRE model is

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{z}_t + \hat{\Pi}\eta_t. \tag{79}$$

Given (79), the same steps are followed to obtain the solution to the system. First, the system in (79) is pre-multiplied by $\hat{\Gamma}_0^{-1}$ to obtain

$$\hat{X}_t = \hat{\Gamma}_1^*\hat{X}_{t-1} + \hat{\Psi}^*\hat{\varepsilon}_t + \hat{\Pi}^*\eta_t, \tag{80}$$

[25] Note that $m = 1$, thus implying that only one equation should be appended. Also, since Farmer et al. (2015) show that the choice of which forecast errors should be redefined as fundamental, it is without loss of generality that we consider the case when $\eta_{2,t}$ is redefined.

where

$$\hat{\Gamma}_1^* \equiv \begin{bmatrix} \Gamma_1^* & \mathbf{0_{4\times 1}} \\ & \\ \mathbf{0_{1\times 4}} & \alpha \end{bmatrix}, \quad \hat{\Psi}^* \equiv \begin{bmatrix} \Psi^* & \mathbf{0_{4\times 1}} \\ & \\ 0 & -1 \end{bmatrix}, \quad \hat{\Pi}^* \equiv \begin{bmatrix} & \Pi_{4\times 2}^* \\ & \\ 0 & 1 \end{bmatrix}.$$

Hence, defining $\hat{\xi}_t \equiv (\xi_t, \omega_t)' = (E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)'$, the equations in (80) can be written as

$$x_t = E_{t-1}(x_t) + \eta_{1,t} \tag{81}$$

$$\pi_t = E_{t-1}(\pi_t) + \eta_{2,t} \tag{82}$$

$$\hat{\xi}_t = \hat{A} \qquad \hat{\xi}_{t-1} + \begin{bmatrix} \tau & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{z}_t + \begin{bmatrix} & A_{2\times 2} \\ & \\ 0 & -1 \end{bmatrix} \eta_t \tag{83}$$

where $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & \alpha \end{bmatrix}$.

Second, the matrix $\hat{A}$ is decomposed using the Jordan decomposition and the system in (83) can be written as

$$\hat{J}^{-1}\hat{\xi}_t = \hat{\Lambda}\hat{J}^{-1}\hat{\xi}_{t-1} + \hat{J}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{z}_t + \hat{J}^{-1} \begin{bmatrix} & A_{2\times 2} \\ & \\ 0 & -1 \end{bmatrix} \eta_t, \tag{84}$$

where

$$\hat{J}^{-1} \equiv \begin{bmatrix} \mathbf{0_{1\times 2}} & 1 \\ & \\ J^{-1} & \mathbf{0_{2\times 1}} \end{bmatrix}, \quad \hat{\Lambda} \equiv \begin{bmatrix} \lambda_3 & 0 \\ 0 & \Lambda \end{bmatrix} = \begin{bmatrix} \lambda_3 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_2 \end{bmatrix}$$

and $\lambda_{1,2}$ are the same as in (75) and $\lambda_3 = \alpha = (1/\psi) < 1$.

Third, since $|\lambda_{1,2}| > 1$ and $\lambda_3 < 1$, then the conditions which guarantee the boundedness of the solution are imposed on the last two equations of (84). This implies

$$\xi_t = \begin{pmatrix} E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = \mathbf{0_{2\times 1}} \tag{85}$$

$$\eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 & 0\\ \kappa & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix} \tag{86}$$

Fourth, combining these restrictions with the first equation of (84) which displays stable dynamics and with (81) and (82), the obtained solution is

$$\omega_t = \alpha\omega_{t-1} + \begin{bmatrix}\frac{\tau\kappa}{1+\kappa\tau\psi} & 1\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix} \tag{87}$$

$$\begin{pmatrix}x_t\\ \pi_t\end{pmatrix} = \eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 & 0\\ \kappa & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}. \tag{88}$$

7.3 Appendix C

In Section 4.2, the NK model is indeterminate and the derivations for the solutions under two alternative representations are provided.

c) To select a unique, bounded rational expectation equilibrium, we follow the solution method suggested by Farmer et al. (2015) when the forecast error for the deviations of inflation from its steady state is included as newly defined fundamental shock. Defining $\tilde{\varepsilon}_t = (\varepsilon_t, \eta_{2,t})'$, then the LRE can be written as

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi_f \tilde{\varepsilon}_t + \Pi_n \eta_{1,t}. \tag{89}$$

The same steps as in Section 7.2 are also applied here. First, by pre-multiplying (89) by Γ_0^{-1}, we obtain the following equations

$$\begin{aligned} x_t &= E_{t-1}(x_t) + \eta_{1,t} &(90)\\ \pi_t &= E_{t-1}(\pi_t) + \eta_{2,t} &(91)\\ \xi_t &= A\xi_{t-1} + \begin{bmatrix}\tau & \tau\left(\psi - \frac{1}{\beta}\right)\\ 0 & \frac{1}{\beta}\end{bmatrix}\tilde{\varepsilon}_t + \begin{bmatrix}1+\frac{\kappa\tau}{\beta}\\ -\frac{\kappa}{\beta}\end{bmatrix}\eta_{1t} &(92)\end{aligned}$$

where the matrix A is the same as for the determinate case as defined in (73) and therefore also its Jordan decomposition delivers the same matrices J and Λ as in (74) and (75) and reported below

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_{1,2} = \frac{(1+\beta+\kappa\tau) \pm \phi}{2\beta} \tag{93}$$

and

$$J^{-1} = \begin{bmatrix} -\frac{\kappa}{\phi} & -\frac{a_2}{2\phi} \\ \frac{\kappa}{\phi} & \frac{\beta+\phi+\kappa\tau-1}{2\phi} \end{bmatrix}.$$

However, the difference with the determinate case is that, while in the latter both roots are outside the unit circle, under indeterminacy it is the case that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. This implies that in the third step the restrictions imposed on the system to guarantee a bounded solution are also distinct from the determinate case. In particular, the restrictions are imposed on the first equation of (92), thus obtaining the following conditions

$$E_t(x_{t+1}) = -\frac{a_2}{2\kappa} E_t(\pi_{t+1}) \tag{94}$$

$$\eta_{1,t} = \begin{bmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \end{bmatrix} \begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix} \tag{95}$$

where $a_1 = (\beta - \phi + \kappa\tau + 1)$, $a_2 = (a_1 - 2)$, $a_3 = (a_1 + 2\phi)$ and $\phi = [(1+\beta+\kappa\tau)^2 - 4\beta(1+\kappa\tau\psi)]^{-1/2}$.

Fourth, using these restrictions, the solution obtained with the methodology of Farmer et al. (2015) is

$$\begin{pmatrix} x_t \\ \pi_t \\ E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = G_{4\times1} E_{t-1}(\pi_t) + H_{4\times2} \begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix} \tag{96}$$

where

$$G_{4\times1} \equiv \begin{pmatrix} -\frac{a_2}{2\kappa} \\ 1 \\ -\frac{a_1 a_2}{4\beta\kappa} \\ \frac{a_1}{2\beta} \end{pmatrix} \qquad H_{4\times2} \equiv \begin{pmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \\ 0 & 1 \\ -\frac{\tau a_2}{a_3} & -\frac{a_2(1+\kappa\tau\psi)}{a_3\kappa} \\ \frac{2\kappa\tau}{a_3} & -\frac{2(1+\kappa\tau\psi)}{a_3} \end{pmatrix}.$$

d) The derivation of the solution provided by the proposed methodology when the model is indeterminate closely follows the one described in Section 7.2 part b). In particular, the first two steps of the solution method are equivalent and recalling the definition of $\hat{\xi}_t \equiv (\xi_t, \omega_t)' = (E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)'$ and $\hat{\varepsilon}_t \equiv (\varepsilon_t, \varepsilon_{\zeta,t})' = (\varepsilon_{R,t}, \varepsilon_{\zeta,t})'$, equation in (84) are reported below in (97)

$$\hat{J}^{-1}\hat{\xi}_t = \hat{\Lambda}\hat{J}^{-1}\hat{\xi}_{t-1} + \hat{J}^{-1}\begin{bmatrix}\tau & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}\hat{\varepsilon}_t + \hat{J}^{-1}\begin{bmatrix}A_{2\times 2} & 0\\ & \\ 0 & -1\end{bmatrix}\eta_t, \tag{97}$$

where

$$\hat{J}^{-1} \equiv \begin{bmatrix}\mathbf{0_{1\times 2}} & 1\\ & \\ J^{-1} & \mathbf{0_{2\times 1}}\end{bmatrix}, \quad \hat{\Lambda} = \begin{bmatrix}\lambda_3 & 0\\ 0 & \Lambda\end{bmatrix} = \begin{bmatrix}\lambda_3 & & 0\\ & \lambda_1 & \\ 0 & & \lambda_2\end{bmatrix}.$$

It is however important to note that under indeterminacy not only $|\lambda_1| > 1$ and $|\lambda_2| < 1$ as in representation c), but also $|\lambda_3| = \alpha = (1/\psi) > 1$. Hence, the third steps imposes restrictions on the first two equations of (97), which result in the following conditions

$$\omega_t = 0 \tag{98}$$

$$E_t(x_{t+1}) = -\frac{a_2}{2\kappa}E_t(\pi_{t+1}) \tag{99}$$

$$\eta_t = \begin{bmatrix}-\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix} \tag{100}$$

Fourth, using these restrictions, the solution of the LRE model for the endogenous variables takes the following form

$$\begin{pmatrix}x_t\\ \pi_t\\ E_t(x_{t+1})\\ E_t(\pi_{t+1})\end{pmatrix} = G_{4\times 1}E_{t-1}(\pi_t) + H_{4\times 2}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}. \tag{101}$$