On Credible Monetary Policies under Model Uncertainty*

Anna Orlik † Ignacio Presno ‡

February 10, 2016

Abstract

This paper studies the design of optimal time-consistent monetary policy in an economy where the planner trusts his own model, while a representative household uses a set of alternative probability distributions governing the evolution of the exogenous state of the economy. In such environments, unlike in the original studies of time-consistent monetary policy, management of households' expectations becomes an active channel of optimal policymaking per se; a feature that our paternalistic government seeks to exploit.

We adapt recursive methods in the spirit of Abreu, Pearce, and Stacchetti (1990) as well as computational algorithms based on Judd, Yeltekin, and Conklin (2003) to fully characterize the equilibrium outcomes for a class of policy games between the government and a representative household that distrusts the model used by the government.

Keywords: monetary policy, government credibility, time consistency, recursive methods, model uncertainty, robust control.

JEL codes: E61, E52, C61, D81.

---

*We are especially grateful to Thomas J. Sargent for his support and encouragement. We thank David Backus, Roberto Chang, Timothy Cogley, David Pearce, Thomas Philippon, Ennio Stacchetti, Sevin Yeltekin, Stanley Zin and seminar participants at Bank of England, Board of Governors of the Federal Reserve System, Carnegie Mellon University, UPF-CREI, and Federal Reserve Bank of San Francisco, for helpful comments.

†Address: Board of Governors of the Federal Reserve System, 20th Street and Constitution Avenue NW Washington, DC 20551, Email: anna.a.orlik@frb.gov.

‡Address: Department of Economics, Universidad de Montevideo, 2544 Prudencio de Pena St., Montevideo, Uruguay 11600. Email: jipresno@um.edu.uy.

1 Introduction

Undoubtedly, the public's inflation expectations greatly influence actual inflation and, therefore, a central bank's ability to achieve price stability. But what do we mean precisely by the "state of inflation expectations"? And, most importantly, what role does monetary policy play in shaping or managing inflation expectations?[1]

In this paper, the central bank's management of private beliefs becomes an integral part of the theory of optimal monetary policymaking.

In our model economy, constructed in the tradition of monetary models by Calvo (1978) and Chang (1998), a representative household derives utility from consumption and real money holdings. The government uses the newly printed money to finance transfers or taxes to households. Taxes and transfers are distortionary. The only source of uncertainty in this economy is a shock that affects the degree of tax distortions through its influence on households' income.

At the heart of this paper lies the assumption that the government has a *single approximating model* that describes the evolution of the underlying shock while a representative household fears that this model might be misspecified. To confront this concern, a representative household contemplates a set of nearby probability distributions or probability models and seeks decision rules that would work well across these models. The household assesses the performance of a given decision rule by computing the expected utility under the worst-case distribution within the set. This worst-case distribution can be seen as the outcome that follows from twisting the approximating model with adequate probability distortions.

The fact that private agents seem unable to assign a unique probability distribution to alternative outcomes has been demonstrated in Ellsberg (1961) and similar experimental studies.[2] Moreover, a lack of confidence in the current models seems to have become apparent during the recent financial crisis[3] as noted in Bernanke (2010):

> Most fundamentally, and perhaps most challenging for researchers, the crisis should motivate economists to think further about modeling human behavior. Most economic researchers continue to work within the classical paradigm that assumes rational, self-interested behavior and the maximization of "expected utility".... An important assumption of that framework is that, in making decisions under uncertainty, economic agents can assign meaningful probabilities to alternative outcomes. However, during the worst phase of the financial crisis, many economic actors—including investors, employers, and consumers—metaphorically threw up

[1] These questions are subject of Bernanke (2007).

[2] See, e.g. Halevy (2007).

[3] See e.g. Caballero and Krishnamurthy (2008) and Uhlig (2010).

> their hands and admitted that, given the extreme and, in some ways, unprecedented nature of the crisis, they did not know what they did not know.

In our model the government follows the above advice to go beyond the expected utility framework. The government recognizes that households are not able or willing to assign a unique probability distribution to alternative realizations of the stochastic state of the economy. The government wants to design optimal monetary policy that explicitly accounts for the fact that households' allocation rules are influenced by how they form their beliefs in light of model uncertainty.

We characterize optimal policy under two timing protocols for the government's choices. First, we work under the assumption that at time zero the government can commit to a policy specifying its actions for all current and future dates and states of nature. Under this assumption, at time zero a government chooses the best competitive equilibrium from the set of competitive equilibria with model uncertainty, i.e. one that maximizes the households' expected lifetime utility but under the government's own unique beliefs. We will refer to such a government as *paternalistic Ramsey planner*.

The competitive equilibrium conditions in our model are represented by the households' Euler equations and an exponential twisting formula for the probability distortions. Using insights from Kydland and Prescott (1980), we express the competitive equilibria in a recursive structure by introducing an adequate pair of state variables. We first need to keep track of the equilibrium (adjusted) marginal utilities to guarantee that the Euler equations are satisfied after each history. Our second state variable is the households' lifetime utility. This variable is needed to express the equilibrium probability distortions in the context of model uncertainty. These two variables summarize all the relevant information about future policies and allocations for households' decisionmaking when the government has the ability to commit. Through the dynamics of the promised marginal utility and households' value, which the government has to deliver in equilibrium, the solution to the government's problem under commitment, the *Ramsey plan*, exhibits history dependence.

Once we abstract from the assumption that the government has the power to commit but instead chooses sequentially, a time inconsistency problem may arise, as first noted by Kydland and Prescott (1977) and Calvo (1978). The government will adhere to a plan only if it is in its own interest to do so. As a consequence, it is urgent to check whether the optimal policies derived by our paternalistic Ramsey planner are time consistent, and, more generally, to characterize the set of *sustainable plans with model uncertainty*.[4] This latter notion should

[4]The notion of a sustainable plan inherits sequential rationality on the government's side, jointly with the fact that households always respond to government actions by choosing from competitive equilibrium

be thought of as an extension of Chari and Kehoe (1990).

Using the government's value as an additional third state variable, an appropriate incentive constraint for the government can then be constructed to complete the formulation of a sustainable plan in a recursive way. This introduces a new source of history-dependence given by the restrictions that the system of households' expectations impose on the government's policy actions in equilibrium.

To our knowledge this paper constitutes the first attempt to characterize the set of all time-consistent outcomes when agents are uncertainty averse in an infinite-horizon model. This feature of our environment provides the government with opportunities to influence households' beliefs about exogenous variables through their expectations of future policies, which have to be confirmed in equilibrium. The management of households' beliefs becomes an active channel of policymaking as the government will exploit this mechanism when designing monetary policies.

Characterizing time-consistent outcomes is a challenging task because any time-consistent solution must include a description of government and market behavior such that the continuation of such behavior after any history is a competitive equilibrium and it is optimal for the government to follow that policy. In this paper, we use insights from the work by Abreu, Pearce, and Stacchetti (1990), Chang (1998), and Phelan and Stacchetti (2001) to compute the sets of equilibrium payoffs as the largest fixed point of an appropriate operator. We also adapt algorithms based on hyperplane approximation methods in the spirit of Judd, Yeltekin, and Conklin (2003) that let us compute the sets in question. The characterization of the entire set of sustainable equilibrium values facilitates the examination of practical policy questions. Our numerical examples suggest that government policies that account for the fact that households contemplate a set of probability distributions may lead to better outcomes.

Although in this paper we restrict attention to the type of models of monetary policymaking that can be cast in the spirit of Calvo (1978), our approach could be applicable to many repeated or dynamic games between a government and a representative household who distrusts the model used by the government.

To our knowledge, there are two papers that try to explore the policymaker's role in managing households' expectations. Karantounias, Hansen, and Sargent (2009) study the optimal fiscal policy problem in Lucas and Stokey (1983) but in an environment where a representative household distrusts the model governing the evolution of exogenous government expenditure. Karantounias, Hansen, and Sargent (2009) apply the techniques of Marcet and Marimon (2009) to characterize the optimal policies when the government has power to

allocations.

commit. Woodford (2003) discusses the optimal monetary policy under commitment in an economy where both the government and the private sector fully trust their own models, but the government distrusts its knowledge of the private sector's beliefs about prices.

The remainder of this paper is organized as follows. Section 2 sets up the model and outlines the assumptions made. In Section 3 we introduce the notion of competitive equilibrium with model uncertainty. In Section 4 we discuss the recursive formulation of the Ramsey problem for the paternalistic government. Section 5 contains the discussion of sustainable plans with model uncertainty. In Section 6 we describe the computational algorithms we have implemented to determine the entire set of equilibrium values to the government and to the representative household, and their promised marginal utilities. We also present some numerical results. Section 7 briefly discusses an alternative hypothesis with both the government and households possibly using distinct sets of models. Finally, Section 8 concludes.

2 Benchmark Model

The model economy is populated by two infinitely lived agents: a representative household (with her evil alter ego, which represents her fears about model misspecification) and a government. The household and the government interact with each other at discrete dates indexed as $t = 0, 1, ...$

At the beginning of each period, the economy is hit by an exogenous shock. The government in our model has a reference or approximating probability model for this shock, which is its best estimate of the economic dynamics. Throughout the paper, we use the terms "probability model" and "probability distribution" interchangeably. While the government fully trusts the probability distribution for the shock, the representative household fears that it is misspecified. In turn, she contemplates a set of alternative probability distributions to be endogenously determined, and seeks decision rules that perform well over this set of distributions. Given her doubts on which model actually governs the evolution of the shock, the household designs decision rules that guarantee lower bounds on expected utility level under any of the distributions.

Let $(\Omega, \mathcal{F}, \Pr)$ be the underlying probability space. Let the exogenous shock be given by s_t, where $s_0 \in \mathbb{S}$ is given (there is no uncertainty at time 0) and $s_t : \Omega \to \mathbb{S}$ for all $t > 0$. The set $\mathbb{S}$ for the shock is assumed to be finite with cardinality S. We assume that s_t follows a Markov process for all $t > 0$, with transition probabilities given by $\pi(s_{t+1}|s_t)$.

Throughout this paper we will refer to the conditional distribution $\pi(s_{t+1}|s_t)$ as the *approximating model*. Let $s^t \equiv (s_0, s_1, ..., s_t) \in \mathbb{S} \times \mathbb{S} \times ... \times \mathbb{S} \equiv \mathbb{S}^{t+1}$ be the history of the realizations of the shock up to t. Finally, we denote by $\mathcal{S}^t \equiv \mathcal{F}\left(s^t\right)$ the sigma-algebra generated by the history s^t.

2.1 The Representative Household's Problem and Fears about Model Misspecification

The households in this economy derive utility from consumption of a single good, $c(s^t)$, and real money balances, $m(s^t)$. The household's period payoff is given by $u\left(c_t\left(s^t\right)\right)+v(m_t\left(s^t\right))$, where the utility components u and v satisfy the following assumptions:

[A1] $u:\mathbb{R}_+ \rightarrow \mathbb{R}$ is twice continuously differentiable, strictly increasing, and strictly concave

[A2] $v:\mathbb{R}_+ \rightarrow \mathbb{R}$ is twice continuously differentiable, and strictly concave

[A3] $\lim_{c\rightarrow 0} u'(c) = \lim_{m\rightarrow 0} v'(m) = +\infty$

[A4] $\exists \overline{m} < +\infty$ such that $v'(\overline{m}) = 0$.

The assumptions [A1]-[A3] are standard. Assumption [A4] defines a satiation level for real money balances.

In this paper we model the representative household as being uncertainty-averse. While the government fully trusts the approximating model $\pi\left(s^t\right)$, the household distrusts it. For this reason, she surrounds it with a set of alternative distributions $\widetilde{\pi}\left(s^t\right)$ that are statistical perturbations of the approximating model, and seeks decision rules that perform well across these alternative distributions. We assume that these alternative distributions, $\widetilde{\pi}\left(s_t\right)$, are absolutely continuous with respect to $\pi\left(s_t\right)$, i.e. $\pi\left(s_t\right)=0 \Rightarrow \widetilde{\pi}\left(s^t\right)=0$, $\forall s^t \in \mathbb{S}^{t+1}$.

By invoking the Radon-Nikodym theorem we can express any of these alternative distorted distributions using a nonnegative $\mathcal{S}^t$-measurable function given by

$$D_t\left(s^t\right)=\begin{cases}\frac{\widetilde{\pi}\left(s^t\right)}{\pi\left(s^t\right)} & \text{if} \pi\left(s^t\right)>0 \\ 1 & \text{if} \pi\left(s^t\right)=0,\end{cases}$$

which is a martingale with respect to $\pi\left(s^t\right)$, i.e. $\sum_{s_{t+1}} \pi\left(s_{t+1}|s_t\right) D_{t+1}\left(s^{t+1}\right)=D_t\left(s^t\right)$. We can also define the conditional likelihood ratio as $d_{t+1}\left(s_{t+1}|s^t\right) \equiv \frac{D_{t+1}\left(s^t, s_{t+1}\right)}{D_t\left(s^t\right)}$ for $D_t\left(s^t\right)>0$. Notice that in case $D_t\left(s^t\right)>0$ it follows that

$$d_{t+1}\left(s_{t+1}|s^t\right)=\begin{cases}\frac{\widetilde{\pi}\left(s_{t+1}|s^t\right)}{\pi\left(s_{t+1}|s_t\right)} & \text{if } \pi\left(s^{t+1}\right)>0 \\ 1 & \text{if } \pi\left(s^{t+1}\right)=0,\end{cases}$$

and that the expectation of the conditional likelihood ratio under the approximating model is always equal to 1, i.e. $\sum_{s_{t+1}} \pi\left(s_{t+1}|s^t\right) d_{t+1}(s_{t+1}|s^t) = 1$.

To express the concerns about model misspecification, we follow Hansen and Sargent (2008) and endow the household with multiplier preferences. In this case, the set of alternative distributions over which the household evaluates the expected utility of a given decision rule is given by an entropy ball. We can then think of the household as playing a zero-sum game against her *evil alter ego*, who is a fictitious agent that represents her fears about model misspecification. The evil alter ego will be distorting the expectations of continuation outcomes in order to minimize the household's lifetime utility. She will do it by selecting a worst-case distorted model $\widetilde{\pi}\left(s^t\right)$, or equivalently, a sequence of probability distortions $\left\{D_t\left(s^t\right), d_{t+1}\left(s_{t+1}|s^t\right)\right\}_{t=0}^{\infty}$.

The representative household ranks contingent plans for consumption and money balances according to

$$V^H = \max_{\{c_t(s^t), m_t(s^t)\}} \min_{\{D_t(s^t), d_{t+1}(s_{t+1})\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) D_t(s^t) \Big\{ \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right))\right]$$

$$+\theta\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) \log d_{t+1}(s_{t+1}|s^t) \Big\} \tag{1}$$

$$D_{t+1}(s^{t+1}) = d_{t+1}(s_{t+1}|s^t) D_t(s^t) \tag{2}$$

$$\sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) = 1, \tag{3}$$

where $m_t \equiv q_t M_t$ is the real money balances, M_t is the money holdings at the end of period t, q_t is the value of money in terms of the consumption good (that is, the reciprocal of the price level), and $\theta \in (\underline{\theta}, +\infty]$ is a penalty parameter that controls the degree of concern about model misspecification. Through the last term, the entropy term, the evil alter ego is being penalized whenever she selects a distorted model that differs from the approximating one. Note that the higher the value of θ, the more the evil alter ego is being punished. If we let $\theta \to +\infty$ the probability distortions to the approximating model vanish, the household and the government share the same beliefs, and expression (1) collapses to the standard expected utility.

Conditions (2) and (3) discipline the choices of the evil alter ego. Condition (2) defines recursively the likelihood ratio D_t. Condition (3) guarantees that every distorted probability is a well-defined probability measure.

The evil alter ego's minimization problem yields lower bounds (in terms of expected utility) on the performance of any of the household's decision rules. The probability distortion

$d(s_{t+1}|s^t)$ that solves this minimization problem satisfies the following exponential twisting formula

$$d(s_{t+1}|s^t) = \frac{\exp\left(-\frac{V^H(s^{t+1})}{\theta}\right)}{\sum_{s_{t+1}\in\mathbb{S}} \pi(s_{t+1}|s_t)\exp\left(-\frac{V^H(s^{t+1})}{\theta}\right)},$$

where $V^H(s^{t+1})$ is the $t+1-$equilibrium value for the household. Condition (2.1) shows how the evil alter ego pessimistically twists the household's beliefs by assigning high probability distortions to the states s_{t+1} associated with low utility for the household, and low probability distortions to the high-utility states. See the Appendix A.1 for the derivation of condition (2.1). Notice from (2.1) that to express the optimal belief distortions chosen by the evil alter ego, we need to know the household's equilibrium values. Using expression (2.1) the expected lifetime utility of the household at time t, in equilibrium, is

$$V(s^t) = u(c(s^t)) + v(m(s^t)) - \beta\theta \log \sum_{s_{t+1}\in\mathbb{S}} \pi(s_{t+1}|s_t)\left(\exp\left(-\frac{V^H(s^{t+1})}{\theta}\right)\right).$$

The representative household takes sequences of prices, $\left\{q_t\left(s^t\right)\right\}_{t=0}^{\infty}$, income, $\left\{y_t\left(s^t\right)\right\}_{t=0}^{\infty}$, taxes or subsidies, $\left\{x_t\left(s^t\right)\right\}_{t=0}^{\infty}$, and the conditional likelihood ratio chosen by her evil alter ego, $\left\{d_{t+1}\left(s_{t+1}|s^t\right)\right\}_{t=0}^{\infty}$, as well as the initial money supply M_{-1}, shock realization s_0 and $D_0 = 1$, as given.

The household then maximizes (1) subject to the following constraints

$$q_t\left(s^t\right) M_t\left(s^t\right) \leq y_t\left(s^t\right) - x_t\left(s^t\right) - c_t\left(s^t\right) + q_t\left(s^t\right) M_{t-1}(s^{t-1}) \tag{4}$$

$$q_t\left(s^t\right) M_t\left(s^t\right) \leq \overline{m}. \tag{5}$$

Condition (4) represents the household's budget constraint, which states that for all $t \geq 0$ and all s^t after-tax income in period t, $y_t - x_t$, together with the value of money holdings carried from last period, must be sufficient to cover the period-t expenditures on consumption and new purchases of money. Condition (5) is introduced for technical reasons, in order to bound real money balances from above.

2.2 Government

In this economy the government chooses how much money, $M_t(s^t)$ to create or to withdraw from circulation. In particular, it chooses a sequence $\{h_t\}_{t=0}^{\infty}$ where h_t is the reciprocal of the gross rate of money growth for all $t \geq 0$, i.e. $h_t \equiv \frac{M_{t-1}}{M_t}$. We make the following assumption on the set of values for the inverse money growth rate,

[A5] $h_t(s^t) \in \Pi \equiv [\underline{\pi}, \overline{\pi}]$ with $0 < \underline{\pi} < \frac{1}{\beta} \leq \overline{\pi}$.

[A5] establishes *ad hoc* bounds on the admissible rates for money creation. A positive lower bound implies that the supply of money has to be positive. The upper bound is set for technical reasons.

The government runs a balanced budget by printing money to finance the transfers to households or destroying the money it collects in the form of tax revenues, x_t,

$$x_t(s^t) = q_t(s^t)\left[M_{t-1}\left(s^{t-1}\right) - M_t(s^t)\right]. \tag{6}$$

Using the definition of m_t and h_t, (6) can be reformulated as

$$x_t(s^t) = m_t(s^t)\left[h_t(s^t) - 1\right]. \tag{7}$$

Notice that from equation (7) $x_t(s^t) \in \mathbb{X} \equiv \left[(\underline{\pi} - 1)\,\overline{m}, (\overline{\pi} - 1)\,\overline{m}\right].$

As in Chang (1998), we assume that taxes and subsidies are distortionary. To model that, we consider an *ad hoc* functional form for households' income, $f : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$, that depends on tax collections in period t and the exogenous shock, s_t, i.e. $y_t(s^t) \equiv f(x_t(s^t), s_t)$. The function $f : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$ is assumed to be at least twice continuously differentiable with respect to its first argument and

[A6] $f(x, s) > 0,\ f_1(0, s) = 0,\ f_{11}(x, s) < 0$ for all $x \in \mathbb{X}$, for all $s \in \mathbb{S}$

[A7] $f(x, s) = f(-x, s) > 0$ for all $x \in \mathbb{X}$, for all $s \in \mathbb{S}$,

where f_1 and f_{11} denote, respectively, the first and second derivative of function f with respect to its first argument. Function f is intended to convey that taxes (and transfers) are distortionary without the need to model the nature of such distortions explicitly. [A6] indicates that it is increasingly costly in terms of consumption to set taxes or to make transfers to households. This assumption will play a key role in the time-inconsistent nature of the *Ramsey plan*, when the government can commit to its announced policies. The symmetry of f given by [A7] implies that taxes and subsidies are equally distortionary.

2.3 The Within-Period Timing Protocol

The timing protocol within each period is as follows. First, the shock realization, $s_t(s^{t-1})$, occurs. Then, the government observes the shock, chooses the money supply growth rate $h_t(s^t)$ and taxes $x_t(s^t)$ for the period, and announces a sequence of future money growth rates and tax collections $\{h_{t+1}(s^{t+1}), x_{t+1}(s^{t+1})\}_{t=0}^{\infty}$. After that, given prices $q_t(s^{t-1})$, the current policy actions $(h_t(s^t), x_t(s^t))$ and their expectations of future policies, the household chooses $M_t(s^{t-1})$, or equivalently real balances $m_t(s^t)$. When making her choice of $m_t(s^t)$, the household can be thought of as playing a zero-sum game against her evil alter ego, who distorts her beliefs' about the evolution of future shock realizations.[5] Then taxes are collected and output is realized, $y_t(s^t) = f(s_t(s^{t-1}), x_t(s^t))$. Finally, consumption $c_t(s^t)$ takes place.

In our economy, the government would want to promote utility by increasing the real money holdings towards the satiation level. In equilibrium, however, this can only be achieved by reducing the money supply over time, which in turn induces a gradual deflationary process along the way. In order to balance its budget the government has to set positive taxes withdrawing money from circulation. Taxes are assumed to be distortionary, and, hence, this has negative effects on households' income.

In this simple framework, as discussed by Calvo (1978) and Chang (1998), the optimal policies for the Ramsey government with the ability to commit would typically be time-inconsistent. A discussion of the source of the time-inconsistency of the *Ramsey plan* is presented in section 4.

3 Competitive Equilibrium With Model Uncertainty

In this section we define and characterize a competitive equilibrium with model uncertainty in this economy. Throughout the rest of the paper we will use bold letters to denote state-contingent sequences.

Definition 3.1. *A government policy in this economy is given by sequences of (inverse) money growth rates* $\boldsymbol{h} = \{h_t(s^t)\}_{t=0}^{\infty}$ *and tax collections* $\boldsymbol{x} = \{x_t(s^t)\}_{t=0}^{\infty}$*. A* price system *is* $\boldsymbol{q} = \{q_t(s^t)\}_{t=0}^{\infty}$*. An allocation is given by a triple of nonnegative sequences of consumption, real balances and income,* $(\boldsymbol{c},\boldsymbol{m},\boldsymbol{y})$*, where* $\boldsymbol{c} = \{c_t(s^t)\}_{t=0}^{\infty}$*,* $\boldsymbol{m} = \{m_t(s^t)\}_{t=0}^{\infty}$*, and* $\boldsymbol{y} = \{y_t(s^t)\}_{t=0}^{\infty}$*.*

Definition 3.2. *Given* M_{-1}, s_0*, a* competitive equilibrium with model uncertainty *is given by an allocation* $(\boldsymbol{c},\boldsymbol{m},\boldsymbol{y})$*, a price system* $\boldsymbol{q}$*, belief distortions* $\boldsymbol{d}$*, and a sequence of households' utility values* $\mathbf{V^H} = \{V_{t+1}^H\}_{t=0}^{\infty}$ *such that for all* t *and all* s^t

(i) given $\mathbf{q}$*, beliefs' distortions* $\mathbf{d}$*, and government's policies* $\mathbf{h}$ *and* $\mathbf{x}$*,* $(\mathbf{m}, \mathbf{V^H})$ *solves households' maximization problem;*

(ii) given $\mathbf{q}$ *and* $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{V^H})$*,* $\mathbf{d}$ *solves the evil alter ego's minimization problem;*

(iii) government's budget constraint holds;

(iv) money and consumption good markets clear, i.e. $c_t(s^t) = y_t(s^t)$ *and* $m_t(s^t) = q_t(s^t)M_t(s^t)$*.*

[5]Since the game between the household and her evil alter ego is zero sum, the timing protocol between their moves do not affect the solution.

Under assumptions [A1-A6] we can prove the following proposition:

PROPOSITION 3.1. *A competitive equilibrium is completely characterized by sequences* $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H})$ *such that for all* t *and all* s^t, $m_t(s^t) \in \mathbb{M}$, $x_t(s^t) \in \mathbb{X}$, $h_t(s^t) \in \Pi$, $d_{t+1}(s^{t+1}) \in \mathbb{D} \subseteq \mathbb{R}_+^S$, *and* $V_{t+1}^H(s^{t+1}) \in \mathbb{V}$ *and*

$$m_t(s^t)\left\{u'(f(x_t(s^t), s_t)) - v'(m_t(s^t))\right\} =$$
$$\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t)\left\{u'(f(x_{t+1}(s^{t+1}), s_{t+1}) h_{t+1}(s^{t+1}) m_{t+1}(s^{t+1})\right\} \quad , \leq \text{ if } m_t = \overline{m} \tag{8}$$

$$d_{t+1}(s_{t+1}|s^t) = \frac{\exp\left(-\frac{V_{t+1}^H(s^{t+1})}{\theta}\right)}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\exp\left(-\frac{V_{t+1}^H(s^{t+1})}{\theta}\right)} \tag{9}$$

$$V_t^H = u\left(f(x_t(s^t), s_t)\right) + v\left(m_t(s^t)\right) - \beta\theta \log \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \exp\left(\frac{-V_{t+1}^H(s^{t+1})}{\theta}\right) \tag{10}$$

$$-x_t(s^t) = m_t(s^t)\left(1 - h_t(s^t)\right). \tag{11}$$

Proof. See Appendix A.1. □

Equation (8) is an Euler equation for real money balances. Equation (9) is simply the exponential twisting formula for optimal probability distortions, rewritten from (2.1). Equation (10), as in (2.1), expresses the household's utility values recursively once the probability distortions chosen by the evil alter ego are incorporated. Finally, equation (11) is the government's balanced budget constraint.

Note that households' transversality condition is not included in the list of conditions characterizing competitive equilibrium. In Appendix A.1. we explain why this is the case.

Formally, let $\mathbb{E} \equiv \mathbb{M} \times \mathbb{X} \times \Pi \times \mathbb{D} \times \mathbb{V}$ and $\mathbb{E}^\infty \equiv \mathbb{M}^\infty \times \mathbb{X}^\infty \times \Pi^\infty \times \mathbb{D}^\infty \times \mathbb{V}^\infty$. We define a set of competitive equilibria for each possible realization of the initial state s_0

$$CE_s = \left\{(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}) \in \mathbb{E}^\infty | \ \ (8)\text{-}(11) \text{ hold and } s_0 = s\right\}.$$

In Appendix A.2, we present an example of a competitive equilibrium sequence.

COROLLARY 3.1. CE_s *for all* $s \in \mathbb{S}$ *is nonempty.*

Proof. See Appendix A.2. □

COROLLARY 3.2. CE_s *for all* $s \in \mathbb{S}$ *is compact.*

Proof. See Appendix A.3. □

Corollary 3.3. *A continuation of a competitive equilibrium with model uncertainty is a competitive equilibrium with model uncertainty, i.e. if* $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_{s_0}$ *then* $\left\{m_t, x_t, h_t, d_t, V_{t+1}^H\right\}_{j=t}^{\infty} \in CE_{s_t}$ *for all* t *and all* $s_0, s_t \in \mathbb{S}$.

Proof. Follows immediately from Proposition 3.1. □

4 Ramsey Problem for a Paternalistic Government: Recursive Formulation

We start by formulating and solving the government's time-zero Ramsey problem. Although the assumption that the government has the ability to commit might be put in question, studying such environment will be useful for two reasons. First, it will allow us to describe the notion of a paternalistic government and to characterize the set of equilibrium values (both for the government and households) that the government can attain with commitment. This set of equilibrium values is interesting as it constitutes a larger set which includes the set of values that could be delivered when the government chooses sequentially. The discrepancy between these two sets sheds some light on the severity of the time-inconsistency problem. Second, as it will become clearer later on, the procedure for solving the Ramsey problem will constitute a helpful step towards deriving a recursive structure for the credible plans.

We assume first that the government sets its policy once and for all at time zero. That is, at time zero it chooses the entire infinite sequence of money growth rates $\{h_t(s^t)\}_{t=0}^{\infty}$ and commits to it. A *benevolent government* in this economy would exhibit households' preference orderings and, hence, maximize households' expected utility under the distorted model given by (1). In our setup, we depart from the assumption of a *benevolent government*, and assume instead that the government is *paternalistic* in the sense that it cares about households' utility but under its own beliefs, which are assumed to be $\pi(s^t)$. The assumption of a paternalistic government implies in turn that the households and the government do not necessarily share the same beliefs when evaluating contingent plans for consumption and real balances. While the government believes that the exogenous shock evolves according to the approximating model $\pi(s^t)$, the households act as if the evolution of the shock is governed by $\widetilde{\pi}(s^t)$.

For a given initial shock realization s_0 and initial M_{-1}, the Ramsey problem that the government solves in our environment therefore consists of choosing $(m, x, h, d) \in CE_{s_0}$ to maximize households' expected utility under the approximating model, i.e.,

$$V_t^G = \max_{(\mathbf{m},\mathbf{x},\mathbf{h},\mathbf{d},\mathbf{V^H})} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right))\right] \quad \text{s.t. (8) - (11).} \qquad (12)$$

We solve the Ramsey problem by formulating it in a recursive fashion. To do so, we need to adopt a recursive structure for the competitive equilibria. It is key then to identify any variables that summarize all relevant information about future policies and future allocations for households' decisionmaking in the current period. From the Euler equation (8) we immediately identify the variables we are after. For time t, history s^t, and the households' choice of real balances $m_t(s^t)$, we need to know the (discounted) expected value of money at $t+1$, defined by the right hand side of equation (8). The expected value of money at $t+1$ can be expressed in terms of the value of money for each shock realization s_{t+1} and the probability distribution households assign to s_{t+1}. Following Kydland and Prescott (1980) and Chang (1998), we designate the value of money as a pseudo-state variable to track.[6] Let $\mu_{t+1}(s^{t+1})$ denote the equilibrium value of money at $t+1$ after history s^{t+1},

$$\mu_{t+1}\left(s^{t+1}\right) \equiv u'(f(x_{t+1}\left(s^{t+1}\right), s_{t+1})(h_{t+1}\left(s^{t+1}\right) m_{t+1}\left(s^{t+1}\right)). \tag{13}$$

We can view $\mu_{t+1}(s^{t+1})$ as the "promised" (adjusted) marginal utility of money after s^{t+1}.

The second ingredient needed to compute the expected value of money at $t+1$ is households' beliefs about s_{t+1}. As shown in Hansen and Sargent (2007), households want to guard themselves against a worst-case scenario by twisting the approximating probability model in accordance to distortions $d_{t+1}(s^{t+1})$. Therefore, the future paths of $h_{t+1}(s^{t+1})$ and $m_{t+1}(s^{t+1})$ influence today's choice of real money balances m_t, not only through their effect on $\mu_{t+1}(s^{t+1})$ but also through the impact they have on the degree of distortion in the beliefs of the representative household, as given by (9).

These probability distortions in equilibrium turn out to be a function of households' continuation values. Therefore, to construct a recursive representation of the competitive equilibria with model uncertainty we need to compute households' utility values $V^H(s^{t+1})$, in addition to $\mu_{t+1}(s^{t+1})$. Together, these can be thought of as device used to ensure that we account for the effects of future policies on agents' behavior in earlier periods.

Let $\Re^2$ be the space of all the subsets of $\mathbb{R}^2$. Moreover, let $\Omega : \mathbb{S} \to \Re^2$ be the value correspondence such that

$$\Omega\left(s=s_0\right)=\Big\{\left(\mu_s, V_s^H\right)\in\mathbb{R}\times\mathbb{R}|\ \mu_s\equiv u'\left[f(x_0\left(s_0\right),s_0)\right]\left[x_0\left(s_0\right)+m_0\left(s_0\right)\right] \text{ and}$$
$$V_s^H = u\left(f(x\left(s_0\right),s_0\right)+v\left(m\left(s_0\right)\right)-\beta\theta\log\sum_{s_1}\pi\left(s_1|s_0\right)\exp\left(\frac{-V_1^H(s_1)}{\theta}\right)$$
$$\text{with } s_0=s \text{ and for some } (\mathbf{m},\mathbf{x},\mathbf{h},\mathbf{d},\mathbf{V^H})\in CE_s\Big\}.$$

For each initial state realization s, the set $\Omega(s)$ is formed by all current (adjusted) marginal

[6] To solve for the Ramsey plan in a dynamic economy with capital accumulation, Marcet and Marimon (2009) instead use the Lagrange multiplier associated with the Euler equations as a pseudo-state variable to guarantee that they are satisfied at every point of time.

utilities and households' values that can be delivered in a competitive equilibrium. Through these two variables, future policies and allocations $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H})$ influence the choice of m_0 for $s_0 = s$. It is straightforward to check that $\Omega(s)$ is nonempty and compact.

Define

$$\Psi\left(s, \mu_s, V_s^H\right) = \Big\{\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_s | \mu_s = u'\left[f\left(x_0\left(s_0\right), s_0\right)\right]\left[x_0\left(s_0\right) + m_0\left(s_0\right)\right] \text{ and}$$
$$V_s^H = u\left(f\left(x\left(s_0\right), s_0\right) + v\left(m\left(s_0\right)\right) - \beta\theta \log \sum_{s_1} \pi\left(s_1|s_0\right) \exp\left(-V_1^H\left(s_1\right)/\theta\right)\Big\}.$$

$\Psi\left(s, \mu_s, V_s^H\right)$ delivers the competitive equilibrium sequences $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right)$ associated with an initial marginal utility μ_s and an initial lifetime utility for the households V_s^H for initial $s_0 = s$. If we know sets $\Omega(s)$ and $\Psi\left(s, \mu_s, V_s^H\right)$, we could solve the Ramsey problem for our paternalistic government in (12) for $s_0 = s$ in two steps as follows. First, we solve the Ramsey problem when the time zero shock realization is s and the initial marginal utility and households' value are μ_s and V_s^H, respectively,

$$V^{G*}(s, \mu_s, V_s^H) = \max_{(\mathbf{m},\mathbf{x},\mathbf{h},\mathbf{d},\mathbf{V^H})} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t)\left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right))\right]. \tag{14}$$
$$s.t. \quad \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in \Psi\left(s, \mu_s, V_s^H\right)$$

Let $\mu = [\mu_1, \mu_2, ..., \mu_S]$ and $V^H = \left[V_1^H, V_2^H, ..., V_S^H\right]$ be the vectors of state-contingent marginal utilities and households' utilities, respectively. Notice that $\mu_s \in [0, \overline{\mu_s}]$ for some $\overline{\mu_s}$, $\forall s \in \mathbb{S}$. Also, given that the period payoffs are bounded, it follows that $V_s^H \in \left[\underline{V}_s^H, \overline{V}_s^H\right]$, for some bounds $\underline{V}_s^H, \overline{V}_s^H$. The primes are used to denote next-period values.

The next proposition formulates the Ramsey problem (14) with a recursive structure that can be solved using dynamic programming techniques.

Proposition 4.1. *$V^{G*}\left(s, \mu_s, V_s^H\right)$ satisfies the following Bellman equation*

$$V^G\left(s, \mu_s, V_s^H\right) = \max_{(m,x,h,\mu',V^{H\prime})} \left[u\left(f\left(x, s\right)\right) + v(m)\right] + \beta \sum_{s'} \pi(s'|s) w_{s'}\left(s', \mu'_{s'}, V_{s'}^{H\prime}\right) \tag{15}$$
$$(m, x, h) \in \mathbb{M} \times \mathbb{X} \times \Pi \text{ and } \left(\mu'_{s'}, V_{s'}^{H\prime}\right) \in \Omega\left(s'\right) \text{ for all } s'$$
$$\mu_s = u'\left[f(x, s)\right]\left[x + m\right] \tag{16}$$
$$V_s^H = u\left(f(x, s)\right) + v\left(m\right) - \beta\theta \log \sum_{s'} \pi\left(s'|s\right) \exp\left(\frac{-V_{s'}^{H\prime}}{\theta}\right) \tag{17}$$
$$-x = m\left[1 - h\right] \tag{18}$$
$$m\left\{u'(f(x, s)) - v'(m)\right\} = \beta \sum_{s'} \pi(s'|s) \frac{\exp\left(-\frac{V_{s'}^{H\prime}}{\theta}\right)}{\sum_{s'} \pi(s'|s) \exp\left(-\frac{V_{s'}^{H\prime}}{\theta}\right)} \mu'_{s'}, \quad \leq \text{ if } m = \overline{m}. \tag{19}$$

Conversely, if a bounded function $V^G : \mathbb{S} \times \Omega(s) \to \mathbb{R}$ satisfies the above Bellman equation, then it is solution of (14).

Proof. Based on the Bellman principle of optimality, this is a straightforward extension of Chang (1998), p. 457, and is left to the reader. □

In the recursive Ramsey problem given by (15) it is clear to see how when maximizing its utility in any period $t > 0$ the government is bounded by its previous-period promises of marginal utility and households' value (μ, V^H). From the households' perspective, these promises were key when choosing real balances at $t-1$. To maximize their utility, the time $t-1$ Euler equation has to hold. Under commitment, these promises must be delivered at t thereby conditioning government's choice in that period. In this way, the government guarantees that households' Euler equation is satisfied in every period. Through the dynamics of the promised marginal utility and households' value, which the government has to manage to deliver in equilibrium, the Ramsey plan exhibits history dependence. Once we have solved the recursive Ramsey problem, the following step has to be undertaken

$$V^{G*}(s) = \max_{(\mu_s, V_s^H)\in\Omega(s)} V^{G*}\left(s, \mu_s, V_s^H\right). \tag{20}$$

In contrast with the other periods, there is no promised (μ_s, V_s^H) to be delivered in the first period. Hence, the government is free to choose the initial vector $\left(\mu_s, V_s^H\right)$.[7]

To solve the recursive problem stated in Proposition 4.1, it is necessary to know in advance the value correspondence Ω. In what follows we provide a procedure for the computation of Ω as the largest fixed point of a specific value correspondence operator, as proposed by Kydland and Prescott (1980).

Let $\mathcal{G}$ be the space of all the correspondences Ω, and let Q live in it. Let the operator $B : \mathcal{G} \to \mathcal{G}$ be defined as follows,

$$\begin{aligned} B(Q)(s) \quad = \quad & \left\{\left(\mu_s, V_s^H\right) \in \mathbb{R}\times\mathbb{R} \middle| \; \exists \left(m, x, h, \mu', V^{H\prime}\right) \in \mathbb{M}\times\mathbb{X}\times\Pi\times Q \text{ such that} \right. \\ & \left. (16)\text{-}(19) \;\; \text{hold}\right\}. \end{aligned}$$

By picking vectors of continuation marginal utilities and households' values $(\mu', V^{H\prime})$ from Q, the operator B computes the set of current marginal utilities and households' values (μ_s, V_s^H) for each shock realization s that are consistent with the competitive equilibrium conditions. The operator B is a monotone operator in the sense that $Q(s) \subseteq Q'(s)$ implies $B(Q)(s) \subseteq B(Q')(s)$.

The next proposition states that the set in question, $\Omega(s)$, is the largest fixed point of the operator B. Moreover, it states that $\Omega(s)$ can be computed by iterating on the operator B until convergence, given that we start from an initial set $Q_0(s)$ that is sufficiently large.

[7] The fact that $\left(\mu_s, V_s^H\right)$ can be set by the government at time 0 explains why we refer to it as *pseudo*-state variables.

Let $Q_0(s) = [0, \overline{\mu_s}] \times \left[\underline{V}_s^H, \overline{V}_s^H\right]$. Clearly, it satisfies $B(Q_0)(s) \subseteq Q_0(s)$. Given the monotonicity property, by applying successively the operator B, we can construct a decreasing sequence $\{Q_n(s)\}_{t=0}^{\infty}$ for each $s \in \mathbb{S}$, where $Q_n(s) = B\left(Q_{n-1}\right)(s)$. The limiting sets are given by $Q_\infty(s) = \cap_{n=0}^{\infty} Q_n(s)$ for $n = 1, 2, ...$

Proposition 4.2.

(i) $Q(s) \subseteq B\left(Q\right)(s) \Rightarrow B\left(Q\right)(s) \subseteq \Omega(s)$

(ii) $\Omega(s) = B\left(\Omega\right)(s)$

(iii) $\Omega(s) = \lim_{n\to\infty} B^{\infty}(Q_0)(s)$.

Proof. Simple extension of the argument in Chang (1998). □

Once we have computed Ω, we can solve the recursive Ramsey problem stated in Proposition (4.1) which clearly yields a Ramsey plan with a recursive representation. The resulting Ramsey plan consists of an initial vector (μ_s, V_s^H), given by the solution to (20), and a five-tuple of functions (h, x, m, μ, V^H) mapping (s, μ_s, V_s^H) into current period's (h, x, m), and next period's state-contingent (μ, V^H), respectively,

$$\begin{aligned}
h_t &= h\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
x_t &= x\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
m_t &= m\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
\mu_{t+1} &= \psi\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
V_{t+1}^H &= \varpi\left(s_t, \mu_t(s_t), V_t^H(s_t)\right).
\end{aligned}$$

As it turns out, the solution to the Ramsey problem is time-inconsistent. In this environment, the government would implement a transitory deflationary process along with a contracting money supply $\{M_t(s^t)\}_{t=0}^{\infty}$ so as to increase the real money holdings towards its satiation level, $\overline{m}$. To achieve this, it would have to collect tax revenues to satisfy its balanced budget constraint (7), which at the same time would entail tax distortions in the form of output costs. At the beginning of time zero, taking prices $\{q_t(s^t)\}_{t=0}^{\infty}$ and taxes $\{x_t(s^t)\}_{t=0}^{\infty}$ as given, the household chooses once and for all her sequence of real balances $\{m_t(s^t)\}_{t=0}^{\infty}$. If the government was allowed to revisit its policy at time $T > 0$, after history s^t, given households' choice $\{m_t(s^t)\}_{t=0}^{\infty}$, the government would find it optimal not to adhere to what the original plan prescribes from then on, $\{M_t(s^t|s^T)\}_{t=T}^{\infty}$, but to deviate to an alternative $\{\widetilde{M}_t(s^t|s^T)\}_{t=T}^{\infty}$ by reducing the money supply more gradually. These incentives arise from the fact that tax distortions are an increasing and convex function of tax collections, as indicated in assumption [A6].

5 Sustainable Plans with Model Uncertainty

From now on, we proceed under the assumption that the government cannot commit to its announced sequence of money supply growth rates. Instead, it will be choosing its policy actions sequentially in each state.[8]

As originally studied by Calvo (1978) and explained in section 4, in this case the government faces a credibility problem. To study the optimal credible policies in this context, we make use of the notion of *sustainable plans*, developed by Chari and Kehoe (1990). The notion of a sustainable plan inherits sequential rationality on the government's side, combined with the fact that households are restricted to choose from competitive equilibrium allocations.[9]

In this section, we extend the notion of sustainable plans of Chari and Kehoe (1990) to incorporate model uncertainty.

Let $h^t = (h_0, h_1, ..., h_t)$ be the history of the (inverse) money growth rates in all the periods up to t. A *strategy for the government* can be defined as $\sigma^G \equiv \{\sigma_t^G\}_{t=0}^{\infty}$, with $\sigma_0^G : \mathbb{S} \to \Pi$ and $\sigma_t^G : \mathbb{S}^t \times \Pi^{t-1} \to \Pi$ for all $t > 0$. We restrict the government to choose a strategy σ^G from the set CE_s^{Π}, where CE_s^{Π} is defined as

$$CE_s^{\Pi} = \left\{ h \in \Pi^{\infty} | \text{ there is some } \left(\mathbf{m}, \mathbf{x}, \mathbf{d}, \mathbf{V^H}\right) \text{ such that } \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_s \right\}.$$

CE_s^{Π} is the set of sequences of money growth rates consistent with the existence of competitive equilibria, given $s_0 = s$. It is straightforward to establish that this set is nonempty and compact.

The restriction above is equivalent to forcing the government to choose after any history h^{t-1}, s^t a period t money supply growth rate from $CE_{s_t}^{\Pi,0}$, where $CE_s^{\Pi,0}$ is given by

$$CE_s^{\Pi,0} = \{h \in \Pi : \text{ there is } \mathbf{h} \in CE_s^{\Pi} \text{ with } h(0) = h\}.$$

An *allocation rule* can be defined as $\alpha \equiv \{\alpha_t\}_{t=0}^{\infty}$ such that $\alpha_t : \mathbb{S}^t \times \Pi^t \to \mathbb{M} \times \mathbb{D} \times \mathbb{X}$ for all $t \geq 0$. The allocation rule α assigns an action vector $\alpha_t(s^t, h^t) = (m_t, x_t, d_t)(s^t, h^t)$ for current real balances, tax collections, and distortions to households' beliefs about the next state s_{t+1}.

[8] We can think instead of this environment as having a sequence of government "administrations" with the time t, history s^t administration choosing only at time t, history s^t government action given its forecasts of how future administrations will act. The time t, history s^t administration intends to maximize the government's lifetime utility only in that particular node.

[9] From a game theoretical perspective, the notion of a sustainable plan entails subgame perfection in a game between a large player (government) and a continuum of atomistic players (households), who cannot coordinate, and are, thus, price-takers.

Definition 5.1. *A government strategy, σ^G, and an allocation rule α, are said to constitute a* sustainable plan with model uncertainty *(SP) if after any history s^t and h^{t-1}*

(i) *(σ^G, α) induce a competitive equilibrium sequence;*

(ii) *given σ^H, it is optimal for the government to follow the continuation of σ^G , i.e. the sequence of continuation future induced by σ^G maximizes*

$$\sum_{j=t}^{\infty} \beta^{j-t} \sum_{s^j|s^t} \pi_j(s^j|s^t) \left[u\left(c_j\left(s^j\right)\right) + v(m_j\left(s^j\right)) \right] \quad \textit{over the set } \; CE_s^{\Pi}.$$

Condition (i) states that after any history s^t, h^t, even if at some point in the past the government has disappointed households' expectations about money growth rates, all economic agents choose actions consistent with a competitive equilibrium. Condition (ii) guarantees that the government attains weakly higher lifetime utility after any history by adhering to σ^G.

Any sustainable plan with model uncertainty (σ^G, α) can be factorized after any history into a current period action profile, a, and a vector $(V^{G\prime}(h), V^{H\prime}(h), \mu'(h))$ of state-contingent continuation values for the government, and for the representative household, and promised marginal utilities, as a function of money growth rate h. The action profile a in our context is given by $a = (\widehat{h}, m(h), x(h), d'(h))$. That is,the action profile a assigns:

(i) an (inverse) money growth rate $\widehat{h}$ that the government is instructed to follow

(ii) a reaction function $m : \Pi \to [0, \overline{m}]$ for the real money holdings chosen by households. If the government adheres to the plan and executes recommended $\widehat{h}$, households respond by acquiring $m(\widehat{h})$ real balances. Otherwise, if the government deviates from the sustainable plan and select any $h \neq \widehat{h}$, households react by selecting $m(h)$.

(iii) a tax allocation rule $x : \Pi \to \mathbb{X}$. Taxes revenues are determined in equilibrium as a residual of money growth and money holdings in order to satisfy the government's budget constraint (6).

(iv) a reaction function $d : \Pi \to \mathbb{D}$ for the beliefs' distortions set by the evil alter ego.

The vector $(V^{G\prime}(h), V^{H\prime}(h), \mu'(h))$ reflects how continuation outcomes are affected by the current choice h of the government through the effect it has on households' expectations and thereby on future prices. Given the timing protocol within the period, households' response or punishment to a government deviation $h \neq \widehat{h}$ consists of an action $m(h)$, typically different from $m(\widehat{h})$, in the same period, followed by subsequent actions and associated future equilibrium prices, the impact of which is captured by $(V^{G\prime}(h), V^{H\prime}(h), \mu'(h))$.

In our context, the sustainable plans combine two sources of history dependence. In addition to the one embedded in the dynamics of the marginal utilities, as in the Ramsey plan, there is a new source of history dependence arising from the restrictions that a system of households' expectations impose on the government's policy actions. As the government after any history is allowed to revisit its announced policy and reset it, households expect that the government will adhere to the original plan only if it is in its own interest to do so.

Let $\mathbb{A}(s)$ be given by

$$\mathbb{A}(s) = \left\{ \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_s | \text{ there is a SP whose outcome is } \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \right\}.$$

Let $\Re^3$ be the space of all the subsets of $\mathbb{R}^3$. We define the value correspondence $\Lambda : \mathbb{S} \longrightarrow \mathcal{R}^3$ as

$$\begin{aligned}
\Lambda(s) \quad = \quad & \Big\{ \left(V_s^G, V_s^H, \mu_s\right) | \text{ there is a } \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in \mathbb{A}(s) \text{ with} \\
& V_s^G = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right))\right], \\
& V_s^H = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) D_t(s^t) \big\{ \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right))\right] \\
& \qquad + \theta\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) \log d_{t+1}(s_{t+1}|s^t) \big\}, \\
& \mu_s = u'\left[f(x_0\left(s_0\right), s_0)\right]\left[x_0\left(s_0\right) + m_0\left(s_0\right)\right] \Big\}.
\end{aligned}$$

For each $s \in \mathbb{S}$, $\Lambda(s)$ constitutes the set of vectors of equilibrium values for the government and the household, and the promised marginal utilities given state s that can be delivered by a sustainable plan. We denote as $\widehat{\mathcal{G}}$ the space of all such correspondences.

Definition 5.2. *For any correspondence $Z \subset \widehat{\mathcal{G}}$, $(a, V^{G\prime}(\cdot), V^{H\prime}(\cdot), \mu'(\cdot))$ is said to be admissible with respect to Z at state s if*

(i) $a = (\widehat{h}, m(h), x(h), d'(h)) \in \Pi \times [0, \overline{m}]^\Pi \times X^\Pi \times \mathbb{R}^\Pi$*;*

(ii) $(V_{s'}^{G\prime}(h), V_{s'}^{H\prime}(h), \mu'_{s'}(h)) \in Z(s') \qquad \forall\ h \in CE_s^{\Pi,0}, s' \in \mathbb{S}$*;*

(iii) (18)-(19) are satisfied;

(iv) $u(f(x(\widehat{h}), s)) + v(m(\widehat{h})) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V_{s'}^{G\prime}(\widehat{h}) \geq$
$u(f(x(h), s)) + v(m(h)) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V_{s'}^{G\prime}(h) \qquad \forall h \in CE_s^{\Pi,0}.$

Condition (i) ensures that a belongs to the appropriate action space. Condition (ii) guarantees that for any h that the government contemplates the vector of continuation values and promised marginal utility for next period's shock s' belongs to the corresponding set $Z(s')$. Condition (iii) imposes the competitive equilibrium conditions in the current period. Finally, condition (iv) describes the incentive constraint for the government in the current period. This incentive constraint deters the government from taking one-period deviations when contemplating money growth rates h other than prescribed $\widehat{h}$. If condition (iv) holds, it follows from the "one-period deviation principle" that there are no profitable deviations at all. A plan is credible if the government finds it is in its own interest to confirm households' expectations about its policy action $\widehat{h}$. Condition (iv) guarantees that this is the case.

In what follows we explain how to compute the equilibrium value sets $\Lambda(s)$. Let $Z \subset \widehat{\mathcal{G}}$. In the spirit of Abreu, Pearce, and Stacchetti (1990) we construct the operator $\widehat{B} : \widehat{\mathcal{G}} \longrightarrow \widehat{\mathcal{G}}$ as follows

$$\widehat{B}(Z)(s) = co\Big\{(V_s^G, V_s^H, \mu_s) | \exists \text{ admissible } (a, V^{G\prime}(\cdot), V^{H\prime}(\cdot), \mu'(\cdot)) \text{ with respect to } Z \text{ at } s:$$
$$a = (\widehat{h}, m(h), x(h), d'(h))$$
$$V_s^G = u(f(x(\widehat{h}), s)) + v(m(\widehat{h})) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V_{s'}^{G\prime}(\widehat{h})$$
$$V_s^H = u(f(x(\widehat{h}), s)) + v(m(\widehat{h})) - \beta\theta \log \sum_{s' \in \mathbb{S}} \pi(s'|s) \exp\left\{ -\frac{V_{s'}^{H\prime}(\widehat{h})}{\theta} \right\}$$
$$\mu_s = u(f(x(\widehat{h}), s))(x(\widehat{h}) + m(\widehat{h}))\Big\}.$$

For each $s \in \mathbb{S}$, $\widehat{B}(Z)(s)$ is the convex hull of the set of vectors (V_s^G, V_s^H, μ_s) that can be sustained by some admissible action profile a and vectors $(V_s^{G\prime}, V_s^{H\prime}, \mu_s')$ of continuation values and marginal utilities in $Z(s')$ for each state s' next period.

We assume that there exists a public randomization device. In particular, we assume that every period an exogenous, serially uncorrelated public signal $\widetilde{X}_t$ is drawn from a $[0, 1]$ uniform distribution. Depending on current actions, this signal will determine which equilibrium will be played next period.

The following propositions are simple adaptations of Abreu, Pearce, and Stacchetti (1990) for repeated games and establish some useful properties of the operator. Together, these properties guarantee that the equilibrium value correspondence Λ is its largest fixed point and can be found by iterating on this operator.

PROPOSITION 5.1. *Monotonicity: $Z \subseteq Z'$ implies $B(Z) \subseteq B(Z')$.*

Proof. The proof is a simple extension of that in Chang (1998). □

PROPOSITION 5.2. *Self-Generation: If $Z(s)$ is bounded and $Z(s) \subseteq B(Z)(s)$, then $B(Z)(s) \subseteq \Lambda(s)$.*

Proof. We need to construct a subgame perfect strategy profile (σ^G, σ^H) such that

(i) for each $s \in \mathbb{S}$ it delivers a lifetime utility value V_s^G to the government, V_s^H to a representative household with an associated marginal promised utility μ_s,

(ii) the associated outcome of the SP satisfies (18)-(19)

(iii) government's incentive constraint holds for every history (s^t, h^{t-1}).

To do so, fix an initial state s and consider any $(V_s^G, V_s^H, \mu_s) \in B(Z)(s)$. Let $(V_0^G, V_0^H, \mu_0) = (V_s^G, V_s^H, \mu_s)$ and define (σ^G, σ^H) recursively as follows.

Let $(V_t^G(h^{t-1}, s^{t-1}, s_t), V_t^H(h^{t-1}, s^{t-1}, s_t), \mu_t(h^{t-1}, s^{t-1}, s_t)) \in Z(s_t)$ be the vector of values and marginal utilities after an arbitrary history (h^{t-1}, s^{t-1}, s_t). Since $Z \subset B(Z)$, for each $s \in \mathbb{S}$ there exists an admissible vector $(\widehat{h}, m(h), x(h), d'(h), V^{G\prime}(h), V^{H\prime}(h), \mu'(h))$ with respect to Z at s. Define $\sigma_t^G(h^{t-1}, (s^{t-1}, s_t)) = \widehat{h}$ and $\widehat{m} = m(h)$. Let $\alpha_t(h^{t-1}, (s^{t-1}, s_t)) = (m(h), m(h)(h-1), d'(h))$ if $h \in CE_{s_t}^{\Pi,0}$ and $= (0, 0, d'^{NM}$ otherwise, where d'^{NM} are the probability distortions corresponding to the nonmonetary equilibrium.[10]

Also, define $(V_{t+1}^G(h^t, s^t, s_{t+1}), V_{t+1}^H(h^t, s^t, s_{t+1}), \mu_{t+1}(h^t, s^t, s_{t+1})) = (V_{s_{t+1}}^{G\prime}(h), V_{s_{t+1}}^{H\prime}(h), \mu'_{s_{t+1}}(h))$ if $h \in CE_{s_{t+1}}^{\Pi,0}$; $(V_{t+1}^G(h^t, s^t, s_{t+1}), V_{t+1}^H(h^t, s^t, s_{t+1}), \mu_{t+1}(h^t, s^t, s_{t+1})) = (V_{s_{t+1}}^{GNM}, V_{s_{t+1}}^{HNM}, \mu_{s_{t+1}}^{NM})$ otherwise. Clearly, $(V_{t+1}^G(h^t, s^t, s_{t+1}), V_{t+1}^H(h^t, s^t, s_{t+1}), \mu_{t+1}(h^t, s^t, s_{t+1})) \in Z(s_{t+1})$. By admissibility, (σ^G, α) is unimprovable and, thus, is subgame perfect. Since $Z(s)$ is bounded for every $s \in \mathbb{S}$, it is straightforward to show that (σ^G, α) delivers (V_s^G, V_s^H, μ_s). Also, admissibility of vectors $(\widehat{h}, m(h), x(h), d'(h), V^{G\prime}(h), V^{H\prime}(h), \mu(h))$ ensures that the equilibrium conditions are satisfied along the equilibrium path. □

Proposition 5.3. *Factorization:* $\Lambda = B(\Lambda)$.

Proof. By the previous proposition, it is sufficient to show that $\Lambda(s)$ is bounded and that $\Lambda(s) \subset B(\Lambda)(s)$. The result follows from the fact that the continuation of a sustainable plan is also a sustainable plan. The boundness of $\Lambda(s)$ follows from (i) the fact that any lifetime utility for the government is the expected discounted sum of one-period bounded payoffs; (ii) any lifetime utility for the household can be bounded by discounted sums of non-stochastic extremal one-period payoffs, and (iii) marginal utilities are determined by continuous functions f, u' over compact sets. □

Proposition 5.4. *If $Z(s)$ is compact for each $s \in \mathbb{S}$, then so is $B(Z)(s)$.*

[10] Even though the continuation outcome in case the government selects h not belonging to $CE_{s_t}^{\Pi,0}$ is irrelevant for the solution (since it cannot occur by assumption), to be rigorous we need to specify the moves after any history. If the government executes h not in $CE_{s_t}^0$ we assume that the economy switches to the nonmonetary equilibrium.

Proof. Let us show first that $B(Z)(s)$ is bounded. Let $\overline{Z}$ be a value correspondence in $\widehat{\mathcal{G}}$. Define the operators $\Upsilon_{i,s} : \widehat{\mathcal{G}} \longrightarrow \mathcal{R}$ for $i = 1, 2$, where $\Re$ is the space of subsets in $\mathbb{R}$,

$$\begin{aligned} \Upsilon_{1,s}(\overline{Z}) &= \left\{ V_s^G : \exists (V_s^G, V_s^H, \mu_s) \in \overline{Z}(s) \right\} \\ \Upsilon_{2,s}(\overline{Z}) &= \left\{ V_s^H : \exists (V_s^G, V_s^H, \mu_s) \in \overline{Z}(s) \right\}. \end{aligned}$$

Boundness of $B(Z)(s)$ follows from having

$$\begin{aligned} \Upsilon_{1,s}(B(Z)) &\subset U_s^0 + \beta \sum_{s'} \pi(s'|s) \Upsilon_{1,s'}(Z) \\ \Upsilon_{2,s}(B(Z)) &\subset U_s^0 - \beta\theta \log \sum_{s'} \pi(s'|s) \exp\left(-\Upsilon_{2,s'}(Z)/\theta\right), \end{aligned}$$

where the sets of one-period payoffs U_s^0 (for current state s), and $\Upsilon_{i,s'}(Z)$ for $i = 1, 2$ are bounded.

Let us show now that $B(Z)(s)$ is closed. Consider any sequence $\left\{ (V^{Gn}, V^{Hn}, \mu^n) \right\}_{n=1}^{+\infty}$ such that $(V_t^{Gn}(s^{t-1}, s_t), V_t^{Hn}(s^{t-1}, s_t), \mu_t^n(s^{t-1}, s_t)) \in B(Z)(s_t) \quad \forall s^{t-1} \in \mathbb{S}^{t-1}, s_t \in \mathbb{S}$ that converges to some (V^{G*}, V^{H*}, μ^*). Fix an arbitrary sequence of states $\{s_t\}_{t=0}^{+\infty}$. We need to show that

$$(V^{G*}(s^{t-1}, s_t), V^{H*}(s^{t-1}, s_t), \mu^*(s^{t-1}, s_t)) \in B(Z)(s_t) \ \forall s^{t-1} \in \mathbb{S}^t, s_t \in \mathbb{S}.$$

For each $(V_t^{Gn}(s^{t-1}, s_t), V_t^{Hn}(s^{t-1}, s_t), \mu_t^n(s^{t-1}, s_t))$, there exists an admissible vector $(\widehat{h}^n, m^n(h), x^n(h), d'^n(h), V^{Gn\prime}(h), V^{Hn\prime}(h), \mu^{n\prime}(h))$ with respect to Z at s. This vector should be indexed by histories of shocks s^t. In particular, $\widehat{h}_t^n(s^t) = \widehat{h}^n$. Since $\{s_t\}_{t=0}^{+\infty}$ is fixed, we slightly abuse the notation and refer to $\widehat{h}_t^n(s^t)$ as just $\widehat{h}_t^n$. Without loss of generality, we assume that $\widehat{h}_t^n$ converges to some $\widehat{h}_t^* \in CE_{s_t}^{\Pi,0}$. In a similar way, for each $h \in CE_{s_t}^{\Pi,0}$, $(m^n(h), x^n(h), d'^n(h), V^{Gn\prime}(h), V^{Hn\prime}(h), \mu^{n\prime}(h)) \longrightarrow (m^*(h), x^*(h), d'^*(h), V^{G\prime}(h)^*, V^{H\prime}(h)^*, \mu'(h)^*)$ where $(m^*(h), x^*(h), d'^*(h)) \in [0, \overline{m}] \times \mathbb{X} \times \mathbb{D}$ and $(V_{s'}^{G\prime}(h)^*, V_{s'}^{H\prime}(h)^*, \mu'_{s'}(h)^*) \in Z(s')$ $\forall s' \in \mathbb{S}$, by compactness of $[0, \overline{m}] \times \mathbb{X} \times \mathbb{D}$ and $Z(s') \quad \forall s' \in \mathbb{S}$. By the continuity of functions u, v, f, u', v', it is straightforward to check that $(m^*(h), x^*(h), d'^*(h), V^{G\prime}(h)^*, V^{H\prime}(h)^*, \mu'(h)^*)$ satisfies conditions (18)-(19). It follows then that $(V^{G*}(s^{t-1}, s_t), V^{H*}(s^{t-1}, s_t), \mu^*(s^{t-1}, s_t)) \in B(Z)(s_t)$. □

6 Computational Algorithm

In this section we describe how to implement the operator $\widehat{B}$ on the computer in order to compute the equilibrium value correspondence Λ. Our computational algorithm is based on an outer approximation of the value sets and is a straightforward adaptation of the approach developed by Judd, Yeltekin, and Conklin (2003).

Several techniques have been applied to find the equilibrium value sets in different environments. Chang (1998) uses an approach based on the discretization of both the space of actions and the space of continuation values and promised marginal utilities. This technique in our case suffers from a severe curse of dimensionality. Instead, the method proposed by Judd, Yeltekin, and Conklin (2003) discretizes only the action space and by solving optimization problems approximates the value sets in question using hyperplanes.[11] In contrast with the other approach, in our case it is necessary to introduce a public randomization device to convexify the value sets.

6.1 Monotone Outer Hyperplane Approximation

We start by discretizing the action space. Let $m_{grid} = [m_1, ..., m_{N_m}]$ be the grid for real balances with N_m gridpoints, such that $m_1 = 0$ and $m_{N_m} = \overline{m}$. Also, we define $h_{grid} = [h_1, ..., h_{N_h}]$, as the grid for money growth rates with N_h gridpoints such that $h_1 = \underline{\pi}$ and $h_{N_h} = \overline{\pi}$.

Consider then a set of D hyperplanes. Each hyperplane is represented by a subgradient $g_i = (g_{l,1}, g_{l,2}, g_{l,3}) \in \mathbb{R}^3$, and a hyperplane level $c_{l,s} \in \mathbb{R}$ for $l = 1, ..., D$. Let $G = \{g_1, ..., g_D\}$ be the vector of subgradients and let $C_s = (c_{1,s}, ..., c_{D,s})$ be the vector of hyperplane levels for state s. For simplicity, we will use the same set of subgradients G in all our approximations. The vector of hyperplane levels, C_s, however, will be state-specific and will be updated after each approximation. The outer approximation of any $W(s) \subset \mathbb{R}^3$ is given by the smallest convex polytope $\widehat{W}(s)$, generated by a set of hyperplanes, that contains $W(s)$. The convex polytope $\widehat{W}(s)$ is determined as the intersection of half-spaces defined by these hyperplanes, i.e.

$$\widehat{W}(s) = \cap_{l=1}^{l=D} \left\{ w \in \mathbb{R}^3 | g_l \cdot w \leq c_{l,s} \right\}. \tag{21}$$

Table 1 displays the algorithm we use to perform the outer approximation.

To initialize the algorithm it is necessary to find a candidate correspondence Z^0 such that for all s, $Z^0(s)$ contains the equilibrium value set $\Lambda(s)$ and $B(Z^0)(s) \subseteq Z^0(s)$. Our candidate

[11] See Fernández-Villaverde and Tsyvinski (2002) for an adaptation of this procedure to characterize the value sets in a dynamic capital taxation model without commitment.

Z^0 is given by the hypercube $[\underline{V}_s^G, \overline{V}_s^G] \times [\underline{V}_s^H, \overline{V}_s^H] \times [\underline{\mu}_s, \overline{\mu}_s]$, where

$$
\begin{aligned}
\underline{V}_s^G &= u(f(\overline{x}, s)) + v(0) + \beta \sum_{s'} \pi(s'|s) \underline{V}_{s'}^G \\
\overline{V}_s^G &= u(f(0, s)) + v(\overline{m}) + \beta \sum_{s'} \pi(s'|s) \overline{V}_{s'}^G \\
\underline{V}_s^H &= u(f(\overline{x}, s)) + v(0) - \beta\theta \log \sum_{s'} \pi(s'|s) \exp\left(-\underline{V}_{s'}^H/\theta\right) \\
\overline{V}_s^H &= u(f(0, s)) + v(\overline{m}) - \beta\theta \log \sum_{s'} \pi(s'|s) \exp\left(-\overline{V}_{s'}^H/\theta\right) \\
\underline{\mu}_s &= 0 \\
\overline{\mu}_s &= u'(f(\overline{x}, s)) \overline{m\pi}.
\end{aligned}
$$

Using the hyperplanes, in Step 0 we compute the initial vector of hyperplane levels C_s^0 corresponding to the outer approximation of each set $Z^0(s)$, denoted by $\widetilde{Z}^0(s)$, and input these in the algorithm. Each of these $Z^0(s)$ will be the set from which the first vectors $(V_{s'}^G, V_{s'}^H, \mu_{s'})$ of continuation values and promised marginal utilities are selected.

In Step 1, in iteration k we compute the convex polytope $\widetilde{Z}^k(s)$ by updating the vector of hyperplane levels C_s^k. To do so, we employ the value correspondence $\widetilde{Z}^{k-1}$ as input, for $s = 1, ..., S$. The set $\widetilde{Z}^k(s)$ is given by the convex hull of the set of vectors of current values (V_s^G, V_s^H, μ_s) that can be sustained by some admissible action profile and continuation values $(V^{G\prime}, V^{H\prime}, \mu')$ such that $(V_{s'}^{G\prime}, V_{s'}^{H\prime}, \mu'_{s'})$ belongs to $\widetilde{Z}^{k-1}(s')$. For the government's incentive constraint we do not need to consider all possible one-period deviations, but only the best one. Moreover, we impose the harshest punishment for the government following any deviation from the previously announced policy. The punishment may not be trivial and has to be determined endogenously, as shown in Step 1, part (a).[12] To compute the worst punishment for each s we undertake a two-step procedure. First, we fix the government's choice of the money growth rate, h, and choose m to minimize the government's value such that the competitive equilibrium conditions are satisfied and the vector of continuation values and promised marginal utilities is picked from $\widetilde{Z}^{k-1}(s')$ for each next period's s'. Second, we select the maximal value from this vector of government's values as function of h and denote it by $\underline{V}_s^G$. This value will be associated with the best deviation for the government for state s. Once we have formulated the government's incentive constraint, we proceed to compute $\widetilde{Z}^k(s)$ for $s = 1, ..., S$.

[12] If we knew the worst value in advance, we would be able to specify the right-hand side of the government's incentive constraint before solving the problem. Having an *ex ante* formulation of the incentive constraint would let us apply techniques in Marcet and Marimon (2009) and solve for the SP associated with the government's highest equilibrium value by deriving the corresponding recursive saddle point functional equation.

We repeat this step until the polytopes, or equivalently the updated vectors of hyperplane levels C_s, attain convergence.

Table 1: Monotone Outer Hyperplane Approximation

Step 0:		Approximate each $Z_0(s) \supset \Lambda(s)$.
		For each $s = 1, ..., S$, and $g_l \in G$, $l = 1, ..., D$, compute
		$c^0_{l,s} = \max g_{l,1} V^G_s + g_{l,2} V^H_s + g_{l,3} \mu_s, \qquad$ such that
		$\quad (V^G_s, V^H_s, \mu_s) \in Z_0(s)$.
		Let $C^0_s = \{c^0_{1,s}, ..., c^0_{D,s}\}$ for $s = 1, ..., S$.
Step 1:		Given C^k_s for $s = 1, ..., S$, update C^{k+1}_s.
		For each $s = 1, ..., S$, and $g_l \in G$, $l = 1, ..., D$,
	(a)	For each pair (m, h), solve
		$P^k_s(m,h) = \min_{(V^{G\prime}, V^{H\prime}, \mu')} u[f(x,s)] + v(m) + \beta \sum_{s' \in \mathbb{S}} \pi(s'\|s) V^{G\prime}_{s'}$,
		such that $\quad m[u'(f(x,s)) - v'(m)] = \beta \sum_{s' \in \mathbb{S}} \pi(s'\|s) d'_{s'} \mu'_{s'}$ with $\leq$ if $m = \overline{m}$
		$x = m(h-1)$
		$g_l \cdot (V^{G\prime}_{s'}, V^{H\prime}_{s'}, \mu'_{s'}) \leq c^k_{l,s'} \qquad$ for $s' = 1, ..., S$, $l = 1, ..., D$.
		Let $P^k_s(m,h) = +\infty$ if no $(V^{G\prime}, V^{H\prime}, \mu')$ satisfies the constraints.
		Let $R^k_s(h) = \min_m P^k_s(m,h)$. Let $\underline{V}^G_s = \max_{h \in \Pi} R^k_s(h)$.
	(b)	For each pair (m, h), solve
		$c^{k+1}_{l,s}(m,h) = \max_{(V^{G\prime}, V^{H\prime}, \mu')} g_{l,1} V^G_s + g_{l,2} V^H_s + g_{l,3} \mu_s, \qquad$ (P1)
		such that $\quad V^G_s = u[f(x,s)] + v(m) + \beta \sum_{s' \in \mathbb{S}} \pi(s'\|s) V^{G\prime}_{s'}$
		$V^H_s = u[f(x,s)] + v(m) - \beta\theta \log \sum_{s' \in \mathbb{S}} \pi(s'\|s) \exp\left\{-V^{H\prime}_{s'}/\theta\right\}$
		$\mu_s = u'[f(x,s)]\,(m + x)$
		$m[u'(f(x,s)) - v'(m)] = \beta \sum_{s' \in \mathbb{S}} \pi(s'\|s) d'_{s'} \mu'_{s'}$ with $\leq$ if $m = \overline{m}$
		$x = m(h-1)$
		$d'_{s'} = \exp\left\{-V^{H\prime}_{s'}/\theta\right\} / \sum_{s' \in \mathbb{S}} \pi(s'\|s) \exp\left\{-V^{H\prime}_{s'}/\theta\right\}$
		$V^G_s \geq \underline{V}^G_s$
		$g_l \cdot (V^{G\prime}_{s'}, V^{H\prime}_{s'}, \mu'_{s'}) \leq c^k_{l,s'} \qquad$ for $s' = 1, ..., S$, $l = 1, ..., D$.
		where $c^{k+1}_{l,s}(m,h) = -\infty$ if no $(V^{G\prime}, V^{H\prime}, \mu')$ satisfies the constraints.
		Let $(V^{G\prime}, V^{H\prime}, \mu')_{l,s}(m,h) \in \mathbb{R}^{S \times 3}$ be the solution to (P1).
	(c)	For each $s = 1, ...S$, and $l = 1, ..., D$, define
		$c^{k+1}_{l,s} = \max_{(m,h)} c^{k+1}_{l,s}(m,h)$
		$(m^*, h^*)_{l,s} = \arg\max_{(m,h)} c^{k+1}_{l,s}(m,h)$
		Update C^{k+1}_s as $C^{k+1}_s = \{c^{k+1}_{1,s}, ..., c^{k+1}_{D,s}\}$ for $s = 1, ..., S$
Step 2:		Stop if $\max_{l,s} \|c^{k+1}_{l,s} - c^k_{l,s}\| < 10^{-6}$; otherwise go to Step 1.

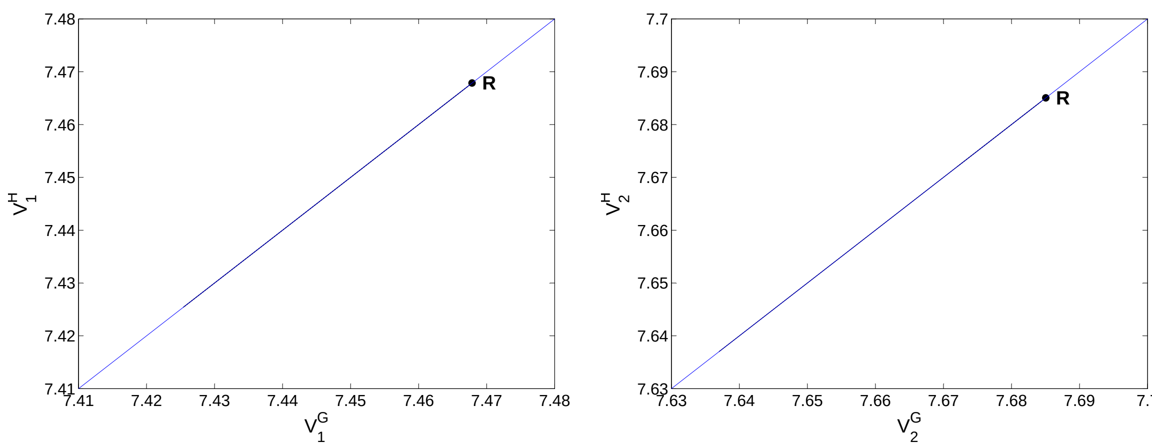
6.2 Numerical Results

In this section we present a numerical example. Assume that $S = 2, N_m = 31, N_h = 8$. We assume the following functional forms and parameter values:

$$\overline{m} = m^f = 30$$
$$\underline{\pi} = 0.75, \quad \overline{\pi} = 2.1$$
$$u(c) = \log c$$
$$v(m) = \frac{1}{500}(m\overline{m} - 0.5m^2)^{0.5}$$
$$f(x, s) = (0.8 + 0.2s)(180 - (0.4x)^2)$$
$$\pi(s' = 1|s = 1) = \pi(s' = 2|s = 2) = 0.75$$

To implement the computational algorithm we use $D = 116$ hyperplanes, with equally spaced subgradients. We assume a discount factor $\beta = 0.3$.[13] Assuming such a high degree of impatience on the part of the government and households lets us observe some intriguing features regarding the sustainability of equilibrium outcomes. It is worth noticing that each equilibrium value can be supported by multiple equilibrium strategies. The characterization of the equilibrium value sets, however, will shed some light on how severe the time-inconsistency issue is with and without uncertainty aversion.

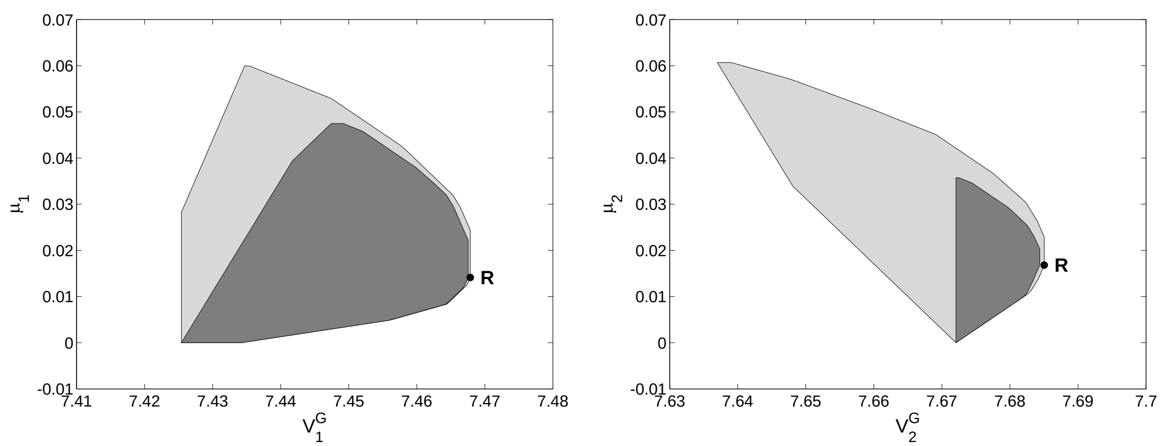
Figure 1: Government's and Households' Equilibrium Values for $\theta = +\infty$ for $s = 1$ (left panel) and $s = 2$ (right panel) with Commitment (light grey area) and without Commitment (dark grey area)



[13]For this numerical example we violate assumption [A5] with respect to having $\frac{1}{\beta} < \overline{\pi}$. We present this example only for illustrative purposes.

We first plot the equilibrium value set for each state s for $\theta = +\infty$ (i.e. households trust the approximating model), both for the case when the government can commit to its announced policies and when it cannot. Figure 1 presents the combinations of government and households' equilibrium values, for $s = 1$ (left panel) and $s = 2$ (right panel), with and without commitment. As expected, these equilibrium values are perfectly aligned along the 45-degree line. In figure 2 we plot the projection of the equilibrium value sets for each s onto the government's value and marginal utilities. The value of the Ramsey plan is marked with an R. Notice that the equilibrium value set without commitment strictly contains the set of values when the government is unable to commit. Without model misspecification, the Ramsey outcome is not sustainable when the government is allowed to choose sequentially. In other words, the Ramsey plan, entailing a gradual deflationary process to bring the real money holdings to their satiation level, is time-inconsistent when $\theta = +\infty$.

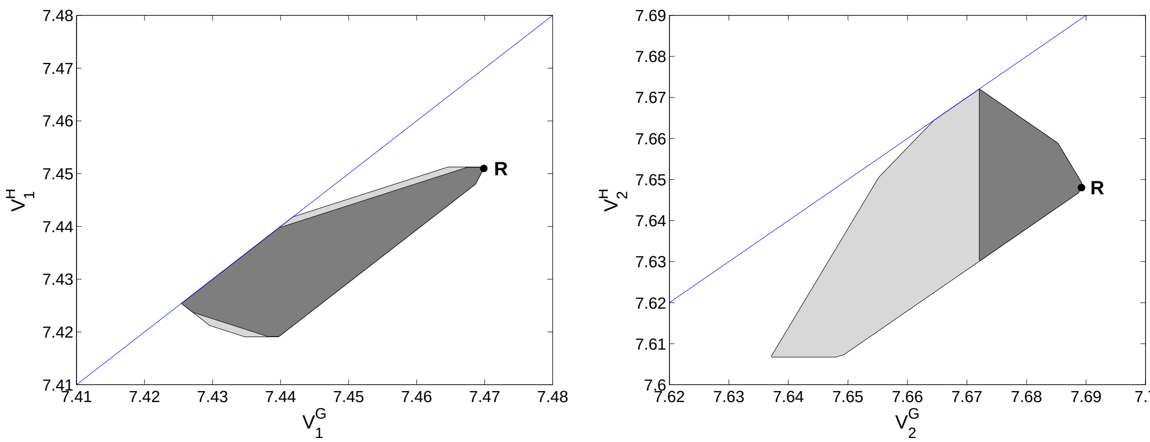
Figure 2: Government's Equilibrium Values and Marginal Utilities for $\theta = +\infty$ for $s = 1$ (left panel) and $s = 2$ (right panel) with Commitment (light grey area) and without Commitment (dark grey area)



Notice also that there is a large range of values, associated with a particularly low utility for the government, that can be delivered only under commitment. These values are associated with monetary policies that involve both alternating monetary contractions and expansions, which end up leaving the money supply practically unaltered, generating negative welfare implications for the households due to the tax distortions incurred along the way.

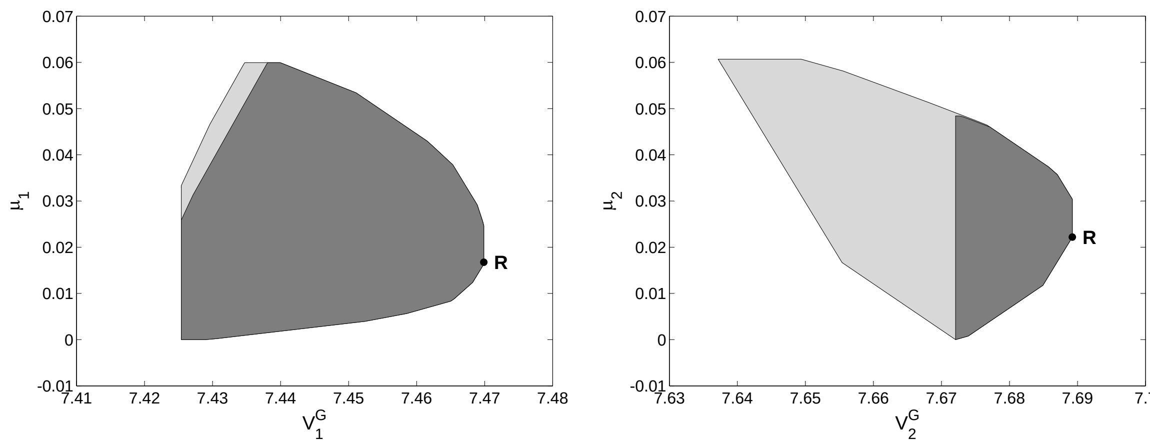
We then compute the equilibrium value sets for $\theta = 0.05$, which, in this context, implies a fairly high degree of model uncertainty. As observed in figure 3, the government and households' values do not typically coincide anymore. Indeed, the set of vectors of equilibrium values is on the semi-hyperplane below the 45-degree line, as the government's values are higher than the households' values.

Figure 3: Government's and Households' Equilibrium Values for $\theta = 0.05$ for $s = 1$ (left panel) and $s = 2$ (right panel) with Commitment (light grey area) and without Commitment (dark grey area)



The most striking feature of the equilibrium results is observed in figure 4. Notice that the set of values without commitment overlaps with the one with commitment to the right for high government values. In contrast with the case without model uncertainty, here the Ramsey plan is credible. While the highest government's values delivered by a SP with model uncertainty are 7.4699 and 7.6892, for $s = 1, 2$, respectively, the corresponding values with expected utility are 7.4675 and 7.6844. In this sense, uncertainty aversion on the households' side has positive welfare implications for the government. The forces that are driving these results are not triggered by the government's incentive constraints and its worst punishment values, which coincide in both economies, but by the dynamics intrinsic to competitive equilibria. With model uncertainty, for the same allocations the households' Euler equations are typically more relaxed (in the sense that their associated Lagrange multiplier is weakly smaller) than with standard expected utility. This follows from the fact that the evil alter ego twists the probability distribution of next period's shock realization by taking away probability mass from those states associated with high utility to the households, and placing it into the low utility states, which are associated with lower current consumption and, hence, higher marginal utility. This way, the right-hand side of the Euler equation becomes larger with model uncertainty. Thereby, lower values of (inverse) money growth rates $h \geq 0$ are consistent with a competitive equilibrium in this environment, which allows for more gradual monetary contractions and deflationary processes.

Figure 4: Government's Equilibrium Values and Marginal Utilities for $\theta = 0.05$ for $s = 1$ (left panel) and $s = 2$ (right panel) with Commitment (light grey area) and without Commitment (dark grey area)



As explained in section 4, the source of time-inconsistency in the Ramsey plan comes from the government's incentives to possibly make the deflationary process even more gradual in order to reduce the tax distortions that come along. Then it is clear to see how, through more gradual deflationary processes, the optimal monetary policies with model uncertainty become credible.

7 Conclusions

In this paper we examine how the optimal monetary policies should be designed when the monetary authority faces households who cannot form a unique probability model for the underlying state of the economy.

Future monetary policies influence households' choice of real balances in the current period by affecting the expected value of money in the coming periods. When households exhibit doubts about model misspecification, the effect of the government's policies on the expected value of money is two-fold. Besides their impact on the value of money for every possible future state of the economy, future policies directly influence the households' beliefs about the evolution of exogenous variables, as households base their decisions on the evaluations of worst-case scenarios. It then becomes key for the government to exploit the management of households' expectations when designing monetary policies.

We study the optimal policies when the monetary authority has the ability to commit to its announced policies and when it does not. Given the high complexity of the environment in consideration, we are not able to derive analytical solutions for the optimal credible policies. We provide, however, a full characterization of the sets of all equilibrium outcomes, both with and without commitment on the government's side. To compute these sets, we implement a computational algorithm based on outer hyperplane approximation techniques proposed by Judd, Yeltekin, and Conklin (2003).

The characterization of the set of all sustainable payoffs may shed some light on how severe the time-inconsistency issue is for the Ramsey plan. As illustrated in our numerical example, the fact that households may have doubts about model misspecification can help mitigate the time-inconsistency of the Ramsey plan.

References

Abreu, Dilip, David Pearce, and Ennio Stacchetti. 1990. "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring." *Econometrica* 58(5): 1041–1063.

Abreu, Dilip, and Yuliy Sannikov. 2011. "An Algorithm for Two Player Repeated Games with Perfect Monitoring." *Princeton Economic Theory Center Working Paper No. 26-2011.* Available at http://www.princeton.edu/econtheorycenter/wps/wp026_2011_Abreu_Sannikov.pdf.

Bernanke, Ben S. 2007. "Inflation Expectations and Inflation Forecasting." Speech delivered at the Monetary Economics Workshop held at the National Bureau of Economic Research Summer Institute, Cambridge, MA, July 10.
Available at http://www.federalreserve.gov/newsevents/speech/bernanke20070710a.htm.

Bernanke, Ben S. 2010. "Implications of the Financial Crisis for Economics." Speech delivered at the Bendheim Center for Finance Tenth Anniversary Conference, Princeton, NJ, September 24.
Available at http://www.federalreserve.gov/newsevents/speech/bernanke20100924a.htm.

Caballero, Ricardo J., and Arvind Krishnamurthy. 2008. "Collective Risk Management in a Flight to Quality Episode." *Journal of Finance* 63(5): 2195–2230.

Calvo, Guillermo A. 1978. "On the Time Consistency of Optimal Policy in a Monetary Economy." *Econometrica* 46(6): 1411–1428.

Chang, Roberto. 1998. "Credible Monetary Policy in an Infinite Horizon Model: Recursive Approaches." *Journal of Economic Theory* 81(6): 431–461.

Chari, V. V., and Patrick Kehoe. 1990. "Sustainable Plans." *Journal of Political Economy* 98(4): 783–802.

Ellsberg, Daniel. 1961. "Risk, Ambiguity and Savage Axioms." *Quarterly Journal of Economics* 75(4): 643–669.

Fernández-Villaverde, Jesús, and Aleh Tsyvinski. 2002. "Optimal Fiscal Policy in a Business Cycle Model without Commitment." *Manuscript, University of Pennsylvania.* Available at http://economics.sas.upenn.edu/~jesusfv/optimal.pdf.

Halevy, Yoram. 2007. "Ellsberg Revisited: An Experimental Study." *Econometrica* 75(2): 503–536.

Hansen, Lars P., and Thomas J. Sargent. 2007. "Recursive Robust Estimation and Control Without Commitment." *Journal of Economic Theory* 136(1): 1–27.

Hansen, Lars P., and Thomas J. Sargent. 2008. *Robustness.* Princeton, NJ: Princeton University Press.

Judd, Kenneth, Sevin Yeltekin, and James Conklin. 2003. "Computing Supergame Equilibria." *Econometrica* 71(4): 1239–1254.

Karantounias, Anastasios G., Lars P. Hansen, and Thomas J. Sargent. 2009. "Managing Expectations and Fiscal Policy." *Federal Reserve Bank of Atlanta Working Paper 2009-29.*

Kydland, Finn, and Edward C. Prescott. 1977. "Rules Rather Than Discretion: The Inconsistency of Optimal Plans." *Journal of Political Economy* 85(3): 473–491.

Kydland, Finn, and Edward C. Prescott. 1980. "Dynamic Optimal Taxation, Rational Expectations and Optimal Control." *Journal of Economic Dynamics and Control* 2: 79–91.

Ljungqvist, Lars P., and Thomas J. Sargent. 2004. *Recursive Macroeconomic Theory,* 2nd ed. Cambridge, MA: The MIT Press.

Lucas, Robert E., and Nancy L. Stokey. 1983. "Optimal Fiscal and Monetary Policy in an Economy Without Capital." *Journal of Monetary Economics* 12(1): 55–93.

Marcet, Albert, and Ramon Marimon. 2009. "Recursive Contracts." *Mimeo, Institut d' Análisi Ecońomica.*

Phelan, Christopher, and Ennio Stacchetti. 2001. "Sequential Equilibria in a Ramsey Tax Model." *Econometrica* 69(6): 1491–1518.

Uhlig, Harald. 2010. "A Model of a Systemic Bank Run." *Journal of Monetary Economics* 57(1): 78–96.

Woodford, Michael. 2003. *Interest and Prices: Foundations of a Theory of Monetary Policy.* Princeton, NJ: Princeton University Press.

A Appendix

A.1 Characterization of the competitive equilibrium sequence

A.1.1 Solving a representative household's maximization problem

Given prices $\{q_t(s^t)\}$, government policies $\{h_t(s^t), x_t(s^t)\}$ and belief distortions $\{D_{t+1}(s^{t+1}), d_{t+1}(s^{t+1})\}_{t=0}^{\infty}$, the households' optimization problem consists of choosing $\{c_t(s^t), M_t(s^t)\}_{t=0}^{\infty}$ and $\{\lambda_t(s^t), \mu_t(s^t)\}_{t=0}^{\infty}$ to maximize and minimize, respectively, the Lagrangian

$$\begin{aligned}\mathcal{L}^H \;=\; & \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) D_t(s^t) \left\{ \left[u\left(c_t\left(s^t\right)\right) + v(q_t\left(s^t\right) M_t\left(s^t\right)) \right] + \right. \\ & -\lambda_t\left(s^t\right) \left[q_t\left(s^t\right) M_t\left(s^t\right) - y_t\left(s^t\right) + x_t\left(s^t\right) + c_t\left(s^t\right) - q_t\left(s^t\right) M_{t-1}\left(s^{t-1}\right) \right] + \\ & \left. -\, \mu_t\left(s^t\right) \left[q_t\left(s^t\right) M_t\left(s^t\right) - \overline{m} \right] \right\}.\end{aligned}$$

Taking FOCs we obtain

$$u'(c_t\left(s^t\right)) = \lambda_t\left(s^t\right) \tag{22}$$

$$\begin{aligned}& D_t(s^t) \left[v'(m_t\left(s^t\right)) q_t\left(s^t\right) - \lambda_t\left(s^t\right) q_t\left(s^t\right) \right] + \\ & \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \lambda_{t+1}\left(s^{t+1}\right) D_{t+1}(s^{t+1}) q_{t+1}\left(s^{t+1}\right) - D_t(s^t) \mu_t\left(s^t\right) q_t\left(s^t\right) = 0.\end{aligned} \tag{23}$$

Substitute equation (22) into (23), use (2) and note that $\frac{q_{t+1}\left(s^{t+1}\right)}{q_t(s^t)} = \frac{m_{t+1}\left(s^{t+1}\right) h_{t+1}\left(s^{t+1}\right)}{m_t(s^t)}$

$$\begin{aligned}v'(m_t\left(s^t\right)) - u'(c_t\left(s^t\right)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{D_{t+1}(s^{t+1})}{D_t(s^t)} u'(c_{t+1}\left(s^{t+1}\right)) \frac{q_{t+1}\left(s^{t+1}\right)}{q_t\left(s^t\right)} \;&\geq\; 0, \\ = 0 \text{ if } m_t\left(s^t\right) < \overline{m}&\end{aligned}$$

$$\begin{aligned}& m_t\left(s^t\right) \left[u'(c_t\left(s^t\right)) - v'(m_t\left(s^t\right)) \right] \\ & \quad -\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}\left(s_{t+1}|s^t\right) u'(c_{t+1}\left(s^{t+1}\right)) m_{t+1}\left(s^{t+1}\right) h_{t+1}\left(s^{t+1}\right) \;\leq\; 0, \\ & \qquad = 0 \text{ if } m_t\left(s^t\right) < \overline{m}.\end{aligned}$$

The above expression is our equilibrium condition, equation (8).

A.1.2 Solving alter ego's minimization problem

Given $c_t(s^t), m_t(s^t)$, the evil alter ego's optimization problem consists of choosing $\{D_t\left(s^t\right), d_{t+1}(s_{t+1}|s_t)\}$ and $\{\phi_{t+1}\left(s^{t+1}\right), \varphi_t\left(s^t\right)\}$ to minimize and maximize, respectively, the Lagrangian

$$\begin{aligned}\mathcal{L}^{AE} &= \sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t(s^t)D_t(s^t)\{[u(c_t)+v(m_t)]+\\ &+\beta\theta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)d_{t+1}(s_{t+1}|s_t)\log d_{t+1}(s_{t+1}|s_t)\}+\\ &-\beta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\phi_{t+1}\left(s^{t+1}\right)\left[D_{t+1}(s^{t+1})-d_{t+1}(s_{t+1}|s_t)D_t(s^t)\right]+\\ &-\varphi_t\left(s^t\right)\left[\sum_{s_{t+1}}\pi(s_{t+1}|s_t)d_{t+1}(s_{t+1}|s_t)-1\right].\end{aligned}$$

The FOCs for $d_{t+1}(s_{t+1}|s_t)$ and $D_t(s^t)$ are respectively given by

$$\beta\theta D_t(s^t)\left[\log d_{t+1}(s_{t+1}|s_t)+1\right]+\beta\phi_{t+1}\left(s^{t+1}\right)D_t(s^t)=\varphi_t\left(s^t\right) \tag{24}$$

$$\begin{aligned}&[u(c_t)+v(m_t)]+\beta\theta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)d_{t+1}(s_{t+1}|s_t)\log d_{t+1}(s_{t+1}|s_t)+\\ &+\beta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\phi_{t+1}\left(s^{t+1}\right)d_{t+1}(s_{t+1}|s_t)=\phi_t\left(s^t\right).\end{aligned} \tag{25}$$

Rearranging (24) leads to

$$\begin{aligned}\log d_{t+1}(s_{t+1}|s_t) &= -1+\frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)}-\frac{\phi_{t+1}\left(s^{t+1}\right)}{\theta}\\ d_{t+1}(s_{t+1}|s_t) &= \exp\left(-1+\frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)}\right)\exp\left(-\frac{\phi_{t+1}\left(s^{t+1}\right)}{\theta}\right).\end{aligned} \tag{26}$$

By condition (3) it has to be the case that

$$\begin{aligned}\exp\left(-1+\frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)}\right)\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\exp\left(-\frac{\phi_{t+1}\left(s^{t+1}\right)}{\theta}\right) &= 1\\ \exp\left(-1+\frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)}\right) &= \frac{1}{\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\exp\left(-\frac{\phi_{t+1}(s^{t+1})}{\theta}\right)}.\end{aligned} \tag{27}$$

Substituting equation (27) back into (26) yields

$$d_{t+1}(s_{t+1}|s_t)=\frac{\exp\left(-\frac{\phi_{t+1}(s^{t+1})}{\theta}\right)}{\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\exp\left(-\frac{\phi_{t+1}(s^{t+1})}{\theta}\right)}. \tag{28}$$

Now we use (24) and impose a respective transversality condition,

$$\lim_{t\to\infty}\beta^t\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\phi_{t+1}\left(s^{t+1}\right)d_{t+1}(s_{t+1}|s_t)=0. \tag{29}$$

It follows that

$$\phi_t\left(s^t\right) = V_t^H\left(s^t\right). \tag{30}$$

Using the above result in equation (28) delivers our equilibrium condition (9)

$$d_{t+1}(s_{t+1}|s_t) = \frac{\exp\left(-\frac{V_{t+1}^H\left(s^{t+1}\right)}{\theta}\right)}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\exp\left(-\frac{V_{t+1}^H\left(s^{t+1}\right)}{\theta}\right)}.$$

A.1.3 On the transversality condition

We will show that the transversality condition,
$\beta^t \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) u'\left[\left(f(x_t\left(s^t\right), s_t\right)\right] m_t\left(s^t\right) h_t\left(s^t\right) \to 0$ as $t \to \infty$ for all t and all s^t, is satisfied if $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}) \in E^\infty$.
Since E is compact, for any $\left(x_t\left(s^t\right), m_t\left(s^t\right), h_t\left(s^t\right), d_{t+1}(s_{t+1}|s^t)\right) \in E$, it must be that $\sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) u'\left[\left(f(x_t\left(s^t\right), s_t\right)\right] m_t\left(s^t\right) h_t\left(s^t\right)$ belongs to a compact interval (due to continuity of u' and f) for every t. Hence, it has to be that
$\sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) u'\left[\left(f(x_t\left(s^t\right), s_t\right)\right] m_t\left(s^t\right) h_t\left(s^t\right)$ is a bounded sequence, and the required sequence indeed converges to zero.

A.2 Example of competitive equilibrium sequences

Assume that $s_t = H, L$ and that the production function is such that $f(0, H) = f(0, L)$. Set $(\mathbf{m}, \mathbf{x}, \mathbf{h}) = \{m^*, 0, 1\}_{t=0}^\infty$ where m^* satisfies the following condition for all t and all s_t

$$u'\left(f(0, s_t)\right)(1-\beta) = v'\left(m^*\right).$$

Then $(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE_s$.

A.3 Proof of Corollary 3.

CE_s for all $s \in \mathbb{S}$ is compact.

Proof. Fix $s_0 \in \mathbb{S}$. Let $\left(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n, \mathbf{d}^n, \mathbf{V}^{\mathbf{H}^n}\right)$ be the sequence from $CE_{s=s_0}$ converging to some sequence $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right)$. We need to show that this limiting sequence belongs to $CE_{s=s_0}$.

$CE_{s=s_0}$ is a nonempty subset of a compact set $\mathbb{E}^\infty$. Since $\mathbb{E}^\infty$ is compact, it is closed, and, hence, $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in \mathbb{E}^\infty$.

The fact that $\left(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n, \mathbf{d}^n, \mathbf{V}^{\mathbf{H}^n}\right) \in CE_{s=s_0}$ implies that equations (8) - (11) are satisfied for each n. Consequently, by continuity of u, v, u', v' and f, $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right)$ satisfy these same equations. It follows then from Proposition 3.1 that $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_{s=s_0}$, which means that $CE_{s=s_0}$ is a closed subset of the compact set. Hence, it is compact. □