Optimal policy with heterogeneous agents and aggregate shocks : An application to optimal public debt dynamics

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Abstract

We show that allocations in incomplete insurance-market economies can be represented as the solution of the program of a constrained planner. This representation allows for solving Ramsey programs in incomplete-market economies with aggregate shocks, and thus determining optimal policies in such setups. We apply this framework to derive optimal public debt and fiscal policy after a technology, a government spending or an uncertainty shock. We find that, for any adverse shock, public debt decreases whereas capital taxes increases on impact. This policy limits the fall in capital after such shocks. Simulations of the optimal solutions can be obtained by simple perturbation methods.

Keywords: Incomplete markets, optimal policy, public debt.

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1 Introduction

Incomplete insurance-market models with aggregate shocks are environments, which allow thinking about the dynamics of heterogeneity and of inequality, in general equilibrium. Such models can improve our understanding of optimal policy designs, by considering rich but relevant tradeoffs between redistribution, insurance and incentives. In these models, agents face incomplete markets and borrowing limits that prevent them from perfectly insuring themselves against idiosyncratic risk, in the tradition of the Bewley-Huggett-Aiyagari literature (Bewley, 1983; Huggett 1993 and Aiyagari 1994). Unfortunately, these models are known to be hard to solve, from both an analytical or numerical point of view, because of the time-varying heterogeneity across agents (see Krusell and Smith, 1998 for a seminal contribution on the simulation of such economies). Due to these difficulties, no analysis of the distortions and optimal policies in these environments with aggregate shocks have been conducted yet. In particular, optimal Ramsey programs have not been solved. In this paper, we propose a tractable representation of incomplete insurance market economies with aggregate shocks, which allows deriving and to simulating such optimal policies. We apply this framework to derive optimal fiscal policy and time-varying optimal debt, as well as their response to various types of shocks. We can therefore discuss tradeoffs between insurance and efficiency in such environments.

The construction of our tractable framework relies on the following observation. When insurance markets are incomplete for idiosyncratic risks, agents with different individual risk realizations have different wealth and consumption levels. As a consequence, even if agents are initially identical, there is an increasing number of heterogeneous wealth levels, as time goes by. Huggett (1993) and Aiyagari (1994) have shown that these economies have a recursive structure in absence of aggregate shocks, provided that a continuous distribution of wealth is introduced as a state variable. We construct an environment where the heterogeneity across agents depends only on a finite, but possibly large, number of consecutive past realizations of the idiosyncratic risk. In our model, for a given (but potentially arbitrarily large) number of periods N, agents with the same realization of their idiosyncratic risk for the previous N periods have the same consumption and wealth. As a consequence, instead of having a continuous distribution of heterogeneous agents in each period, the economy is characterized by a finite number of heterogeneous consumption and wealth levels.

The interest of this framework relies on the five following properties. First, the equilibrium allocation is determined as the solution of a central planner allocation, where the planner faces some explicit constraints on its ability to transfer resources across agents. We can thus derive a recursive formulation of our equilibrium with aggregate shocks, which simplifies the derivation of equilibrium conditions.

Second, we show that the central planner allocation characterizing our equilibrium can be expressed as the solution of a decentralized optimization program. More precisely, we exhibit a tax system in a decentralized market economy with aggregate shocks, such that the competitive equilibrium exactly coincides with the equilibrium of the planner economy.

Third, we show that the allocations of the planner economy and of the decentralized economy coincide with each other, not only for any finite $N-$period individual histories, but also for infinitely long histories, i.e., when $N \to \infty$, provided that certain conditions hold. As a consequence, the equilibrium of the planner economy can be made arbitrarily close to general competitive equilibrium (with incomplete insurance markets as studied in Krusell and Smith, 1998) whenever these conditions are fulfilled. We show that these conditions are satisfied in many setups. As an additional result, this proves the existence of a recursive equilibrium in an incomplete insurance-market model with aggregate shocks.

Fourth, using the finite-state equilibrium representation, we can solve the Ramsey program with aggregate shocks and derive optimal policies in a time-varying environment. This enables us to obtain responses of policies to various shocks, such as a technology, a government spending or an uncertainty or inequality shock.

Finally, the finite equilibrium structure simplifies to a large extent the simulation of our model. Instead of having to track a continuous distribution of agents for wealth and consumptions, we only nee to account for a finite (even if possibly large) number of different agent types, where all agents of the same type have the same consumption level and the same asset holdings. We can therefore use standard numerical techniques for the equilibrium simulation. In particular, provided that shocks are not too large, standard perturbation methods can be used.

We apply this framework to investigate the design of optimal fiscal policy in an incomplete market environment, when the government can issue public debt and must rely on distorting taxes on capital and labor to finance public spending. This framework can first be seen as a extension to an incomplete insurance market environment of standard optimal taxation models with a representative agent (Stockey and Lucas, 1983; Aiyagari, Marcet, Sargent and Seppala 2002; Farhi 2012 among many others). Second, it can be seen as the solution of a Ramsey program in a rather standard incomplete market economy with aggregate shocks, first studied by Krusell and Smith (1998) and analyzed in Heathcote (2005) among others (see the literature review below for reference to recent and important contributions). Considering incomplete insurance markets deeply affects optimal public debt determination. Indeed, agents use both public debt and the capital stock for self-insurance motives, which generates a well-defined optimal level of public debt, when there is no aggregate shocks (Aiyagari and McGrattan, 1998 and Acikgoz, 2013 for a recent analysis).

In this environment, we consider three types of agregate shocks. The first one is a technology shock, which affects the productivity of both capital and labor. The second one is a shock to the persistence of the idiosyncratic risk, which can be regarded as an uncertainty or inequality shock. The third one is a shock on public spending that affects the financing needs of the government. We find that public debt and taxes are mean-reverting after any transitory aggregate shock. In addition, when the economy is hit by a negative shock, we find that optimal taxes increase to reduce public debt. This optimal outcome tends to reduce the fall in aggregate capital after a negative shock. The amplitude of this general pattern varies with the shock considered. In addition, we find that capital taxes are more volatile than labor taxes, what is also the result of models featuring a representative agent (Aiyagari, Marcet, Sargent and Seppala, 2002 among others). Finally, we find that capital taxes generally increase after a negative shock.

This paper contributes first to the literature on the theory of incomplete insurance-market economies with aggregate shocks. We propose a tractable model which generalizes a number of previous works. Some environments have been provided to show equilibrium existence and to derive theoretical properties when insurance markets are incomplete. First, a class of no trade equilibria has been studied, relying of permanent idiosyncratic shocks (Constantinides and Duffie, 1996). This framework is used for instance in Heathcote, storesletten and Violante (2014). Krusell, Mukoyama and Smith (2011) study a class of no-trade equilibrium in an economy without capital and with a tight-enough credit constraint. Recently, a class of small-trade equilibrium has been provided, based on quasi-linear utility function (Challe, Le Grand and Ragot, 2013; Challe and Ragot, 2014 or Le Grand and Ragot, 2015).[1] While these models

[1]Quasi-linearity has been used in other setups to reduce heterogeneity. Lagos and Wright (2005) consider

allow for the derivation of the properties of the equilibrium allocation, they rely on specific assumptions about the shape of the utility function or the volume of assets. The environment we propose is valid for general utility functions and arbitrary asset quantities. In addition, we can prove that a recursive equilibrium exists when the wealth of agents is introduced as a state variable, as in Krusell and Smith (1998) simulation strategy (see Miao, 2006, for a discussion).

Second, our paper contributes to the literature on optimal policies in incomplete insurance-market models without aggregate shocks (Aiyagari, 1995; Aiyagari and McGrattan, 1998). Recent papers, such as Acikgoz (2013), solve for the Ramsey program in these economies without aggregate shocks. Extending the analysis to economies with aggregate shocks allows considering a wide set of new economic problems, such as temporary change in uninsurable risk, or in technology.

Finally, this paper contributes to the vast literature on optimal time-varying fiscal policy, when the planner has distorting instruments, and cannot thus simply implement the first-best allocation. Seminal contributions consider a complete-market representative agents (Barro, 1979; Lucas and Stockey, 1983). More recent contributions consider incomplete market for the aggregate risk, introducing non state-contingent public debt (Aiyagari, Marcet, Sargent and Seppala, 2002; Farhi, 2012). Recently, Bhandari, Evans, Golosov and Sargent (2013) consider an environment with a finite number of agents facing idiosyncratic risk. They consider a Ramsey problem where the planner has access to non-distorting tools (positive or negative lump-sum transfers) and show that in this environment, public debt is irrelevant, which is a form of Ricardian equivalence. In the current paper, we do not allow the planner to use lump-sum transfers, what makes the level of public a relevant policy variable.

The rest of the paper is organized as follows. In Section 2, we present our economy and in particular the structure of aggregate and idiosyncratic shocks. We describe the central planner problem and derives the associated allocation in Section 3. We then show in Section 4 how the central planner allocation can be decentralized by a well-chosen lump-sum tax-system. We then take advantage in Section 5 of the finite equilibrium structure to solve the Ramsey program.

a model with linear disutility in labor in order to reduce heterogeneity in the time-dimension (every agent has the same marginal utility of consumption at the end of each period). Kiyotaki and Moore (2005, 2008), and Miao and Wang (2015) consider that entrepreneurs face idiosyncratic investment opportunities with a constant return-to-scale. This constant marginal productivity of idiosyncratic investments reduces the heterogeneity among entrepreneurs. Dang, Holmstrom, Gorton, and Ordoñez (2014) introduce a piecewise linear utility function to model the urgency to consume a certain amount of goods.

Finally, conclusions are given in Section 7.

2 The environment

We consider a discrete-time economy populated by a continuum of agents, distributed on a segment J following a non-atomic measure μ. The segment is of length 1: $\mu(J) = 1$.[2] In each period t, the economy has two goods: a consumption-capital good and labor. The period utility function $u(c,l)$ is increasing and concave in consumption c, decreasing in labor l and is twice continuously differentiable. The discount factor is $\beta \in [0,1]$. In period 0, each household ranks consumption and labor streams, denoted respectively as $(c_t)_{t\geq 0}$ and $(l_t)_{t\geq 0}$, according to:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} u(c_t, l_t),$$

where the expectation $\mathbb{E}_0$ is taken with respect to both the idiosyncratic and aggregate risk. As is standard in the incomplete-insurance market literature, we will consider a special class of utility function which does not exhibit wealth effect for the labor supply. We consider GHH utility function of the form[3]:

$$u(c,l) = u\left(c - \frac{l^{1+1/\varphi}}{1+1/\varphi}\right). \tag{1}$$

2.1 Aggregate risk and production

Time is discrete, indexed by $t \geq 0$. The aggregate shock s_t in period t, can take S different values in the state space $\mathcal{S} \equiv \{1, ..., S\}$. The history of aggregate shocks up to date t is denoted $s^t \equiv \{s_0, ..., s_t\}$. The aggregate risk process (s_t) is a first-order markov chain with transition density $P(\cdot|\cdot)$ where $P(s'|s)$ denotes the transition probability from state s to state s'. We denote as $s^t \succeq s^{t-1}$ that s^t is a possible continuation of the history s^{t-1}: in other words, they are identical over the first $t-1$ periods. We assume that the aggregate state at date 0, s_0, is deterministic and the period-0 probability that history s^t will occur at date t is denoted $m(s^t) \equiv P(s_1|s_0)P(s_2|s_1)\ldots P(s_t|s_{t-1})$.

[2] Among others, Feldman and Gilles (1985) have identified issues when applying the law of large number to a continuum of random variables. Green (1994) describes a construction of the sets J_i and of the measures ℓ_i to ensure that our statements hold. Feldman and Gilles (1985), Judd (1985), and Uhlig (1996) also propose other solutions to this issue. From now on, we assume that the law of large numbers applies.

[3] Altough we focus on this utility function which does not exhibit wealth effect on the labor (see below), results are valid for general utility functions.

2.2 Idiosyncratic risk

Agents face time-varying idiosyncratic risk. At the beginning of each period, agents face an idiosyncratic labor productivity shock $e_t \in \mathcal{E} \equiv \{0, ..., E\} \in \mathbb{R}_+^{L+1}$ that follows a discrete first-order Markov process with transition matrix $M(s_t)$. The probability $M_{e,e'}(s_t)$, $e, e' = 0, ..., E$ is the probability for an agent to switch from individual state $e_t = e$ at date t to state $e_{t+1} = e'$ at date $t+1$, when aggregate state is s_t in period t. To cover the various cases that can be found in the literature, we assume that households in state $e \in \{1, \ldots, E\} = \mathcal{E} - \{0\}$ have a labor productivity $n(e)$, whereas households in state $e = 0$ have a zero labor productivity but earn a home production δ. The former agents can be considered as employed workers with various productivities, while the latter can be considered as unemployed workers.

At period t, we denote as $\eta_{e,t}(z^t) \in [0, 1]$ the share of the population in state $e \in \mathcal{E}$. For instance, $\eta_{0,t}$ is the share of the population in idiosyncratic state 0 at period t. Assuming that the law of large numbers holds in a continuum, these shares evolve as, for $t \geq 1$:

$$\eta_{e,t}(s^t) = \sum_{e'=0}^{\mathcal{E}} M_{e,e'}(s_{t-1})\eta_{e',t-1}(s^{t-1}),\ z^t \succeq z^{t-1}. \tag{2}$$

2.3 Production and assets

In any period $t \geq 1$, a production technology with constant-returns-to-scale (CRS) $F(K_{t-1}, L_t, s_{t-1})$ transforms capital K_{t-1} and labor L_t into output. Capital must be installed one period before production, and the productivity may depend on the aggregate state at the date of capital installation. Labor L_t is measured in efficient units. This formulation allows for capital depreciation, which is subsumed by the production function $F(K_{t-1}, L_t, s_{t-1})$. The production function is smooth in K_{t-1} and L_t and satisfies the standard Inada conditions. The good is produced by a profit-maximizing representative firm. We denote as $\tilde{w}_t$ the before-tax real wage rate in period t and as $\tilde{r}_t$ the real before-tax rental rate of capital between period $t-1$ and period t. Profit maximization yields in each period $t \geq 1$:

$$\tilde{r}_t = F_K(K_{t-1}, L_t, s_{t-1}),$$
$$\tilde{w}_t = F_L(K_{t-1}, L_t, s_{t-1}).$$

In each period t, the government has to finance a public good $G_t \equiv G(s_t)$, which is possibly stochastic. The public good can be financed either by levying distorting taxes on capital income τ_t^k or on labor income τ_t^l or by issuing an amount B_t of a riskless one period debt.[4] Capital income are made of interest payments of all interest-bearing assets. Labor income are equal to the real wage multiplied by aggregate labor supply, $\tilde{w}_t L_t$. Debt issuance mechanically implies that in addition to the public good, the government also needs to reimburse the maturing public debt and the attached interest. In equilibrium, taxes τ_t^l, τ_t^k and debt B_t should be understood as a function of the history of shocks s^t. To simplify the exposition, we omit this dependence except when needed for clarifying the timing. Importantly, as the productivity of the firms is known in period $t-1$, the return on the capital in period t is riskless and known in period $t-1$. As a consequence, due to no-arbitrage opportunities, the returns on public debt and on capital are identical. We then deduce that the period t budget constraint of the government can be expressed as follows:

$$G_t + (1 + \tilde{r}_t)B_{t-1} \leq \tau_t^l \tilde{w}_t L_t + \tau_t^k \tilde{r}_t (K_{t-1} + B_{t-1}) + B_t. \tag{3}$$

We now denote as $r_t \equiv (1 - \tau_t^k)\tilde{r}_t$ and $w_t \equiv (1 - \tau_t^l)\tilde{w}_t$ the after-tax real interest and real wage rate respectively. Using the CRS property of the production function that implies that $F(K, L, s) = KF_K(K, L, s) + NF_L(K, L, s)$, we deduce that the budget constraint (3) can be simplified as follows

$$G_t + r_t K_{t-1} + w_t L_t + (1 + r_t)B_{t-1} \leq F(K_{t-1}, L_t, s_{t-1}) + B_t. \tag{4}$$

Finally, if C_t^{tot} denotes the total agents' consumption in period t, the resource constraint of the economy can be expressed as follows

$$G_t + C_t^{tot} + K_t \leq F(K_{t-1}, L_t, s_{t-1}) + \eta_{0,t}(s^t)\delta \tag{5}$$

In equation (5), the right-hand side collects all resources, which are made of produced goods and of goods obtained from home production.

[4] The question of the optimal mix of these financing tools will be the focus of the second part of the paper and in particular of the Ramsey program studied in Section 5.

3 The quasi-planner economy

In general, the previous economy features a growing heterogeneity over time, because agents with different realizations for the uninsurable idiosyncratic risk will choose different consumption levels and different amounts of wealth. The heterogeneity can be represented by a time-varying continuous distribution of wealth, which prevents any analytical characterization of the equilibrium and raises considerable computational challenges. We first present our environment in which the agents heterogeneity is summarized by only a finite number of agents types. A planner will allocate the resources among agents while facing some contraints to transfer resources across agents. Due to these constrains, we qualify the planner of *quasi-planner*. In Section 4, we show why this environment is interesting, and notably how it can be decentralized.

3.1 The islands

Our environment relies on the concept of islands. In a nutshell, island enables resource pooling and risk sharing for agents in the same island, but prevents transfers across agents of different islands. More specifically, an island gathers all agents sharing the same history for the last N periods. We denote $e^N \in \mathcal{E}^N$ an history of individual risk realizations of length N. The set of possible individual risk realizations $\mathcal{E}$ being of cardinal E, there are therefore $(E+1)^N$ different islands, indexed by $e^N \in \mathcal{E}^N$. At the beginning of every period, an agent on island $\hat{e}^N \in \mathcal{E}^N$ faces an idiosyncratic shock $e \in \mathcal{E}$ and will therefore be endowed with the history $e^N \in \mathcal{E}^N$. The agent will move to the e^N-island and will take his own resources of island $\hat{e}^N$ when transferring to island e^N.

The quasi-planner aims at maximizing an utilitarian welfare criterion, i.e. the sum of individual welfares. The key assumption is that the central planner can transfer resources between agents of the same island, but cannot transfer resources across islands. The utilitarian welfare criterion implies that the quasi-planner treats agents symetrically on each island and will allocate the same consumption level and the same end-of-period saving to all agents of the same island. The quasi-planner thus chooses at each period t the consumption level $c_t(e^N)$, the labor supply $l_t(e^N)$ and the end-of-period saving $a_t(e^N)$ of agents in every island $e^N \in \mathcal{E}^N$ and at all dates t. We assume that the quasi-planner faces a borrowing constraint $a_t(e^N) \geq -\bar{a}$ in all periods. This important assumption will further limit the ability of the quasi-planner to share risks across agents.

The term "quasi-planner" is justified by two assumptions. First, as said before, the quasi-planner cannot transfer resources across islands and, second, the quasi-planner is price-taker. The quasi-planner will indeed consider prices $(r_t)_t$ and $(w_t)_t$ as given. We discuss the implication of this assumption later, when commenting the optimality of the allocation, referring to Davila et al. (2012).

Size of the islands.

Using the law of large numbers, we can deduce the size of the e^N-island from the transition matrix $M(s_t)$. Note first that a N-period idiosyncratic history $e^N \in E$ can be written as a sequence $e^N = (e_{N-1}, ..., e_0)$. We start with computing the probability $\Pi_t\left(\hat{e}^N, e^N, s_t\right)$ that an household having experienced history $\hat{e}^N$ in period t, experiences history e^N at period $t+1$. This probability can be expressed as the probability to switch from the state $\hat{e}_0$ at t to state e_0 at $t+1$, provided that histories $\hat{e}^N$ and e^N are compatible. More formally:

$$\Pi_t\left(\hat{e}^N, e^N, s_t\right) = 1_{e^N \succeq \hat{e}^N} M_{\hat{e}_0, e_0}\left(s_t\right),$$

where $1_{e^N \succeq \hat{e}^N} = 1$ if e^N is a possible continuation of history $\hat{e}^N$. We now turn to the island size. At any date t, we denote as $S_t\left(e^N\right)$ the size of e^N-island, which is equal to the measure of households experiencing an individual $N-$period history e^N at period t. More formally, $S_t\left(e^N\right) = J(i \in [0,1] | e^{i,t,N} = e^N)$. The measure $S_t\left(e^N\right)$ evolves according to the following dynamics, for $t \geq N$:

$$S_{t+1}(e^N) = \sum_{\hat{e}^N \in \mathcal{E}^N} S_t(\hat{e}^N)\Pi_t\left(\hat{e}^N, e^N, s_t\right). \tag{6}$$

Equation (6) simply states that agents on island e^N at date t come from island $\hat{e}^N \in \mathcal{E}^N$ at date $t-1$ with probability $\Pi_t\left(\hat{e}^N, e^N, s_t\right)$.

To simplify the exposition of the model, we assume that in period 0, households enter the economy having already experienced a N-period history $e^N \in \mathcal{E}^N$. As a consequence, the measure $S_0\left(e^N, s^0\right)$ is considered as given (with $\sum_{e^N \in \mathcal{E}^N} S_0(e^N, s^0) = 1$). Moreover, agents having the same initial N-period history e^N enters the economy with the same initial wealth denoted $a_0(e^N)$. The law of motion (6) is thus valid from period 1 onwards.[5]

[5] As will become clear after the presentation of the economy, these assumptions are made without loss of generality.

Timing of events.

The timing of events is the following. Consider an agent with history $\hat{e}^N = (\hat{e}_{N-1}, \ldots, \hat{e}_0)$ at date $t-1$, living on $\hat{e}^N$-island at the end period $t-1$, with an end-of-period asset holding $a_{t-1}(\hat{e}^N, z^{t-1})$. The timing is as follows:

1. At the beginning of period t, she learns her new individual status e_t and aggregate risk realization z_t at date t. She is therefore endowed with the $N-$period history $e^N = (\hat{e}_{N-2}, .., \hat{e}_0, e_t)$ (history is shifted by one period).

2. She moves to the e^N-island with her individual asset holding $a_{t-1}(\hat{e}^N)$.

3. The period t per capita beginning-of-period resources of the e^N-island is equally shared among all agents of the island, who have pooled all their resources in common:

$$\tilde{a}_t(e^N) = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S_{t-1}(\tilde{e}^N)}{S_t(e^N)} \Pi_{t-1}\left(\tilde{e}^N, e^N, s_t\right) a_{t-1}(\tilde{e}^N). \tag{7}$$

   As the initial level of wealth $a_0(e^N)$ is given by initial conditions, equation (7) is true for $t \geq 1$ onwards.

4. The central planner is deciding the common consumption of island inhabitants $c_t(e^N)$ and their new individual asset holding $a_t(e^N)$.

3.2 Recursive formulation

3.2.1 Program of the quasi-planner

We now provide a recursive formulation for the program of the planner (We justify the existence of a recursive formulation below).[6] The program of the quasi-planner depends on the beginning-of-period per capita wealth in each island $\tilde{a}(e^N)_{e^N \in \mathcal{E}^N}$ and on a set of state variables that are taken as given by the central planner. We denote z this vector, which is necessary for the quasi-planner to form rational expectations. We specify the vector z below.

The program of the planner is

[6]Obviously, a sequential equilibrium can be presented, at the cost of more notations. As the equilibrium definitions coincide, we directly focus on the recursive equilibrium.

$$V\left(\tilde{a}(e^N)_{e^N\in\mathcal{E}^N}, z\right) = \max_{(a(e^N),c(e^N),l(e^N))_{e^N\in\mathcal{E}^N}} \sum_{e^N\in\mathcal{E}^N} S'(e^N)u\left(c(e^N), l(e^N)\right) + \beta\mathbb{E}V\left(\tilde{a}'(e^N)_{e^N\in\mathcal{E}^N}, z'\right) \tag{8}$$

$$a'(e^N) + c(e^N) = wn(e_0^N)l(e^N) + \delta 1_{e_0^N=0} + (1+r)\tilde{a}(e^N), \text{ for all } e^N \in \mathcal{E}^N, \tag{9}$$

$$c(e^N), l(e^N) \geq 0, a'(e^N) \geq -\bar{a}, \text{ for all } e^N \in \mathcal{E}^N, \tag{10}$$

$$S_0\left(e^N\right) \text{ and } a_0(e^N) \text{ are given.} \tag{11}$$

and subject to the laws of motion, for all $e^N \in \mathcal{E}^N$:

$$\tilde{a}'(e^N) = \sum_{\tilde{e}^N\in\mathcal{E}^N} \frac{S(\tilde{e}^N)}{S'(e^N)}\Pi\left(\tilde{e}^N, e^N, z\right)a'(\tilde{e}^N) \tag{12}$$

$$S'(e^N) = \sum_{\tilde{e}^N\in\mathcal{E}^N} S(\tilde{e}^N)\Pi\left(\tilde{e}^N, e^N, z\right) \tag{13}$$

First note that we use the after-tax real interest rate r and wage rate w in the budget constraints of the quasi-planner on each island (equations (9). In these equations, the term $wn(e_0^N)l(e^N)$ is the per capita labor income on island e^N: it is equal to the wage rate multiplied by the productivity $n(e_0^N)$ and the labor supply $l(e^N)$ on this island. The equations (10) include non-negativity constraints for consumption and the credit constraint in each island, as well as credit constraints. The existence of a recursive formulation of the problem is straightforward, as the problem is concave and the additional constraints (12) and (13) are linear. Note $\eta(e^N)$ as the Lagrange multiplier of the credit constraint $a'(e^N) \geq \bar{a}$ on island e^N. Solving this program yields, for all $e^N \in \mathcal{E}^N$

$$u_c\left(c(e^N), l(e^N)\right) + \eta(e^N) = \beta\mathbb{E}\sum_{\hat{e}^N\in\mathcal{E}^N} \Pi\left(e^N, \hat{e}^N, z'\right)u_c\left(c'(\hat{e}^N), l'(e^N)\right)(1+r') \tag{14}$$

$$w(z)n(e_0^N)u_c\left(c(e^N), l(e^N)\right) = -u_l\left(c(e^N), l(e^N)\right) \tag{15}$$

$$\eta(e^N)(a'(e^N) + \bar{a}) = 0 \text{and } \eta(e^N) \geq 0 \tag{16}$$

The Lagrange coefficient $\eta(e^N)$ is zero when $a'(e^N) > -\bar{a}$. From equation (15), we deduce that if $n(e_0^N) = 0$, then agents are unemployed in the island e^N. There is no labor supply $l(e^N) = 0$, and the per capita income on this island is home production δ. When $n(e_0^N) > 0$, we deduce the labor supply from equation (15) using the expression of the utility function (9). We obtain:

$$l(e^N) = \left(wn(e_0^N)\right)^{\varphi} \text{ if } n(e_0^N) > 0 \tag{17}$$

$$l(e^N) = 0 \text{ if } n(e_0^N) = 0 \tag{18}$$

3.2.2 Market equilibria and government budget constraint

Before turning to the recursive equilibrium definition, we provide the expression of aggregate quantities in the economy. To aggregate, we simply sum over all islands $e^N \in \mathcal{E}^N$ of size $S(e^N)$ the individual quantities. The total labor supply amounts to

$$L = \sum_{e^N \in \mathcal{E}^N} n(e^N) S(e^N) l(e^N), \tag{19}$$

while the end-of-period saving of all agents, denoted A can be expressed as

$$A \equiv \sum_{e^N \in \mathcal{E}^N} S(e^N) a'(e^N) = \sum_{e^N \in \mathcal{E}^N} S(e^N) \tilde{a}'(e^N). \tag{20}$$

The last equality stems from the definition of $\tilde{a}'(e^N)$ and the law of motion of the share of agents in each island (equations (12) and (13)). In words, the transfer of wealth across islands does not affect the total amount of wealth.

If we denote $s_-$is the aggregate state realization in the previous period and B' the public debt level in the next period, we deduce that the government budget constraint is

$$G + rK + wN + (1+r)B \leq F(K, L, s_-) + B'. \tag{21}$$

Finally, the financial market equilibrium can simply be written as

$$A = B' + K' \tag{22}$$

We finally express the factor prices as well as taxes:

$$\begin{cases} \tilde{r} & = F_K(K, L, s_-) \\ \tilde{w} & = F_N(K, L, s_-) \end{cases} \tag{23}$$

and:

$$\begin{cases} \tau^K & = 1 - \frac{r}{\tilde{r}} \\ \tau^L & = 1 - \frac{w}{\tilde{w}} \end{cases} . \tag{24}$$

3.2.3 Recursive Equilibrium definition

To form rational expectations, the information set is $z = \{S(e^N), B, s_-\}$. Initial conditions are the initial size and wealth of islands $\left(S_0(e^N), a_0(e^N)\right)_{e^N \in \mathcal{E}^N}$, and thus the initial capital stock $K = \sum_{e^N \in \mathcal{E}^N} S_0(e^N), a_0(e^N)$, the initial public debt B_0 and the initial state s_{-1}. For a given fiscal policy $\tau^K(z)$ and $\tau^L(z)$, a recursive equilibrium is a set $\{\left(c(e^N), l(e^N), \tilde{a}(e^N), a'(e^N), \eta(e^N), S'e^N)\right)_{e^N \in \mathcal{E}^N}, w, r, L, A, B', K', \tilde{r}, \tilde{w}\}$ satisfying the $6(E+1)^N + 8$ equations (9) and (12)-(24).

The main interest of the previous setup is that we can provide conditions to show that it converges when N grows to a full-fledge incomplete insurance-market economy with aggregate shocks. To prove this property, the next section provides a decentralization of the previous allocation.

4 The decentralized economy

In this section, we explain how we can decentralize the allocation of the quasi-planner by choosing a proper transfer system. The environment is the one described in Section 24, and is thus obviously the same as for the quasi-planner program in Section 2. In this decentralized economy, agents maximize their intertemporal welfare using both public debt shares and claims on the capital stock to self-insure against both idiosyncratic and aggregate risks. Financial markets are incomplete for both risks, because agents have just a safe asset (capital or public debt) to self-insure. The decentralization relies on a particular additional taxation scheme. In every period, agents pay or receive a lump-sum transfer, which is contingent on their individual history over the previous N periods. In equilibrium, this transfer will ensure that agents sharing the same

$N-$period history $e^N \in \mathcal{E}^N$ will have the same after-transfer beginning-of-period wealth. This taxation scheme allows reducing the heterogeneity among agents, since it "pools" together all agents with the same history e^N. We now provide a detailed and formal construction of this decentralized equilibrium, where we will in particular show that decentralized Euler equations match those of the quasi-planner of Section 3.

4.1 Recursive economy

As in the quasi-planner economy, we construct a recursive equilibrium to simplify the exposition.[7] The economy is now populated by agents who are expected-utility maximizers. The value function of an agent will depend on three state variables: (i) the personal history, (ii) the beginning-of-period and before-tax wealth a and (iii) other (exogenous) state variables used by the agent to form her rational expectations that we denote $\tilde{z}$. Let us be a bit more specific about personal history as a state variable. Consider an agent that enters the period with the history $\tilde{e}^N \in \mathcal{E}^N$. Before making any decision, she learns the realization of her idiosyncratic shock $e \in \mathcal{E}$ for the current period. She is now endowed with the $N+1-$period history $e^{N+1} = (\tilde{e}^N, e) = (e^N_N, e^N_{N-1}, \ldots, e^N_1, e)$. This $N+1-$period history will drive the tax $T(e^{N+1})$ payment she has to make. This transfer $T(e^{N+1})$ also depends on state variables $\tilde{z}$. The agent maximizes her intertemporal welfare by choosing the current consumption c, labor effort l and asset holding a'. Her value function denoted V can be expressed as follows

The agent with history $\tilde{e}^N$ learns (by assumption) the realization of her idiosyncratic shock $e \in \mathcal{E}$ at the beginning the period, before making her choices. The agent is now endowed with the $N+1-$period history $e^{N+1} = (\tilde{e}^N, e)$. She will have to pay a after distorting-tax interest rate and wage rate are denoted as r and w, as before. Its value function can be written as

$$V(a, e^{N+1}, \tilde{z}) = \max_{a',c,l} u(c,l) + \beta \mathbb{E} \sum_{e' \in \mathcal{E}} M_{e,e'}\,(\tilde{z})\, V(a', (e^N_{N-1}, ..., e, e'), \tilde{z}') \tag{25}$$

$$a' + c + T(e^{N+1}) = wn(e)l + \delta 1_{e=0} + (1+r)a \tag{26}$$

$$c, l \geq 0, a' \geq -\bar{a} \tag{27}$$

[7] We justify the existence of a recursive equilibrium below. In particular the state space will be finite in equilibrium. Once again, one can study the sequential representation of this economy and show that it delivers the same equilibrium condition.

In the value function (25), the expectation operator $\mathbb{E}$ is taken with respect to the future aggregate shock $\tilde{z}'$ only. The expectation with respect to the idiosyncratic shock is written explicitly in the sum $\sum_{e' \in \mathcal{E}} M_{e,e'}(\tilde{z})$, where $M_{e,e'}(\tilde{z})$ is the probability to switch from the current idiosyncratic state $e \in \mathcal{E}$ to $e' \in \mathcal{E}$ in the next period, when the current aggregate state is $\tilde{z}$. In this case, the next-period $N+1$ history will be $(e_{N-1}^N, ..., e, e')$ (note that e^{N+1} will be shifted by one period). Equation (26) is the budget constraint of the agent, where r and w denote the after-tax interest rate and wage rate. Agent's resources are made of asset payoffs $(1+r)a$ and of income earnings, which are $wn(e)l$ when employed (i.e., $e \in \mathcal{E} - \{0\}$) and δ when not (i.e., $e = 0$). The agent use these resources to consume, purchase assets and pay taxes $T(e^N)$. Positivity constraints for consumption and labor, as well as borrowing constraints are provided in equations (27). Denote as $\tilde{\eta}(a, e^{N+1})$ the Lagrange coefficient of the credit constraint $a' \geq -\bar{a}$.

The solution to the maximization program (25)–(27) are the policy rules denoted $c = c(a, e^{N+1}, \tilde{z}), a' = a'(a, e^{N+1}, \tilde{z}), l = l(e^{N+1}, \tilde{z})$ and the multiplier $\tilde{\eta}(a, e^{N+1})$ satisfying the following first order conditions:

$$u_c(c,l) + \tilde{\eta} = \beta \mathbb{E} \sum_{e' \in \mathcal{E}} M_{e,e'}(\tilde{z}) \, u_c(c', l') \, (1 + r'), \tag{28}$$

$$u_l(c,l) + wn(e) u_c(c,l) = 0, \tag{29}$$

$$\eta(a' + \bar{a}) = 0 \text{ and } \eta \geq 0. \tag{30}$$

Using the first order condition (29) and the expression of the utility function in equation (9), we deduce that the labor supply depends on whether the productivity is zero or not and it can be expressed as follows:

$$l(e^{N+1}, \tilde{z}) = (wn(e))^{\varphi} \text{ if } e \in \mathcal{E} - \{0\}, \tag{31}$$

$$l(e^{N+1}, \tilde{z}) = 0 \text{ if } e = 0, \tag{32}$$

We can again interpret agents with $e \in \mathcal{E} - \{0\}$ and $n(e) > 0$ as employed agents and agents with $e = 0$ and $n(e) = 0$ as unemployed ones.

4.2 Taxation scheme

The tax expression. We can now construct the taxation scheme following a guess-and-verify strategy. The tax transfer is constructed such that all agents with the same individual history over the last N periods will have the same after-tax weath. Consider agents with a beginning-of-period history $\tilde{e}^N \in \mathcal{E}^N$ who experience an idiosyncratic risk realization $e \in \mathcal{E}$ in the current period. The measure of agents with history $\tilde{e}^N$ is $S(\tilde{e}^N)$, while the measure of agents with history $e^N \in \mathcal{E}^N$ is $S'(e^N)$. The relationship between $S(\tilde{e}^N)$ and $S'(e^N)$ is given by the equation (13). We now assume that for any $\tilde{e}^N \in \mathcal{E}^N$, agents having the history $\tilde{e}^N$ have the same beginning-of-period wealth $a(\tilde{e}^N)$. For agents having transited to the history e^N to hold the same asset quantity denoted $\tilde{a}(e^N)$, the following equality must hold:

$$\tilde{a}(e^N) = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S(\tilde{e}^N)}{S'(e^N)} \Pi\left(\tilde{e}^N, e^N, \tilde{z}\right) a(\tilde{e}^N), \text{ for all } e^N \tag{33}$$

This is identical to the pooling equation defining $\tilde{a}(e^N)$ in the quasi-planner economy (12). The tax transfer borne by an agent having the current $N+1$- period history $e^{N+1} = (\tilde{e}^N, e)$ has to mimic this pooling operation and is equal to

$$T(e^{N+1}) = (1+r)\left(\tilde{a}(e^N) - a(\tilde{e}^N)\right) \tag{34}$$

In words, the tax removes the remuneration of the beginning of period wealth of agents having history $\tilde{e}^N$ and adds the remuneration of the average wealth $\tilde{a}(e^N)$ of agents having the current $N-$period history $e^N = (\tilde{e}_{N-2}, ...\tilde{e}_0, e)$.

A balanced taxation scheme. We can show that the taxation scheme is balanced within the period, because it mainly realizes a redistribution among agents with the same current history e^N. The following lemma summarizes this result

Lemma 1 (Balanced taxation scheme) *In each period t, we have*

$$\forall e^N \in \mathcal{E}^N, \sum_{\tilde{e} \in E} S'(\tilde{e}, e^N) T(\tilde{e}, e^N) = 0.$$

Proof. First, from the definition of $\tilde{a}(e^N)$ in equation (33), we obtain for any history $e^N = (e^N_{N-1}, \ldots, e^N_1, e^N_0) \in \mathcal{E}^N$:

$$\begin{aligned} S'(e^N)\tilde{a}(e^N) &= \sum_{\hat{e}^N \in \mathcal{E}^N} S(\hat{e}^N)\Pi\left(\hat{e}^N, e^N, \tilde{z}\right) a(\hat{e}^N) \\ &= \sum_{\hat{e} \in E} S(\hat{e}, e^N_{N-1}, ..., e^N_1) M_{e^N_1, e^N_0}(\tilde{z}) a(\hat{e}, e^N_{N-1}, \ldots, e^N_1). \end{aligned}$$

Therefore, for a given $N-$period history $e^N \in \mathcal{E}^N$, we obtain (noting that $\sum_{\tilde{e} \in E} S'(\tilde{e}, e^N) = S'(e^N)$):

$$\begin{aligned} \sum_{\tilde{e} \in E} S'(\tilde{e}, e^N) T(\tilde{e}, e^N) &= (1+r)\left[\sum_{\tilde{e} \in E} S'(\tilde{e}, e^N)\left(\tilde{a}(e^N) - a(\tilde{e}, e^N_{N-1}, \ldots, e^N_1)\right)\right] \\ &= (1+r)\left[S'(e^N)\tilde{a}(e^N) - \sum_{\tilde{e} \in E} S'(\tilde{e}, e^N) a(\tilde{e}, e^N_{N-1}, \ldots, e^N_1)\right] \\ &= 0, \end{aligned}$$

which concludes the proof. ■

Agents are tax-takers. All agents consider the lumpsum tax T as given and thus do not internalize the effect of their choice on this transfer. Indeed, there is a continuum –with positive mass $S(\tilde{e}^N)$– of agents with history $\tilde{e}^N$ and each individual agent with history $\tilde{e}^N$ is atomistic in this set. As a consequence, an individual agent has no effect on the aggregate wealth of agents with history $\tilde{e}^N$. We can therefore consider that agents are tax-takers and that the taxation has no direct influence on the Euler equation (28).

The impact of tax on agents' wealth. We consider the impact of tax $T(e^{N+1})$ for an agent with history $e^{N+1} = (\tilde{e}^N, e) \in \mathcal{E}^{N+1}$ and beginning-of-period wezalth $a(\tilde{e}^N)$. Her budget constraint (26) can be expressed using tax expression (34) as follows:

$$a' + c = wn(e_0)l + \delta 1_{e=0} + (1+r)\tilde{a}(e^N) \tag{35}$$

As a consequence, the beginning-of-period and after-tax wealth of agents with history e^{N+1} only depends on the current $N-$period history e^N. Moreover, as can be seen from (25), agents with the same $N-$period history e^N are endowed with the same expected continuation utility if they save the same amount a'. Therefore, agents with the same current $N-$period history e^N behave similarly: they consume the same level, they supply the same labor quantity and hold the same wealth.

Comparison with the quasi-planner economy. The key outcome of the construction of this truncated economy is that this economy maps the quasi-planner environment. First, Euler equations (28)–(30) defining consumption level, labor supply and borrowing constraints are the same as (14)–(16). Second, the definitions of $\tilde{a}(e^N)$ in equations (12) and (33) are identical. Finally, the budget constraints are identical for both economies, as it appears from equations (9) and (35).

4.3 Production and budget constraint of the government

Before turning to the equilibrium definition, we provide the definition of aggregate quantities in this economy. Since individual behaviors are similar in both the truncated and the quasi-planner economies, the production sector and the budget constraint of the government have also the same expression as in Section 3. In particular, total labor in efficient unit is

$$L = \sum_{e^N \in \mathcal{E}^N} n(e^N)(e^N)l(e^N), \tag{36}$$

while the end-of-period wealth denoted A is

$$A \equiv \sum_{e^N \in \mathcal{E}^N} S(e^N)a'(e^N) = \sum_{e^N \in \mathcal{E}^N} S(e^N)\tilde{a}'(e^N). \tag{37}$$

The last equality stems from the definition of $\tilde{a}'(e^N)$ and the law of motion of the share of agents in each island (equations (12) and (13)). In words, the transfer of wealth across islands does not affect the total amount of wealth. The financial market equilibrium can be expressed as

$$A = B' + K'. \tag{38}$$

Note that because of Walras law, the financial market equilibrium implies the capital good market equilibrium. The government budget constraint is

$$G + rK + wN + (1+r)B \leq F(K, N, s_-) + B', \tag{39}$$

where, as previously, z_- denotes the previous period value of the aggregate state and B' is the next period value of public debt.

Factor prices are

$$\begin{aligned} \tilde{r} &= F_K(K, N, s_-), \\ \tilde{w} &= F_N(K, N, s_-). \end{aligned} \tag{40}$$

Finally, the distorting taxes are

$$\begin{aligned} \tau^K &= 1 - \tfrac{r}{\tilde{r}} \\ \tau^L &= 1 - \tfrac{w}{\tilde{w}} \end{aligned} \tag{41}$$

4.4 Equilibrium definition

Initial conditions are the initial size and wealth of the different populations with history $e^N \in \mathcal{E}^N$, $\Big(S_0(e^N), a_0(e^N)\Big)_{e^N \in \mathcal{E}^N}$, and thus the initial capital stock $K = \sum_{e^N \in \mathcal{E}^N} S_0(e^N), a_0(e^N)$, the initial public debt B_0 and the initial state s_{-1}. To form rational expectations, the information set is $\tilde{z} = \{(S(e^N), a(e^N))_{e^N \in \mathcal{E}^N}, B, s_-\}$. The state vector now includes the beginning-of-period distribution of wealth, given by by the vector $(S(e^N), a(e^N))_{e^N \in \mathcal{E}^N}$, which is necessary for agents to form expectations about next period prices (since it notably affects taxation).

For a given fiscal policy $\tau^K(\tilde{z})$, $\tau^L(\tilde{z})$, a recursive equilibrium is a set of variables $\{\Big(c(e^N), l(e^N), \tilde{a}(e^N), a'(e^N), \eta(e^N), S'e^N)\Big)_{e^N \in \mathcal{E}^N}, T'(e^{N+1})_{e^{N+1} \in \mathcal{E}^{N+1}}, w, r, N, A, B', K', \tilde{r}, \tilde{w}\}$ satisfying equations from (28) to (41).

We can now state the main result justifying the construction of this truncated economy.

Proposition 1 *For the same initial conditions, the set of equilibria of the quasi-planner economy is the same as the set of equilibria of the decentralized economy.*

Quasi-planner and truncated economies coincide with each other.. We have therefore proved that the quasi-planner economy of Section 3 can be decentralized by a well-chosen taxation scheme.

5 Ramsey problem

5.1 Problem formulation

We now turn to the resolution of Ramsey program in this incomplete market economy with aggregate shocks, where we take advantage of our limited heterogeneity equilibrium to solve the Ramsey program. The Ramsey program consists for the government to choose at date 0 a sequence of taxes on capital and labor as well as a path of public debt level that maximizes the aggregate welfare of the economy, assuming that individual agents behave rationally and subject to constraints on the economy-wide resources and on the government budget. In other words, it consists in finding the paths for taxes and public debt that selects the competitive equilibrium associated to the largest aggregate welfare. The following definition formalizes this statement.

Definition 1 (Ramsey program) *Given initial conditions about the wealth distribution* $\left(S_0(e^N), a_0(e^N)\right)_{e^N \in \mathcal{E}^N}$*, the initial public debt* B_0 *and the initial state aggregate state* s_{-1}*, the Ramsey program consists in choosing, at date* 0*, paths for capital and labor taxes* $(\tau_t^k, \tau_t^l)_{t\geq 0}$ *and for public debt* $(B_t)_{t\geq 0}$ *that maximizes the aggregate welfare among the set of competitive equilibria defined by equations (25)–(27), subject to aggregate resource constraints (36)–(38) and to the government budget constraint (39).*

In other words, the Ramsey program consists in setting taxes and public debt at $t = 0$, such that the government maximizes the aggregate welfare while internalizing individual agents objectives and constraints.

Following our result that shows that the quasi-planner equilibrium coincides with the truncated economy, the Ramsey program can be expressed as finding paths for taxes and public debt for maximizing the quasi-planner program subject to resource constraints and government budget constraints. Moreover, given the expression of distorting taxes τ_t^k and τ_t^l in equation (24) (or equivalently in (24) for the truncated economy), it is equivalent for the government to decide the post-tax interest rate $(r_t)_{t\geq 0}$ and the post-tax wage rate $(w_t)_{t\geq 0}$ instead of distorting taxes. In consequence, we can formalize the Ramsey program as follows:

$$\max_{\left(r_t,w_t,B_{t+1},(a_{t+1,e},c_{t,e},l_{t,e})_{e\in E^N}\right)_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t m(s^t) \sum_{e\in\mathcal{E}^N} S_{t,e} u\left(c_{t,e}, l_{t,e}\right), \tag{42}$$

$$B_{t+1} + F(K_t, L_t) = G_t + (1+r_t)B_t + (r_t+\delta)K_t + w_t L_t, \tag{43}$$

$$a_{t+1,e} + c_{t,e} = w_t n(e_0) l_{t,e} + \delta 1_{e_0=0} + (1+r_t)\tilde{a}_{t,e}, \text{ for all } e^N, \tag{44}$$

$$l_{t,e} = n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}, \tag{45}$$

$$u_c\left(c_{t,e}, l_{t,e}\right) + \eta_{t,e} = \beta \mathbb{E}_t \mathbb{E}_{e'|e} u_c\left(c_{t+1,e'}, l_{t+1,e'}\right)(1+r_{t+1}), \tag{46}$$

$$\eta_{t,e} a_{t+1,e} = 0, \tag{47}$$

$$\tilde{a}_{t,e} = \sum_{e'\in\mathcal{E}^N} \Pi_{t+1}\left(e', e\right) \frac{S_{t-1,e'}}{S_{t,e}} a_{t,e'}, \tag{48}$$

$$S_{t+1,e} = \sum_{e'\in\mathcal{E}^N} \Pi_{t+1}\left(e', e\right) S_{t,e'}, \tag{49}$$

$$A_t = \sum_{e\in\mathcal{E}^N} S_{t,e} a_{t,e}, \tag{50}$$

$$L_t = \sum_{e\in\mathcal{E}^N} S_{t,e} l_{t,e}, \tag{51}$$

$$K_{t+1} = A_t + B_{t+1}, \tag{52}$$

$$c_{t,e}, l_{t,e} \geq 0,\ a_{t+1,e} \geq -\underline{a}. \tag{53}$$

where all constraints (43)–(53) should be understood, unless specified, for all $s^t \in \mathcal{S}^t$ and all $e^t \in \mathcal{E}^t$. We denote by the subscript e the dependence in $N-$period history. The operator $\mathbb{E}_t$ is the expectation on future aggregate states s^{t+1} conditional on the current state being s^t. The operator $\mathbb{E}_{e'|e}$ is the expectation on future individual $N-$period histories denoted e' conditional on the current individual history being e.

Let us comment the Ramsey program (42)–(53). Equation (42) is the objective and corresponds to the maximization of the aggregate welfare with an additive criterion. Maximization devices are on one hand individual quantities of consumption, labor and asset holdings and one the other hand the public debt and post-tax interest and wage rates. Equation (43) is the government budget constraint, while the individual budget constraint is given in equation (44). The labor supply of agents is specified in equation (45). Remember that it corresponds to the explicit solution of the first-order condition determining labor supply in the individual program.

Together with Euler equation (46), equation (45) guarantees that the Ramsey outcome is a competitive equilibrium. Note that the Euler equation (46) is not restricted to only hold with equality for unconstrained agents, because of the presence of the coefficient η_t, which is the Lagrange multiplier of the borrowing constraint. The multiplier η_t appears in the slackness condition (47). Equation (48) defines the pooling operation that occurs when agents transit from islands to others. The dynamics of island sizes is specified in equation (49). Equations (50) and (51) provide the aggregation for individual wealth and labor supply, while the financial market clearing is given by equation (52). Finally, positivity constraints and borrowing constraints appear in equation (53).

5.2 Simplification of the Ramsey program

In this Section, we simplify the formulation of the Ramsey program exposed in equations (42)–(53). For sake of clarity, we will drop in the remainder the "accounting" equalities (49)–(51).

There are two steps in our simplification –that follows Acikgoz (2013). First, we substitute some expressions: labor supply using equation (45), savings $\tilde{a}$ using (48), capital using (52). Second, we express the Lagrangian and include the individual Euler equation and credit constraints (46) and (47). To do the latter, we denote $\beta^t m(s^t) S_{t,e} \lambda_{t,e}$ the Lagrange multiplier of the Euler equation of agent e^N in state s^t and $\beta^t m(s^t) S_{t,e} \theta_{t,e}$ the one for the credit constraint of agent e^N in state s^t. The Ramsey program (42)–(53) can be rewritten as

$$\max_{(r_t,w_t,B_{t+1},a_{t+1},c_t,l_t)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e} u\left(c_{t,e}, n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}\right) \tag{54}$$

$$-\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e}\lambda_{t,e}\left(u_c\left(c_{t,e}, l_{t,e}\right) + \eta_{t,e}\right.$$

$$\left. - \beta\mathbb{E}_t \sum_{\hat{e}\in\mathcal{E}^N} \mathbb{E}_{e'|e} u_c\left(c_{t+1,e'}, l_{t+1,e'}\right)(1 + r_{t+1})\right)$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e}\theta_{t,e}\eta_{t,e}a_{t+1,e}$$

$$\text{s.t.} \quad B_{t+1} + F(A_t - B_t, L_t) = G_t + (1 + r_t)B_t \tag{55}$$

$$+ (r_t + \delta)(A_t - B_t) + w_t L_t$$

$$a_{t+1,e} + c_{t,e} = w_t n(e_0)^{\varphi+1} w_t^{\varphi} 1_{e_0>0} + \delta 1_{e_0=0} + (1 + r_t) \sum_{\hat{e}\in\mathcal{E}^N} \frac{S_{t-1,\hat{e}}}{S_{t,e}} \Pi_t\left(\hat{e}, e\right) a_{t,\hat{e}} \tag{56}$$

where we have additionally dropped the flow equation (49) and positivity constraints (53).

We maximize the expression (54) subject to the government budget constraint (55) and individual budget constraints (56) –that should hold for any $e^N \in \mathcal{E}^N$.

We now define for all $e^N \in \mathcal{E}^N$ and $s^t \in \mathcal{S}^t$:

$$\Lambda_t(e^N, s^t) = \frac{\sum_{\hat{e}^N\in E^N} S_{t-1}(\hat{e}^N, s^{t-1})\lambda_{t-1}(\hat{e}^N, s^{t-1}) T\left(\hat{e}^N, e^N, s^t\right)}{S_t(e^N, s^t)}, \tag{57}$$

which can be interpreted as an average Lagrange multiplier, where the quantities $\lambda_{t-1}(\hat{e}^N, s^{t-1})$ for any $\hat{e}^N \in E^N$ are weighted the size of population transiting from state $\hat{e}^N$ to e^N. We also introduce $\lambda_0 = 0$ for sake of simplicity.

Finally, we can notice that we have $\lambda_{t+1}(e^N, s^{t+1}) = 0$ if $a_{t+1}(e^N, s^{t+1}) = 0$. We can therefore drop Lagrange multipliers η and θ that are in fact redundant.

The following lemma summarizes our simplification of the Ramsey program.

Lemma 2 (Simplified Ramsey program) *The Ramsey program in equations (54)–(53) can be simplified into:*

$$\max_{\left(r_t,w_t,B_{t+1},(a_{t+1,e},c_{t,e},l_{t,e})_{e\in\mathcal{E}^N}\right)_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in E^N} S_{t,e} \left(u\left(c_{t,e}, n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}\right)\right. \tag{58}$$

$$\left. + u'\left(c_{t,e}, n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}\right)\left(\Lambda_{t,e}(1+r_t) - \lambda_{t+1,e}\right)\right)$$

$$s.t.\ \ \lambda_{t,e} = 0 \ \textit{if}\ a_{t+1,e} = 0, \tag{59}$$

$$B_{t+1} + F(A_t - B_t, L_t) = G_t + (1+r_t)B_t \tag{60}$$

$$+ (r_t + \delta)(A_t - B_t) + w_t L_t,$$

$$a_{t+1,e} + c_{t,e} = w_t n(e_0)^{\varphi+1} w_t^{\varphi} 1_{e_0>0} + \delta 1_{e_0=0} \tag{61}$$

$$+ (1+r_t) \sum_{\hat{e}\in\mathcal{E}^N} \frac{S_{t-1,\hat{e}}}{S_{t,e}} \Pi_t\left(\hat{e}, e\right) a_{t,\hat{e}}.$$

The proof is relegated in Appendix.

5.3 Ramsey program first-order conditions

We finally compute the first order conditions associated to the Ramsey program (58)–(61). The following proposition summarizes our findings:

Proposition 2 (Ramsey program first-order conditions) *First order conditions associated to the Ramsey program can be expressed as follows*

$$u_{cc}(c_{t,e}, l_{t,e})\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right) = \beta \mathbb{E}_t \mathbb{E}_{e'|e}(1+r_{t+1}) u_{cc}(c_{t+1,e}, l_{t,e})\left(\Lambda_{t+1,e'}(1+r_{t+1}) - \lambda_{t+1,e'}\right)$$

$$+ \beta \mathbb{E}_t\left[\mu_{t+1}(r_{t+1} + \delta - F_K)\right] \tag{62}$$

$$\mu_t = \beta \mathbb{E}_t\left[\mu_{t+1}\left(F_K + 1 - \delta\right)\right] \tag{63}$$

$$\mu_t A_t = \sum_e S_{t,e} u_c(c_{t,e}, l_{t,e}) \Lambda_{t,e} \tag{64}$$

$$+ \sum_e S_{t,e} \tilde{a}_{t,e}\left(u_c(c_{t,e}, l_{t,e}) + u_{cc}(c_{t,e}, l_{t,e})\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right)$$

$$\mu\left(L_t + w_t L'_{t,w} - F(K_t, L_t) L'_{t,w}\right) = \sum_{e\in\mathcal{E}^N} S_{t,e} n(e_0)\left(1 - \varphi w_t^{\varphi} n(e_0)^{\varphi}\right) * \tag{65}$$

$$\left(u_c(c_{t,e}, l_{t,e}) + u_{cc}(c_{t,e}, l_{t,e})\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right) \tag{66}$$

Proof. The proof is relegated in Appendix. ■

Equation (62) is the *modified* Euler equation. It equalizes the marginal cost of saving more today to the marginal benefit of $(1+r_{t+1})$ consumption units in the next period. Note that there is no direct effect (through marginal utilities u') in the equation, simply because the individual Euler equation (46) still holds and direct effect terms cancel out. In this modified Euler equation, today's cost is the (internalized) saving price channel (through the term in u''). The expected benefit in the next period reflects the price channel again (term in u'') as well as the impact on the governement budget constraint: larger asset suplply means higher capital and thus larger interest payments on one side but also larger production on the other side.

Equation (63) is the Euler equation for the multiplier of the government budget constraint. This equation sets equal the marginal benefit of a larger debt today to the expected cost of the next period. The benefit is a more slack government budget constraint today because a larger debt issuance, while the cost is the larger debt repayment. This equation appears in similar fashion in absence of aggregate risk in Acikgoz (2013).

Equation (64) equalizes the marginal costs and benefits of raising the after-tax interest rate or equivalently the tax on capital. The benefit is simply a more slack government budget constraint and the benefit is all the larger when the interest-bearing asset supply is large (since it is the tax base). The cost is first the direct impact on individual budget constraints –this is the term proportional to a, which is the individual tax base– and second the distorsion impact on individual saving decisions.

Equation (65) is similar to equation (64) for the after-tax wage or or equivalently the tax on wages. The benefit is the impact of the tax on the government budget constraint. Note that the direct positive effect (proportional to the labor supply) is dampened by a distorsion on the labor supply: raising taxes affcets the tax base. The cost is only made of the direct impact on the individual budget constraints.

6 A numerical application

We now provide a numerical application to investigate the properties of optimal public debt and taxes. Following the literature, we identify the uninsurable risk to the unemployment risk. We thus consider two idiosyncratic states, that we qualify of employment and of unemployment. As there are two types of agents, we denote by e the variable referring to employed agents, and by

u the variable referring to unemployed agents. Unemployed agents get the home production δ, while employed agents have a labor productivity $n(e)$. To uncover the trade-offs, we consider the simple case where there are two islands, $N = 1$. As a consequence, all employed agents will consume and save the same amount and all unemployed agents will consume and save the amount. The simple environment already exhibits a rich and non obvious time-varying optimal policy. In addition, the analysis of this equilibrium makes it explicit how perturbation method can be used to numerically solve for this equilibrium.

6.1 Assumptions and model equations

The utility function is CRRA $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ if $\sigma \neq 1$ and $u(c) = \log(c)$, otherwise. We denote $c_{t,e}$ and $c_{t,u}$ the consumption of employed and unemployed agents, while $a_{t,e}$ and $a_{t,u}$ denote asset holdings. We construct our equilibrium with a guess-and-verify strategy. We assume that unemployed agents are always credit-constrained, such that $a_{t+1,u} = 0$ -(we set $\overline{a} = 0$). We check that it is indeed the case, by showing that this property holds in steady sate as well as in transitory dynamics. This will notably require the aggregate shock to be not too volatile. This assumption also implies that $\tilde{a}_{t,e} = \alpha_{t-1} S_{t-1,e} a_{t,e} / S_{t,e}$ and $\tilde{a}_{t,u} = (1-\alpha_{t-1}) S_{t-1,e} a_{t,e} / S_{t,u}$. The dynamics of households consumption is determined by the three equations

$$u'\left(c_{t,e}\right) = \mathbb{E}_t\left(\alpha_t u'\left(c_{t+1,e}\right) + \left(1-\alpha_t\right) u'\left(c_{t+1,u}\right)\right)\left(1+r_{t+1}\right) \tag{67}$$

$$a_{t+1,e} + c_{t,e} = w_t + (1+r_t)\alpha_t S_{t-1,e} a_{t,e} / S_{t,e}, \tag{68}$$

$$c_{t,u} = \delta + (1+r_{t-1})(1-\alpha_t) S_{t-1,e} a_{t,e} / S_{t,u}, \tag{69}$$

The dynamic of public debt is

$$G_t + r_t K_{t-1} + w_t L_t + (1+r_t)B_{t-1} = F(K_{t-1}, L_t, s_{t-1}) + B_t. \tag{70}$$

$$\tilde{r}_t = F_K(K_{t-1}, L_t, s_{t-1}), \tag{71}$$

$$\tilde{w}_t = F_L(K_{t-1}, L_t, s_{t-1}). \tag{72}$$

$$L_t = S_{t,e} n(e) \left(n(e) w_t\right)^{\varphi} \tag{73}$$

$$S_{t,e} a_{t+1,e} = B_t + K_t \tag{74}$$

$$\tau_t^K = 1 - \frac{r_t}{\tilde{r}_t} \tag{75}$$

$$\tau_t^L = 1 - \frac{w_t}{\tilde{w}_t} \tag{76}$$

$$S_{t+1,e} = \alpha_t S_{t,e} + (1-\rho_t) S_{t,u} \tag{77}$$

$$S_{t+1,u} = (1-\alpha_t) S_{t,e} + \rho_t S_{t,u} \tag{78}$$

Before characterizing the optimal fiscal policy in this environment, we start with noting that $\lambda_t(u) = 0$, since credit constraints bind for unemployed agents. The optimal fiscal policy is characterized by the following equations

$$\begin{aligned} u''(c_{t,e}) \left((1+r_t)\Lambda_{t,e} - \lambda_{t,e}\right) \lambda_{t,e} = {} & \beta \mathbb{E}_t \alpha_t (1+r_{t+1}) u''(c_{t+1,e}) \\ & \times \left((1+r_{t+1})\Lambda_{t+1,e} - \lambda_{t+1,e}\right) \\ & + \beta \mathbb{E}_t (1-\alpha_t)(1+r_{t+1}) u''(c_{t+1,u})(1+r_{t+1})\Lambda_{t+1,u} \\ & + \beta \mathbb{E}_t \mu_{t+1} \left(F_K(K_t, L_{t+1}, s_t) - r_{t+1}\right) \end{aligned} \tag{79}$$

$$\mu_t = \beta \mathbb{E}_t \mu_{t+1} \left(F_K(K_t, L_{t+1}, s_t) + 1\right) \tag{80}$$

$$\begin{aligned} \mu_t S_{t,e} a_{t+1,e} = {} & S_{t,e} u'(c_{t,e}) \Lambda_{t,e} + S_{t,u} u'(c_{t,u}) \Lambda_{t,u} \\ & + \alpha_{t-1} a_{t,e} S_{t,e} \left(u'(c_{t,e}) + u''(c_{t,e}) \left((1+r_t)\Lambda_{t,e} - \lambda_{t,e}\right)\right) \end{aligned} \tag{81}$$

$$\tag{82}$$

$$+ (1-\rho_{t-1}) a_{t,e} S_{t,u} \left(u'(c_{t,u}) + u''(c_{t,u})(1+r_t)\Lambda_{t,u}\right) \tag{83}$$

$$\mu_t \left(1 + \varphi \frac{w_t - F_N(K_{t-1}, L_t, s_{t-1})}{w_t}\right) L_t = S_{t,e} n(e) \left(u'(c_{t,e}) + u''(c_{t,e}) \left((1+r_t)\Lambda_{t,e} - \lambda_{t,e}\right)\right) \tag{84}$$

$$\Lambda_{t,e} = \alpha_t \lambda_{t-1,e} S_{t-1,e} / S_{t,e} \tag{85}$$

$$\Lambda_{t,u} = (1-\rho_t) \lambda_{t-1,e} S_{t-1,e} / S_{t,u} \tag{86}$$

The condition for credit constraints to be binding for unemployed households is

$$u'\left(c_{t,u}\right) > \mathbb{E}_t\left[\left(\left(1-\alpha_t\right)u'\left(c_{t+1,e}\right) + \rho_t u'\left(c_{t+1,u}\right)\right)\left(1+r_{t+1}\right)\right] \tag{87}$$

In this economy we consider three shocks. The first one is a shock on the idiosyncratic risk. We introduce a shock on the transition α_t such that the unemployment rate temporarily increases. This shock will help to identify optimal fiscal policy after a increase in uncertainty at the agents level. The second one shock is a technology shock affecting the production function. The third shock is public spending shock. We want to investigate the shape of the fiscal multiplier when optimal fiscal policy is implemented. The state is thus $s_t = (s_t^\alpha, s_t^a, s_t^g)$.

Uncertainty shock. The first exogenous process affects transition probabilities. Assume that

$$s_t^\alpha = (1-\varphi_\alpha)\bar{\alpha} + \varphi_\alpha s_{t-1}^\alpha + \epsilon_t^\alpha \tag{88}$$

and the transitions are

$$\alpha_t = s_t^\alpha \tag{89}$$

$$\rho_t = \bar{\rho} \tag{90}$$

Technology shock. Technology is Cobb-Douglas with the following production function $F(K, L, s) = \Psi(s)K^\kappa L^{1-\kappa}$., where $\Psi(s) = e^s$. where z follows

$$s_t^a = \varphi_a s_t^a + \epsilon_t^a \tag{91}$$

$$\Psi_t = e^{s_{t-1}^a} \tag{92}$$

Public spending shock. The process for public spending is assumed to be auto-regressive:

$$s_t^g = (1-\varphi_g)\bar{G} + \varphi_g s_{t-1}^g + \epsilon_t^g \tag{93}$$

$$G_t = s_t^g \tag{94}$$

$\tilde{r}$	τ^K	τ^L	B/Y	λ_e	μ
4.17	14.7%	9.4%	17%	0.011	11.8

Table 1: Steady-state outcome of the model

For the law of motion the aggregate state described by equations (88), (91) and (93) and for the processes $\{\alpha_t, \rho_t, \Psi_t, G_t\}$ given by the four equations (89),(90),(92),(94), for given initial capital wealth $a_0(e)$, initial public debt B_{-1}, initial promise $\lambda_{-1}(e)$, state s_{-1}, and initial population shares $S_0(e) = \bar{S}$ and $S_0(u) = 1 - \bar{S}$, an equilibrium is a set of 18 variables $\{c_t(e), c_t(u), a_{t+1}(e), r_t, w_t, G_t, K_t, L_t, B_t, \tilde{r}_t, \tilde{w}_t, \tau_t^K, \tau_t^L, S_t(e), S_t(u), \Lambda_t(e), \lambda_t(e), \mu_t\}$satisfying the 18 equations (67) to (86), and the inequality constraint (87).

The steady state is defined as an equilibrium where $\epsilon_t^\alpha = \epsilon_t^g = \epsilon_t^a = 0$ and where all real variables are constant.

6.2 Parametrization

The period is a year. There are 7 parameters in the model. We first assume a log-utility function $\sigma = 1$. The subjective discount factor is set $\beta = 0.96$, to obtain a real interest close to 4%. The estimation of the elasticity of the labor supplies varies in the literature. We use $\varphi = 1$, and we check that the dynamics below don't change for alternative values $\varphi \in (0.3, 2.0)$.

The capital share in production is $\kappa = 0.36$. The steady state value for public spending $\bar{G} = 0.063$ is set such that public spending in GDP is 40%, which is the value for the US in 2010. Concerning labor market transitions, the values are set to reproduce the annualized transitions in the US job market (Challe and Ragot, 2014). It implies $\bar{\alpha} = 0.95$ and $\bar{\rho} = 0.05$, i.e. a steady-state unemployment rate equal to 5%. As a benchmark, we set home production to $\delta = 0$. It implies a fall in consumption for agents switching from employment to unemployment of 14%. This last value is the direct measure of the lack of insurance in our model. This figure is well inside the range of available estimates (which range between 7 and 30 percent, see, e.g., Cochrane, 1991; Gruber, 1997; Chodorow-Reich and Karabarbounis 2015).

The steady state outcomes of the model are summed up in the following table.

The steady-state (before tax) real interest rate is 4.17%. The tax on capital is 14.7% and the steady-state tax on labor is 9.4%. Optimal debt over GDP is 17%. These fiscal values are much lower than the US average, which is 27% for tax on labor and 40% for the tax on capital.

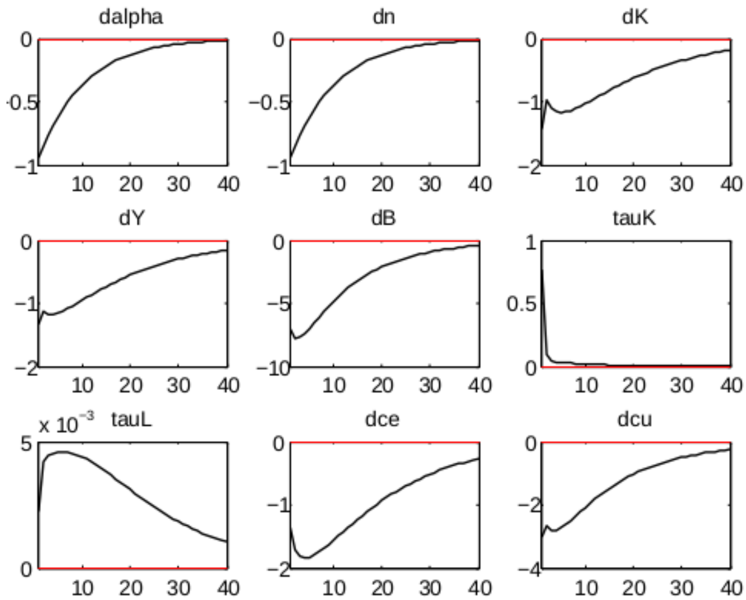

Figure 1: Impulse responses after a negative shock on α

This low value of public debt over GDP was attained in 1930.

Finally, we set persistence parameters to $\varphi_\alpha = \varphi_a = \varphi_g = 0.9$ for the three shocks.

6.3 Fiscal policy after an uncertainty shock

The following graph plots the path of the real variables after a negative shock to α.

We denote dx for the percentage deviation of the variable x from its steady-state value. For example, *dalpha* is the percentage change in α. Both α and employment n falls from 1% on impact and goes back progressively to their steady state value. This generates a fall in output Y and capital K of roughly 1% on impact. The three policy variables, public debt and tax rate on capital and labor are then plotted. Public debt decreases on impact, whereas capital tax increases sharply for a small period. The labor tax increases persistently for much smaller amount. After such a policy, consumption of employed agents falls by less than 1.8% whereas the consumption of unemployed agents falls by 2.8% .

What the the effect of this optimal fiscal policy? As capital is a fixed factor during the period, the sharp increase in capital tax is a way to get some resources to induce a fall in public debt, by 7%. This fall reduces the "crowding-out" effect of public debt and reduces the fall in capital. To see this, we simulated the economy fixing the capital tax to its steady state value,

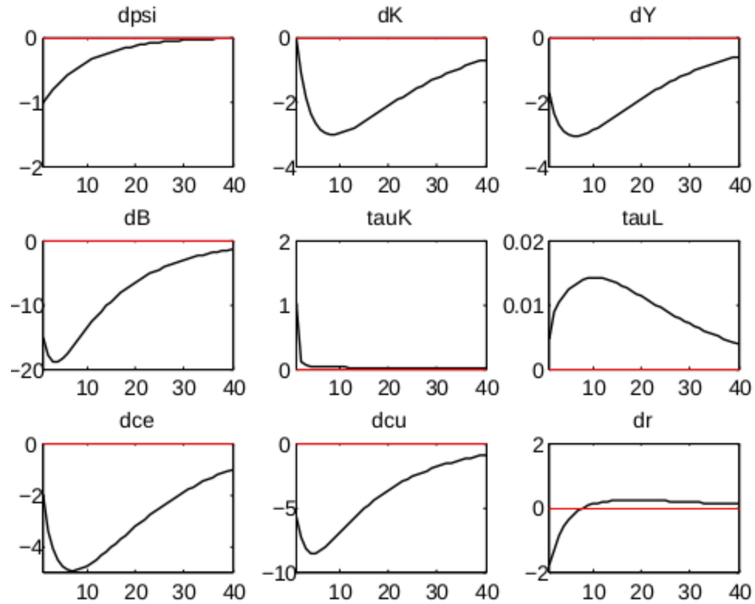

Figure 2: Impulse responses after a negative technology shock

and optimizing only on labor tax. In this case, the reduction in public debt is only 3.8% and capital and output falls by 1.7% (instead of 1%). In this alternative economy, the consumption of employed households falls by 2.9%. After a negative shock on employment, optimal fiscal policy reduces the fall in capital by finding resources to reduce public debt.

6.4 Fiscal policy after a negative technology shock

The following graph plots key variables after a negative technology shock.

The fall of TFP of 1% generates a fall in capital and output of 3.1%. Note that as there is no change in the transitions, employment does not change, but labor supply falls on impact. As before, capital tax increases a lot on impact because capital is a fixed factor at the beginning of the period. This allows a sharp reduction in public debt of roughly 20% (debt over GDP falls from 17% to 13.5% of debt over steady-state GDP), what reduces the fall in capital after the negative technology shock. The consumption of employed agents falls by 5%, whereas the consumption of unemployed agents falls by 8%. As before, keeping the capital tax equal to its steady-state value, one finds a much smaller fall in public debt (14%), and a larger fall in output, equal to 3.9%.

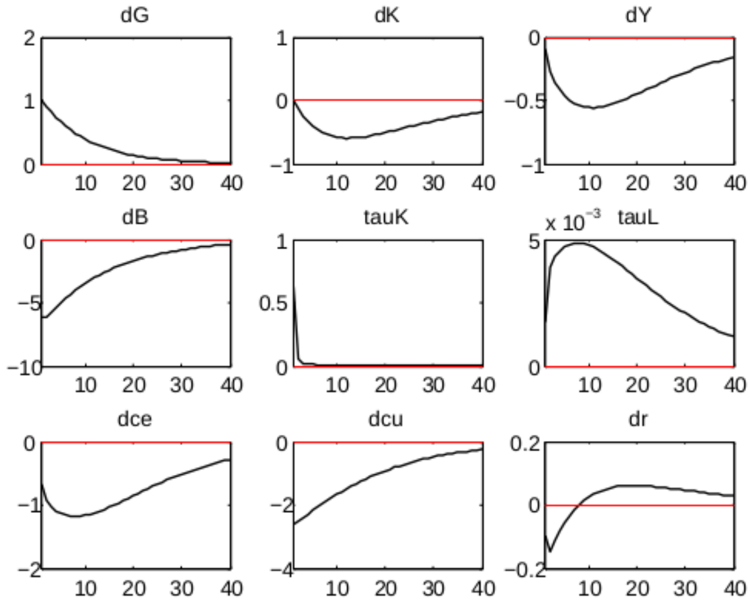

Figure 3: Impulse responses after a negative public spending shock

6.5 Fiscal policy after consumption spending shock

Finally, the last graph plots the effect of an increase of 1% in public spending financed by distorting taxes.

An increase in public spending increases distorting taxes and thus reduces output by 0.5%. As before the sharp increase in capital taxes allows for reducing public debt. This, in turn limits the fall in capital. Keeping distorting taxes on capital fixed, one finds a fall in output and consumption twice as big.

6.6 Concluding remarks on optimal fiscal policy

From this experience, we can conclude that optimal monetary policy after a adverse shock aims at containing the fall of capital by reducing public debt. This is obtained by an initial increase in capital taxes. Additionally, one finds that labor taxes are very smooth whereas capital taxes are very volatile. This result confirms the one found in Aiyagari, Marcet, Sargent and Seppala (2002), Farhi (2012) who considers an economy with a representative agent. Considering incomplete insurance market does not alter this general result.

Incomplete insurance market generates a well defined optimal long-run level of public debt, which is used as a store of value by private agents. As a consequence, both taxes and public debt are mean-reverting after a transitory shock. In particular, public debt does not exhibit a unit root component as in Aiyagari, Marcet, Sargent and Seppala (2002). As capital and public debt compete as self-insurance device in a incomplete insurance-market economy, one finds that public debt falls after a negative shock to limit its “crowding out” effect. To our knowledge, this is a new finding about optimal debt management in the business cycle.

7 Concluding remarks

We have proved that the competitive equilibrium in an incomplete insurance market economy with aggregate shocks can be represented as the allocation of quasi-planner using limited fiscal instruments. The gain of this representation is that it generates a finite state-space equilibrium, in which the Ramsey outcome can be studied for various types of instruments and various types of aggregate shocks. This opens the possibility to study optimal policies in these environments.

We use this framework to study optimal fiscal policy, when distorting taxes on capital and labor and public debt are available. Compared to representative agent economy, the main distinguishing feature of the incomplete insurance-market economy, is that the optimal level of public debt is well defined, as agents hold both public debt and capital for self-insurance motives against idiosyncratic shocks. We find that all taxes and public debt are mean reverting, and that public debt decreases after a transitory negative shock, to reduce the fall in the capital stock.

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Appendix

A Proof of Lemma 2

We start from the following program

$$\max_{(r_t,w_t,B_{t+1},a_{t+1},c_t,l_t)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e} u\left(c_{t,e}, l_{t,e}\right) \tag{95}$$

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e}\lambda_{t,e} \left(u_c\left(c_{t,e}, l_{t,e}\right) - \beta\eta_{t,e}\mathbb{E}_t \sum_{\hat{e}\in\mathcal{E}^N} T_{t+1}\left(e,\hat{e}\right) u_c\left(c_{t+1,\hat{e}}, l_{t+1,\hat{e}}\right)\left(1+r_{t+1}\right)\right)$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e}\theta_{t,e}\eta_{t,e}a_{t+1,e}$$

$$\text{s.t.} \qquad B_{t+1} + F(A_t - B_t, L_t) = G_t + (1+r_t)B_t + (r_t+\delta)(A_t - B_t) + w_t L_t \tag{96}$$

$$a_{t+1,e} + c_{t,e} = w_{t,e} n(e_0) + (1+r_t) \sum_{\hat{e}\in\mathcal{E}^N} \frac{S_{t-1,\hat{e}}}{S_{t,e}} \Pi_t\left(\hat{e}, e\right) a_{t,\hat{e}} \tag{97}$$

We start with rewriting the Ramsey objective of equation (54):

$$\max_{(r_t,w_t,B_{t+1},a_{t+1},c_t,l_t)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e} u\left(c_{t,e}, l_{t,e}\right) \tag{98}$$

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e}\lambda_{t,e} \left(u_c\left(c_{t,e}, l_{t,e}\right) - \beta\eta_{t,e}\mathbb{E}_t \sum_{\hat{e}\in\mathcal{E}^N} T_{t+1}\left(e,\hat{e}\right) u_c\left(c_{t+1,\hat{e}}, l_{t+1,\hat{e}}\right)\left(1+r_{t+1}\right)\right)$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e\in\mathcal{E}^N} S_{t,e}\theta_{t,e}\eta_{t,e}a_{t+1,e}$$

The second term of the second line in equation (54) can be simplified as

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t+1} \sum_{e\in\mathcal{E}^N} S_{t,e}\lambda_{t,e}\mathbb{E}_t \sum_{\hat{e}\in\mathcal{E}^N} T_{t+1}\left(e,\hat{e}\right) u_c\left(c_{t+1,\hat{e}}, l_{t+1,\hat{e}}\right)\left(1+r_{t+1}\right) =$$

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \mathbb{E}_t\beta^{t+1} \sum_{e\in\mathcal{E}^N} \sum_{\hat{e}\in\mathcal{E}^N} S_{t,e}\lambda_{t,e} T_{t+1}\left(e,\hat{e}\right) u_c\left(c_{t+1,\hat{e}}, l_{t+1,\hat{e}}\right)\left(1+r_{t+1}\right) =$$

$$\sum_{t=0}^{\infty} \mathbb{E}_0\beta^{t+1} \sum_{e\in\mathcal{E}^N} \frac{\sum_{\hat{e}\in\mathcal{E}^N} S_{t,\hat{e}}\lambda_{t,\hat{e}} T_{t+1}\left(\hat{e},e\right)}{S_{t+1,e}} S_{t+1,e} u_c\left(c_{t+1,e}, l_{t+1,e}\right)\left(1+r_{t+1}\right) =$$

$$\sum_{t=0}^{\infty} \mathbb{E}_0\beta^{t+1} \sum_{e\in\mathcal{E}^N} \Lambda_{t+1,e} S_{t+1,e} u_c\left(c_{t+1,e}, l_{t+1,e}\right)\left(1+r_{t+1}\right) \tag{99}$$

where $\Lambda_{t,e}$ is defined in equation (57). We can remark that we can impose without loss of generality that $\lambda_{-1} = 0$. We obtain that (99) becomes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t+1} \sum_{e \in \mathcal{E}^N} S_{t,e} \lambda_{t,e} \mathbb{E}_t \sum_{\hat{e} \in \mathcal{E}^N} T_{t+1}(e, \hat{e})\, u_c(c_{t+1,\hat{e}}, l_{t+1,\hat{e}})(1 + r_{t+1}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e \in \mathcal{E}^N} \Lambda_{t,e} S_{t,e} u_c(c_{t,e}, l_{t,e})(1 + r_t) \tag{100}$$

Using equation (100), the Ramsey program (98) becomes:

$$\max_{(r_t, w_t, B_{t+1}, a_{t+1}, c_t, l_t)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e \in \mathcal{E}^N} S_{t,e}\left(u(c_{t,e}, l_{t,e}) + (\Lambda_{e,t}(1 + \tilde{r}_t) - \lambda_{e,t})\, u_c(c_{t,e}, l_{t,e})\right) \tag{101}$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e \in \mathcal{E}^N} S_{t,e} \theta_{t,e} \eta_{t,e} a_{t+1,e}$$

We deduce then the Ramsey program (58)–(61).

B Proof of Proposition2

We compute the first-order conditions of the simplified Ramsey program (58)–(61). Denoting $\beta^t m(s^t) \mu_t$ the Lagrange multiplier associated to the government budget constraint, we obtain the following expression for the Lagragian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in E^N} S_{t,e}\left(u\left(c_{t,e}, n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}\right)\right. \tag{102}$$

$$\left. + u_c\left(c_{t,e}, n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}\right)(\Lambda_{t,e}(1 + r_t) - \lambda_{t+1,e})\right)$$

$$- \mu_t \beta^t \left(G_t + (1 + r_t) B_t + (r_t + \delta)(A_t - B_t) + w_t L_t - B_{t+1} - F(A_t - B_t, L_t)\right),$$

where $c_{t,e} = n(e_0)^{\varphi+1} w_t^{\varphi+1} 1_{e_0>0} - a_{t+1,e} + (1 + r_t) \sum_{\hat{e} \in \mathcal{E}^N} \frac{S_{t-1,\hat{e}}}{S_{t,e}} \Pi_t(\hat{e}, e)\, a_{t,\hat{e}}$.

Derivative with respect to $a_{t+1,e}$. We compute the derivative of the Lagrangian (102) wrt $a_{t+1,e}$. Note that $c_{t,e}$ obviously depends on $a_{t+1,e}$ but this is also the case of any $c_{t+1,\tilde{e}}$ where $s^{t+1} \succ s^t$ and $\tilde{e} \in E^N$:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial a_{t+1,e}} =&\beta^t m(s^t) S_{t,e} \frac{\partial c_{t,e}}{\partial a_{t+1,e}} \left(u_{c,e} + u_{cc,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right) \\
&+ \sum_{s^{t+1}} \beta^{t+1} m(s^{t+1}) \sum_{\tilde{e} \in E^N} S_{t+1,\tilde{e}} \frac{\partial c_{t+1,\tilde{e}}}{\partial a_{t+1,e}} \left(u_{c,\tilde{e}} + u_{cc,\tilde{e}}\left(\Lambda_{t+1,\tilde{e}}(1+r_{t+1}) - \lambda_{t+1,\tilde{e}}\right)\right) \\
&- \beta^t m(s^t) S_{t,e} \mu_t (r_t + \delta - F_K),
\end{aligned}
$$

where $A_t = \sum_{e \in \mathcal{E}^N} S_{t,e} a_{t+1,e}$, $u_{c,t,e} = u_c(c_{t,e}, l_{t,e})$ and $u_{cc,t,e} = u_{cc}(c_{t,e}, l_{t,e})$. We deduce that $\frac{\partial \mathcal{L}}{\partial a_{t+1e}} = 0$ iff

$$
\begin{aligned}
0 =& \frac{\partial c_{t,e}}{\partial a_{t+1,e}} \left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right) \\
&+ \beta \mathbb{E}_t \sum_{\tilde{e} \in E^N} \frac{S_{t+1,\tilde{e}}}{S_{t,\tilde{e}}} \frac{\partial c_{t+1,\tilde{e}}}{\partial a_{t+1,e}} \left(u_{c,t+1,\tilde{e}} + u_{cc,t+1,\tilde{e}}\left(\Lambda_{t+1,\tilde{e}}(1+r_{t+1}) - \lambda_{t+1,\tilde{e}}\right)\right) \\
&- \mu_t (r_t + \delta - F_K)
\end{aligned}
\tag{103}
$$

Noting that

$$
\begin{aligned}
\frac{\partial c_{t,e}}{\partial a_{t+1,e}} &= -1 \\
\frac{\partial c_{t+1,\tilde{e}}}{\partial a_{t+1,e}} &= (1+r_{t+1}) \frac{S_{t,e}}{S_{t+1,\tilde{e}}} \Pi_{t+1}(e, \tilde{e})
\end{aligned}
$$

and (103) becomes

$$
\begin{aligned}
&\left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right) - \mu_t (r_t + \delta - F_K) = \\
&\beta \mathbb{E}_t \mathbb{E}_{\tilde{e}|e} (1+r_{t+1}) \left(u_{c,t+1,\tilde{e}} + u_{cc,t+1,\tilde{e}}\left(\Lambda_{t+1,\tilde{e}}(1+r_{t+1}) - \lambda_{t+1,\tilde{e}}\right)\right).
\end{aligned}
$$

This last equation can be simplified using Euler equation $u_{c,e} = \beta \mathbb{E}_t \mathbb{E}_{\tilde{e}|e}(1+r_{t+1}) u_{c,\tilde{e}}$ and we obtain

$$
\begin{aligned}
&u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right) - \mu_t (r_t + \delta - F_K) = \\
&\beta \mathbb{E}_t \mathbb{E}_{\tilde{e}|e} (1+r_{t+1}) u_{cc,t+1,\tilde{e}}\left(\Lambda_{t+1,\tilde{e}}(1+r_{t+1}) - \lambda_{t+1,\tilde{e}}\right),
\end{aligned}
\tag{104}
$$

which is the modified Euler equation.

Derivative with respect to B_{t+1}. Computing the derivative of the Lagrangian (102) with respect to B_{t+1} yields:

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = \beta^t m(s^t)\mu_t - \beta^{t+1}\sum_{s^{t+1}} m(s^{t+1})\mu_{t+1}\left(F_K + (1+r_{t+1}) - (r_{t+1}+\delta)\right)$$

We deduce the Euler equation for μ_t:

$$\mu_t = \beta \mathbb{E}_t \mu_{t+1}\left(F_K + 1 - \delta\right). \tag{105}$$

Derivative with respect to r_t. Computing the derivative of the Lagrangian (102) with respect to r_t yields:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial r_t} = \beta^t m(s^t)\sum_{e\in E^N} S_{t,e}\left(u_{c,t,e}\Lambda_{t,e} + \frac{\partial c_{t,e}}{\partial r_t}\left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t+1,e}\right)\right)\right)\\ - \beta^t m(s^t)\mu_t A_t\end{aligned}$$

Noting that

$$\frac{\partial c_{t,e}}{\partial r_t} = \sum_{\hat{e}\in\mathcal{E}^N} \frac{S_{t-1,\hat{e}}}{S_{t,e}}\Pi_t\left(\hat{e},e\right)a_{t,\hat{e}},$$

we obtain that $\frac{\partial \mathcal{L}}{\partial r_t} = 0$ iff:

$$\begin{aligned}0 = \sum_{e\in\mathcal{E}^N} S_{t,e}\left(u_{c,t,e}\Lambda_{t,e} + \sum_{\hat{e}\in\mathcal{E}^N}\frac{S_{t-1,\hat{e}}}{S_{t,e}}\Pi_t\left(\hat{e},e\right)a_{t,\hat{e}}\left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t+1,e}\right)\right)\right)\\ - \mu_t A_t\end{aligned}$$

and we finally obtain:

$$\begin{aligned}\mu_t A_t = \sum_{e\in\mathcal{E}^N} S_{t,e}u_{c,t,e}\Lambda_{t,e}\\ + \sum_{e\in\mathcal{E}^N}\left(\sum_{\hat{e}\in\mathcal{E}^N} S_{t-1,\hat{e}}\Pi_t\left(\hat{e},e\right)a_{t,\hat{e}}\right)\left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t+1,e}\right)\right)\end{aligned} \tag{106}$$

Derivative with respect to w_t.

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in E^N} S_{t,e} \left(u\left(c_{t,e}, n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}\right)\right.$$

$$-u_c\left(c_{t,e}, n(e_0)^{\varphi} w_t^{\varphi} 1_{e_0>0}\right)\left(\Lambda_{t,e}(1+r_t) - \lambda_{t+1,e}\right))$$

$$-\mu_t \beta^t \left(G_t + (1+r_t)B_t + (r_t+\delta)(A_t - B_t) + w_t L_t - B_{t+1} - F(A_t - B_t, L_t)\right),$$

Computing the derivative of the Lagrangian (102) with respect to w_t yields:

$$\frac{\partial \mathcal{L}}{\partial w_t} = \beta^t m(s^t) \sum_{e \in E^N} S_{t,e} \left(\frac{\partial c_{t,e}}{\partial w_t} \left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right)\right)$$

$$+ \beta^t m(s^t) \sum_{e \in \mathcal{E}^N} S_{t,e} \left(\frac{\partial l_{t,e}}{\partial w_t} \left(u_{l,t,e} + u_{cl,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right)\right)$$

$$- \beta^t m(s^t) \mu_t \left(L_t + w_t L'_{t,w} - F(K_t, L_t) L'_{t,w}\right)$$

Noting that $u_{t,e} = u(c_{t,e} - \frac{l_{t,e}^{1+1/\varphi}}{1+1/\varphi})$ and $l_{t,e} = (w_t n(e_0))^{\varphi}$ we have $u_{l,t,e} = -l_{t,e}^{1/\varphi} u_{c,t,e} = -w_t n(e_0) u_{c,t,e}$,

$$\frac{\partial c_{t,e}}{\partial w_t} = n(e_0), \qquad \frac{\partial l_{t,e}}{\partial w_t} = \varphi w_t^{\varphi-1} n(e_0)^{\varphi}$$

we obtain:

$$\mu_t \left(L_t + w_t L'_{t,w} - F(K_t, L_t) L'_{t,w}\right) = \sum_{e^N \in \mathcal{E}^N} S_{t,e} n(e_0) \left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right)$$

$$- \sum_{e \in \mathcal{E}^N} S_{t,e} n(e_0) \left(\varphi w_t^{\varphi} n(e_0)^{\varphi} \left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right)\right)$$

$$= \sum_{e \in \mathcal{E}^N} S_{t,e} n(e_0) \left(1 - \varphi w_t^{\varphi} n(e_0)^{\varphi}\right) \left(u_{c,t,e} + u_{cc,t,e}\left(\Lambda_{t,e}(1+r_t) - \lambda_{t,e}\right)\right)$$

Alternatively, we could consider $u(C_{t,e})$ with $C_{t,e} = c_{t,e} - \frac{l_{t,e}^{1+1/\varphi}}{1+1/\varphi}$:

$$\frac{\partial C_{t,e}}{\partial w_t} = \frac{\partial c_{t,e}}{\partial w_t} - \frac{\partial l_{t,e}}{\partial w_t} l_{t,e}^{1/\varphi}$$

$$= n(e_0) - \varphi w_t^{\varphi} n(e_0)^{1+\varphi}$$

$$= n(e_0) \left(1 - \varphi w_t^{\varphi} n(e_0)^{\varphi}\right)$$