Island Matching

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Abstract

The purpose of this paper is to use Shimer’s (2006) mismatch structure to create a synthesis of the Lucas-Prescott island model and the Diamond-Mortensen-Pissarides matching model of unemployment. The wage and employment are determined by an auction on each island. All unmatched agents are randomly assigned to another market at the beginning of each period. The number of unmatched jobs is determined by free entry. The model has a dynamic equilibrium solution which is unique and efficient in the limit as the number of workers and job per market become large. When calibrated to the recently observed unemployment and vacancy rates, the model fits the vacancy-unemployment relationship well and implies a log linear relationship between the job finding rate and the vacancy-unemployment relationship. Finally, the model’s implied responses to productivity shocks that are much larger than the canonical equilibrium model of unemployment.

1 Introduction

The purpose of this paper is to use Shimer’s (2006) mismatch structure to create a synthesis of the Lucas-Prescott island model and the Diamond-Mortensen-Pissarides matching model of unemployment. The distinguishing feature of the model is that all matched workers and jobs randomly search submarket, called "islands in the literature" at the beginning of each period. As the characteristics of each island depends only on the agents that visit
it, random independent assignment is an equilibrium search strategy in the special case of identical workers and jobs and trivial mobility costs. The paper’s title emphasizes the synthesis.

As is Lucas and Prescott (1974), price and quantity in each location are determined by an ex post auction in the model. Specifically, the number of job-worker matches that form in a particular location in any period is determined by the number on the short side of the market and the prevailing wage for new matches is the common reservation wage of the agents on the long side. Finally, the number of unmatched jobs in the economy is determined by a free entry condition while the total number of workers is given.

The search equilibrium is both unique and efficient if the matching process exhibits constant returns to scale in the sense that the expected number of matches increases in proportion to the average numbers of unmatched workers and jobs holding their ratio constant for reasons anticipated by Mortensen (1982) and explicitly pointed out by Kennes et al. (2006).\textsuperscript{1} Although the matching process implied by the random assignment of workers and jobs to islands exhibits increasing returns in general, the required condition for efficiency holds as an approximation when the average numbers of workers and jobs per island are large. In this case, the private and social incentives are (almost) aligned when agents on the short side of the market obtain all of match surplus.

Except for the way in which wages are determined, the model is also closely related to the original Diamond-Mortensen-Pissarides formulation of job-worker matching as summarized in the first chapter of Pissarides (2000). Specifically, the explicit matching process generates a negative relationship between vacancies and unemployment (a Beveridge curve) and a positive relationship between the job finding rate and the ratio of vacancies to unemployment. Indeed, given parameters chosen to match the U.S. average unemployment and vacancy rates observed in the last six years, the Beveridge curve implied by the model and observed unemployment rates explains 90\% of the variation in vacancy rates observed over that same period. Furthermore, the implicit relationship between the job finding rate and the vacancy-unemployment ratio is log linear with an elasticity of about 0.485, two facts consistent with the literature on the estimation of empirical match-

\textsuperscript{1}In other words, exante observation of the wages that will prevail in each island, as assumed in directed search models, is not needed to internalize search externalities.
ing functions reviewed by Petrongolo and Pissarides (2000).

The model also has important implications for the volatility of unemployment and vacancies. Although only about 40% of the volatility in the vacancy-unemployment ratio is explained given the parameter values suggested by Mortensen and Nagypál (2006), this figure is much larger than that reported by Shimer (2005) for the canonical matching model. Furthermore, if one assumes that the wage is determined by non-cooperative bargaining instead of by auction, along the lines suggested by Hall and Milgrom (2005), the volatility in the ratio of vacancies to unemployed implied by the model matches almost exactly that computed from Shimer’s (2005) time series data on the U.S. labor market aggregates.

2 The Matching Process

In the standard search equilibrium framework, the matching function is a black box that relates the number of unemployed workers and vacant jobs to the flow of matches that form assumed to be homogenous of degree one (See Pissarides (2000)). In this paper and in Shimer (2006), the function is the outcome of a more primitive assumption about how matching takes place. That specification follows.

The economy is composed of a continuum of workers and employers and a continuum of islands where exchange takes place. Time is divided into discrete periods of equal length denoted as $t = 1, 2, \ldots$. Let $M$ and $N$ represent measures of unmatched workers and unmatched jobs per island respectively at the beginning of any period. Under the assumption that participants are assigned independently across markets, Shimer (2006) demonstrates that the joint probability that there are $i$ workers and $j$ jobs on any island are independent Poisson variables. Formally,

$$
\pi(i, j; M, N) = \frac{e^{-(M+N)}M^iN^j}{i!j!}.
$$

(1)

Furthermore,

$$
\frac{\partial \pi(i, j; M, N)}{\partial M} = \pi(i - 1, j; M, N) - \pi(i, j; M, N) \quad (2)
$$

$$
\frac{\partial \pi(i, j; M, N)}{\partial N} = \pi(i, j - 1; M, N) - \pi(i, j; M, N).
$$
In other words, the derivative of the probability that there are \(i\) workers (\(j\) jobs) in any market with respect to the average number of workers (jobs) per market is equal to the change in the probability induced by the addition of the marginal worker (job).

As the short side determines the match outcome in each island, the average number of matches created per island is given by

\[ F(M, N) \equiv E\{\min(i, j)\} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \min(i, j)\pi(i, j; M, N) \quad (3) \]

As an implication of equations (1) and (2), Shimer (2006a) demonstrates that

\[ \frac{\partial F}{\partial M} = F_M(M, N) = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \pi(i, j; M, N) = \Pr\{i > j\} > 0 \quad (4) \]
\[ \frac{\partial F}{\partial N} = F_N(M, N) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \pi(i, j; M, N) = \Pr\{i < j\} > 0. \]

In other words, the partial derivative \(F_N(M, N)\) is the share of the islands with unemployed workers while \(F_M(M, N)\) is the fraction of islands with vacant jobs. In this paper, we refer to \(F(M, N)\) as the structural matching function implied by the matching process. A characterization of its properties follow:

**Proposition 1** The matching function \(F(M, N)\) is concave in \(M\) and \(N\) (\(F_{MM} < 0\) and \(F_{MM} < 0\)). Furthermore,

\[ F_{MN}(M, N) = 1 - F_N(M, N) - F_M(M, N) = \sum_{i=0}^{\infty} \pi(i, i; M, N) > 0 \quad (5) \]

where

\[ \lim_{M+N \to \infty} F_{MN}(M, N) = 0. \quad (6) \]

**Proof.** See the Appendix. □

The next result, a corollary of Shimer’s (2006a) Proposition 3, implies that the matching process is exhibits increasing returns. However, equation (6) implies that the matching function is approximately linearly homogenous when the number of workers and jobs per island are large.
Proposition 2 A one percent increase in both $M$ and $N$ increases the number of matches by more than one percent. Formally, the matching function exhibits increasing returns in the sense that

$$\frac{MF_M}{F} + \frac{NF_N}{F} - 1 = F_{MN}(M, N) \frac{E\{i \mid i = j\}}{E\{\min(i, j)\}} > 0$$

where

$$E\{i \mid j = i\} = \frac{\sum_{i=0}^{\infty} i\pi(i, i; M, N)}{\sum_{i=0}^{\infty} \pi(i, i; M, N)}.$$

Proof. See the Appendix. $\blacksquare$

Still, the graph in Figure 1 suggests that the speed of convergence can be slow

3 Market Equilibrium

Suppose that all the available unmatched workers and jobs are independently assigned to islands at the beginning of each period. Let $1 > s > 0$ represent the exogenous probability that an existing job-worker match is destroyed
within a period. The workers who lose their jobs during the period plus those not matched by the end are available to be randomly assigned to islands at the beginning of the next period. Hence, the change in the average number of unmatched workers per market that enter the matching process is given by the law of motion

\[ M_{t+1} - M_t = s(L - U_t) - F(M_t, N_t) \]  

(8)

where \( L \) represents the average number of workers per island.

The assumption that all unmatched workers and jobs are allocated randomly across the islands is easily justified in the case of homogenous workers and jobs given no reallocation cost. Specifically, if all other unmatched workers (jobs) at the end of a period were to stay in their respective islands, then the expected return to search in a randomly selected alternative island is higher. Hence, staying is not a symmetric non-cooperative Nash equilibrium. However, if all search by selecting an island at random, then the return to search is the same in all islands which in turn implies that independent assignment is a Nash equilibrium. It is this assumption of continual reallocation that distinguishes this model in this paper from Shimer’s (2006) mismatch model.

The determination of the number of unmatched jobs in the economy is determined by free entry as in the standard formulation. (See Pissarides (2000, Chapter 1)). Assume that workers and employers are risk neutral and discount future income at the common rate \( r \) per period. A match produces output of value \( p \) per period. An unemployed worker alone can produce output at rate \( z \). Obviously, gain from trade require that \( p > z \). Let \( S_t \) represent the joint surplus value of a match at the beginning of period \( t \) and let \( R_t \) represent the flow value of unemployment, the worker’s reservation wage. As the flow surplus earned by a match is the difference \( p - R_t \), the expected discounted present value of the future match flow is the solution to

\[ S_{t-1} = \frac{p - R_t + (1 - s)S_t}{1 + r} \]  

(9)

given that revenue accrues at the end of the period.

The wage on any island is the outcome of an ex post auction. In the event that there are fewer workers than jobs on the island, all the workers become employed and the wage is bid up to the value of match surplus. Similarly, employers obtain the surplus if there are fewer jobs than workers. In the case
of equal numbers, I choose to allocate the surplus to the workers. (As we will note later, this specification is not innocuous. However, the analysis of any alternative assumption follows the same logic.)

Employers create new unmatched jobs up to point where the expected return is to the cost of posting a vacancy, assumed to be a given constant $c$, as in Pissarides (2000). Because an unfilled job once matched earns the match surplus if and only if allocated to an island where the number of unmatched jobs is less than or equal to the number of unmatched workers, unmatched jobs are created up to the point where

$$c \geq F_N(M, N)S_t \text{ with equality holding if } N > 0$$

where $F_N(M, N)$ is the fraction of islands with fewer jobs than workers.

An unmatched worker receives the value of home production $z$ plus the possibility of becoming employed at the end of the period. As employment has value net of continued search only if the worker is in a market with an excess number of jobs, the reservation wage is the sum of current income and expected flow value of continued search. Formally,

$$R_t = z + [1 - F_N(M_t, N_t)]S_t$$

where $1 - F_M$ is the fraction of markets in which the number of jobs is at least as large as the number of workers seeking them.

Given an arbitrary initial value of $M$, a search equilibrium is a time path $\{M_t, N_t, S_t, R_t\}_0^\infty$ satisfying equations (7)-(10) and the transversality condition $\lim_{t\to\infty} \{(1 + r)^{-t}S_t\} = 0$.

4 Equilibrium Dynamics

The existence of at least one search equilibrium can be demonstrated with the following graphical argument. First, solve the free entry condition, equation (10) for $N_t$. Since $F_N(M, N)$ is increasing in $M$ and decreasing in $N$ from (4), the solution for the number of unmatched jobs posted per island, denoted as $N(S, M)$, is positive if $S > c$ and zero otherwise since $F_N(M, 0) = 1$, and is increasing in both the value of a match and the number of unmatched workers.

Using (11) to eliminate $R_t$ in equation (9), one obtains the difference equation

$$S_t - S_{t-1} = (r + s + [1 - F_N(M_t, N(S_t, M_t))]S_t - (p - z).$$

(12)
which, together with the law of motion characterized by equation (8), form a system of difference equations in $S$ and $M$. An equilibrium is any solution time path consistent with an initial value of $M$ that and transversality.

Since $F(M, N)$ is positive and increasing in both of its arguments, it follows from (4) and the properties of $N(S, M)$ that the singular curve representing steady state condition $\Delta M = 0$ can be described by a strictly negatively sloped curve for values of $S > c$ that has no intercept on the $S$ axis, as represented in Figures 1 and 2. Because $N(S, M) = 0$ or all $S \leq c$ and $F(M, 0) = 0$, $M = L$ for these values of $S$. Finally, because the right side of (12) is decreasing in $M_t$, $M_{t+1} - M_t < (>)0$ to the right (left) of the curve singular curve as indicated by the direction arrow.

Because the share of the markets with more workers than jobs, $F_N(M, N)$ lies in the unit interval, the curve defined by $\Delta S_t = 0$ is bounded above by $(p - z)/(r + s)$ and below by $(p - z)/(1 + r + s)$. This fact and the properties of the $\Delta M = 0$ singular curve implies that the two singular curves must intersect at least once if $p > z$. Finally, because the right side of equation (8) is increasing in $S$, $\Delta S > (>)0$ at points above (below) the curve as indicated by the directional arrows in the phase diagrams portrayed in Figures 1 and 2. As a corollary, it follows that $p - z > (r + s)c$ is necessary for a steady solution with positive employment ($L > M$) and $(p - z) \geq c(1 + r + s)$ is sufficient. The first condition reflects that fact that the flow surplus must cover the amortized cost of job creation and the second condition is sufficient for that purpose.

Because $F(M, N)$ exhibits increasing returns, $F_{MM}F_{NN} - F_{MN}F_{NM} < 0$ which in turn implies

\[
\frac{\partial F_M(M, N(S, M))}{\partial M} = F_{MM} + F_{MN} \frac{\partial N(S, M)}{\partial M} = F_{MM} - F_{MM} \frac{F_{NN}}{F_{NN}} \frac{F_{NN} - F_{MN}F_{NM}}{F_{NN}} > 0.
\]

Hence, the singular curve $\Delta S = 0$ also has a negative slope as illustrated in Figures 1 and 2.

There are two cases. In the case of a single steady state, illustrated in Figure 2, the steady state solution is unique and it is a saddle point. Hence, the only solution consistent with any initial value of $M$ that satisfies the transversality conditions is described by the unique trajectory that converges from the initial value to the steady state. Multiple steady states, as illustrated in Figure 3, are also a possibility. Furthermore, if there is more than one, the
number are odd and the first and last ones are saddle points. Hence, multiple equilibria exist that can exhibit quite complex dynamics.\textsuperscript{2}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Unique Equilibrium Case}
\end{figure}

\section{Social Planner’s Problem}

The planner chooses the number of unmatched jobs in each period to maximize the expected present value of the difference between market and home production. The social value function, $V(M)$, associated with the problem solves the following Bellman equation:

$$V(M_t) = \max_N \left\{ \frac{(p - z)(L - M_t) + V(M_{t+1}) - cN}{1 + r} \right\}.$$ 

where

$$M_{t+1} = M_t + s(L - M_t) - F(M_t, N_t)$$

\textsuperscript{2}See Mortensen (1989) for an analysis of the canonical model when the matching function exhibits increasing returns. The related case of increasing returns in production is treated in Mortensen (1999).
The FOC and Euler equations are

$$-F_N(M_t, N_t) V'(M_{t+1}) - c = 0$$  \hspace{1cm} (13)

and

$$V'(M_t) = \frac{V'(M_{t+1}) (1 - s - F_M(M_t, N_t)) - (p - z)}{1 + r}.$$  \hspace{1cm} (14)

By solving the Euler equation forward in time, it follows that the solution has the property that $V'(M_t) < 0$ for all $t$. Hence, the sufficient second order condition for an optimal choice of $N$, $-V'(M)F_{NN} < 0$, is satisfied. The fact that the Bellman equation is a contraction on the space of continuous and bounded functions implies that a unique solution exists. The following welfare theorem relates the equilibrium to the solution to the planner’s problem.

**Proposition 3** A unique equilibrium solution exists and it is approximately efficient when $M$ and $N$ are large.

**Proof.** See the Appendix. □
6 The Beveridge Curve

Following Shimer (2006), the number of unemployed workers and vacant jobs are defined as those that were not matched at the beginning the period. Hence, the unemployment and vacancy rates are

\[ u = \frac{M - F(M, N)}{L} \]  

and

\[ v = \frac{N - F(M, N)}{L - M + N} \]  

where \( L \geq M \) is the average number of workers per market and \( L - M + N \) is the total number of jobs per market. Shimer (2006) shows that observations on the unemployment and vacancy rates tie down the number of unmatched workers and jobs conditional on the total average number of workers per market, \( L \). Furthermore, variation in the number of unmatched jobs induce a negative relationship between the vacancy rate and the unemployment rate given the number of unmatched workers.

Although \( M \) is a state variable which is fixed at a moment of time, it responds over time to changes in the number of unmatched jobs, induced say by shocks to match productivity, according to the law of motion characterized in equation (8). Indeed, in steady state the number of workers hired and job filled is equal to the separations flow. Formally, \( \Delta M = 0 \) requires that \( M \) adjusts to equate the flow of hires to the separation flow.

\[ F(M, N) = s(1 - u)L. \]  

This fact suggests the following strategy for calibrating the model.

First, choose period length. Given this choice, use data on observed separations to set the value of the separation rate per period \( s \) to its average observed value. Given this number and the average observed unemployment rate and vacancy rate, use equation (15)-(17) to determine \( M, N \), and \( L \). Then, one can compute the values of the endogenous variables \( v, u, \) and \( M \) associated with different values of \( N \).

The choice of a matching period length is not totally arbitrary in this model. Specifically, in the U.S. case the length must be consistent with the fact that the average duration of an unemployment spell is in the order of one quarter and that the median spell length is considerably shorter. With
these facts in mind, a period length of one month suggests itself as a base line case. Shimer’s (2005) estimate of $s$ is 10% per quarter or 3.33% per month. The average vacancy rate reported in the JOLTS data over the 72 month period from December 2000 to November 2001 inclusive was 2.51% while the monthly average of standard CPS measure of the (non-farm) unemployment rate over the same period was 5.29%. Given these choices, equations (15)-(17) imply $M = 1.612$, $N = 0.768$, and $L = 13.816$. These numbers suggest that a "island" might be considered equivalent to a firm of about median size which receives 1.6 or so applicants per month seeking jobs that become available with frequency 0.768 per month.

The raw data on vacancy and unemployment rates over the last six years, reported in the data Appendix, are plotted as the scatter of points illustrated in Figure 4. The plot represents a well defined empirical Beveridge curve. The vacancy-unemployment relationship obtained by varying $N$ between 0.7 and 0.9 is illustrated as the solid curve in the figure. As anyone can see, the fit of the model is remarkable. Indeed, the percent of variance in the vacancy rate explained by the implied curve is 91%.

Figure 4: U.S. Beveridge Curve (12/2000-11/2006)
Fortunately, the shape of the model’s Beveridge curve is relatively invariant to choices of the length of the matching period. This fact is illustrated in Figure 4 where the implied relationship between the vacancy and unemployment rate are drawn under the assumption that the period length is a quarter and a week as well as a month.

Of course, Shimer’s (2006) mismatch model can also match time series data on unemployment and vacancy rates. In that model, the labor market is viewed as a collection of segmented markets for different occupations and regions. Given this interpretation, he argues that cross market mobility is quite small or non-existent. In this environment, the total number of workers per island ($L$ in our notation) is $M$ which he regards as fixed. He calibrates his model by setting $M$ and $N$ to match observed vacancy and unemployment rate averages and then varies $N$ holding $M$ fixed to generate a Beveridge curve. Although $M$ in this paper is endogenous, it doesn’t change that much.
as \( N \) varies. Hence, although the interpretations of the Beveridge curve are quite different in the two models, the explanations for the fit to the data are very similar from a technical point of view.

7  The Job Finding Rate

The job finding rate, defined as the ratio of the hires flow to the number of unemployed workers, is an empirical measure of unemployment spell hazard. As documented in Petrongolo and Pissarides (2001), the empirical literature on the matching function suggests that the job finding rate is well described as a log linear function of the vacancy-unemployment ratio with a elasticity in the range of 0.3 to 0.5.

In the model under study, the job finding rate per period is simply the ratio \( F(M,N)/U \) and the vacancy-unemployment ratio is \( V/U \) where

\[
U = M - F(M,N) \tag{18}
\]
and

\[
V = N - F(M,N). \tag{19}
\]

Since \( V/U \) and \( F(M,N)/U \) both increase as \( N \) increases, a positive implicit relationship exists between the two variables, one that Shimer (2006) calls the "reduced form" matching function. Indeed, log-log relationship obtained when \( N \) is increased from 0.7 and 0.9 is illustrated in Figure 5. Obviously, the relationship is very close to linear over this range. Furthermore, the slope (elasticity) is 0.485, a number near the upper end of Petrongolo-Pissarides "plausible range" of estimates.\(^3\) Hence, the model provides a solid micro foundation for empirical matching functions estimated in the literature.

8  Volatility

In a now famous paper, Shimer (2005) argues that the standard matching model can explain at most 10% of the observed volatility in vacancies and unemployment. In his more recent paper, Shimer (2006) shows that his model of mismatch unemployment does much better in this dimension. In

\(^3\)Although Shimer's mismatch model also implies a nearly log linear relationship, his implied elasticity is only about 0.2.
this section, I show that the island matching model can explain over 40% of the volatility given reasonable parameter values. However, if the wage is set as the outcome of a symmetric non-cooperative bargaining game with delay rather than unemployment is the credible outside option then the volatility of the vacancy-unemployment ratio is virtually identical to that found in Shimer’s (2005) post WWII time series data.

One implication of matching models that differentiate them from Shimer’s mismatch model is that shocks to the match separation or job destruction rate, $s$, induce variation in both vacancies and unemployment as well as shocks to productivity, represented in the model by the parameter $p$. Indeed, the steady state values of the number of unmatched workers and jobs, $M$ and $N$, are determined by the following equations:

$$c = \frac{F_N(M, N)(p - z)}{r + s + 1 - F_N(M, N)} \quad \text{(free entry)} \quad (20)$$

$$F(M, N) = s(L - M + F(M, N)) \quad \text{(steady state)} \quad (21)$$

Without loss of generality, one can normalize the base line value of match productivity per period at $p = 1$. Given the normalization, $c$ and $z$ are
expressed in units of output per period. Given a period length of one month, accepted values of the interest rate and baseline separation rate are \( r = 0.004 \) and \( s = 0.033 \). In his work, Shimer (2005, 2006) sets the opportunity cost of employment, \( z \), equal to 0.4 (40% of market output). Hagadorn and Manovskii (2005), Hall (2006), and Mortensen and Nagypál (2006) argue for larger values. As in the last of these papers, I set \( z = 0.7 \). Because \( M, N, \) and \( L \) are determined by the steady state condition and the observed average values of the unemployment and vacancy rates over the 12/2000 to 11/2006 period as discussed above, the free entry condition evaluated at these benchmark values can be used to tie down the cost of vacancy posting. The implied value if \( c = 0.231 \), equal to about one week of match output. Given all these parameter values, one can now use equations (20) and (21) to compute the responses of all the endogenous variables to variation in both the productivity and separation rates.

The central variable of interest in the literature on labor market volatility is the vacancy-unemployment ratio. At the baseline parameter values, the elasticity of the ratio with respect to \( p \), computed using equations (18) and (19), is 8.61.\(^4\) This measure of the response to productivity variation is much larger than the value of 3.43 obtained using the version of the canonical matching model studied by Shimer (2005) given the same parameter values and very much larger than the 1.72 number obtained when Shimer’s preferred value for the opportunity cost of employment \( z = 0.4 \) is assumed.

As Mortensen and Nagypál (2006) argue, the elasticity of the vacancy-unemployment ratio is not the only parameter needed to explain volatility given that separation shocks also occur and are known to be negatively correlated with productivity shocks as documented by Shimer (2005). Surprisingly, the computed value of the elasticity of the vacancy-unemployment ratio with respect to \( s \) is actually small and positive, equal to 0.468. Given these elasticity values and the standard deviation of log productivity (\( \sigma_p = 0.02 \)), the standard deviation of the log of the separation rate (\( \sigma_s = 0.075 \)), and the correlation between the two (\( \rho_{ps} = -0.524 \)) computed from U.S. post WWII data reported in Shimer (2005) the implied standard deviation of the

\(^4\)This is a comparative static result, and as such, is not the response one would see to a known transitory shock under rational expectation. However, the evidence in Shimer (2005) suggest that productivity shocks are quite permanent, a fact that justifies the use of the number as an approximation to the dynamic response.
The vacancy-unemployment ratio is

\[
\sigma_{V/U} = \left( \frac{\partial \ln(V/U)}{\partial \ln p} \right)^2 \sigma_p^2 + \frac{\partial \ln(V/U)}{\partial \ln p} \frac{\partial \ln(V/U)}{\partial \ln s} \rho_p \sigma_p \sigma_s \right) \left( \frac{\partial \ln(V/U)}{\partial \ln s} \right)^2 \sigma_s^2 = 0.166.
\]

This number is 43% of the standard deviation of the vacancy-unemployment ratio (\(\sigma_{V/U} = 0.382\)) in Shimer's data.

Given the analysis in Mortensen and Nagypál (2006), one might suspect that the dependence of the realized wage on the workers' reservation wage, attributable to the assumption that wages are determined at auction, substantially dampens the effects of shocks on vacancy creation, as it does in the canonical model. To see this point, write the free entry condition as

\[
c = F_N(M,N)(p - R) \quad \text{where} \quad R = z + (1 - F_N(M,N)) \left( \frac{p - R}{r + s} \right)
\]

where \(F_N(M,N)\) is the probability that the number of jobs available in any island is less than the number of workers. Given that \(F_N(M,N)\) is concave and \(M\) is fixed, at least in the short run, the direct effect of an increase in \(p\) is an increase in \(N\). The reservation wage rises with \(p\) both directly and because the probability of being on the short side of the market increases with \(N\). Hence, the dependence of the wage in the model on the value of the search option dampens the effect overall effect of a productivity shock.

If instead the wage is determined through a process of bilateral bargaining, then the search option is not a credible threat in the bargaining game except in states in which the wage outcome would otherwise imply that the value of continued employment is less than the value of searching for an alternative as originally pointed out by Binmore, Rubinstein and Wolinsky (1986) and more recently suggested by Milgrom and Hall (2005).

To justify a bargaining setting, suppose that matching takes place as above but that the wage is determined only after job and worker meet and leave the market. In other words, the number of matches is determined by the number on the short side of the market where the specific workers or jobs on the long side are chosen at random. Under this assumption, rent sharing replaces the competitive outcome. Assuming that the worker can generate
value at rate $z$ while bargaining but neither can search for an alternative until
next period, the unique perfect wage outcome of a symmetric alternating offer
game is given by
\[ w = \max \left( z + \frac{1}{2} (p - z), R \right). \] (22)
In other words, the pair split the flow surplus equally provided that the result
exceeds the flow value of the worker’s search option.

Because the value of a filled job to the employer is the present value of
the profit flow and the ex ante probability of filling any job in the matching
process is the ratio of the matching rate to the number of unmatched jobs
at the beginning of the period, the free entry condition is
\[ c = \frac{F(M, N)}{N} \times \frac{p - w}{r + s}. \] (23)
Similarly, because the probability of becoming employed for the worker is the
ratio of those matched to the number of unmatched workers at the beginning
of the period, the reservation wage solves
\[ R = z + \frac{F(M, N)}{M} \times \frac{w - R}{r + s}. \] (24)
Hence, there is no indirect dampening effect of a productivity shock on the
wage in states in which $z + \frac{1}{2}(p - z) > R$.

Since in fact, the variation in productivity is small in Shimer’s (2005)
data, the constraint in hardly ever binding even when $z$ is set at 0.7. Hence,
the response elasticities to shocks in both $p$ and $s$ are much different than
when the wage is determined at auction. Indeed, at the baseline parameter
values, the elasticity of the vacancy-unemployment ratio with respect to $p$ is
13.42 and with respect to $s$ is $-2.77$, negative a relatively large. Given these
values and Shimer’s (2005) statistics $\sigma_p = 0.02, \sigma_s = 0.075$, and $\rho_{ps} = -0.524$,
the implied standard error of the log vacancy-unemployment ratio is
\[ \sigma_{V/U} = \left( \left( \frac{\partial \ln(V/U)}{\partial \ln p} \right)^2 \sigma_p^2 + \frac{\partial \ln(V/U)}{\partial \ln p} \frac{\partial \ln(V/U)}{\partial \ln s} \rho_{ps} \sigma_p \sigma_s \right)^{\frac{1}{2}} + \left( \frac{\partial \ln(V/U)}{\partial \ln s} \right)^2 \sigma_s \right)^{\frac{1}{2}} = 0.380, \]
almost exactly matching the value of 0.382 reported in Shimer (2005).
9 Conclusion

To be completed.

References


10 Appendix

10.1 Proof of Proposition 1

By applying equations (1) and (2), one obtains the following:

\[ \frac{\partial F_N}{\partial N} = F_{NN}(M, N) = \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} \pi(i, j - 1; M, N) - \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \pi(i, j; M, N) \]

\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} \pi(i, j - 1; M, N) - \sum_{i=1}^{\infty} \sum_{j'=1}^{i} \pi(i, j' - 1; M, N) \]

\[ = - \sum_{i=1}^{\infty} \pi(i, i - 1, M, N) < 0 \]

\[ \frac{\partial F_M}{\partial M} = F_{MM}(M, N) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \pi(i, j; M, N) - \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \pi(i - 1, j; M, N) \]

\[ = \sum_{i=1}^{\infty} \sum_{j=0}^{i} \pi(i, j; M, N) - \sum_{i'=0}^{\infty} \sum_{j=0}^{i'} \pi(i', j; M, N) \]

\[ = - \sum_{i'=0}^{\infty} \pi(i', i' + 1; M, N) = - \sum_{j=1}^{\infty} \pi(j - 1, j, M, N) < 0 \]

\[ \frac{\partial F_N}{\partial M} = F_{NM}(M, N) \]

\[ = \sum_{i=0}^{\infty} \left[ \sum_{i'=0}^{\infty} \sum_{j=0}^{i'-1} \pi(i', j; M, N) - \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \pi(i, j; M, N) \right] \]

\[ = \sum_{i=0}^{\infty} \left[ \sum_{i'=0}^{\infty} \sum_{j=0}^{i'} \pi(i', j; M, N) - \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \pi(i, j; M, N) \right] \]

\[ = \sum_{i=0}^{\infty} \pi(i, i; M, N) = \sum_{j=0}^{\infty} \pi(j, j; M, N) = F_{MN}(M, N) = \frac{\partial F_M}{\partial N} > 0 \]
Note that

\[ F_{MN}(M, N) = \sum_{i=0}^{\infty} \frac{e^{-(M+N)M^iN^i}}{i!i!} \leq \max_{(x_1, x_2) \geq 0} \left\{ \sum_{i=0}^{\infty} \frac{e^{-(M+N)M^iN^i}}{i!i!} |x_1 + x_2 = M + N} \right\} \]

\[ = \sum_{i=0}^{\infty} \frac{e^{-(M+N)(\frac{M+N}{2})^i(\frac{M+N}{2})^j}}{i!i!} = F_{MN} \left( \frac{M + N}{2}, \frac{M + N}{2} \right). \]

An application of the limit operator in Mathcad yields \( \lim_{x \to \infty} F_{MN}(x, x) = 0. \)

10.2 Proof of Proposition 2

Define,

\[ x \equiv \frac{M - F(M, N)}{M}, \]

Shimer (2006a, footnote 7) claims that

\[ \frac{\partial x}{\partial \ln M} + \frac{\partial x}{\partial \ln N} = -\frac{N}{M} \sum_{i=1}^{\infty} \pi(i, i-1; M, N). \]

Since the definition implies

\[ \frac{\partial x}{\partial \ln M} = -\left( F_M(M, N) - \frac{F(M, N)}{M} \right) \]

and

\[ \frac{\partial x}{\partial \ln N} = -\frac{N}{M} F_N(M, N), \]

it follows that

\[ \frac{MF_M(M, N)}{F(M, N)} + \frac{NF_N(M, N)}{F(M, N)} - 1 \]

\[ = \frac{N}{F(M, N)} \sum_{i=1}^{\infty} \pi(i, i-1; M, N) = \frac{N}{F(M, N)} \sum_{i=1}^{\infty} \frac{e^{-(M+N)M^iN^i}}{i!} \]

\[ = \sum_{i=1}^{\infty} \frac{e^{-(M+N)\min(i,j)M^iN^i}}{i!i!} = F_{MN}(M, N) \frac{E\{i | i = j\}}{E\{\min(i, j)\}}. \]

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10.3 Proof of Proposition 3

Letting $S_t = -V'(M_{t+1})$, equation (12) can be written as

$$S_{t-1} = \frac{p - [z + S_tF(M_t, N_t)] + S_t(1 - s)}{1 + r}$$

Using equation (9) to eliminate $R_t$ in (11) yields the same expression. Of course, the FOC, equation (13), and the free entry condition, equation (10), are equivalent if and only if $F(M, N) = 0$ which holds approximately when $M$ and $N$ are large under the hypothesis. Specifically, as $F(M, N)$ becomes small, the slope of the $\Delta S = 0$ curve flattens and tends to zero. Hence, the equilibrium solution is unique for all $F(M, N)$ sufficiently small as illustrated in Figure 1. Furthermore, the necessary and sufficient condition for efficiency hold in the limit under the hypothesis.