Take-it-or-leave-it contracts in many-to-many matching markets*

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Abstract

We study a class of sequential non-revelation mechanisms where hospitals make simultaneous take-it-or-leave-it offers to doctors that either accept or reject them. We show that the mechanisms in this class are equivalent. They (weakly) implement the set of stable allocations in subgame perfect equilibrium. When all preferences are substitutable, the set of equilibria of the mechanisms in the class forms a lattice. Our results reveal a first-mover advantage absent in the model without contracts. We apply our findings to centralize school admissions problems, and we show obtaining pairwise stable allocations is possible through the immediate acceptance mechanism.

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Keywords: Many-to-many, contracts, ultimatum games.

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1 Introduction

In this paper, we study markets involving many-to-many relationships that can be expressed as contracts (see Hatfield and Milgrom, 2005). Relevant examples of these markets are the market for part-time workers or the allocation of specialty training slots for junior doctors in the UK. We find that simple mechanisms where agents on one side of the market make take-it-or-leave-it offers to the agents on the other side of the market are a robust way to implement stable allocations in this setting. We apply our results to the centralization of school admissions in a realistic many-to-many framework.

This finding is important. Many-to-many markets with contracts are a challenge for market design for several reasons. First, the nature of the contractual process makes these markets particularly complex. Also, centralizing the allocation of contracts in this context is difficult because no strategy-proof mechanism (even for one side of the market) exists that is able to generate stable allocations.

Many decentralized procedures share a simple structure. The agents on one side of the market (hospitals) simultaneously make offers to the agents on the other side (doctors) that either accept or reject them. This market structure is simple enough to promote participation while preventing the coordination problems that might arise in this setting and often disrupting the decentralized mechanism (see Triossi, 2009, and Romero-Medina and Triossi, 2014).

In this paper, we analyze the class of what we call take-it-or-leave-it offers mechanisms or TOM. This class includes all mechanisms such that in the first stage, hospitals make simultaneous offers to doctors, and then groups of doctors sequentially accept or reject the offers. The order in which doctors choose can be arbitrary and even endogenous to the play, which is history dependent.

To make the exposition more transparent and to guarantee the comparability of our findings with previous results, we start by presenting the simplest mechanism in the TOM family, the simultaneous acceptance mechanism or SAM. This procedure has two stages. In the first stage, hospitals simultaneously offer contracts to doctors. In the second stage, doctors simultaneously select from among the offers received in the first stage.

Although simple, the SAM mimics the decentralized procedures used in labor markets and college or school admissions. Therefore, our analysis contributes to identifying the basic forces at work in these settings, and captures the relevant interactions among hospitals and doctors.

We consider the subgame perfect equilibria (SPE from now on) in pure strategies of the game induced by the SAM. In this game, doctors have a unique best response: to accept the favorite set of contracts they are offered. We prove each SPE outcome of the game is a pairwise stable allocation in this market without assumptions on the preferences of the agents. The converse is not true. Even when the preferences are substitutable, there are stable allocations that cannot be achieved as an SPE outcome of the SAM. This finding uncovers an important strategic difference introduced by the use of contracts in matching markets. Indeed, Echenique and Oviedo (2006) find that in many-to-many matching markets without contracts and with substitutable preferences, the set of stable matching coincides with the set of SPE outcomes of the SAM. The logic of our results is simple: without contracts, matching markets assign agents to agents. When contracts are available, each hospital can negotiate the details of the relationship with its counterparts. Therefore, ceteris paribus, each hospital will offer only its preferred contract from among the ones a doctor is willing to accept. However, we prove that if enough competitive pressure is present, the SAM implements the set of stable allocations, generalizing the results in Echenique and Oviedo (2006).

Our findings highlight a problem that is both theoretical and conceptual. Because the set of SPE outcomes can be a strict subset of the set of stable allocations, the existence of stable allocations does not guarantee the game has an SPE in pure strategies.

To tackle this issue, we start by observing that the sequential game induced by the SAM is equivalent to a simultaneous game Γ where only hospitals play, and the outcome is obtained by replacing the doctors with their unique best response (see Baron and Kalai, 1993). Then we prove an SPE in pure strategies exists when hospitals have substitutable preferences and doctors have unilateral substitutable preferences. This result extends the existence of pairwise stable allocations beyond the case, previously analyzed in the literature, of substitutable preferences (see Blair, 1988, Pepa Risma, 2015, and Hatfield and Kominers, 2017).

We also show that if both sides of the market have substitutable preferences, the sets of the SPE outcomes of the game constitute a lattice. The lattice structure reflects an opposition of interests between the two sides of the markets, within the equilibrium outcomes. This opposition of interests is consistent with the one found in the set of stable allocations (see Blair, 1988, and Pepa Risma, 2015). We also find a first-mover advantage absent in the model without contracts. In fact, when the preferences of both sides of the market are substitutable and satisfy the law of aggregate demand (see Hatfield and Milgrom, 2005), the hospital-optimal stable allocation is an SPE outcome while the doctor-optimal stable allocation might not be an equilibrium outcome of the game.

After completing the study of the SAM, we extend our findings to the entire TOM class, by proving all games in this class are equivalent. The set of SPE outcomes of the game induced by the SAM coincides with the set of SPE outcomes of the game induced by any of the mechanisms within the TOM class.

To conclude the paper, we consider the possibility of building a centralized assignment procedure able to result in stable allocations when the preferences of one side of the market are known and can be interpreted as priorities. This scenario exists, for instance, in school admissions problems. We find the immediate acceptance mechanism guarantees the implementation of stable allocations in many-to-many environments with responsive school priorities because the game induced by the mechanism is equivalent to the game induced by the SAM. If schools' priorities are not responsive, the immediate acceptance mechanism fails to implement stable allocations. However, if priorities are at least substitutable, the one shot immediate acceptance mechanism yields stable allocations. This result also relies on the equivalence of the one-shot immediate acceptance mechanism and the SAM.

This paper is rooted in the recent matching literature on sequential mechanisms, both in many-to-many and many-to-one markets. In that sense, our analysis on the SAM extends previous results on many-to-many matching markets without contracts in Sotomayor (2004) and Echenique and Oviedo (2006) to the framework of many-to-many matching markets with contracts. Our analysis of sequential acceptances also extends the result on Romero-Medina and Triossi (2014) from the many-to-one to the many-to-many framework (see also Klaus and Kljin, 2016).

Although many-to-many relations are common in bilateral markets, the strategic interaction of agents in decentralized many-to-many markets with contracts has not been fully analyzed. We have already mentioned the specialty training followed by junior doctors in the UK, where doctors have to arrange separate medical residencies and training positions with several hospitals (Roth, 1991) and part-time lecturers. However, we find additional examples in other labor markets, for example, the market for school teachers in countries such as Argentina, Chile, and Italy. In these countries, teachers can work simultaneously in more than one school under different labor conditions. Outside labor markets, we can mention relationships between health insurers and health care providers and the ones between car producers and auto parts suppliers as relevant examples of many-to-many markets with contracts.

The many-to-many framework can also be used to model the application stage of decentralized many-to-one markets. For example, Yenmez (2015) models the college admissions problem in the United States as a many-to-many matching model with contracts where applicants apply to several colleges and receive several acceptances before investing in reaching a final decision. In this model, students get different acceptance packages with different tuition fees and financial aid packages (scholarships, loans, grants, and work opportunities).

The school admissions problem has been traditionally studied as a many-to-one problem. However, families may have more than one child, and students can be admitted in different conditions. In this case, the school admissions problem becomes an example of a market with contracts that fits in the scope of our analysis (see also Hatfield and Kominers, 2017).

Most of the markets we have mentioned are either totally or partially decentralized. But some of them are centralized using a revelation mechanism as the admission procedure in many school districts. However, the centralized procedures in place either ignore or underplay the many-to-many aspect of these markets. The analysis of decentralized procedures allows us to better understand the problem. In particular, we consider the different options available to the designer in a realistic many-to-many school admissions problem with contracts. We provide alternative mechanisms to build a centralized clearing house.

The paper is organized as follows. Section 2 introduces the model and notation. Section 3 presents the SAM and the implementation results for this mechanism. Section 4 extends our results to the class of TOM. Section 5 studies the possibility of centralizing the TOM family in markets with priorities on one side of the market. Finally, Section 6 concludes. The proofs are in the appendix.

2 The Model

In our model, a set of doctors seeks positions at different hospitals. The (finite) sets of hospitals and doctors are denoted by H and D, respectively. The set of agents will be denoted by $N = H \cup D$. There is a finite set X of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. We assume each agent can sign multiple contracts. The null contract is denoted by $\emptyset$. An allocation is a set of contracts $Y \subseteq X$. Let Y be an allocation and let $N' \subseteq N$. Let $Y_{N'} = \{y \in Y \mid \{y_H, y_D\} \cap N' \neq \emptyset\}$ be the set of contracts that belong to Y and involve a member of N'. With abuse of notation, for all $n \in N$, we will use Y_n instead of $Y_{\{n\}}$.

For each $h \in H$, $\succ_h$ is a strict preference relation on $\{Y \subseteq X \mid x_H = h \; \forall \; x \in Y\}$. A contract is acceptable if it is strictly preferred to the null contract, and unacceptable if it is strictly worse than the null contract. The set of contracts that are acceptable to h is denoted by $A(\succ_h) = \{x \in X \mid x \succ_h \emptyset\}$. A preference profile $\succ_h$ defines a choice function $C_h(\cdot)$. Formally, for each $h \in H$ and $Y \subseteq X$, we define $C_h(Y)$, the chosen set in Y, as $C_h(Y) = \max_{\succ_h} \{Z \subseteq Y_h\}$. Let $C_H(Y) = \bigcup_{h \in H} C_h(Y)$ be the set of contracts chosen from Y by some hospital. Preference relations are extended to allocations in a natural way: for all allocations Y, Z, $Y \succ_h Z$ means $Y_h \succ_h Z_h$. For each $d \in Y$, $\succ_d$, $A(\succ_d)$, C_d, Y_d, and C_D are defined in the same way.

Each choice functions C_n is derived by a strict preference relation $\succ_n$, for all $n \in N$; then it satisfies IRC.[1] Thus, for every $Y \subseteq X$ and every $z \in X \setminus Y$,

$$z \notin C_n(Y \cup \{z\}) \implies C_n(Y \cup \{z\}) = C_n(Y).$$

We define $\succ_H = (\succ_h)_{h \in H}$, $\succ_D = (\succ_d)_{d \in D}$ and $\succ = \left((\succ_h)_{h \in H}, (\succ_d)_{d \in D}\right)$. The quadruple $M = (H, D, X, \succ)$ is called a matching market. We could model many-to-one matching markets by assuming no doctor finds an allocation

[1] Sönmez and Aygün (2013) present a detailed analysis of this condition and its implications.

where she signs more than one contract to be acceptable. Formally, $(H, D, X, \succ)$ is a many-to-one matching market if, for all allocations Y and for $d \in D$ such that $|Y_d| > 1$, we have $\emptyset \succ_d Y$.

We then define two partial orders $\succ_{HB}$ and $\succ_{DB}$ on the set of allocations, that are usually called Blair's orders.

Definition 1 *Let Y and Z be allocations.*
(i) The allocation Y is preferred to the allocation Z according to Blair's partial order for hospitals, or $Y \succ_{HB} Z$ if $C_h(Y_h \cup Z_h) = Y_h$ for all $h \in H$.
(ii) The allocation Y is preferred to the allocation Z according to Blair's partial order for doctors, or $Y \succ_{DB} Z$ if $C_d(Y_d \cup Z_d) = Y_d$ for all $d \in D$.

We assume each doctor can sign at most one contract with the same hospital, and vice versa. This assumption is called the "unitarity assumption" (see Kominers, 2012) and it is common in the literature (but see Pepa Risma, 2015, and Hatfield and Kominers, 2017). We model the **unitarity assumption** (UA) by assuming the allocations where an agent $n \in N$ signs more than one contract with the same counterpart are not acceptable to n. Formally, we assume that if $Y \subseteq X$ is an allocation, and there exist $y, z \in Y_h$, $y \neq z$ for some $h \in H$ (resp. $y, z \in Y_d$ $y \neq z$ for some $d \in D$) with $y_D = z_D$ (resp. $y_H = z_H$), then $\emptyset \succ_h Y$ (resp. $\emptyset \succ_d Y$).

2.1 Stability and substitutability

Stability is a key concept in market design. Gale and Shapley first introduced it in their 1962 seminal paper. Theoretical and empirical findings suggest markets that achieve stable outcomes are more successful than markets that do not achieve stable outcomes (see Roth and Sotomayor, 1990, and Abdulkadiroğlu and Sönmez, 2013).

Stable allocations are identified by two requirements. The first requirement is individual rationality. An allocation is individually rational if no agent wants to unilaterally cancel any of the assigned contracts.

Definition 2 *An allocation Y is individually rational for agent $n \in N$ if $C_n(Y) = Y_n$.*

The second requirement is that the allocation must not be "blocked." Intuitively, a coalition blocks an allocation when the members of the coalition can profitably renegotiate the contracts of the allocations.

A coalition of agents can block a given allocation in a variety of forms.

Definition 3 *Let Y be an allocation for matching market M. A set of agents $N' = H' \cup D'$, where $H' \subseteq H$ and $D' \subseteq D$:*

- *Pairwise blocks Y if $H' = \{h\}$, $D' = \{d\}$ and $x \in X \setminus Y$ exists such that $x_D = d$, $x_H = h$ and $x \in C_h(Y \cup \{x\}) \cap C_d(Y \cup \{x\})$.*
- *Blocks (Hatfield and Kominers, 2017) Y if a set of contracts $Z \neq \emptyset$ exists such that*
  (i) $Z \cap Y = \emptyset$;
  (ii) $Z_{N'} = N'$;
  (ii) for all $j \in N'$, $Z_j \subseteq C_j(Z \cup Y)$.
- *Strongly blocks (Hatfield and Kominers, 2017) Y if a set of contracts $Z \neq \emptyset$ exists such that*
  (i) $Z \cap Y = \emptyset$;
  (ii) $Z_{N'} = N'$;
  (ii) for all $j \in N'$, an individually rational $T_j \supseteq Z_j$ exists such that $T_j \succ_j Y_j$.

The previous blocking conditions imply the following stability concepts.

Definition 4 *Let Y be an allocation for matching market M.*

- *Y is pairwise stable if it is individually rational and no coalition exists that pairwise blocks it. The set of pairwise stable allocations is denoted by $\mathcal{PS}(M)$.*

- *Y is stable if it is individually rational and no coalition exists that blocks it. The set of stable allocations is denoted by $\mathcal{S}(M)$.*

- *Y is strongly stable if it is individually rational and no coalition exists that strongly blocks it. The set of strongly stable allocations is denoted by $\mathcal{SS}(M)$.*

As we move from pairwise stable allocation to strongly stable allocation, the set of potential blocking coalitions enlarges. Therefore, the set of surviving allocations shrinks. Thus, we have $\mathcal{SS}(M) \subseteq \mathcal{S}(M) \subseteq \mathcal{PS}(M)$.[2]

The set of pairwise stable, stable, and strongly stable allocations may be empty. The literature has focused on preference restrictions that guarantee the existence of stable allocations by avoiding complementarities among contracts. Substitutability is a key condition for the existence of stable allocations. Next, we formally define substitutable preferences, and we present the concepts of unilateral and strong substitutability. Unilateral substitutability guarantees the existence of stable allocations in many-to-one matching markets at the time that allows some complementarities among the agents (see Hatfield and Kojima, 2010 and Sönmez and Switzer, 2013).[3] The condition of strong substitutability is a strengthening of substitutability and was defined by Echenique and Oviedo (2006) and studied in Klaus and Waltz (2009) and Hatfield and Kominers (2017) in the case of matching with contracts.

Next, we present formally the different concepts of substitutability from the weakest to the strongest.

Definition 5 *The preferences of hospital h, $\succ_h$ are unilaterally substitutable if there does not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such*

[2]Klaus and Waltz (2009) present alternative stability concepts. They introduce weak setwise stable allocations, setwise stable allocations, and strongly setwise stable allocations. Hatfield and Kominers (2017) establish connections between these conditions and the concepts of stable and strongly stable allocations.

[3]Hatfield and Kojima (2010) also present the weaker concept of bilateral substitutable preferences.

that $z_D \notin Y_D$, $z \notin C_h(Y \cup \{z\})$ and $z \in C_h(Y \cup \{x, z\})$.

The preferences of hospital h are unilaterally substitutable if, whenever h rejects the contract z and that is the only contract with z_D available, it still rejects the contract z when the choice set expands. Unilateral substitutable preferences are defined in the same way for doctors.

Definition 6 *The preferences of hospital h, $\succ_h$ are substitutable if there do not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $z \notin C_h(Y \cup \{z\})$ and $z \in C_h(Y \cup \{x, z\})$.*

The preferences of hospital h are substitutable if the addition of a contract to the choice set never induces a hospital to accept a contract it previously rejected. Substitutable preferences are defined in the same way for doctors.

Definition 7 *The preferences of hospital h, $\succ_h$ are strongly substitutable if, for all set of contracts $Y, Z \subseteq X$ such that $C_h(Y) \succ_h C_h(Z)$, we have $Z \cap C_h(Y) \subseteq C_h(Z)$.*

The preferences of hospital h are strongly substitutable if, whenever h chooses a contract y from a set of contracts Y and $y \in Z$, where $C_h(Y)$ is a better set than $C_h(Z)$, then h chooses y from Z as well. Strongly substitutable preferences are defined in the same way for doctors.

In the paper, we also employ an additional condition called the "law of aggregate demand."

Definition 8 *Let $n \in N$. The preferences of agent n, $\succ_n$ satisfy the law of aggregate demand if, for all $Z \subseteq Y \subseteq X$, $|C_n(Z)| \leq |C_n(Y)|$.*

If the preferences of an agent satisfy the law of aggregate demand and new contracts become available, the agent will choose a (weakly) larger number of contracts.

2.2 Subgame perfect implementation

An **extensive-form matching mechanism** is an array $G = (N, X, I, S, g)$, where N is the set of players, I is the set of histories, and S is the strategy space. More precisely, $S = \prod_{n \in N} S_n$, where $S_n = \prod_{i \in I} S_n^i$ for all $n \in N$. Set $S^i = \prod_{n \in N} S_n^i$. Histories and strategies are linked by the following property: $S^i = \{s^i \mid (i, s^i) \in I\}$. An initial history $i^0 \in I$ exists, and every history $i \in I$ is represented by a finite sequence $(i^0, s^1, ..., s^{r-1}) = i^r$. If $i^{r+1} = (i^r, s^r)$, history i^{r-1} precedes history i^r and that history i^r precedes history i^{r+1}. The set $W = \{w \in I \mid \text{there is no } i \in I \text{ proceeding } w\}$ is the set of terminal histories. Given the initial history, every strategy profile $s \in S$ defines a unique terminal history w_s. The outcome function $g : W \to X$ specifies an outcome allocation for each terminal history, and hence for each strategy profile s. With abuse of notation, we use $g(s)$ to denote $g(w_s)$. Given $\succ$, $(G, \succ)$ constitutes an extensive-form game. Every $i \in I \setminus W$ identifies a subgame $G(i) = (N, I(i), S(i), g_i, \succ)$, where i is the initial history, $I(i) = \{i' \in I \mid i' \text{ precedes } i\}$ and $S(i) = \prod_{i' \in I(i)} S^{i'}$. Let $s \in S(i)$. Given the initial history i, strategy s specifies a unique terminal history, w_s. The outcome function is defined by $g_i(s) = g(w_s)$. Given $s \in S$ and $i \in I$, let $s(i) \in S(i)$ be the strategy prescribed by s once i is reached. Formally, if $s = \left(s^{i'}\right)_{i' \in I}$, then $s(i) = \left(s^{i'}\right)_{i' \in I(i)}$. With abuse of notation, we will identify a subgame $G(i)$ with its initial history i.

An SPE is a strategy profile that induces a Nash equilibrium in every subgame. Formally, s^* is an SPE if for all $i \in I$ and for all $n \in N$: $g_i(s^*(i)) \succeq_n g_i\left(s_n', s_{-n}^*(i)\right)$ for all $s_n' \in S_n(i)$. An allocation $g(s^*)$ is called an SPE outcome of $(G, \succ)$, and the set of SPE outcomes of $(G, \succ)$ is denoted by $SPE(G, \succ)$. Let $\mathcal{M}$ be a set of matching markets, and let $\Phi : \mathcal{M} \twoheadrightarrow X$ be a correspondence. An extensive-form matching mechanism G **implements** Φ in SPE if, for all $M \in \mathcal{M}$, $SPE(G, \succ) = \Phi(M)$, which is if every SPE outcome of $(\Gamma, \succ)$ belongs to $\Phi(H, D, X, \succ)$ and for all contracts $x \in \Phi(M)$, an SPE of $(G, \succ)$ exists yielding x as outcome.

An extensive-form matching mechanism G **weakly implements** Φ in SPE if, for all $M \in \mathcal{M}$, $SPE(G, \succ) \neq \emptyset$ and $SPE(G, \succ) \in \Phi(M)$, which is if every SPE outcome of $(G, \succ)$ belongs to $\Phi(M)$. Throughout the paper, we consider only equilibria in pure strategies.

3 Simultaneous Acceptance Mechanism

In this section, we analyze the **simultaneous acceptance mechanism**. The SAM is a natural extension of the mechanism studied in Sotomayor (2004) and Echenique and Oviedo (2006) to many-to-many matching markets with contracts. The game has two stages. In the first stage, hospitals simultaneously offer contracts to doctors. In the second and final stage simultaneously, each doctor chooses from among the offers she receives, if any. The contracts accepted in the second stage are enforced as an outcome of the mechanism.

The SAM is described by the following procedure:

1. **Offers.** Each hospital h offers contracts to some doctors. Let $X_1(h) \subseteq X_h$ be the set of contracts offered by hospital h. If h does not make any offer, then $X_1(h) = \varnothing$. For all $d \in D$, let $X_1(d) = \left(\bigcup_{h \in H} X_1(h)\right)_d$ be the set of offers received by doctor d.

2. **Choice.** Each doctor selects a set of contracts from among the ones she was offered. Let $X_2(d) \subseteq X_1(d)$ be the set of offers d selects.

In the first stage of the game, the strategy set of hospital h is 2^{X_h}.[4] Every subgame $i \in I \setminus (\{i^0\} \cup W)$ where doctors have to play is completely characterized by sets of contracts proposed by each hospital. Formally, $i \in I \setminus (\{i^0\} \cup W)$ is characterized by $\{X_1^i(h)\}_{h \in H}$, where $X_1^i(h)$ is the set of contracts that hospital proposed in the history preceding z. Let $X_2^i(d)$

[4]Let Y be a set. By $2^Y = \{Z \mid Z \subseteq Y\}$, we denote the set of all of its subsets.

be the choice of doctor d at i. For all $i \in I \setminus (\{i^0\} \cup W)$ and every doctor $d \in D$, let $X_1^i(d) = \left(\bigcup_{h \in H} X_1^i(h)\right)_d$ be the set of offers doctor d receives in this subgame. We will use $\{X_2^i(d)\}_{d \in D}$ to denote the profile of strategies of the doctors at subgame $i \in I \setminus (i^0 \cup W)$.

A strategy is given by $s = \left((X_1(h))_{h \in H}, (X_2^i(d))_{d \in D}^{i \in I \setminus (\{i^0\} \cup W)} \right)$.

3.1 Results

We first characterize doctors' optimal behavior. At the second stage of the game, doctors have a unique best response, namely, to accept the best set of contracts from among the ones being offered. Formally:

Lemma 1 *Consider the game induced by the SAM when preferences are $\succ$. Then doctors have a unique best response: $X_2^{*i}(d) = C_d(X_1^i(d))$ for all $d \in D$ and all $i \in I \setminus (\{i^0\} \cup W)$.*

First, we show that any SPE of the game is pairwise stable regardless of the agents' preferences.

Proposition 1 *All SPE outcomes of the game induced by the SAM are pairwise stable allocations.*

If all agents have substitutable preferences, the set of pairwise stable allocations coincides with the set of stable allocations (see Hatfield and Kominers, 2017). If one side of the market has strongly substitutable preferences and the other side of the market has substitutable preferences, the set of pairwise stable allocations coincides with the set of strongly stable allocations (see Hatfield and Kominers, 2017). Therefore, our results partially extend the findings of Echenique and Oviedo (2006) to the framework of matching with contracts.

Corollary 1 **(i)** *If $\succ_H$ and $\succ_D$ are substitutable, all the SPE outcomes of the game induced by the SAM are stable allocations.*

(ii) *If $\succ_H$ (resp. $\succ_D$) are strongly substitutable and $\succ_D$ (resp. $\succ_H$) are substitutable, all the SPE outcomes of the game induced by the SAM are strongly stable allocations.*

Proof. The first claim follows from our Proposition 1 and Proposition 2 in Hatfield and Kominers (2017).

The second claim follows from our Proposition 1 and Theorem 3 in Hatfield and Kominers (2017). ■

In the case of non-substitutable preferences, we can find SPE outcomes that are pairwise stable, but not stable as the following example shows.

Example 1 *Let us assume $H = \{h_1, h_2\}$ and $D = \{d_1, d_2\}$. Let x_r and $\tilde{x}_r$ denote contracts between d_1 and h_r, $r = 1, 2$. Let z_r and $\tilde{z}_r$ denote contracts between d_2 and h_r, $r = 1, 2$. Assume the preferences of the agents are the following:*

$\succ_{h_r}$: $\{x_r, z_r\}, \{\tilde{x}_r\}, \{\tilde{z}_r\}, \{x_r\}, \{z_r\}, \ r = 1, 2;$

$\succ_{d_1}$: $\{x_1\}, \{\tilde{x}_2\}, \{\tilde{x}_1\}, \{x_2\};$

$\succ_{d_2}$: $\{z_1\}, \{\tilde{z}_1\}, \{\tilde{z}_2\}, \{z_2\}.$

The preferences of the hospitals are not substitutable. The SAM yields $\{\tilde{x}_2, \tilde{z}_1\}$ as SPE outcome, which is pairwise stable but not stable.

3.1.1 Markets with and without contracts

We can define a many-to-many matching market without contracts as a market where $|X_h \cap X_d| = 1$ for all $h \in H$ and $d \in D$. Echenique and Oviedo (2006) analyze the SAM in this framework (see also Sotomayor, 2004). Without contracts, the SAM implements the set of stable allocations in SPE when both sides of the market have substitutable preferences (Echenique and Oviedo, 2006, Theorem 7.1). In addition, if one side of the market has strongly substitutable preferences, the SAM implements the set of strongly stable allocations in SPE (Echenique and Oviedo, 2006, Theorem 7.2).

Both results rely on the fact that, in the model without contracts, hospitals only choose whom to make the offer to. Contracts introduce new strategic considerations. With contracts, hospitals can also renegotiate the terms of the collaboration with the doctors. Intuitively, a hospital can always offer a doctor the worst conditions (e.g., the lowest salary) she is willing to accept. Therefore, hospitals benefit from a first-mover advantage. The following example shows the set of SPE allocations of the game induced by SAM does not include all stable allocations.

Example 2 *Let us assume* $H = \{h\}$ *and* $D = \{d\}$. *Let* $x_i, x'_i, \tilde{x}$ *denote contracts between hospital h and doctor d. Assume the preferences of the agents are the following:*
$\succ_d$: $\{x\}, \{x'\}, \{\tilde{x}\}$;
$\succ_h$: $\{\tilde{x}\}, \{x'\}, \{x\}$.
We can assume that, for example, x_i, x'_i *and* $\tilde{x}$ *are contracts that pay a salary of \$200,000, \$175,000, and \$150,000 a year, respectively, and all other contract terms are identical.*

In Example 2, the hospital prefers to pay less and the doctor prefers to be paid more. Only the \$150,000 contract is an SPE outcome of the SAM mechanism where the hospital makes the offer. Only the \$200,000 contract is an SPE outcome of the SAM mechanism where the doctor makes the offer.[5] The set of SPE allocations depends on who is making the offers. Therefore, Example 2 highlights another difference that emerges from the

[5] One might conjecture the set of stable allocations is the union of the SPE outcomes of the game where hospitals make offers and of SPE outcomes of the game where doctors make offers. This conjecture is not true. Notice that in Example 2, the \$175,000 contract is not an SPE outcome of any of the two games.

use of contracts. When no contracts exist, the set of SPE allocations is independent of who makes the offers.

Notice that in Example 2, the contracts where the doctor is paid \$175,000 and \$200,000 are unilaterally renegotiable by hospital h, when making the offers. Unilateral renegotiation of contracts undoubtedly plays a role in shaping the set of SPE outcomes. Still, this property does not fully characterize implementable allocations. Indeed, allocations exist that can be unilaterally renegotiated by a hospital and are SPE outcomes as shown by the following example.

Example 3 *Let us assume* $H = \{h_1, h_2\}$ *and* $D = \{d_1, d_2\}$*. Let* x_1 *and* $\tilde{x}_1$ *denote contracts between* h_1 *and* d_1*. Let* x_2 *denote a contract between* h_2 *and* d_1*. Let* z *denote a contract between* h_2 *and* d_2*. Assume the preferences of the agents are the following:*
$\succ_{h_1}$: $\{\tilde{x}_1\}, \{x_1\}$;
$\succ_{h_2}$: $\{x_2\}, \{z_2\}$;
$\succ_{d_1}$: $\{x_1\}, \{x_2\}, \{\tilde{x}_1\}$;
$\succ_{d_2}$: $\{z_2\}$.
A unique stable allocation $\{x_1, z_2\}$ *exists. The allocation could be unilaterally renegotiated by hospital* h_1 *by offering* $\tilde{x}_1$ *instead of* x_1*. However, the strategies* $X_1(h_1) = \{x_1, \tilde{x}_1\}$*,* $X_1(h_2) = \{x_2, z_2\}$ *jointly with the fact that a doctor selects the best set of contracts among the ones she was offered are an* SPE *yielding* $\{x_1, z_2\}$ *as an outcome of the game induced by* SAM*.*

Examples 2 and 3 highlight the differences that emerge from the use of contracts. These differences lie in the structure both of the market and of the mechanism. Each hospital has to negotiate the nature of the relationship with the doctors, and the mechanism provides the hospitals with a first-mover advantage. In this case, the threat of other hospitals' counteroffers helps in increasing competition and sustaining stable outcomes as shown in

Example 3.[6] The idea that the potential entry of new competitors helps in sustaining efficient outcomes is not new to economics, and bears relation to the concept of contestable markets (see Baumol et al., 1982). We thus provide the following definition.

Definition 9 *The market* $(H, D, X, \succ)$ *satisfies contestability if, for any individually rational allocation* Y*, such that there exist* $x \in X \setminus Y$ *and* $y \in Y$ *such that* $x \in C_h(Y \cup \{x\}) \cap C_d(Y \setminus \{y\} \cup \{x\})$*, where* $h = x_H$*,* $d = x_D$*, then there exists a contract* $x' \in X_d$*,* $x' \in C_d(Y \setminus \{y\} \cup \{x, x'\})$*,* $x \notin C_d(Y \setminus \{y\} \cup \{x, x'\})$*,* $x' \notin C_d(Y \cup \{x, x'\})$*.*

The essence of the contestability condition is the existence of the threat of a deviation that introduces a potential competitive pressure and allows for full implementability of the set of stable allocations.

Proposition 2 *Assume the market satisfies contestability, and the preferences of the agents are substitutable; then every stable allocation is an* SPE *outcome of the game induced by the* SAM*. Therefore, under contestability, the* SAM *implements the set of stable allocations in* SPE*.*

In the absence of contracts, each allocation is an agreement only on the identities of the counterparts. Thus, the contestability condition holds emptily, and Proposition 2 extends Theorems 7.1 and 7.2 in Echenique and Oviedo (2006).

Alcalde et al. (1998) prove the implementability of stable allocations in SPE in a many-to-one matching model with money a la Kelso and Crawford (1982). They use a mechanism that is very similar to the SAM. Their model satisfies contestability, because they assume that at least two firms exist, each firm finds every worker acceptable, and firms can make arbitrarily high

[6] Hatfield and Kominers (2017) prove the theoretical possibility of implementing the set of stable allocations in NE, although employing Maskin mechanisms.

offers. These assumptions allow them to sustain SPE, preventing unilateral deviation with the threat of a sufficiently high offer. We cannot extend these assumptions in our framework, because the set of contracts is finite, and we do not assume contracts between every firm and worker are feasible or acceptable. Therefore, the contestability condition is more demanding in our framework.

3.1.2 Equilibrium existence

In general, as Example 2 shows, not every stable allocation is an SPE outcome of the game induced by SAM. Therefore, the existence of stable allocations is not able to guarantee the existence of an SPE in pure strategies of the game induced by SAM.

We will prove the existence of equilibria directly, without relying on previous existence results, by using a lattice theoretical argument. To simplify the analysis, considering the normal form game Γ, where the set of the players is H, the strategy space of the hospital h is $S_h = 2^{X_h}$, and the outcome function is $g\left((S_h)_{h\in H}\right) = C_D\left(\bigcup_{h\in H} S_h\right)$, $\Gamma = \left(H, \succ_H, \left(2^{X_h}\right)_{h\in H}, g\right)$ is useful. Thus, from Lemma 1, it follows directly that a one-to-one correspondence exists between the NE of Γ and the SPE of the game induced by the SAM.

Lemma 2 *The strategy profile* $\left((S_h^*)_{h\in H}, (S_d^{*i})_{d\in D}^{i\in I\setminus(\{i^0\}\cup W)}\right)$ *is an equilibrium of the game induced by the* SAM *if and only if* $(S_h^*)_{h\in H}$ *is a Nash equilibrium of* Γ.

Let S_{-h} be a strategy profile for all the hospitals but h. Let $F_h(S_{-h}) = \left\{x \in X_h \mid x \in C_D\left(\bigcup_{h'\neq h} S_{h'} \cup \{x\}\right)\right\}$ be the set of contracts that would be accepted if they were offered by h, when the other hospitals offer contracts in $\bigcup_{h'\neq h} S_{h'}$. Let $R_h(S_h, S_{h-h}) = \left\{x \in X_h \mid x \notin C_D\left(\bigcup_{h'\in H} S_{h'} \cup \{x\}\right)\right\}$ be the set of contracts of agent h that would be rejected if they were offered by h jointly with the contracts in $\bigcup_{h'\in H} S_{h'}$. Notice $F_h(S_{-h}) = X_h \setminus$

$R_h(\emptyset, S_{h-h})$. Let us define $br_h(S_{-h}) = C_h(F_h(S_{-h}))$ and $BR_h(S_{-h}) = br_h(S_{-h}) \cup R_h(br_h(S_{-h}), S_{-h})$. Finally, set $BR_H\left((S_h)_{h\in H}\right) = (BR_h(S_{-h}))_{h\in H}$.

We first characterize the structure of the best response correspondence of game Γ, by proving $br_h(S_{-h}) = C_h(F_h(S_{-h}))$ and $BR_h(S_{-h}) = br_h(S_{-h}) \cup R_h(br_h(S_{-h}), S_{-h})$ are the minimal and the maximal best response, respectively.

Lemma 3 *Let $(\succ_d)_{d\in D}$ be a profile of preferences for doctors. Then Y_h is a best response to S_{-h} in Γ if and only if*

$$br_h(S_{-h}) \subseteq Y_h \subseteq BR_h(S_{-h}).$$

The result on Lemma 3 characterizes the hospitals best responses in game Γ when the SAM is used.

The next step is to prove the existence of equilibrium when hospitals have substitutable preferences and doctors have unilateral substitutable preferences. The strategy of proof is to provide an increasing selection of the best response correspondence and apply the Tarski's Fixed Point Theorem. First, we order the strategy space using the product of the natural set order, $\subseteq$. Because the minimal best response function $br_h(\cdot)$ is non-increasing, a natural choice would be the maximal best response $BR_h(\cdot)$. However, $BR_h(\cdot)$ is not increasing as the following example shows.

Example 4 *Let us assume $H = \{h_1, h_2\}$ and $D = \{d_1, d_2\}$. Let x_i, x_i' denote contracts between hospital h_i and doctor d_1, for $r = 1, 2$. Let y_i denote contracts between hospital h_i and doctor d_2, for $r = 1, 2$. Assume the preferences of the agents are the following:*

$\succ_{d_1}$: $\{x_1', x_2\}, \{x_1, x_2'\}, \{x_1, x_2\}, \{x_1', x_2'\}, \{x_1'\}, \{x_1\}, \{x_2'\}, \{x_2\}$;

$\succ_{d_2}$: $\{y_1\}, \{y_2\}$;

$\succ_{h_1}$: $\{y_1, x_1\}, \{y_1, x_1'\}, \{x_1\}, \{x_1'\}\{y_1\}$;

$\succ_{h_2}$: $\{y_2\}, \{x_1'\}, \{y_1\}$.

Notice $\succ_x$ is unilaterally substitutable but not substitutable, because $x_1 \in C_x(\{x_1, x_1', x_2'\})$ but $x_1 \notin C_x(\{x_1, x_1'\})$.
Now consider the following strategies for hospital h_2, $S_{h_2} = \{\emptyset\}$ and $S'_{h_2} = \{x_2'\}$. We have $F_{h_1}(S_{h_2}) = \{y_1, x_1, x_1'\}$ and $BR_{h_1}(S_{h_2}) = \{x_1, x_1', y\}$. We have $F_{h_1}\left(S'_{h_2}\right) = \{y, x_1, x_1'\}$ and $BR_{h_1}\left(S'_{h_2}\right) = \{x_1, y\}$. Then $BR_{h_1}\left(S'_{h_2}\right) \subseteq BR_{h_1}(S_{h_2})$, but $S_{h_2} \subseteq S'_{h_2}$. It follows that $BR_{h_1}(\cdot)$ is not monotonic.

Although $BR_H(\cdot)$ is not increasing, in general, it can be shown that $Br_h(S_{-h}) = br_h(S_{-h}) \cup (X_h \setminus F_h(S_{-h}))$ is an increasing selection of the best response correspondence when $\succ_H$ are substitutable and $\succ_D$ are unilaterally substitutable. This finding allows us to prove the following result.

Proposition 3 *Assume $\succ_H$ are substitutable and $\succ_D$ are unilaterally substitutable. Then the game induced by the SAM has a SPE. Therefore, the SAM mechanisms weakly implement the set of pairwise stable allocations in SPE.*

Proposition 3 also provides a new existence result for stable allocations in many-to-many matching markets with contracts, which extends Hatfield and Kominers (2017).

Corollary 2 *Assume $\succ_H$ are substitutable and $\succ_D$ are unilateral substitutable. Then the set of pairwise stable allocations is non-empty.*

Consider the two following algorithms:

ho **Algorithm**

Step 0**:**
$(X_0)_h = \emptyset$ for all $h \in H$;

Step $r \geq 1$**:**
$(X_{r+1})_h = Br_h\left((X_r)_{-h}\right)$ for all $h \in H$.
Set $X^{ho} = \bigcup_{h\in H} (X_{\bar{r}})_h$, where $\bar{r} = \min\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\}$.

hp **Algorithm**

Step 0**:**

$(X_0)_h = X_h$ for all $h \in H$;

Step $r \geq 1$**:**

$(X_{r+1})_h = Br_h\left((X_r)_{-h}\right)$ for all $h \in H$.

Set $X^{hp} = \bigcup_{h \in H} (X_{\bar{r}})_h$, where $\bar{r} = \min\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\}$.

The monotonicity of the operator $Br_h(\cdot)$ used to prove Proposition 3 implies both algorithms stop in a finite number of steps at pairwise stable allocations. Notice that if the agents on one side of the market have unilateral substitutable preferences, pairwise stability and stability are not equivalent, as shown by the following example.

Example 5 *Let us assume* $H = \{h\}$ *and* $D = \{d_1, d_2, d_3\}$*. Let* x_r *be a contract between* h *and* d_1*, let* y_r *be a contract between* h *and* d_2*, and let* z_r *be a contract between* h *and* d_3*, for* $r = 1, 2$*. Assume the preferences of the agents are the following:*

$\succ_h$: $\{x_1, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_2, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_1, y_1, z_1\}, \{x_2, y_1, z_2\},$
$\{x_2, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_2, y_1\}, \{x_1, y_1\}, \{y_1, z_2\}, \{y_1, z_1\},$
$\{y_1\}, \{x_1\}, \{x_2\}, \{z_1\}, \{z_2\};$

$\succ_{d_1}$: $\{x_1\}, \{x_2\}$;

$\succ_{d_2}$: $\{y_1\}, \{y_2\}$;

$\succ_{d_3}$: $\{z_1\}, \{z_2\}$.

Preferences $\succ_h$ *are unilateral substitutable but not substitutable, because* $z_1 \notin C_h(\{y_1, z_1, z_2\})$ *but* $z_1 \in C_h(\{x_1, x_2, y_1, z_1, z_2\})$*. The allocation* $Y = \{y_1, x_2, z_2\}$ *is pairwise stable but not stable, because it is blocked by* $N' = \{h, d_1, d_3\}$ *through* $Z = \{x_1, z_1\}$*.*

When one side of the market has unilateral substitutable preferences, the set of pairwise stable allocations might not be a lattice. Moreover, under the same assumptions, the set of SPE is not even a lattice with respect to the joint preferences of the agents or Blair's order (see Blair, 1988), as can be seen in the following example that we borrow from Hatfield and Kojima (2010).

Example 6 *Let us assume* $H = \{h\}$ *and* $D = \{d_1, d_2\}$. *Let* x_1, x_2, x_3 *denote the contracts between* d_1 *and* h, *and let* y_1, y_2, y_3 *denote the contracts between* d_2 *and* h. *Assume the preferences of the agents are the following:*
$\succ_h$: $\{x_1, y_3\}, \{x_3, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_3, y_2\}, \{x_2, y_3\}, \{x_2, y_1\}, \{x_1, y_2\},$
$\{x_1, y_1\}, \{x_3\}, \{y_3\}, \{x_2\}, \{y_2\}, \{x_1\}, \{y_1\}$;
$\succ_{d_1}$: $\{x_2\}, \{x_1\}, \{x_3\}$;
$\succ_{d_2}$: $\{y_2\}, \{y_1\}, \{y_3\}$.
This market contains three pairwise stable allocations, $X_1 = \{x_2, y_2\}$, $X_2 = \{x_3, y_1\}$, *and* $X_3 = \{x_1, y_3\}$, *that are also* SPE *outcomes. However, as observed in Hatfield and Kojima (2010),* $\{X_1, X_2, X_3\}$ *is not a lattice with respect to the order induced by* $\succ_D$ *and not even with respect to the Blair's order.*

3.1.3 The structure of the set of SPE outcomes

In this section, we restrict our attention to markets where the preferences of both sides of the market are substitutable. We first prove that, under this assumption, the maximal best response $BR_h(\cdot)$ is increasing.

Lemma 4 *Assume* $\succ_H$ *and* $\succ_D$ *are substitutable. Then the maximal best response function* $BR_h(\cdot)$ *is increasing: if* $S_h \subseteq S'_h$ *for all* $h \in H$, *then* $BR_h(S_{-h}) \subseteq BR_h(S'_{-h})$ *for all* $h \in H$.

Lemma 4 implies the set of fixed points of $BR_H(\cdot)$ is a non-empty lattice.

To apply this result to our environment, we prove all SPE outcomes of the game induced by the SAM are generated by the set of fixed points of the maximal best response.

Lemma 5 *Assume $\succ_H$ and $\succ_D$ are substitutable. The allocation Y is an NE outcome of Γ if and only if a strategy profile $(S_h)_{h\in H}$ exists such that $BR_H\left((S_h)_{h\in H}\right) = (S_h)_{h\in H}$ and $C_D\left(\bigcup_{h\in H} S_h\right) = Y$.*

Lemma 5 does not extend to every selection of the best response correspondence; for instance, it does not extend to the minimal best response, as shown by the following example.

Example 7 *Consider Example 3. The unique stable allocation $\{x_1, z_2\}$ is not a fixed point of the minimal best response, $(br_h(\cdot))_{h\in H}$, because $br_h(\{z_2\}) = \tilde{x}_1$. Thus, $\{x_1, z_2\}$ cannot be obtained from a fixed point of the minimal best response.*

Lemma 5 shows the structure of the fixed points of $BR_H(\cdot)$ reflects the structure of the set of SPE outcomes and allows us to prove the set of SPE outcomes of the SAM mechanism is a lattice according to Blair's orders. Furthermore, an opposition of interests within the set of SPE allocations emerges. Given two SPE outcomes Y and Z, if Y dominates Z, according to hospitals' Blair's order, Z dominates Y according to doctors' Blair's order.

Proposition 4 *Let us assume $\succ_H$ and $\succ_D$ are substitutable.*

(i) The set of SPE of the game induced by the SAM is a non-empty lattice with Blair's orders $\succ_{HB}$ and $\succ_{DB}$.

(ii) Let Y, Z be SPE outcomes. Then $Y \succ_{HB} Z$ if and only if $Z \succ_{DB} Y$.

As in the case of Proposition 3, Proposition 4 also provides the following algorithms to compute pairwise stable allocations.

HO **Algorithm**

Step 0:

$(X_0)_h = \emptyset$ for all $h \in H$;

Step $r \geq 1$**:**

$(X_{r+1})_h = BR_h\left((X_r)_{-h}\right)$ for all $h \in H$.

Set $X^{HO} = \bigcup_{h \in H} (X_{\bar{r}})_h$, where $\bar{r} = \min\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\}$.

HP **Algorithm**

Step 0**:**

$(X_0)_h = X_h$ for all $h \in H$;

Step $r \geq 1$**:**

$(X_{r+1})_h = BR_h\left((X_r)_{-h}\right)$ for all $h \in H$.

Set $X^{HP} = \bigcup_{h \in H} (X_{\bar{r}})_h$, where $\bar{r} = \min\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\}$.

Lemmas 4 and 5 imply the outcome of the HO algorithm coincides with the best (resp. worst) SPE outcome for hospitals (resp. doctors), and the HP algorithm coincides with the worst (resp. best) SPE outcome for hospitals (resp. doctors).[7] A DO algorithm and a DP algorithm can be defined symmetrically considering the game where the doctors make the offers. Future research should compare the efficiency of these algorithms and of the cumulative offer mechanism (see Hatfield and Kominers, 2017).

The existence of contracts provides hospitals with a first-mover advantage as shown in Example 2. This effect shrinks the set of SPE allocations through unilateral deviations, and hurts doctors more than hospitals. Indeed, although the doctor-optimal stable allocation can be excluded from the set of SPE outcomes, as we have seen in Example 2, the hospital-optimal stable allocation is always an SPE outcome.

Proposition 5 *Assume $\succ_D$ are substitutable and $\succ_H$ are substitutable and satisfy the law of aggregate demand. Then the hospital-optimal stable allocation is an SPE outcome.*

[7] The same result does not hold with the *ho* and the *hp* algorithm, because, in general, the set of fixed points of $(Br_h(\cdot))_{h \in H}$ does not coincide with the set of NE outcomes of the game Γ.

Consider a many-to-one situation where each hospital can hire at most one doctor. If $\succ_D$ does not satisfy the law of aggregate demand, the hospital-optimal stable allocation is not strategy-proof for hospitals (see Hatfield and Milgrom, 2005). However, Proposition 5 implies that it is an NE outcome of the SAM.

When preferences are substitutable and the preferences of hospitals satisfy the law of aggregate demand, the outcome of the HO algorithm coincide with the hospital-optimal stable allocation. When no contracts exist, the SAM fully implements the set of stable allocations (see Echenique and Oviedo, 2006), and the outcome of the HP algorithm coincides with the doctor-optimal stable allocation.

4 Take-It-Or-Leave-It Offers Mechanisms

We now introduce the class of **take-it-or-leave-it offers mechanisms**. This class extends the SAM by relaxing the assumption that the agents accept the proposals they receive simultaneously. The TOM are such that, in a first stage, hospitals make simultaneous offers to doctors. Then groups of doctors sequentially accept or reject the offers they received. Doctors in the same group choose simultaneously. The order of choice can be arbitrary and/or endogenous to the play, which is history dependent. The contracts accepted in the choice stage are enforced as an outcome of the mechanism.

The **take-it-or-leave-it offers mechanisms** are described by the following procedure:

Offers. Each hospital h offers contracts to some doctors. Let $X_1(h) \subseteq X_h$ be the set of contracts offered by hospital h. If h does not make any offer, then $X_1(h) = \varnothing$. For all $d \in D$, let $X_1(d) = \left(\bigcup_{h \in H} X_1(h)\right)_d$, be the set of offers received by doctor d.

Choice. At node $i \in I$, a subset of doctors $D^i \subseteq D$ that did not choose before selects a set of contracts from among the ones she was offered. Let $X_2^i(d) \subseteq X_1(d)$ be the set of offers selected by doctor $d \in D^i$. The procedure follows until all doctors have chosen.

Formally, a TOM is an extensive-form mechanism $\mathcal{T} = (N, X, I, S, g)$ with the following characteristics. At the initial history i^0, $S^{i^0} = \prod_{h \in H} 2^{X_h}$.[8] Let i be a successor of i^0, $i = \left(i^0, (X_1^i(h))_{h \in H}\right)$ and let $X_1^i(d) = \left(\bigcup_{h \in H} X_1^i(h)\right)_d$. For any non-initial or terminal node $i \in I \setminus (\{i^0\} \cup W)$, there exists $D^i \subseteq D$ such that, if i' proceeds i, then $D^{i'} \cap D^i = \emptyset$ and $\bigcup_{i \in I \setminus (\{i^0\} \cup W)} D^i = D$. For all $i \in I \setminus (\{i^0\} \cup W)$, let $i^1(i)$ be the unique successor of i^0 preceding i. At each $i \in I \setminus (\{i^0\} \cup W)$, the strategy space is $\prod_{d \in D^i} 2^{X^i(d)}$, where $X^i(d) = X_2^{i^1(i)}(d)$.

Notice that in a TOM, the strategy space of a doctor d depends only on the offer she receives at Stage 0. In her turn, every doctor has a strictly dominant strategy, which is to accept the best offers that she receives. Let $\mathcal{T}$ be a TOM, and consider the game induced by $\mathcal{T}$ when preferences are $\succ$.

Lemma 6 *Doctors have a unique best response $X_2^{*i}(d) = C_d\left(X_1^{i^1(i)}(d)\right)$ for all $d \in D$, for all $i \in I \setminus (\{i^0\} \cup W)$.*

The result is analogous to Lemma 1. Notice the result implies the set of SPE outcomes coincides with the set of NE outcomes of game Γ. Therefore, we have

Proposition 6 *The set of SPE outcomes of $\mathcal{T}$ coincides with the set of SPE outcomes of the SAM.*

In particular, all the results of Section 3 extend to all the mechanisms on the class of TOM.

[8] Notice we are abusing notation. In order to be completely consistent with the definition of an extensive-form mechanism, we should write, for instance, $S^{i^0} = \prod_{h \in H} 2^{X_h} \times \prod_{d \in D} \{d\}$, meaning that doctors, at this stage, do not have any choice to make.

Corollary 3 *Let $(H, D, X, \succ)$ be a matching market and let $\mathcal{T}$ be a TOM. Then*

a) All SPE outcomes of the game induced by the $\mathcal{T}$ are pairwise stable allocations.

b) If preferences are substitutable, all SPE outcomes of the game induced by $\mathcal{T}$ are stable allocations.

c) If $\succ_H$ are substitutable and $\succ_D$ are unilateral substitutable, the game induced by the $\mathcal{T}$ has an SPE.

d) If $\succ$ are substitutable, the set of SPE outcomes of the game induced by the $\mathcal{T}$ outcomes is a non-empty lattice with respect to Blair's order.

e) If $\succ_H$ are substitutable and satisfy the law of aggregate demand and $\succ_D$ are substitutable, the hospital-optimal stable allocation is an SPE outcome of the game induced by $\mathcal{T}$.

5 Centralized Markets

In many markets, clearing houses are already in use, as is the case, for instance, with the school admission procedures in place in many school districts. As mentioned in the introduction, a many-to-many matching market with contracts is arguably the most accurate representation of school admission models given that a significant number of families have more than one child and different arrangements between schools and families are possible (e.g., different tuition fees, schedules, lunch options, or grants). However, school admission problems have been modeled as a many-to-one matching market, allowing for minor adjustments on school priorities to favor siblings.

Let us now consider the problem of centralizing the assignment of students to schools as a many-to-many matching market with contracts. The objective is to provide a centralized mechanism able to implement stable allocations. Because no strategy-proof, stable revelation mechanism exists in this framework, we pursue the stability of the Nash equilibrium outcomes.

To maintain the concordance with the rest of the paper, we call the agents on each side of the market hospitals and doctors, respectively. As is usual in this case (see, e.g., Ergin and Sönmez, 2006, and Haeringer and Klijn, 2009), we assume the hospital priorities are known to doctors, and we consider strategic behavior only among the latter.

Formally, the problem is characterized by the following:

- a set of hospitals H,
- a set of doctors D,
- a vector of quotas $(q_h)_{h \in H}$, where q_h is a positive integer that represents the number of doctors hospital $h \in H$ can sign,
- a set of contracts X,
- a strict priority structure $\succ_H$, where, for all $h \in H$, $\succ_h$ is a strict order over $X_h \cup \{\emptyset\}$,
- for all doctors $d \in D$, a strict preference profile $\succ_d$ over $X \cup \{\emptyset\}$.

We need to extend hospitals' priorities over doctors to priorities over sets of doctors. To do so, we assume that priorities over a set of doctors are responsive to priorities over individual doctors. Formally, we say the profile $\tilde{\succ}_H$ is **responsive** to the priority structure $\succ_H$ with a vector of quotas $(q_h)_{h \in H}$ if for all h and all $Y_h \subseteq X_h$, $x, z \in X \setminus Y$: (i) if $|Y_h| < q_h$, then $Y \cup \{x\} \tilde{\succ}_h Y \cup \{z\}$ if and only if $x \succ_h z$, (ii) $Y \cup \{x\} \tilde{\succ}_h Y$ if and only if $x \succ_h A(\succ_h)$, and (iii) if $|Y_h| > q_h$, then $\emptyset \tilde{\succ}_h Y$.

Responsive priorities are substitutable and satisfy the law of aggregate demand, so our previous results extend to this framework.

Contrary to the previous sections, from now on, we consider TOM mechanisms where doctors make the proposals, because we are interested in designing an admission mechanism for doctors.

It is a well-known fact (see Roth and Sotomayor, 1990) that, in a many-to-many framework, no stable revelation mechanism exists where truth-telling is a dominant strategy for the agents on either side of the market. Thus, we turn our attention to the stability of Nash equilibrium outcomes.

A natural candidate for the centralization is the doctor-optimal stable mechanism. However, it may produce unstable allocations as NE outcomes (see Haeringer and Klijn, 2009). An alternative to the doctor-optimal stable mechanism is the DO algorithm. However, even if the DO algorithm does not coincide with the cumulative offer algorithm, its outcome coincides with the doctor-optimal stable allocation when preferences are substitutable and satisfy the law of aggregate demand.

Alternatively, we can focus on the incentives provided by the so-called immediate acceptance mechanism (see Alcalde, 1996), also known as the Boston mechanism (see Abdulkadiroğlu and Sönmez, 2003).

First, let us define the **immediate acceptance mechanism** for the many-to-many matching markets with contracts that we analyze. The strategy space of doctor d is the set of strict rational preferences over 2^{X_d}, $\mathcal{L}\left(2^{X_d}\right)$. Given a preference for doctor d, $\succ_d$ over 2^{X_d} and an integer r, $1 \leq r \leq \left|2^{X_d}\right|$, let $Y^r_{\succ_d}$ be the r^{th} ranked acceptable set according to $\succ_d$, when one exists. Let $Y^r_{\succ_d}$ be empty otherwise. Formally, $Y = Y^r_{\succ_d}$ if and only if $Y \subseteq X_d$, $Y \succ_d \emptyset$ and $|\{Z \subseteq X_d \mid Z \succ_d Y\}| = r - 1$. Set $Y^r_{\succ_d} = \emptyset$ if $|\{Z \subseteq X_d \mid Z \succ_d Y\}| = r - 1 \Longrightarrow \emptyset \succeq_d Y$.

The following procedure describes the **immediate acceptance mechanism.**

- Step 1: Only the top acceptable choices of the doctors at $\succ_D$ are considered. Set $A^1 = \bigcup_{d \in D} Y^1_{\succ_d}$ and set $Y^1 = C_H\left(A^1\right)$. Contracts in Y^1 are signed. Every doctor who has signed a contract and every doctor who has proposed the empty set, that is, every $d \in Y^1_D \cup \left\{d \in D \mid Y^1_{\succ_d} = \emptyset\right\}$, is removed from the market. Let $D^2 = D \setminus Y^1_D$ be the set of remaining doctors.

- Step r, $r \geq 2$: Only the r^{th} choices of doctors in D^r are considered, and hospitals decide which contracts to add to the ones selected at step $r-1$, compatibly with the already signed contracts. Set $A^r = \bigcup_{d \in D^{r-1}} Y^r_{\succ_d}$, for all $h \in H$, set $Y^{r,h} = \max_{\succ_h} \left\{ Y \subseteq X_h \mid Y_h^{r-1,h} \subseteq Y \subseteq Y^{r-1,h} \cup A_h^r \right\}$. Finally set $Y^r = \bigcup_{h \in H} Y^{r,h} \cup Y^{r-1}$. Contracts in Y^r are signed. Every doctor who has signed a contract and every doctor who has proposed the empty set, that is, every $d \in Y_D^r \cup \left\{ d \in D^r \mid Y^r_{\succ_d} = \emptyset \right\}$, is removed from the market. Let $D^{r+1} = D \setminus Y_D^r \cup \left\{ d \in D^{r-1} \mid Y^r_{\succ_d} = \emptyset \right\}$ be the set of remaining doctors. The procedure stops at $r^* = \min \{ r \mid D^r = \emptyset \}$. Let $Y = Y^{r^*}$ be the final outcome.

Let $IA(\succ_D, \succ_H)$ be the outcome of the algorithm when doctors submit preference profile $\succ_D$ and the profile of hospital priority is $\succ_H$. The game induced by the immediate acceptance mechanism is $\mathcal{IA} = \left(D, \succ_D, \mathcal{L}\left(2^{X_d}\right), IA(\cdot, \succ_H) \right)$. We consider the game Γ introduced in Section 3.1.2, where doctors make proposals.

Proposition 7 *Assume $\succ_D$ are substitutable and $\succ_H$ are responsive; then the set of NE outcomes of Γ coincides with the set of NE outcomes of $\mathcal{IA}$.*

Thus, the set of NE outcomes of the immediate acceptance mechanism is a non-empty lattice of stable allocations that includes the doctor-optimal stable allocation.[9]

Unfortunately, in situations where priorities are not responsive (e.g., in cases involving budget constraints; see, e.g., Mongell and Roth, 1986, and Abizade, 2016, or when students are ranked using scores systems that give extra points to siblings simultaneously entering a new school), the immediate acceptance mechanism may fail to implement even individually rational allocations.

Example 8 *Let us assume $H = \{h_1, h_2\}$ and $D = \{d_1, d_2, d_3, d_4\}$.*

[9] The result follows from Corollary 1, Proposition 4, and Proposition 5.

Priorities and preferences are the following:

$\succ_{h_1}$: $\{d_1, d_3\}, \{d_1 d_2, d_3\}, \{d_2, d_3\}, \{d_1, d_2\}, \{d_1\}, \{d_3\}, \{d_2\};$

$\succ_{h_2}$: $\{d_4\}, \{d_1\};$

$\succ_{d_1}$: $\{h_2\}, \{h_1\};$

$\succ_{d_2}$: $\{h_1\};$

$\succ_{d_3}$: $\{h_1\};$

$\succ_{d_4}$: $\{h_2\};$

The outcome of truth-telling in the immediate acceptance algorithm results in the following allocation: $Y = \{\{h_1, d_1, d_2, d_3\}, \{h_2, d_4\}\}$, *which is not individually rational, because* h_1 *would like to fire* d_2*. However, truth-telling is an* NE *because any agent but* d_1 *is assigned to her preferred hospital, but* d_1 *has no profitable deviations.*

The failure of the immediate acceptance mechanism in implementing individually rational allocations relies on the multiple-round structure of the mechanism. Therefore, we consider the game derived from the immediate acceptance mechanism, where the first-round allocation, Y^1, is the final one. We call this mechanism **the one-shot immediate acceptance mechanism**. Formally, let $\succ_D = (\succ_d)_{d \in D}$ be a profile of preferences for doctors, and let $(\succ_h)_{h \in H}$ be hospital priorities. Define $OS\left((\succ_d)_{d \in D}, \succ_H\right) = C_H\left(\bigcup_{d \in D} Y^1_{\succ_d}\right)$. The game induced by the one-shot immediate acceptance mechanism is $\mathcal{OS} = \left(D, \succ_D, \mathcal{L}\left(2^{X_d}\right), OS\left(\cdot, \succ_H\right)\right)$.

Notice the structure of $\mathcal{OS}$ is very similar to the structure of game Γ, where doctors make the proposal. Indeed, the Nash equilibrium outcomes of the two games coincide.

Proposition 8 *The set of* NE *outcomes of the game induced by the one-shot immediate acceptance mechanism coincides with the set of* NE *outcomes of game* Γ*. Then*

a) All NE *outcomes of* $\mathcal{OS}$ *are pairwise stable allocations.*

b) If preferences are substitutable, all NE outcomes of $\mathcal{OS}$ are stable allocations.

c) If $\succ_D$ are substitutable and $\succ_H$ are unilateral substitutable, OS has an NE.

d) If $\succ$ are substitutable, the set of NE outcomes of $\mathcal{OS}$ is a non-empty lattice with respect to the Blair's order.

e) If $\succ_D$ are substitutable and satisfy the law of aggregate demand and $\succ_H$ are substitutable, the hospital-optimal stable allocation is an NE outcome of $\mathcal{OS}$.

The immediate acceptance mechanism and the one-shot immediate acceptance mechanism are equivalent when priorities are responsive. However, when one relaxes the assumption of complete information, the one-shot immediate acceptance mechanism makes an agent more likely to end without signing any contract, preventing an efficient allocation from being achieved. The immediate acceptance mechanism allows information to be revealed and agents' preferences to be expressed along the different stages of the mechanism as the following example shows.

Example 9 *Let $H = \{h_1, h_2\}$ and let $D = \{d_1, d_2\}$. Doctors only know their own preferences. The priority of the hospitals are $\succ_{h_1}=\succ_{h_2}$: $d_1 \succ d_2$. Each hospital has quota $q = 1$. Each doctor can be of two independent types a or b. The preferences of the doctors are represented by the following utility functions: $u_{d_i}(d_1 \mid a) = u_{d_i}(d_2 \mid b) = 2$, $u_{d_i}(d_2 \mid a) = u_{d_i}(d_1 \mid b) = 1$, and $u_{d_i}(\emptyset \mid a) = u_{d_i}(\emptyset \mid b) = 0$. Doctor d_1 is of type a with probability p, and doctor d_2 is of type a with probability $\frac{1}{2}$.*

Assume the immediate acceptance mechanism is employed. Let $p \leq \frac{1}{2}$. A unique Bayesian Nash equilibrium exists where both doctors reveal their preferences. The payoff of doctor d_1 is 2 and the payoff of doctor d_2 is $2 - p$. Now let $p > \frac{1}{2}$. A unique Bayesian Nash equilibrium exists where doctor d_1

reveals her preferences and doctor d_2 ranks her second option as the first one and her first option as the second one. The payoff of doctor d_1 is 2 and the payoff of doctor d_2 is $1 + p$.

Assume the one-shot immediate acceptance mechanism is employed. Then, in any Bayesian Nash equilibrium, both doctors reveal their first options. The payoff of doctor d_1 is 2, the interim payoff of doctor d_2 of type a is $2 - 2p$, and the interim payoff of doctor d_2 of type b is $2p$. Furthermore, doctor d_1 is left without any position with strictly positive probability.

Both when $p \leq \frac{1}{2}$ and when $p > \frac{1}{2}$, the one-shot immediate acceptance mechanism always yields a strictly higher interim expected payoff than the immediate acceptance mechanism to doctor d_2.

6 Conclusions

In this paper, we have studied a simple mechanism called the simultaneous acceptance mechanism or SAM. The SAM is a take-it-or-leave-it mechanism where hospitals make their offers simultaneously and then doctors accept or reject them simultaneously. The mechanism is well known and mimics real-world environments, allowing us to explore the allocative implications of the use of contracts in many-to-many matching markets.

The SAM generalizes previous results in many-to-many matching markets (Sotomayor, 2004; Echenique and Oviedo, 2006) to the many-to-many matching markets with contracts environment. The SAM also allows us to prove the existence of pairwise stable allocations when one side of the market has substitutable preferences and the other side has unilaterally substitutable preference. This extends the existence result provided by Hatfield and Kominers (2017). Moreover, when restricted to the many-to-one case, the procedures coincide with the mechanisms presented in Alcalde et al. (1998) and Alcalde and Romero-Medina (2000), unifying their implementation results, with and without transferable utility.

We extend our findings on the SAM to the general class of take-it-or-leave-it offers mechanisms or TOM. All the mechanisms in this newly defined class weakly implement the set of pairwise stable allocations in SPE when the doctors' preferences are unilaterally substitutable and hospitals' preferences are substitutable. The SAM is not the only mechanism in the TOM class that has been previously analyzed in the literature. Another interesting element of the TOM class is the mechanism where doctors accept proposals one at a time in an order that has been previously established at the beginning of the game. This member of the TOM class extends the mechanisms introduced by Romero-Medina and Triossi (2014) (see also Klaus and Klijn, 2016) to many-to-many matching markets with contracts.

The inability of the mechanism in the TOM class to fully implement the set of pairwise stable allocation when the preferences are unilaterally substitutable gives us insight on the strategic limitation of the market mechanism. These limitations are important, because a class of simple sequential mechanisms is able to perform well in a complex environment.

Finally, we apply our findings to the school admissions problem. We show the extension of the school choice problem to the many-to-many case is not trivial, because we can no longer rely on strategy-proof mechanisms. However, we can guarantee the allocation of stable allocations in equilibria of the immediate acceptance mechanism as long as schools have responsive priorities. If the priorities of the schools are not responsive, the immediate acceptance mechanism can produce allocations that are not individually rational. In this case, the one-shot immediate acceptance mechanism guarantees the implementation of stable allocations.

References

[1] Abdulkadiroğlu, A. and Sönmez, T., (2003). "School Choice: A Mechanism Design Approach." American Economic Review, 93 (3), 729–747.

[2] Abdulkadiroğlu, A. and Sönmez, T., (2013). "Matching Markets: Theory and Practice." in D. Acemoğlu, M. Arello, and E. Dekel (eds), Advances in Economics and Econometrics, Vol. 1, Cambridge, 3–47.

[3] Abizada, A., (2016). "Stability and incentives for college admissions with budget constraints." Theoretical Economics, 11 (2), 735–756

[4] Alcalde, J., (1996). "Implementation of Stable Solutions to Marriage Problems." Journal of Economic Theory, vol. 69 (1), 240–254.

[5] Alcalde, J., Pérez-Castrillo, D., and Romero-Medina, A., (1998). "Hiring procedures to implement stable allocations." Journal of Economic Theory, 82 (2), 469–480.

[6] Alcalde, J. and Romero-Medina, A., (2000). "Simple mechanisms to implement the core of college admissions problems." Games and Economic Behavior, 31 (2), 294–302.

[7] Baron, D. and Kalai, E., (1993). "The Simplest Equilibria of a Majority Division Game." Journal of Economic Theory, 61 (2), 290–301.

[8] Baumol, W. J., Panzar, J. C. and Willig, R. D., (1982). Contestable Markets and the Theory of Industry Structure. New York: Harcourt Brace Jovanovick, Inc.

[9] Blair, C., (1988). "The lattice structure of the set of stable matchings with multiple partners." Mathematics of Operations Research, 13 (4), 619–628.

[10] Echenique, F. and Oviedo, J., (2006). "A theory of stability in many-to-many matching markets." Theoretical Economics, 1 (2), 233–273.

[11] Ergin, H. and Sönmez, T., (2006). "Games of School Choice under the Boston Mechanism." Journal of Public Economics, 90 (1–2), 215–237.

[12] Gale, D. and Shapley, L., (1962). “College Admissions and the Stability of Marriage.” American Mathematical Monthly, 69 (1), 9–15.

[13] Haeringer, G. and Klijn, F., (2009). “Constrained school choice.” Journal of Economic Theory, 144 (5), 1921–1947.

[14] Hatfield, J. W. and Kojima, F., (2010). “Substitutes and stability for matching with contracts.” Journal of Economic Theory, 145 (5), 1704–1723.

[15] Hatfield, J. W. and Kominers, S., (2017). “Contract Design and Stability in Many-to-Many Matching.” Games and Economic Behavior, 101, 78–97.

[16] Hatfield, J. W. and Milgrom, P. R., (2005). “Matching with Contracts.” American Economic Review, 95 (4), 913–935.

[17] Kelso, A. S. Jr. and Crawford, V. P., (1982). “Job Matching, Coalition Formation, and Gross Substitutes.” Econometrica, 50 (6), 1483–1504.

[18] Klaus, B. and Klijn F., (2016). “Non-Revelation Mechanisms for Many-to-Many Matching: Equilibria versus Stability.” Cahiers de recherches économiques du DEEP No. 16.07.

[19] Klaus, B. and Walzl, M., (2009). “Stable many-to-many matchings with contracts.” Journal of Mathematical Economics, 45 (7–8), 422–434.

[20] Kominers, S., (2012). “On the Correspondence of Contracts to Salaries in (Many-to-Many) Matching.” Games and Economic Behavior, 75 (2), 984–989.

[21] Mongell, S. J. and Roth, A. E., (1986). “A Note on Job Matching with Budget Constraints.” Economics Letters, 21 (2), 135–138.

[22] Pepa Risma, E., (2015). “A deferred acceptance algorithm with contracts.” Journal of Dynamics and Games, 2 (3–4), 289–302.

[23] Romero-Medina A. and Triossi M., (2014). "Non-revelation mechanisms in many-to-one markets." Games and Economic Behavior, 87, 624–630.

[24] Roth, A. E., (1991). "A natural experiment in the organization of entry level labor markets: regional markets for new physicians and surgeons in the UK." American Economic Review, 81 (3), 415–440.

[25] Roth, A. E. and Sotomayor, M., (1990). Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis, Cambridge, Cambridge University Press.

[26] Sönmez, T. and Switzer T. B., (2013) "Matching with (Branch-of-Choice) Contracts at the United States Military Academy." Econometrica, 81 (2), 451–488.

[27] Sönmez, T. and Aygün O., (2013). "Matching with Contracts: Comment." American Economic Review, 103 (5), 2050–2051.

[28] Sotomayor, M., (2004). "Implementation in the many-to-many matching market." Games and Economic Behavior, 46 (1), 199–212.

[29] Triossi, M., (2009). "Hiring mechanisms, application costs and stability." Games and Economic Behavior, 66 (1), 566–575.

[30] Yenmez, M. B., (2015). "College Admissions." (January 4, 2015). GSIA Working Paper No. 2014-E24. Available at SSRN: https://ssrn.com/abstract=2539818 or http://dx.doi.org/10.2139/ssrn.2539818.

Appendix

Proof of Proposition 1. Assume s^* is an SPE, and let Y be the outcome from s^*. We will show by contradiction that Y is a pairwise stable allocation.

We first prove Y is an individually rational allocation. The proof of the claim is by contradiction. Assume Y is not an individually rational allocation for agent $n \in N$. Let $n = h \in H$; then $C_h(Y_h)$ is a profitable deviation, from IRC, yielding a contradiction. Let $n = d \in D$; in this case, the contradiction follows from Lemma 1.

We conclude the proof by showing Y is not pairwise blocked. By contradiction, assume a hospital h, a doctor d, and a contract $x \in X \setminus Y$ exist with $x_D = d$, $x_H = h$ such that $x \in C_h(Y \cup \{x\}) \cap C_d(Y \cup \{x\})$.

First, we prove $x \in C_d\left(\bigcup_{s^*_{h'D}=d} \{s^*_{h'}\} \cup \{x\}\right)$. Set $Z = \bigcup_{s^*_{h'D}=d} \{s^*_{h'}\}$. From Lemma 1, $C_d(Z) = Y_d$. By contradiction, assume $x \notin C_d\left(\bigcup_{s^*_{h'D}=d} \{s^*_{h'}\} \cup \{x\}\right) = C_d(Z \cup \{x\})$. From $x \in C_d(Y_d \cup \{x\})$, it follows that $C_d(Z \cup \{x\}) \succ_d C_d(X_d \cup \{x\})$. However, because $x \notin C_d(Z \cup \{x\})$, $C_d(Z \cup \{x\}) = C_d(Z) = X_d$, yielding a contradiction.

Set $T = \{y \mid y \in s^*_h, y \notin C_h(Y \cup \{x\}), y_D \neq d\}$. T is the set of offers that h made in equilibrium to doctors different than d who don't have contracts in $C_h(Y \cup \{x\})$. Consider the following deviation for h, $s_h = (s^*_h \cup \{x\}) \setminus T$. In the subgame induced by this deviation, doctor d is offered the contracts in $\bigcup_{s^*_{h'D}=d} \{s^*_{h'}\} \cup \{x\}$. From IRC and Lemma 1, it follows that the deviation is profitable to h, yielding a contradiction. ■

Proof of Proposition 2. Let Y be a stable allocation. We will construct an equilibrium yielding Y as the outcome. If $br_h(Y_{-h}) = Y_h$ for all $h \in H$, the proof is complete. Otherwise, let $T_h \neq Y_h$ be such that $T_h = br_h(Y_{-h})$. Then $T_h = C_h(Y_h \cup T_h)$, and thus $C_h(Y_h \cup \{t\})$ is a profitable deviation for h for all $t \in T_h \setminus Y_h$. Let $T = \left(\bigcup_{h \in H} T_h\right) \setminus Y$.

Let $t \in T$. Because Y is pairwise stable, preferences are substitutable and satisfy UA. $d \in D$, $y \in Y$ exist such that $y_H = t_H = h$, $y_D = t_D = d$ such that $t \in C_h(Y \cup \{t\}) \cap C_d((Y \setminus \{y\}) \cup \{t\})$ and $y \notin C_d(Y \cup \{t\})$. Because the market is contestable, $t' = t'(t) \in X$ and $h' \neq h$ exist such that $t' \in C_d((Y \setminus \{y\}) \cup \{t, t'\})$, $t \notin C_d((Y \setminus \{y\}) \cup \{t, t'\})$, and $t' \notin C_d((Y) \cup \{t, t'\})$. Set $Y' = Y \cup \bigcup_{t \in T} t'(t)$. Let $S_h = Y'_h$ for all $h \in H$. Observe that, by construction, $Y_h = br_h(S_{-h})$. Furthermore, $t'(t) \in R_h\left(Y'_h, Y'_{-h}\right)$ because Y is stable and $\succ_D$ are substitutable. It follows that $(S_h)_{h\in H}$ is an NE of Γ yielding Y as the outcome, which completes the proof of the claim. ■**Proof of Lemma 3.** (i) First we show $br_h(S_{-h})$ is a best response to S_{-h}, and for any best response Y_h, $br_h(S_{-h}) \subseteq Y_h$. Notice $g(br_h(S_{-h}), S_{-h}) = br_h(S_{-h})$. Let Y_h be a best response and let $Y'_h = C_D\left(\bigcup_{h'\neq h} S_{h'} \cup Y_h\right)_h = (g(Y_h, S_{-h}))_h$. To complete the proof of the claim, we show $br_h(S_{-h}) = Y'_h \subseteq Y_h$. Let $x \in Y'_h$. Because $(\succ_d)_{d\in D}$ satisfy UA, $x \in C_{x_D}\left(\bigcup_{h'\neq h} S_{h'} \cup \{x\}\right)$. Thus, $Y'_h \subseteq \left\{x \in X_h \mid x \in C_D\left(\bigcup_{h'\neq h} S_{h'} \cup \{x\}\right)\right\}$. In particular, $br_h(S_{-h}) \succeq_h Y'_h$. The set Y_h is a best response to S_{-h}, so $Y'_h \succeq_h br_h(S_{-h})$. Because preferences are strict, $br_h(S_{-h}) = Y'_h$.

(ii) Now, we show that if Y_h is a best response to S_{-h}, then $Y_h \subseteq BR_h(S_{-h})$. Observe that Y_h is a best response to S_{-h} if and only if $\left[C_D\left(\bigcup_{h'\neq h} S_{h'} \cup Y_h\right)\right]_h = br_h(S_{-h})$. Let Y_h be a best response. From part (i) of the proof, we have $Y_h = br_h(S_{-h}) \cup Z_h$ for some $Z_h \subseteq X_h$, $Z_h \cap br_h(S_{-h}) = \emptyset$. From IRC, it follows that $z \notin \left\{x \in X_h \mid x \in C_D\left(\bigcup_{h'\neq h} S_{h'} \cup br_h(S_{-h}) \cup \{z\}\right)\right\}$ for all $z \in Z_h$; therefore, $Y_h \subseteq BR_h(S_{-h})$.

(iii) Let $br_h(S_{-h}) \subseteq Y_h \subseteq BR_h(S_{-h})$. We can write $Y_h = br_h(S_{-h}) \cup Z_h$, where $Z_h \subseteq \left\{x \in X_h \mid x \notin C_D\left(\bigcup_{h'\neq h} S_{h'} \cup br_h(S_{-h}) \cup \{x\}\right)\right\}$ and $Z_h \cap br_h(S_{-h}) = \emptyset$. By contradiction, assume $C_D\left(\bigcup_{h'\neq h} S_{h'} \cup Y_h\right)_h \neq br_h(S_{-h})$. Therefore, $z \in Z_h$ exists such that $z \in C_{z_D}\left(\bigcup_{h'\neq h} S_{h'} \cup br_h(S_{-h}) \cup Z_h\right)$. The UA and IRC imply $C_d\left(\bigcup_{h'\neq h} S_{h'} \cup br_h(S_{-h}) \cup Z_h\right) =$
$C_d\left(\bigcup_{h'\neq h} S_{h'} \cup br_h(S_{-h}) \cup \{z\}\right)$. In particular,
$z \in C_{z_d}\left(\bigcup_{h'\neq h} S_{h'} \cup br_h(S_{-h}) \cup \{z\}\right)$, yielding a contradiction. ■

The following Lemmas 7 and 8 will be used in the proof of Proposition 3. In particular, Lemma 7 will be repeatedly used in the proof of Lemma 8.

Lemma 7 *(i) $R_h(S_h, S_{-h})$ is increasing in S_h, for all S_{-h}: if $S'_h \subseteq S_h$, then $R_h(S'_h, S_{-h}) \subseteq R_h(S_h, S_{-h})$.*

(ii) $Br_h(S_{-h}) = br_h(S_{-h}) \cup (X_h \setminus F_h(S_{-h}))$ *is a best response.*

(iii) Assume $\succ_D$ *are unilaterally substitutable. Let* $h \in H$*. If* $S'_{h'} \subseteq S_{h'} \subseteq X_{h'}$ *for all* $h' \in H \setminus \{h\}$*, then* $F_h(S_{-h}) \subseteq F_h\left(S'_{-h}\right)$*.*

(iv) Assume $\succ_D$ *are unilaterally substitutable and* $\succ_H$ *are substitutable. Let* $h \in H$*. If* $S'_{h'} \subseteq S_{h'} \subseteq X_{h'}$*, for all* $h' \in H \setminus \{h\}$*, then* $br_h\left(S'_{-h}\right) \subseteq Br_h(S_{-h})$*.*

(v) Assume $\succ_D$ *is substitutable. Assume* $S'_{h'} \subseteq S_{h'} \subseteq X_{h'}$ *for all* $h' \in H$*; then* $R_h\left((S'_{h'})_{h' \in H}\right) \subseteq R_h\left((S_{h'})_{h' \in H}\right)$ *for all* $h \in H$*.*

(vi) Assume $\succ_D$ *and* $\succ_H$ *are substitutable. If* $S'_{h'} \subseteq S_{h'} \subseteq X_{h'}$ *for all* $h' \in H \setminus \{h\}$*, then* $br_h\left(S'_{-h}\right) \subseteq BR_h(S_{-h})$*.*

Proof. (i) Let $S'_h \subseteq S_h$, and let $x \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S'_h \cup \{x\}\right)$. We prove by contradiction that $x \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)$. Assume $x \in C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)$. The UA and IRC imply $\left(C_d\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)\right) = \left(C_d\left(\bigcup_{h' \neq h} S_{h'} \cup S'{}_h \cup \{x\}\right)\right)_h$ for all d, reaching a contradiction. Therefore, $R_h(S'_h, S_{-h}) \subseteq R_h(S'_h, S_{-h})$.

(ii) Notice $X_h \setminus F_h(S_{-h}) = R_h(\emptyset, S_{-h})$. From (i) $R_h(\emptyset, S_{-h}) \subseteq R_h(br_h(S_{-h}), S_{-h})$; then $br_h(S_{-h}) \cup (X_h \setminus F_h(S_{-h})) \subseteq BR_h(S_{-h})$, which implies the claim.

(iii) Let $x \in F_h(S_{-h})$; then $x \in C_d\left(\bigcup_{h' \neq h} S_{h'} \cup \{x\}\right)$, where $d = x_D$. Notice $h \notin \left[\bigcup_{h' \neq h} S'_{h'}\right]_H$. Because $\succ_d$ is unilaterally substitutable, $x \in C_d\left(\bigcup_{h' \neq h} S'_{h'} \cup \{x\}\right)$.

(iv) We have $br_h\left(S'_{-h}\right) = C_h\left(F_h\left(S'_{-h}\right)\right) \cap F_h(S_{-h}) \cup \left(C_h\left(F_h\left(S'_{-h}\right)\right) \setminus F_h(S_{-h})\right)$. From (iii) $F_h(S_{-h}) \subseteq F_h\left(S'_{-h}\right)$. Therefore, the substitutability of $\succ_h$ implies $C_h\left(F_h\left(S'_{-h}\right)\right) \cap F_h(S_{-h}) \subseteq C_h(F_h(S_{-h}))$, which concludes the proof of the claim.

(v) The claims follows directly from the definition of substitutability of $\succ_D$.

(vi) From (iv), $br_h\left(S'_{-h}\right) \subseteq Br_h(S_{-h}) \subseteq BR_h(S_{-h})$. ■

The next result will be used in the proof of Propositions 3 and 4.

Lemma 8 *Assume $\succ_D$ are unilaterally substitutable and $\succ_H$ are substitutable. Let $h \in H$. If $S'_{h'} \subseteq S_{h'} \subseteq X_{h'}$, for all $h' \in H \setminus \{h\}$. Then $Br_h\left(S'_{-h}\right) \subseteq Br_h\left(S_{-h}\right)$.*

Proof. (i) The proof of the result follows from (iii) and (iv) of Lemma 7 above. ■

Proof of Proposition 3. The result follows directly from Lemmas 8 and 7. ■

Proof of Lemma 4. To complete the proof, it suffices to show $R_h\left(br_h\left(S'_{-h}\right), S'_{-h}\right) \subseteq R_h\left(br_h\left(S_{-h}\right), S_{-h}\right)$, or equivalently, $X_h \backslash R_h\left(br_h\left(S_{-h}\right), S_{-h}\right) \subseteq X_h \backslash R_h\left(br_h\left(S'_{-h}\right), S'_{-h}\right)$. We prove the claim by contradiction. Let $x \in C_D\left(br_h\left(S_{-h}\right) \cup \bigcup_{h' \neq h} S_{h'} \cup \{x\}\right) \cap X_h$, and assume $x \notin C_D\left(br_h\left(S'_{-h}\right) \cup \bigcup_{h' \neq h} S'_{h'} \cup \{x\}\right)$. From Lemma 7 (vi), we have $br_h\left(S'_{-h}\right) \subseteq BR_h\left(S_{-h}\right) = br_h\left(S_{-h}\right) \cup R_h\left(br_h\left(S_{-h}\right), S_{-h}\right)$. From substitutability, $x \notin C_D\left(br_h\left(S_{-h}\right) \cup R_h\left(br_h\left(S_{-h}\right), S_{-h}\right) \cup \bigcup_{h' \neq h} S_{h'} \cup \{x\}\right)$. From IRC it follows that$C_D\left(br_h\left(S_{-h}\right) \cup R_h\left(br_h\left(S_{-h}\right), S_{-h}\right) \cup \bigcup_{h' \neq h} S_{h'} \cup \{x\}\right) =$
$= C_D\left(br_h\left(S_{-h}\right) \cup R_h\left(br_h\left(S_{-h}\right), S_{-h}\right) \cup \bigcup_{h' \neq h} S_{h'}\right) =$
$C_D\left(br_h\left(S_{-h}\right) \cup \bigcup_{h' \neq h} S_{h'}\right)$. It follows that $x \notin C_D\left(br_h\left(S_{-h}\right) \cup \bigcup_{h' \neq h} S_{h'} \cup \{x\}\right)$, reaching a contradiction. ■

Proof of Lemma 5. The set of fixed points of BR is a subset of the set of Nash equilibrium of Γ, so it suffices to show that any NE outcome is the outcome of a fixed point of BR. Let X' be an SPE outcome, and let $(Y_h)_{h \in H}$ be an NE of Γ yielding X' as outcome.

Let $h \in H$. We have $Y_h = X'_h \cup Z_h$, where $Z_h \subseteq R_h\left(X'_h, X'_{-h}\right)$ (see Lemma 3). Notice $X'_h = br_h\left(Y_{-h}\right) \subseteq BR_h\left(Y_{-h}\right)$. It follows that $Y_h \subseteq BR_h\left(Y_{-h}\right)$ for all $h \in H$. Consider the sequence $T^0 = Y$, $T^{k+1} = \left(\left(BR_h\left(T^k_{-h}\right)\right)_{h \in H}\right)$ for all $k \geq 0$. Notice that, by construction, $br_h\left(T^k_{-h}\right) = X'_h$ for all $h \in H$. Because $T^0 \subseteq T^1$ and BR is increasing, the sequence $\left(T^k\right)_{k \geq 0}$ is increasing in k. Because X is finite, $K \geq 0$ exists such that $T^K = T^s$ for all $s \geq K$. It follows that T is a fixed point of BR yielding X' as an outcome. ■

Proof of Proposition 4. (i) First, we show two preliminary results.

(a) Let $(A_h)_{h\in H}$ and $(B_h)_{h\in H}$ be fixed points of BR such that $A_h \subseteq B_h$ for all $h \in H$. We show $X^B = C_D\left(X^A \cup X^B\right)$, where $X^A = g\left((A_h)_{h\in H}\right)$ and $X^B = g\left((B_h)_{h\in H}\right)$. Let $A = \bigcup_{h\in H} A_h$ and $B = \bigcup_{h\in H} B_h$ and notice $A \subseteq B$. We have $X^B = C_D(B) = C_D(A \cup B)$. Because $X^A \cup X^B \subseteq A \cup B$, we have $X^B = C_D\left(X^A \cup X^B\right)$.

(b) Let $(A_h)_{h\in H}$ and $(B_h)_{h\in H}$ be fixed points of BR, and let $X^A = g\left((A_h)_{h\in H}\right)$ and $X^B = g\left((B_h)_{h\in H}\right)$. Assume $X^B = C_D\left(X^A \cup X^B\right)$. We show $A_h \subseteq B_h$ for all $h \in H$. Notice $X^A = C_H\left(X^A \cup X^B\right)$ (see Pepa Risma, 2015). Let $x \in X^A \setminus X^B$. We prove $x \in BR_h(B_{-h})$. Let $h = x_H$ and let $d = x_D$. By substitutability of $\succ_h$, $x \in C_h\left(X^B \cup \{x\}\right)$. From the pairwise stability of X^B $x \notin C_d\left(X^B \cup \{x\}\right)$. Because $\succ_d$ are substitutable, we have $x \notin C_d\left(X^B \cup \bigcup_{h'\neq h} B_{h'}\right)$; thus, $x \in R_h\left(X^B, B_{-h}\right) \subseteq BR_h(B_{-h})$.
Now let $x \in R_h\left(X^A, A_{-h}\right) \cap X_h$ and let $d = x_D$. We have
$C_d\left(X^A \cup \bigcup_{h'\neq h} A_{h'} \cup \{x\}\right) = X^A_h$ so $X^A_d = C_d\left(X^A \cup \{x\}\right)$. From the substitutability of $\succ_d$, we obtain $x \notin C_d\left(X^A \cup X^B \cup \{x\}\right) = X^B$ as $C_d\left(X^A \cup X^B \cup \{x\}\right) = X^B$ from IRC. It follows that $x \notin C_d\left(X^B \cup \{x\}\right)$. Again, the substitutability of $\succ_d$ implies $x \notin C_d\left(X^B \cup \bigcup_{h'\neq h} B_{h'}\right)$; thus, $x \in R_h\left(X^B, B_{-h}\right) \subseteq BR_h(B_{-h})$.
It follows that $A_h = X^A_h \cup R_h\left(X^A_h, A_{-h}\right) \subseteq X^B_h \cup R_h\left(X^B_h, B_{-h}\right)$ for all $h \in H$.[10]

The claim follows from (a) and (b) and Lemma 5. Notice the set of fixed points of BR is a non-empty lattice from the Tarski's Fixed Point Theorem.

(ii) The claim follows from (i) and Pepa Risma, 2015). ■

Proof of Proposition 5. Let Y be the hospital-optimal stable allocation. We will construct an equilibrium yielding Y as an outcome. If $br_h(Y_{-h}) = Y_h$ for all $h \in H$, the proof is complete. Otherwise, let $T_h \neq Y_h$ be such that $T_h = br_h(Y_{-h})$. Then $T_h = C_h(Y_h \cup T_h)$, and thus $C_h(Y_h \cup \{t\})$ is a prof-

[10]See Lemma 3

itable deviation for h for all $t \in T_h \setminus Y_h$. Let $T = \left(\bigcup_{h\in H} T_h\right) \setminus Y$.
Let $t \in T$. Because Y is pairwise stable, and preferences are substitutable and satisfy UA, $d \in D$, $y \in Y$ exist such that $y_H = t_H = h$, $y_D = t_D = d$ such that $t \in C_h(Y \cup \{t\}) \cap C_d((Y \setminus \{y\}) \cup \{t\})$ and $y \notin C_d(Y \cup \{t\})$.
Preferences $\succ_H$ satisfy the law of aggregate demand; thus, $|C_h(Y)| \le |C_h(Y \cup \{t\})|$. Because Y is individually rational and $y \notin C_h(Y \cup \{t\})$, then $C_h(Y \cup \{t\}) = (Y_h \setminus \{y\}) \cup \{t\}$. Observe that $C_d((Y \setminus \{y\}) \cup \{t\}) = (Y_d \setminus \{y\}) \cup \{t\}$, because preferences are substitutable and $C_d(C_d(Y \cup \{t\})) = Y$.
Next consider $Z = (Y \setminus \{y\}) \cup \{t\}$. Because preferences are substitutable, allocation Z is individually rational and $Z \succ_H Y$. In particular, Z is not stable. Because Y is stable, $t' = t'(t) \in X$ and $h' \neq h$ exist such that $t' \in C_{h'}(Y \cup \{t\}) \cap C_d((Y_d \setminus \{y\}) \cup \{t, t'\})$ and $t \notin C_d((Y_d \setminus \{y\}) \cup \{t, t'\})$.
Set $Y' = Y \cup \bigcup_{t\in T} t'(t)$. Let $S_h = Y'_h$ for all $h \in H$. Observe that, by construction, $Y_h = br_h(S_{-h})$. Furthermore, $t'(t) \in R_h\left(Y'_h, Y'_{-h}\right)$ because Y is stable and $\succ_D$ are substitutable. It follows that $(S_h)_{h\in H}$ is an NE of Γ yielding Y as an outcome, which completes the proof of the claim. ■

Proof of Proposition 7. Let (S^*_d) be a NE of Γ when the preferences of the students are given by $\succ_D$. For all $d \in D$, let $\succ^*_d$ such that $Y^1_{\succ^*_d} = S^*_d$ and $Y^r_{\succ^*_d} = \emptyset$ for all $r > 1$. Because (S^*_d) is an NE of Γ , $\succ^*_D$ is an NE of the game induced by the immediate acceptance mechanism.

Now let $\succ^*_D$ be an NE of $\mathcal{IA}$ yielding allocation Y as the outcome. For all $d \in D$ and $h \in H$, let r_d be the step of the algorithm where doctor d was removed. For all $d \in D$, let $S^*_d = Y^r_d$. We prove $(S^*_d)_{d\in D}$ is an NE of Γ yielding Y as the outcome.

Let Z be the outcome of $(S^*_d)_{d\in D}$ in Γ. By construction, $Z \subseteq Y$. Because $\succ_H$ is responsive, Y is individually rational and $Z = Y$.

By contradiction, assume $(S^*_d)_{d\in D}$ is not an NE of Γ. Then $d \in D$ exists such that $C_d\left\{x \mid x \in C_H\left(\bigcup_{d'\neq D} Y_{d'} \cup \{x\}\right)\right\} \neq Y_d$. Let $\succ'_d$ such that $Y^1_{\succ'_d} = C_d\left\{x \mid x \in C_H\left(\bigcup_{d'\neq D} Y_{d'} \cup \{x\}\right)\right\}$ and $Y^r_{\succ'_d} = \emptyset$ for all $r > 1$. The deviation $\succ'_d$ is a profitable deviation from strategy $\succ^*_d$ when all other agents play $\succ^*_{-d}$

in game $\mathcal{IA}$, which yields a contradiction. ∎

Proof of Proposition 8. We first prove that if $(\succ_d^*)_{d\in D}$ is an NE of $\mathcal{OS}$, then $\left(Y^1_{\succ_d^*}\right)_{d\in D}$ is an NE of Γ. Notice $OS\left((\succ_d^*)_{d\in D}\right) = g\left(\left(Y^1_{\succ_d^*}\right)_{d\in D}\right)$. The proof is by contradiction. Assume $\left(Y^1_{\succ_d^*}\right)_{d\in D}$ is not an NE of Γ. Then $d' \in D$ and $Y' \subseteq X_{d'}$ exist such that $\left(g\left(Y', \left(Y^1_{\succ_d^*}\right)_{d\neq d'}\right)\right)_{d'} \succ_{d'} \left(g\left((A(\succ_d^*))_{d\in D}\right)\right)_{d'}$, where g is the outcome function of Γ. Let $\succ'_{d'}$ be any preference profile for d' where the set Y' is ranked first, which is $Y^1_{\succ'_{d'}} = Y'$. Then $\succ'_{d'}$ is a profitable deviation for d' in game $\mathcal{OR}$, yielding a contradiction.

Next, we prove that if $(S_d)_{d\in D}$ is an NE of Γ yielding allocation Y as the outcome, then an NE of Γ^*, $\left((\succ_d^*)_{d\in D}\right)$ exists yielding $g\left((S_d)_{d\in D}\right)$. For all $d \in D$, let $\succ_d^*$ be any preference profile on 2^{X_h} such that S_d is ranked first, which is $Y^1_{\succ_d^*} = S_d$ for all $d \in D$. Then $OS\left((\succ_d^*)_{d\in D}\right) = g\left((S_d)_{d\in D}\right)$. We next prove by contradiction that $\left((\succ_d^*)_{d\in D}\right)$ is an NE of Γ^*. Assume $(\succ_d^*)_{d\in D}$ is not an NE of Γ^*. Then $d' \in D$ and $\succ'_{d'}$ exist such that $\left(OS\left(\succ'_{d'}, (\succ_d^*)_{d\neq d'}\right)\right)_{d'} \succ_{d'} \left(OS\left((\succ_d^*)_{d\in D}\right)\right)_{d'}$. It follows that $\left(g\left(Y^1_{\succ_{d'}}, S_{-d}\right)\right)_{d'} \succ_{d'} \left(g\left((S_d)_{d\in D}\right)\right)_{d'}$, yielding a contradiction.

Thus, claim (a) follows from Proposition 1, claim (b) follows from Corollary 1, claim (c) follows from Proposition 3, claim (d) follows from Theorem 4, and claim (e) follows from Proposition 5. ∎