Euler equation with a latent trend

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Abstract

We consider a model of intertemporal arbitrage of nonstationary stochastic output in which accumulated assets are nonnegative. The presence of a deterministic trend in stochastic output can induce singular behavior in the nonstationary Euler equation, posing a challenge to econometric estimation. The productivity trend is implicit only in the jumps expected when the non-negativity constraint is binding. The predictor does not satisfy standard conditions for proofs of consistency. We prove strong consistency of an estimator of key parameters, that takes account of interactions between the parameters, in contrast to econometric approaches where the trend is estimated in a preliminary stage.

Keywords: Euler equation, trend, strong consistency.

JEL: C13, C18, C51.

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1 Introduction

Empirical work in economics often involves estimation of Euler equations with trending marginal value. However, when intertemporal arbitrage is active, the behavior of expectations of marginal values obeys the intertemporal Euler condition, regardless of the trend. In this paper we consider a dynamic model of asset accumulation via intertemporal arbitrage of output with a deterministic trend subject to a random productivity shock with finite support. Marginal value is bounded above, implying an occasionally binding non-negativity constraint on accumulated assets. The productivity trend is implicit in the nonstationary Euler equation; its effects are revealed in jumps in expected marginal value when accumulated assets are zero. Such a trend can induce singular behavior in the Euler equation, posing a challenge to econometric estimation of its parameters.

In introducing the effects of time we consider a productivity trend such that marginal value M_t is related to detrended marginal value m_t by a simple exponential trend, $M_t = \lambda^t m_t$, with $\lambda > 0$. Thus in this model, described in Section 2, there is a trending threshold $\lambda^t m^*$ in the Euler equation, such that:

$$E_t M_{t+1} = (1+r)\min\{\lambda^t m^*, M_t\} \tag{1}$$

where M_{t+1} and M_t are marginal values at periods $t+1$ and t respectively, r is the interest rate, and E_t is the expectation conditional on information at time t.[1]

Remark 1. Equation (1) implies that the constant relative marginal value trend (induced by the deterministic growth in productivity) is implicit in the expected relative marginal value change only in the jumps expected when current assets are zero. These jumps occur at random (when available supply is sufficiently low), and their magnitude depend on current marginal value, calendar time t, the trend parameter λ, and a detrended threshold m^* which itself depends upon λ.

To ensure identification in the empirical regression, we work with relative marginal values M_{t+1}/M_t. The conditional expectation for relative marginal value is:

$$E_t\left(\frac{M_{t+1}}{M_t}\right) = (1+r)\min\left\{\frac{\lambda^t m^*}{M_t},\ 1\right\} \tag{2}$$

We prove strong consistency of the least squares estimators of the trend parameter, the threshold marginal value, and the interest rate. Denote by (λ_0, m_0^*, r_0) the true (unknown) values of these three parameters. Equation (2) implies the following regression:

[1]For a buffer stock model of saving with nonnegative liquidity constraints, if the utility function is of the constant relative risk aversion family with risk aversion coefficient ρ, then equation (1) can be written as $E_t C_{t+1}^{-\rho} = (1+r)\min\{\lambda_0^t C^{*-\rho}, C_t^{-\rho}\}$, where C_t and C_{t+1} are consumption at times t and $t+1$ respectively, and C^* is a threshold consumption level. In this case the *i.i.d.* shocks to income are proportional to the deterministic income trend. For a model of a storable commodity with non-negativity constraints on stocks, M_t is the price of the commodity at period t. In this case, since prices are observed directly, the structure of the predictor is robust to the specification of consumption demand.

$$\frac{M_{t+1}}{M_t} \;=\; f_t(\lambda_0, m_0^*, r_0) \;+\; \xi_{t+1} \tag{3}$$

where $\{\xi_{t+1}\}_{t\in\mathbb{N}}$ is a martingale difference sequence, and the predictor $f_t(\lambda, m^*, r)$ is:

$$f_t(\lambda, m^*, r) \;=\; (1+r)\min\left\{\left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\; 1\right\}.$$

Strong consistency of least squares estimators in non-linear regression models has been studied extensively (see for example Jennrich 1969, Wu 1981, Lai 1994, Skouras 2000). The predictors in model (3) have singular asymptotic behavior depending on whether $\lambda < \lambda_0$, or $\lambda > \lambda_0$. They do not satisfy the conditions for consistency in Jennrich (1969), Wu (1981), Lai (1994) or Skouras (2000), nor the conditions for the uniform laws of large numbers of Andrews (1987), or of Pötscher and Prucha (1989, 1994).

2 The model

We specify an underlying persistent exogenous output disturbance as a deterministic trend. We assume output is exogenous. In our model this output trend is assumed to be such that it implies a deterministic exponential trend in the marginal value M_t :

$$M_t = \lambda_0^t\, m_t \tag{4}$$

where $\lambda_0 > 0$ and m_t is detrended marginal value. Trending marginal value M_t and detrended marginal value m_t are given by $M_t = M(C_t)$ and $m_t = M(c_t)$, respectively, where M is the marginal value function, and C_t and c_t denote consumption and detrended consumption, respectively.

Assume that M is strictly decreasing, and satisfies:

$$\frac{M''M}{(M')^2} = \kappa \tag{5}$$

where κ is a constant.[2] Integrating (5) we obtain $M(C) = (A + BC)^{\frac{1}{1-\kappa}}$ if $\kappa \neq 1$, and $M(C) = e^{A+BC}$ if $\kappa = 1$, where A and B are constants. Thus our specification includes linear, log-linear, and iso-elastic marginal value function functions as particular cases, given appropriately chosen values for A and B.

Assume that detrended output h_t are *i.i.d.*, with $EM(h) < \infty$, where E denotes the expectation with respect to h.

Begin with the case in which asset accumulation is by assumption infeasible. In this case consumption C_t equals output H_t, detrended consumption c_t equals detrended output h_t, and

[2] M has the form of a derivative of a Hyperbolic Absolute Risk Aversion (HARA) utility function. For a detailed discussion of HARA functions, see the Appendix in Carroll and Kimball (1996).

therefore detrended marginal value is $m_t = M(h_t)$. Equation (4) then implies that the logarithm of the marginal value of output follows a linear trend with $i.i.d.$ shocks:

$$M(H_t) = \lambda_0^t M(h_t) \tag{6}$$

Equation (6) implies that $E_t M_{t+1} = E_t M(H_{t+1}) = \lambda_0^{t+1} EM(h)$, where E denotes the expectation with respect to h.

We now allow for non-negative asset accumulation. We assume there is a constant interest rate $r_0 > 0$. Expected value maximization implies that, when assets are positive,

$$E_t M_{t+1} = (1 + r_0) M_t,$$

thus the percentage spread between the expectation of marginal value in the next period and current marginal value is r_0, regardless of the output trend. However, the assumption of bounded $EM(h)$ implies the eventual occurrence of zero assets.

Let Z_t denote total accumulated output at time t. Price obeys the following Euler condition:

$$M(C_t) \;=\; \max\Big\{M(Z_t),\ \frac{1}{1+r_0} E_t M(C_{t+1})\Big\},\ \text{s.t.} \tag{7}$$

$$Z_{t+1} \equiv Z_t - C_t + H_{t+1},\ \forall\, t \in \mathbb{N} \tag{8}$$

The model has a normalized representation, with a stationary rational expectations equilibrium. More in detail, equations (7) and (8) have the following normalized counterparts, with detrended total accumulated output z_t implicitly defined by $M(Z_t) = \lambda_0^t M(z_t)$:

$$M(c_t) \;=\; \max\Big\{M(z_t),\ \frac{\lambda_0}{1+r_0} E_t M(c_{t+1})\Big\},\ \text{s.t.} \tag{9}$$

$$z_{t+1} \equiv \lambda_0^{\kappa-1}(z_t - c_t) + h_{t+1}{}^{3} \tag{10}$$

If $\lambda_0 < 1 + r_0$, then standard arguments imply the existence of a stationary detrended marginal value function m :

$$m_t = m(z_t) \;=\; \max\Big\{M(z_t),\ \frac{\lambda_0}{1+r_0} E_t\, m(z_{t+1})\Big\} \tag{11}$$

The proof of Theorem 4.3 in Mendelson and Amihud (1982) can be used to show that m is non-negative, continuous, strictly decreasing, and that the following complementary inequalities hold:

$$\begin{aligned} m(z) = M(z), &\quad \text{for} \quad z \le M^{-1}(m_0^*), \\ m(z) > M(z), &\quad \text{for} \quad z > M^{-1}(m_0^*), \end{aligned}$$

[3]Equation (10) is obtained from (8) using the following property of M: for any $K > 0$ and $y \in \mathbb{R}$ that allows M to be well defined, $M^{-1}(KM(y)) = \left[K^{1-\kappa}(By + A) - A\right]/B$ if $\kappa \neq 1$, and $M^{-1}(KM(y)) = \ln(K)/B + y$ if $\kappa = 1$.

where $m_0^* \equiv \left(\dfrac{\lambda_0}{1+r_0}\right) Em(\omega) \in \mathbb{R}$.

Equation (11) implies the following autoregression for detrended marginal values:

$$E_t m(z_{t+1}) \;=\; \left(\frac{1+r_0}{\lambda_0}\right)\min\Big\{m_0^*,\; m(z_t)\Big\} \tag{12}$$

Remark 2. Equation (12) implies that the detrended marginal value process given $\lambda_0 \neq 1$ differs from the marginal value process if λ_0 was equal to 1, because the value of the trend parameter affects the location of the threshold marginal value m_0^*.

Multiplying (12) by λ_0^{t+1}, we obtain the autoregression expressed in terms of observable marginal values:

$$E_t M_{t+1} \;=\; (1+r_0)\min\Big\{\lambda_0^t m_0^*, M_t\Big\} \tag{13}$$

3 Strong consistency of estimators

For estimation that uses information on marginal values, in this section we prove strong consistency of nonlinear least squares estimation of three key parameters of the model: the trend parameter, λ_0, the detrended threshold marginal value, m_0^*, and the interest rate, r_0.

For our consistency proof, we now add the assumption that the distribution of detrended output h_t has an absolutely continuous part with continuous and strictly positive derivative, on the interior of its support $[\underline{h}, \overline{h}]$, with $-\infty < \underline{h} < \overline{h} < \infty$. The proof of the Theorem in Bobenrieth, Bobenrieth and Wright (2008) can then be used to show that the detrended marginal value process $\{m_t\}_{t\in\mathbb{N}}$ is aperiodic and positive Harris recurrent, implying that it has a unique invariant distribution which is a global attractor. We assume that the invariant distribution for the detrended marginal value process has support $[\underline{m}, \overline{m}]$, with $0 < \underline{m} < \overline{m} < \infty$.[4]

Equation (13) implies:

$$M_{t+1} \;=\; (1+r_0)\min\Big\{\lambda_0^t m_0^*,\; M_t\Big\} \;+\; e_{t+1}, \text{ where } E_t(e_{t+1}) = 0 \tag{14}$$

The parameters of interest are $\theta = (\lambda, m^*, \gamma)$, where $\gamma \equiv 1 + r$. We assume that the parameter space Θ is compact.

Our objective is to estimate θ_0 using least squares. Note that if $0 < \lambda_0 < 1$, we cannot identify θ_0 in (14). Indeed, for $\mu \neq \theta_0$, there exists a ball $B(\mu)$ centered at μ such that $g_t(\theta) \equiv \gamma \min\{\lambda^t m^*,\; M_t\}$ satisfies:

$\inf_{\theta\in B(\mu)} \sum_{t=1}^{T}\{g_t(\theta) - g_t(\theta_0)\}^2 \leq \sum_{t=1}^{+\infty} \lambda_0^t\, \kappa < \infty$, where $\kappa < \infty$.

To avoid this problem, we divide the regression model (14) by M_t :

[4] For linear consumption demand, Bobenrieth and Bobenrieth (2010) specify an upper bound for $\overline{h}$ to guarantee $m_t \geq \underline{m} > 0$, for all t.

$$\frac{M_{t+1}}{M_t} \;=\; \gamma_0 \min\left\{\frac{\lambda_0^t m_0^*}{M_t},\; 1\right\} \;+\; \xi_{t+1} \tag{15}$$

where $\xi_{t+1} \equiv \dfrac{e_{t+1}}{M_t}$.

Define the predictor $f_t(\theta) = \gamma \min\left\{\left(\dfrac{\lambda}{\lambda_0}\right)^t \dfrac{m^*}{m_t},\; 1\right\}$.

Therefore, the normalized regression model is:

$$Y_{t+1} \;=\; f_t(\theta_0) \;+\; \xi_{t+1}, \quad \text{where} \quad Y_{t+1} = \frac{M_{t+1}}{M_t} \;= \lambda_0 \left(\frac{m_{t+1}}{m_t}\right) \tag{16}$$

Given $\mu \neq \theta_0$, let $A_T \equiv \inf\limits_{\theta \in B(\mu)} \sum\limits_{t=1}^{T} \{f_t(\theta) - f_t(\theta_0)\}^2$.

Remark 3. The predictors do not satisfy the Lipschitz conditions for consistency in Wu (1981). Moreover, $f_t(\theta)$ does not satisfy the Lipschitz conditions of Andrews (1988), the continuity-type smoothness conditions of Pötscher and Prucha (1989, 1994), or the Lipschitz L_1-conditions $(3.10) - (3.11)$ of Skouras (2000). The predictor is non-differentiable at the $2-$dimensional set

$$S_t \equiv \left\{(\lambda, m^*, \gamma) \;:\; \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t} = 1\right\},$$

and therefore does not satisfy the conditions for consistency in Lai (1994).

Using the fact that the marginal value process has a unique invariant distribution which is a global attractor, our next result establishes that A_T diverges to infinity, thus identifying θ_0. Furthermore, the rate of divergence is at least T :

Theorem 1. *Given $\mu \neq \theta_0$, there exists a constant $b > 0$, and a $T_1 \in \mathbb{N}$, $T_1 = T_1(\{m_t\}_{t \in \mathbb{N}})$, such that:*

$$A_T \;\geq\; bT, \quad \textit{for all} \quad T \geq T_1, \quad \textit{with probability one.}$$

Proof of Theorem 1. Appendix A.

Define $\hat{\theta}_T$ to be the least squares estimator of θ_0, that is,

$$\hat{\theta}_T \;\equiv\; \text{Argmin}_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \left(Y_{t+1} - f_t(\theta)\right)^2 .$$

Our next result establishes that the least squares estimator for θ_0 is strongly consistent.

Remark 4. The term $(\lambda/\lambda_0)^t$ in the predictor implies that the objective function in the least squares minimization does not converge uniformly in the parameter space; our objective function does not satisfy the unform convergence condition of Jennrich (1969).

Theorem 2. *$\hat{\theta}_T$ is strongly consistent, that is,*

$$\lim_{T\to\infty} ||\hat{\theta}_T - \theta_0|| = 0, \qquad a.s.$$

Proof of Theorem 2. Appendix B.

4 Conclusion

When a deterministic productivity trend is included in a model of intertemporal arbitrage subject to non-negativity constraints, the predictor of marginal value displays two distinct modes of behavior. The expected marginal value change is either the gross rate of interest times the current marginal value, or is a jump equal to the difference between the current marginal value and a time-related threshold.

The predictors in this model have singular asymptotic behavior. They do not satisfy sets of conditions presented in the literature, including Jennrich (1969), Wu (1981), Andrews (1988), Pötscher and Prucha (1989, 1994), Lai (1994), and Skouras (2000), which are used in standard proofs of consistency. Our results make it possible to use an estimator that takes account of the interactions between estimators of the trend parameter, the interest rate and the threshold marginal value, in contrast to econometric approaches where the trend parameter is estimated in a preliminary stage.

Predictors presenting the challenges addressed here are encountered, for example, in empirical models of consumption or price behavior that recognize trends in stochastic productivity or consumer demand, as well as occasionally binding liquidity constraints.

Acknowledgments

We thank participants at the Workshop on Electricity, Energy and Commodities Risk Management 2013 at The Fields Institute for Research in Mathematical Science,Toronto, Canada, and Ivar Ekeland, for useful comments and discussion. Juan Bobenrieth is associate professor at Departamento de Matemática, Universidad del Bío-Bío. Eugenio Bobenrieth is professor at Departamento de Economía Agraria and Instituto de Economía, Pontificia Universidad Católica de Chile, and research fellow at Finance UC. Brian Wright is professor at Department of Agricultural and Resource Economics, University of California, Berkeley and member, Giannini Foundation. Work on this article was supported by the Energy Biosciences Institute, and by CONICYT/Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Project 1130257. Eugenio Bobenrieth's research for this paper was initiated when he was a professor at Universidad de Concepción, Chile. Eugenio Bobenrieth acknowledges partial financial support from Vicerrectoría Académica, Pontificia Universidad Católica de Chile, from The Fields Institute for Research in Mathematical Science, and from Project NS 100046 of the Iniciativa Científica Milenio of the Ministerio de Economía, Fomento y Turismo, Chile. We acknowledge the excellent research assistance of Di Zeng.

Appendix A. Proof of Theorem 1. Let $\mu \equiv (\lambda_\mu, m^*_\mu, \gamma_\mu) \neq \theta_0$. Consider the non-trivial case where $(\lambda_\mu, \gamma_\mu) = (\lambda_0, \gamma_0)$. Then, $m^*_\mu \neq m^*_0$. Without loss of generality we assume $m^*_\mu > m^*_0$. For m^* close enough to m^*_μ, for appropriately chosen values of m_t on its ergodic support such that $\dfrac{m^*}{m_t} > 1 > \dfrac{m^*_0}{m_t}$, and for γ close enough to $\gamma_\mu = \gamma_0$, we have:

$$|f_t(\theta) - f_t(\theta_0)| = \left|\gamma \min\left\{\left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1\right\} - \gamma_0 \min\left\{\frac{m^*_0}{m_t},\ 1\right\}\right|$$

$$= \left|\gamma \min\left\{\left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1\right\} - \gamma_0 \frac{m^*_0}{m_t}\right| \quad \geq \quad \left\{\frac{\gamma_0}{2}\right\} \left|\min\left\{\left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1\right\} - \frac{m^*_0}{m_t}\right|.$$

If $\lambda > \lambda_0$, then:

$$|f_t(\theta) - f_t(\theta_0)| \ \geq \ \left\{\frac{\gamma_0}{2}\right\} \left|1 - \frac{m^*_0}{m_t}\right| \geq a_1 > 0, \quad \text{where } a_1 \text{ is a constant.}$$

If $\lambda \leq \lambda_0$, then $|f_t(\theta) - f_t(\theta_0)| \ \geq \ a_1$, or:

$$|f_t(\theta) - f_t(\theta_0)| \ \geq \ \left\{\frac{\gamma_0}{2}\right\} \left|\left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t} - \frac{m^*_0}{m_t}\right| \ \geq \left\{\frac{\gamma_0}{2\overline{p}}\right\} \left|\left(\frac{\lambda}{\lambda_0}\right)^t m^* - m^*_0\right|.$$

A straightforward calculation shows for arbitrary constant $0 < \phi < 1$, we have that for all $t \in \mathbb{N}$, except for a finite number of t :

$$\left|\left(\frac{\lambda}{\lambda_0}\right)^t m^* \ - \ m^*_0\right| \ \geq \ \phi m^*_0.$$

In fact:

$$\left(\frac{\lambda}{\lambda_0}\right)^t m^* \;-\; m_0^* \;\geq\; \phi m_0^* \iff t \leq \frac{\ln\left((\phi+1)\frac{m_0^*}{m^*}\right)}{\ln\left(\frac{\lambda}{\lambda_0}\right)},$$

and

$$\left(\frac{\lambda}{\lambda_0}\right)^t m^* \;-\; m_0^* \;\leq\; -\phi m_0^* \iff t \geq \frac{\ln\left((-\phi+1)\frac{m_0^*}{m^*}\right)}{\ln\left(\frac{\lambda}{\lambda_0}\right)}.$$

Choosing small enough $\phi \in (0,1)$ and small enough radius of the ball $B(\mu)$, by the ergodicity of the marginal value process $\{m_t\}_{t\in\mathbb{N}}$, we conclude that there exists a constant $b > 0$, and a $T_1 \in \mathbb{N}$, $T_1 = T_1(\{m_t\}_{t\in\mathbb{N}})$, with:

$$T \geq T_1 \implies \frac{1}{T} \inf_{\theta\in B(\mu)} \sum_{t=1}^{T} \{f_t(\theta) - f_t(\theta_0)\}^2 \;\geq\; b > 0. \qquad Q.E.D.$$

Appendix B. Proof of Theorem 2. Let $\mu \;\equiv\; (\lambda_\mu, m_\mu^*, \gamma_\mu) \;\neq \theta_0$, and $B(\mu)$ a ball centered at μ. The strong consistency of $\hat{\theta}_T$ follows from the following uniform strong law of large numbers:[5]

$$\lim_{T\to\infty} \frac{1}{A_T} \sup_{\theta\in B(\mu)} \left| \sum_{t=1}^{T} \xi_{t+1}\{f_t(\theta) - f_t(\theta_0)\} \right| \;=\; 0, \qquad a.s. \tag{17}$$

Considering the facts that γ is in a bounded set, that $A_T \geq bT$ (Theorem 1), and that $\{\xi_{t+1}\}_{t\in\mathbb{N}}$ is a martingale difference sequence, it suffices to prove:

$$\lim_{T\to\infty} \frac{1}{T} \quad \sup_{(m^*,\lambda)\in B(m_\mu^*,\lambda_\mu)} \left| \sum_{t=1}^{T} \xi_{t+1} \;\min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\; 1 \right\} \right| \;=\; 0, \qquad a.s. \tag{18}$$

where $B(m_\mu^*, \lambda_\mu) \;\equiv\;]m_\mu^* - \zeta, m_\mu^* + \zeta[\; \times\;]\lambda_\mu - \zeta, \lambda_\mu + \zeta[$.

We divide the proof of (18) in two Lemmata.

For any given $T \in \mathbb{N}$, consider in the rectangle $B(m_\mu^*, \lambda_\mu)$, a grid of $\lceil 4\,\zeta\, T^4 \rceil$ dots,[6] defined by:

$$B^{(T)}(m_\mu^*, \lambda_\mu) \;\equiv\; \left\{ \left(m_\mu^* + \frac{i}{T^2},\; \lambda_\mu + \frac{j}{T^2} \right) \;:\; i,\; j \in \{0, \pm 1, \pm 2, \cdots, \pm\lceil \zeta T^2 \rceil\} \right\}.$$

[5]See for example, Wu (1981, Lemma 1, p. 504), Lai (1994, p. 1927), and Skouras (2000, pp. 878-879).

[6]For $x \in \mathbb{R}$, we denote by $\lceil x \rceil$ the integer part of x, that is, the greatest integer $\leq$ x.

Lemma B.1 presents the proof of (18) for $(m^*, \lambda) \in B^{(T)}(m^*_\mu, \lambda_\mu)$.[7] Lemma B.2 extends the result for $(m^*, \lambda) \in B(m^*_\mu, \lambda_\mu)$.

Lemma B.1.

$$\lim_{T\to\infty} \sup_{(m^*,\lambda)\in B^{(T)}(m^*_\mu,\lambda_\mu)} \frac{1}{T} \left| \sum_{t=1}^{T} \xi_{t+1} \min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1 \right\} \right| = 0, \quad \text{a.s.} \tag{19}$$

Proof of Lemma B.1.

Since $\{\xi_{t+1}\}_{t\in\mathbb{N}}$ is a martingale difference sequence, we conclude that for any given (m^*, λ) :

$$\left\{ \xi_{t+1} \min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1 \right\} \right\}_{t\in\mathbb{N}}$$

is a martingale difference sequence. Observing that this martingale sequence is bounded by a finite constant $\bar{\xi}$. Using Azuma's inequality (Azuma, 1967) we conclude that for any (m^*, λ), for any $\rho > 0$, and for all $T \in \mathbb{N}$:

$$\text{Prob}\left[\frac{1}{T} \left| \sum_{t=1}^{T} \xi_{t+1} \min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1 \right\} \right| \geq \rho \right] \leq 2 \exp\left[\frac{-\rho^2 T}{2\, \bar{\xi}^2} \right] \tag{20}$$

where the upper bound in (20) is independent of (m^*, λ). Since there are $\lceil 4\, \zeta\, T^4 \rceil$ points in $B^{(T)}(m^*_\mu, \lambda_\mu)$, (20) implies that for any $\rho > 0$, and for all $T \in \mathbb{N}$:

$$\text{Prob}\left[\max_{(m^*,\lambda)\in B^{(T)}(m^*_\mu,\lambda_\mu)} \frac{1}{T} \left| \sum_{t=1}^{T} \xi_{t+1} \min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1 \right\} \right| \geq \rho \right] \leq 8\, \zeta\, T^4\, e^{-\frac{\rho^2 T}{2\, \bar{\xi}^2}} \tag{21}$$

From (21) and the Borel-Cantelli Lemma, we conclude that with probability one:

$$\sup_{(m^*,\lambda)\in B^{(T)}(m^*_\mu,\lambda_\mu)} \frac{1}{T} \left| \sum_{t=1}^{T} \xi_{t+1} \min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1 \right\} \right| \to 0 \qquad Q.E.D.$$

Lemma B.2.

$$\sup_{(m^*,\lambda)\in B(m^*_\mu,\lambda_\mu)} \frac{1}{T} \left| \sum_{t=1}^{T} \xi_{t+1} \min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1 \right\} \right| \to 0, \quad \text{a.s.} \quad (\text{as } T \to \infty).$$

[7] This partition technique is presented in Kundu (1993), for a trigonometric regression model with Gaussian innovations.

Proof of Lemma B.2. Let $\varphi_t(\lambda, m^*) \equiv \min\left\{ \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t},\ 1 \right\}$. First, note that there exist finite positive constants $\alpha,\ \beta,$ such that for any $(\lambda_1, m_1^*),\ (\lambda_2, m_2^*)$:

$$|\varphi_t(\lambda_1, m_1^*) - \varphi_t(\lambda_2, m_2^*)| \ \leq \ t\,\alpha|\lambda_1 - \lambda_2| \ + \ \beta\,|m_1^* - m_2^*| \tag{22}$$

Indeed,

i) If $\varphi_t(\lambda_1, m_1^*) = \left(\frac{\lambda_1}{\lambda_0}\right)^t \frac{m_1^*}{m_t}$, and $\varphi_t(\lambda_2, m_2^*) = \left(\frac{\lambda_2}{\lambda_0}\right)^t \frac{m_2^*}{m_t}$, then applying the mean value theorem to $g_t(\lambda, m^*) \equiv \left(\frac{\lambda}{\lambda_0}\right)^t \frac{m^*}{m_t}$, we conclude:

$$|\varphi_t(\lambda_1, m_1^*) - \varphi_t(\lambda_2, m_2^*)| \ \leq \ \left(\frac{t}{\underline{\lambda}}\right) \left(\frac{\overline{m^*}}{\underline{m^*}}\right) \Big|\lambda_1 - \lambda_2\Big| \ + \ \left(\frac{1}{\underline{m^*}}\right) \Big|m_1^* - m_2^*\Big|,$$

where $\underline{\lambda},\ \underline{m^*},\ \overline{m^*},$ denote the minimum and the maximum values of the corresponding parameters.

ii) If $\varphi_t(\lambda_1, m_1^*) = \left(\frac{\lambda_1}{\lambda_0}\right)^t \frac{m_1^*}{m_t}$, and $\varphi_t(\lambda_2, m_2^*) = 1$, by continuity of $\phi_t(\sigma) \equiv \left(\frac{\lambda_1 + \sigma(\lambda_2 - \lambda_1)}{\lambda_0}\right)^t \left(\frac{m_1^* + \sigma(m_2^* - m_1^*)}{m_t}\right)$, and the fact that $\phi_t(0) \leq 1 \leq \phi_t(1)$, there exists $(\tilde{\lambda}, \tilde{m^*})$ between (λ_1, m_1^*) and (λ_2, m_2^*), with $1 = \left(\frac{\tilde{\lambda}}{\lambda_0}\right)^t \frac{\tilde{m^*}}{m_t}$. We now repeat the argument in *i)* to show (22).

For any given $(\lambda, m^*) \in B(m_\mu^*, \lambda_\mu)$, and any given $T \in \mathbb{N}$, choose a point $(\lambda^{(T)}, m^{*(T)})$ in the grid $B^{(T)}(m_\mu^*, \lambda_\mu)$ such that:

$$|m^* - m^{*(T)}| \leq \frac{1}{T^2}, \quad |\lambda - \lambda^{(T)}| \leq \frac{1}{T^2} \tag{23}$$

Then:

$$\frac{1}{T}\left|\sum_{t=1}^{T} \xi_{t+1}\,\varphi_t(\lambda, m^*)\right| = \frac{1}{T}\left|\sum_{t=1}^{T} \xi_{t+1}\ \big(\varphi_t(\lambda, m^*) - \varphi_t(\lambda^{(T)}, m^{*(T)}) + \varphi_t(\lambda^{(T)}, m^{*(T)})\big)\right|$$

$$\leq \frac{1}{T}\sum_{t=1}^{T} \bar{\xi}\,|\varphi_t(\lambda, m^*) - \varphi_t(\lambda^{(T)}, m^{*(T)})| \ + \ \frac{1}{T}\left|\sum_{t=1}^{T} \xi_{t+1}\,\varphi_t(\lambda^{(T)}, m^{*(T)})\right|$$

By (22) and (23), the first term goes to zero uniformly in $(\lambda, m^*) \in B(m_\mu^*, \lambda_\mu)$, and by Lemma B.1, the second term goes to zero uniformly in $B^{(T)}(m_\mu^*, \lambda_\mu)$. *Q.E.D.*

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