

OPTIMAL MERGER RULE UNDER INCOMPLETE INFORMATION

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ABSTRACT. The purpose of this project is to characterize the optimal merger rule when a regulator does not have precise information about the firms' cost function. We assume that the regulator: maximizes consumer surplus, can either block or accept a proposed merger and sets divestitures (transfers of capital) from the merged firms to its competitors. We propose a mechanism design approach, without the usual quasi-linear payoff assumption. We first characterized the incentive-compatible rules, and then find the optimal one. The complete information case is also presented as a benchmark. For the case when $n - 1$ firms merge in one, we show the errors that regulator incurs because the firms' private information. First, among approved mergers, some will decrease consumer surplus and/or require less divestitures than the optimal one (under-fixing problem). Second, among non-approved mergers, some will increase consumer surplus and/or require not optimal amount of divestiture. In this case, there will be over-fixing problem, but also under-fixing problems.

Date: First draft: June 18, 2015. Current draft: July 5, 2016 .

Acknowledgment: I am grateful to Srihari Govindan and Paulo Borelli for their guidance and encouragement. I also thank seminar participants at University of Rochester for helpful discussions and suggestions. All errors are my own.

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1. INTRODUCTION

In a market with few competitors (oligopoly) society's welfare might be largely affected, positively or negative, by a firm's merger. On one side, a merger reduces the number of firms active in the market, reducing the competition between them and then affecting negatively the consumers. On the other side, a union of firms may allow firms to improve the production technology, obtaining efficiencies that reduce the cost of production, affecting positively the production as a whole. In general, there exist a trade-off from society's perspective that balance any reduction in competition against the possibility of productivity improvements (Williamson, 1968).

Most countries have regulations to deal with this problem, blocking mergers that are detrimental for the society. For instance, since 1989 the European Union (EU) has had a Merger Regulation. Since that date up to 2000, 1,500 mergers have been notified to and reviewed by the European Commission (EC) (Motta, 2004) ¹

A crucial issue is the asymmetric information that exists between firms and regulator. Firms have better information about a lot of factors that affect the evaluation of the merger (production cost, demand, efficiency gains, future merger possibilities, etc.). Since both efficiencies gains and competitions loss from the merger are directly related with the society's benefits and costs, it is crucial to extract as much as possible accurate information from the firms. Looking from approval, merging firms might tend to overestimate the efficiencies due to the merger. On the other hand, for opposite reasons, rival firms that might be affected negatively by the merger have the incentive to underestimate the efficiencies from the merger. Thus, from the regulator point of view, a key question is how to design a methodology to extract accurate information from the firms, and take an informed decision.²

One realistic option that the regulator has available is to use *remedies* that might possibly screen between mergers (Rey, 2003) and use as a condition to approve a particular merge (Motta and Vasconcelos, 2003)³. Remedies work establishing conditions to the merged firms

¹Only thirteen mergers have been blocked over the same period. The growth in the workload of the EC has been raised from 63 notifications in 1991 to 292 notifications in 1999.

²A possible solution is to rely on independent studies. In Neven, Nutall and Seabright (1993) is suggested that regulator should create a special unit of auditors specializing in estimating efficiency gains

³For instance, a 40 out of 345 mergers reviewed by the EC in the year 2000 were approved after remedies according to the EC report on Competition Policy, 2001

such that any potential hurt to the consumers is avoided. There are two types of possible remedies. The first, *structural remedies*, modify the allocation of property rights: they include transfers of an entire (or partial) ongoing business, or transfers of assets, which are essential for the production. The second, *behavioral remedies*, set constraints on the merged firms' property rights: they consist of engagements by the merging parties not to abuse certain assets available to them, or to enter into specific contractual arrangements.⁴

We will focus on structural remedies because from the regulator perspective these are easier to monitor and verify. For example, in 2011 two airline companies, LAN (from Chile) and TAM (from Brazil), merge in one creating the largest airline in South America. The merger was approved by the regulator in Chile subject to certain conditions such as the divestment of airport slots in the Santiago-Sao Paulo route, among others. That will give the opportunity to an existing firm, or a new one, to compete against the new firm. In another example, in 2012, two gas station chains in Chile (ENEX-Shell and Terpel) proposed a merger. The regulator condition the approval to the divestiture of 61 stations out of 500 that the new firm will have together.

Analysis of mergers in the economic literature has focused on aggregate surplus and consumer surplus. However, current U.S. law as well as the Department of Justice (DOJ) and Federal Trade Commission (FTC) Horizontal Merger Guidelines are closest to the consumer surplus standard. If this is the case, it is even more important to have an accurate estimation of efficiencies.⁵

In this paper, the main aim is to characterize the *optimal merger rule* using divestitures (structural remedy) as a tool for screening, in an environment where there exists asymmetric information between firms and the regulator (antitrust authority) about the production technology and the regulator follow a consumer surplus standard. We assume a standard Cournot competition (competition in quantities) with heterogeneous firms. From a theoretical perspective, we use a mechanism design approach, with a regulator designing an optimal mechanism to elicit information from the firms, in order to take an informed decision. As

⁴*Behavioral remedies* need continuous monitoring from the authorities, whereas *structural remedies* do not. Structural remedies might be riskier, as they are not reversible

⁵Besanko and Spulber (1993) argue that it may be better to focus in consumer surplus standard even though the true objective is the aggregate surplus. Committing to a consumer surplus criteria make that -in equilibrium- any merger proposed raises aggregate surplus.

is the case in an oligopoly competition, the outcome of the competition depends on the marginal cost of all firms. Thus, we are facing an interdependent value problem. Furthermore, divestitures are not assumed to affect linearly firms payoffs in order to be realistic. In sum, we model the situation as a mechanism design problem with interdependent values, and without the usual quasi-linear assumption in transfers. In an asymmetric information context, the power of the regulator to require remedies, in particular, transfers of capital between firms is crucial. Merging parties have no incentive to transfer assets to a firm that will be competitive in the future. Thus, the firms criteria to pick the receiver of the transfer are different than the regulators criteria.

In general, there are two types of effects due a merger. The first one, *unilateral effects*, -which is the focus in this project- considers the possibility that the new firm unilaterally exercises market power and increase prices. The second one, *coordinated effects*, consider the possibility that a merger might favor a collusion in the industry. Model mergers and their effects are complicated since we need a specification of the synergies obtained by the merger. We assume a simple way considering firms that own capital, and a merge is just the combination of units of capital, that make the new firms more productive.

We characterize an incentive compatible merger rule and find that they have the same flavour than VCG mechanism in the sense that from a particular firm point of view, information report only affects through the merged decision and divestitures are fully determined by the report of all the rival firms. In particular, when the merge has only one rival firm, divestitures do not depend on any information. Using this we find the optimal merger rule from the regulator perspective, and show existence. Then, we compare this rule with the optimal one in the complete information case.

The article proceeds as follows. We discuss the actual literature in horizontal merger in Subsection 1.1. We describe our model in Section 2. In Section 3 we derive the optimal merge rule when there are no information issues between firms and regulator (benchmark). In section 4 we characterize merger rules that are incentive compatible and derive the optimal one.

1.1. Related Literature. The seminal work that study profit and welfare effects of horizontal mergers is [Farrell and Shapiro \(1990a\)](#). In a Cournot environment, it formalized Williamson trade-off and showed under what conditions cost improvements are sufficient for

a merger to reduce the price. This is important for the regulator that follows a consumer surplus criterion. It also derives a sufficient condition when the regulator uses an aggregate surplus criterion for the increasing of the welfare when it is assumed that the merger proposed is profitable. In a related work in the same context, [Farrell and Shapiro \(1990b\)](#) assume cost functions that depend on units of capital owned by the firms, and showed how a competitive output changes when transfers of capital between firms are considered. This will be useful when we characterize the complete information case in this project.

A different perspective considers a policy-making approach and uses the enforcement aspects of antitrust. This literature started with [Besanko and Spulber \(1993\)](#). In their model, the regulator cannot observe the efficiencies due the merger, although the firms integrating the merger can observe this. They showed that it might be better to commit to a consumer surplus criterion even though the true welfare objective is aggregate surplus. The intuition is that after the pre-commitment to a rule that maximize expected consumer surplus, self-selection by the merging firms increase the average quality of the proposal mergers (profitability is positively correlated with efficiencies that the merger generates), making the regulator more willing to approve mergers in a sequential equilibrium where firms anticipate this fact.

A more recent literature in the same spirit of focusing in merger rules is [Nocke and Whinston \(2013\)](#) and [Nocke and Whinston \(2010\)](#). The first one considered a static Cournot setting, when the merger proposed is endogenous (a pivotal firm may choose after a bargaining between firms which feasible merger to propose). Here the regulator knows everything from the firms but the set of feasible merger from which the pivotal firm picked the finally proposed merger. The regulator can commit ex-ante to its merger rule. They focused in a consumer surplus criterion. They characterized the optimal policy and showed that the last impose a tougher standard on mergers involving larger merger partners (in terms of their premerger market share). Specifically, the minimal acceptance increase in consumer surplus is strictly positive for all but the smallest merger partner and is larger the greater is the merger partner's premerger share. The intuition is that the different incentives between pivotal firm and regulator make that proposed mergers to be not necessarily the best for consumers. Thus, the regulator rejects some consumer surplus-improving larger mergers to induce firms to propose instead better smaller ones. In the second one, the authors consider

a dynamic environment without informational issues, where the possible mergers do not overlap. This paper showed under which conditions the optimal dynamic policy -that wants to maximize discounted expected consumer surplus- is a completely myopic policy.

In the same vein, part of the literature considers transfers of units of capital as remedies for merger policy, but without informational issues. In [Vasconcelos \(2010\)](#) a four-oligopoly model is analyzed, where synergies are possible through the union of units of capital. He assumed a consumer surplus criterion and showed that an over-fixing problem associated with remedial divestitures may emerge. Under this, a firm may abstain from proposing a socially desirable merger, anticipating an over-divestiture to obtain the merger approval.

Finally, there is literature that studies the effect of new firms entering the market. Intuitively, the possibility of post-merger entry reduces the set of profitable mergers. If we are interested in a consumer surplus standard, the possibility of new entry increases the likelihood that a given merger will lower price. In [Werden and Froeb \(1998\)](#) it is showed that mergers that lead firms entry in the future are rarely profitable in the absence of efficiencies. Thus, profitable mergers will be heavily weighted towards mergers that reduce costs. In a recent paper, [Pesendorfer \(2005\)](#), a repeated game with endogenous merger and entry of firms is studied. Two properties are established. First a merger for monopoly may not be profitable (Because the entrant of new firms and the no future possibility to merge). Second, a merger in a no concentrated industry can be profitable (when future expected mergers exist).

It is important to realize that most of the literature considers environments where the regulator knows the production technology. This project intends to recognize the informational issue that exists between regulator and firms, assuming that the marginal cost is not known. In addition, the fact that all firms' information affect the outcome has not been recognized, which this project try to explore. From a theoretical perspective, this project solves a no trivial case of mechanism design problems, since it is not assumed that transfers are quasi-linear. Thus, particular arguments for this context are studied.

2. MODEL

Consider a set $I = \{1, \dots, n\}$ of firms. Each firm is characterized by parameters $(\theta_i, k_i) \in [\theta, 1] \times K$, $K \subset \mathbb{R}_+$. Each firm produces one good with constant marginal costs technology,

given by $c(\theta_i, k_i) = \theta_i (\bar{k} - k_i)$, with $\bar{k} \in \mathbb{R}$ a constant sufficiently large compared with k_i ⁶. This setup corresponds to the simplest case considered in the seminal works [Farrell and Shapiro \(1990a\)](#) and [Farrell and Shapiro \(1990b\)](#) for horizontal mergers, and allow us to capture in the simplest way the trade-off in this problem⁷. We interpret the parameter θ_i as a measure of productivity (the smaller is this value, the more productive is the firm) for a given amount of capital, and k_i as the unit of capitals owned by firm i .⁸ Denote $\theta \times k = (\theta_1, \dots, \theta_N) \times (k_1, \dots, k_N) \in \Theta \times \mathcal{K}$, where $\Theta \equiv [\underline{\theta}, 1]^{|I|}$ and $\mathcal{K} \equiv K^{|I|}$. Firms compete in quantities (Cournot competition), with an inverse demand function $P(Q) = 1 - Q$.

Under the previous assumptions⁹, there is a “well behaved”¹⁰ Nash equilibrium for any (θ, k) . The comparative statics for different (θ, k) are well studied in [Farrell and Shapiro \(1990a\)](#).

We think a merger as a change in the structure of the economy, in particular we will consider a new pair (θ', k') based in the initial (θ, k) such that it captures the idea of a group of firms merging. From now on, we explicitly write each variable in equilibrium as a function of (θ, k) . Denote firm i 's payoff as $\pi_i(\theta, k) = [P(Q(\theta, k)) - c(\theta_i, k_i)] q_i(\theta, k)$ with $Q(\theta, k) = \sum_{i \in I} q_i(\theta, k)$. Note that on equilibrium, all firms parameters affect each firm payoffs.

We say a *merger* is any subset of firms $M \subsetneq I$. We think a merger as the union of all firms in the set M . The new firm parameters, denoted by (θ_M, k_M) , will depend in the parameters of the firms in M . We assume there is an exogenous merging technology $\mu : [\underline{\theta}, 1]^{|M|} \rightarrow [0, 1]$ that gives the new firm productivity parameter. Thus (θ_M, k_M) is the result of $\theta_M = \mu(\theta|_M)$ (where $\theta|_M$ is the restriction of θ to the parameters from subset M), the new productivity parameter, and $k_M = \sum_{i \in M} k_i$, the sum of the units of capital from firms in M . Consider WLOG that $M = \{(|I| - |M| + 1), \dots, |I|\}$, thus $I = \{1, \dots, (|I| - |M|)\} \cup M$. What a

⁶We will assume $\bar{k} \in \left(1, \frac{N}{N-1}\right)$ and $k_i = \bar{k} - 1$, for every $i \in I$. This ensure that for any transfer of capital within firms, the marginal cost will be positive.

⁷We rule out scale effects. Firms want to merge only because synergies that affect marginal costs.

⁸We think a situation where firms must have some units of capital to operate, and the total amount in the economy is limited. That creates the oligopoly environment.

⁹More general, this is true for any demand function such that: For any $Q > 0$ such that $P(Q) > 0$, the following two conditions hold (i) $P'(Q) < 0$, (ii) $P'(Q) + QP''(Q) < 0$. and (iii) $\lim_{Q \rightarrow \infty} P(Q) = 0$

¹⁰“Well behaved” means that if you have (θ, k) and (θ', k') such that $(\theta_i, k_i) = (\theta'_i, k'_i)$ for each $i \neq j$ and $\theta_j < \theta'_j$ or $k_j > k'_j$, then in (θ', k') compared to (θ, k) , both firm j and total production increase.

merger creates is a new set of firms $I' = \{1, \dots, (|I| - |M| + 1)\}$, with vector of parameters (θ', k') such that $(\theta'_i, k'_i) = (\theta_i, k_i)$, every $i \leq (|I| - |M|)$ and $(\theta'_i, k'_i) = (\mu(\theta|_M), \sum_{j \in M} k_j)$ for $i = (|I| - |M| + 1)$.

For a given a set of firms I , their characteristics (θ, k) and a merge M , we are interested in compare two equilibrium scenarios: with and without merge.

Denote firm i 's difference in profit due the merger $\Delta\pi_i(\theta, k)$, with $i \in I'$. Thus we have the following:

$$\Delta\pi_i(\theta, k) = \begin{cases} \pi_M(\theta', k') - \sum_{j \in M} \pi_j(\theta, k) & \text{if } i = M \\ \pi_i(\theta', k') - \pi_i(\theta, k) & \text{if } i \in I \setminus M. \end{cases}$$

Besides the firms, there is an Antitrust Authority (AA) that has the ability to block a merger and/or propose remedies¹¹. These remedies refer to the ability to require transfers of capital from the merged firms to the rest of firms. Assume that AA's objective is to maximize the change in consumer surplus¹² $\Delta CS(\theta, k) = CS(\theta', k') - CS(\theta, k)$.

Regarding information, we assume that the vector of capitals k is perfectly observable by the firms and the AA. The only source of incomplete information is the vector θ . AA can not observe θ , and the firms only observe its own parameter θ_i (the merger firms can perfectly observe theirs parameters θ_i but not the ones from firms outside the merger). We think θ as a random variable with cumulative distribution function F and density f . The support of θ is $[\underline{\theta}, 1]^{|I|}$. We also assume that $f_i(\theta_i) > 0$, every $i \in I$ and $\theta_i \in [\underline{\theta}, 1]$, and that θ_i is i.i.d. across firms. The function F will encompass all the previous information of the AA about each firm productivity.

We impose the following assumptions over μ :

Assumption 2.1. μ is a continuous function such that: $\mu : [\underline{\theta}, 1]^{|M|} \rightarrow [0, \underline{\theta}]$

This assumption is saying that any merger will give a productivity parameter θ_M smaller than any other firm parameter that does not belong to the merger. This is important for three reasons. First, we ensure that any divestiture will increase consumer surplus. So, we are focusing only in cases when divestitures are useful. Second, this condition will be

¹¹We will focus on structural remedies

¹²Change in consumer surplus is defined by $\Delta CS^M(\theta, k) = CS(\theta', k') - CS(\theta, k)$ with $CS(\theta, k) = \int_0^{Q(\theta, k)} P(x) dx - Q(\theta, k)P(Q(\theta, k))$

important to show existence of an optimal mechanism later. And third, it will ensure that there exists a merger that is profitable for at least some set of parameters. We think that a necessary conditions for a group of firms to propose a merger is to be profitable. This leads to the following assumption:

Assumption 2.2. Any proposed merger M is always profitable, $\Delta\pi_M(\theta, k) \geq 0$

For any proposed merger M , since the difference in payoffs is decreasing in divestitures, we can define $\bar{\delta}(\theta)$ as the amount of divestitures that make indifferent the group of firms to propose the merger and not do it ($\pi_M(\theta, \bar{\delta}) = 0$). Note that there is a natural bound in the amount of divestitures for any vector θ , $\bar{\delta} = \sup_{\theta \in [0,1]^I} \bar{\delta}(\theta)$.

Assumption 2.3. μ is such that for any proposed merger M , there exist a vector $(\delta_i)_{i \in I \setminus M}$ such that $\sum_{i \in I \setminus M} \delta_i = \bar{\delta}(\theta)$ and $\pi_i(\theta, \delta_i) \geq 0$ for $i \in I \setminus M$

This assumption says that it is always possible to divide the divestitures between the firms outside the merger such that they are at least indifferent about the merger.

From a theoretical point of view, we follow a mechanism design approach, with the AA designing a rule that elicit the information from the firms, to take an informed decision. Firms care about their profits that could be from a merger situation or not. Contrary to the standard mechanism design assumptions, here the AA has the ability to set budget balanced¹³ transfers among firms that will not enter linear in profits. Moreover, from the designer perspective, we will have interdependent values, since profits in any case depend in others information. Additionally, we will have a type-dependent participation constraint, since in case of no-merge, firms obtain profits which depend on information. However, the simple structure of the problem will allow us to overcome this problem. As usual in mechanism design, we use the revelation principle to focus only in direct mechanisms. More formally, we have the following:

Fix a merger M . For simplicity, assume that $k_i = k_j = \bar{k} - 1$, every $i, j \in I$ ¹⁴, so in the case of no merge the marginal costs of firm i will be $c(\theta_i) = \theta_i$.

Definition 2.4. A *merger rule* is a pair of functions (x, δ) such that:

¹³The amount of divestiture is totally divided in the firms nor merging.

¹⁴Thus $(\bar{k} - k_i) = 1$, every $i \in I$

$$x : \Theta \rightarrow \{0, 1\}$$

$$\delta = (\delta_i)_{i \in I'} \text{ and } \delta_i : \Theta \rightarrow \mathbb{R}_+$$

The first function is the *decision* of allowing ($x = 1$) or blocking ($x = 0$) the merger. The second one are the divestitures received (given) by each non-merged firm (the merger). Note that a merger rule induce the following pair (θ', k') in case of merger: $(\theta'_i, k'_i) = (\theta_i, k_i + \delta_i)$, every $i \leq (|I| - |M|)$ and $(\theta'_i, k'_i) = (\mu(\theta|_M), \sum_{j \in M} k_j - \delta_M)$ for $i = (|I| - |M| + 1)$. Note that the vector k' will only depend on δ , so from now on we will write payoffs as a function only on δ . Firm i 's payoffs induced by a merger rule (x, δ) will be $x(\hat{\theta})\Delta\pi_i(\theta, \delta(\hat{\theta})) + \pi_i(\theta)$, where $\hat{\theta}$ is vector of reports, and $\pi(\theta)$ is the profits in case of no merge. As usual, we will ensure that all firms to participate in the mechanism. We assume that the outside option to not participate is the no merger case, in which case firms get $\pi_i(\theta)$. Using this we have the following definitions:

Definition 2.5. A merger rule (x, δ) is *incentive compatible* (IC) if truth telling is an ex post Nash equilibrium; that is if, for all $i \in I'$, $\theta \in \Theta$, and $\hat{\theta}_i \in [0, 1]$

$$x(\theta)\Delta\pi_i(\theta, \delta(\theta)) \geq x(\hat{\theta}_i, \theta_{-i})\Delta\pi_i(\theta, \delta(\hat{\theta}_i, \theta_{-i}))$$

Note that in the previous definition we are implicitly assuming that the merger firms report jointly to the AA. A merger report will consist on a vector θ_M , that gives the productivity parameter for each firm in the merge.

Definition 2.6. A merger rule (x, δ) is *individually rational* (IR) if each firm, conditional on his type, is willing to participate; that is if, for all $i \in I'$, $\theta \in \Theta$

$$x(\theta)\Delta\pi_i(\theta, \delta(\theta)) \geq 0$$

Definition 2.7. A merger rule (x, δ) is *feasible* (F) if the amount of capital taken from the merger is divided exactly to the rest of firms; that is if, for all $i \in I'$, $\theta \in \Theta$

$$\sum_{i \in I \setminus M} \delta_i(\theta) = \delta_M(\theta)$$

Thus, the AA problem is to select over all possible IC, IR and F merger rules, the one that maximize the expected value of the difference in consumer surplus.

In sum, the timing of the game is the following:

t=0: θ is drawn from F .

t=1: A subset of firms M decide to propose a merger (exogenous).

t=2: AA commit to a merger rule (x, δ) .

t=3: All firms report to the AA and a merger decision and divestitures are implemented.

t=4: Firms compete in quantities.

Note that we are not studying the merger formation procedure. We are just considering the optimal merger rule for any possible proposed merger.

3. COMPLETE INFORMATION CASE

In this section, we study as a benchmark the case when the AA knows θ . When the AA has total information about firms characteristics, given a proposed merger, the optimal merger rule decision is easy: Accept the merger whenever the difference in consumer surplus is positive considering divestitures. The possibility to set transfers from the merged firm to the other firms could help potentially to change the difference in consumer surplus from a negative to a positive value. This is the intuition explored in the next results. Formally, the AA problem is the following:

$$\begin{aligned} \max_{x(\theta), \delta(\theta)} \quad & x(\theta) \Delta CS(\theta, \delta(\theta)) \\ \text{subject to:} \quad & (IR), (F) \end{aligned}$$

Before the results, it is useful to have an example that gives the intuition behind the results.

Example. Consider 3 firms with parameters $\theta = (\frac{1}{8}, \frac{1}{8}, \frac{2}{8})$. Assume that $\bar{k} = 2$ and each firm has one unit of capital. Thus, the marginal costs will be $c = (\frac{1}{8}, \frac{1}{8}, \frac{2}{8})$. A Cournot competition will have output quantities $q = (\frac{2}{8}, \frac{2}{8}, \frac{1}{8})$ with total $Q = \frac{5}{8}$. Profits will be $\pi = ((\frac{2}{8})^2, (\frac{2}{8})^2, (\frac{1}{8})^2)$. There are two things to notice from this example. First, lower marginal cost implies higher produced quantity. Second, profit is an increasing function in the produced quantity on equilibrium. Let suppose now firm 1 and 2 merge in one, denoted by M . Assume the merger parameter is $\theta_M = \frac{1}{16}$. In this case the new firm will have two units of capital, while the third firm will have only one. Without divestitures, we will have $c' = (0, \frac{1}{4})$, $q' = (\frac{5}{12}, \frac{2}{12})$, $Q' = \frac{7}{12}$ and $\pi' = ((\frac{5}{12})^2, (\frac{2}{12})^2)$. Note that the merger is profitable for the merger $\pi'_M > \pi_1 + \pi_2$ and the firm 3 $\pi'_3 > \pi_3$, but consumer surplus decreases $Q' < Q$. Consider now that the AA require a divestiture of $\delta = \frac{4}{7}$ from the merger to firm 3. If this is

the case, the new output will be $c'' = (\frac{1}{56}, \frac{6}{56})$, $q'' = (\frac{20}{56}, \frac{15}{56})$, $Q'' = \frac{5}{8}$ and $\pi'' = ((\frac{20}{56})^2, (\frac{15}{56})^2)$. In this case, the merger is still profitable for the merger and the other firm, but consumer surplus does not change $Q'' = Q$. Whenever a merger decrease consumer surplus $Q' < Q$ but there exists a positive δ such that $Q'' \geq Q$ and it is still profitable for the merger and the firm 3, we say the *merge can be fixed* by the AA requiring a divestiture of δ units of capital from the merger to the rival firm.

Important things to note is that the merger profit decreases in the third case compared with the second one $\pi''_M < \pi'_M$ and the firm 3 profit increases $\pi''_3 > \pi'_3$. In general, the merger profit is decreasing in divestitures, and the receiver firm profit is increasing. Using this fact, the AA could set $\delta = \frac{4}{7} + \varepsilon$ such that it is still profitable for both firms and the consumer surplus increase strictly $Q'' > Q$. Thus, in the case of complete information the idea behind to how to set optimally δ is simple. Later, with incomplete information, The AA will be required to do it in an incentive compatible way, but the ideas behind are the same.

A more general case we have when there are more firms. Suppose there exists one firm that does not receive divestitures. For that firm the marginal cost will not change, thus profits will be affected only through the total output of the economy, more precisely the price. In the case that consumer surplus increases, the price will decrease, and the same will happen with the profit. Thus, the merger is profitable for this firm only when the consumer surplus decreases (price increases). Since we are interested in cases when all firms are willing to participate, the AA must either give divestitures to all the firms or at most keep the consumer surplus unchanged.

Most of the intuition from the example is summarized in the following lemma.

Lemma 3.1. *Consider a feasible merger rule (x, δ) . Then:*

- (i) *Merger profit is decreasing in any divestiture.*
- (ii) *Firm i profit is increasing in divestitures to itself, but decreasing in divestitures to other firms.*
- (iii) *Consumer surplus is increasing in any divestiture. The increment is higher, the higher is the receiver firm parameter θ_i .*¹⁵

¹⁵In the case that $c(\theta_i, k_i) = \frac{\theta_i}{k+k_i}$ this is also true until a certain value of divestiture determined by the parameters. But we can always consider functions μ such that the amount of divestitures that gives zero gains to the merger is smaller than that value. Thus, the same result applies.

This last property is the key to get the optimal merger rule. The AA will try to transfer units of capital to more unproductive firms in general, but that affects negatively other firms, so the AA must compensate in some cases giving some units to those firms to satisfy individual rationality condition.

The analysis of the optimal merger rule can be divided in three. First, for some set of parameters θ , the difference in consumer surplus without any divestiture is positive. Then, the merger must be approved. Second, for other set the difference in consumer surplus without any divestiture is negative. But, using the divestitures, we can *fix the merger*, making it at least zero. Thus, again, the merger must be approved. Finally, for the rest of parameters θ , the difference in consumer surplus without any divestiture is negative, and even after divestitures it is negative. In those cases, the merger must be denied.

Formally, denote $\Delta^+ = \{\theta \in \Theta : \Delta CS(\theta, 0) \geq 0\}$, $\Delta^- = \{\theta \in \Theta : \Delta CS(\theta, 0) < 0\}$. Define $\bar{\delta}(\theta)$ as the amount of divestitures that gives zero payoff to the merger, $\Delta \pi_M(\theta, \bar{\delta}) = 0$. From our assumption that any merger is profitable, this value is always positive. Define $\delta^*(\theta)$ as the amount of divestitures that keep the consumer surplus unchanged, $\Delta CS(\theta, \delta^*) = 0$. Finally, denote $\Phi = \{\theta \in \Theta : \bar{\delta}(\theta) \geq \delta^*(\theta)\}$, the set of states where the amount of divestiture taken from the merger could fix the merger.

Definition 3.2. A merger M such that $\theta \in \Delta^-$ can be fixed if there exists $\delta > 0$ such that $\Delta CS(\theta, \delta) \geq 0$ and $\Delta \pi_i(\theta, \delta) \geq 0, i \in I'$.

Proposition 3.3. A merger M such that $\theta \in \Delta^-$ can be fixed if and only if $\theta \in \Phi$.

Using the previous results we can state the optimal merger policy depending in the number of firms merging:

Proposition 3.4. In the case $|M| = n - 1$, the optimal merger rule (x, δ) among feasible and individually rational is:

$$x(\theta) = \begin{cases} 1 & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

$$\delta_i(\theta) = \begin{cases} \bar{\delta}(\theta) & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

The intuition behind the optimal merger rule in this case can be summarized in the following: Since consumer surplus is increasing in divestitures, and the merger is profitable, extract units of capital from the merger up to the point where the gains for the merger firms is zero. Given this increment in consumer surplus, authorize all the merger when the difference in consumer surplus is at least zero. These mergers can be divided in two groups. The first one, the difference in consumer surplus is at least zero without divestitures. And the second one, the difference in consumer surplus without divestitures is negative, but with the divestitures this difference is at least zero.

For the other case, we need to consider that any divested unit to firm i will decrease $\Delta\pi_{-i}$, so some individual rationality restriction may be binding. Thus, it may be not feasible to give all the units of capital from the merger to the most unproductive firm. For the next proposition, denote $h = \operatorname{argmax}_{j \in I \setminus M} \theta_j$, as the most unproductive firm.

Proposition 3.5. *In the case $|M| < n - 1$, the optimal merger rule (x, δ) among feasible and individually rational is:*

$$x(\theta) = \begin{cases} 1 & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

$$\delta_i(\theta) = \begin{cases} \delta^*(\theta) + z_h(\theta) & \text{if } i = h \text{ and } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ z_i(\theta) & \text{if } i \neq h \text{ and } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi]. \\ 0 & \text{in other case.} \end{cases}$$

Where $(z(\theta))_{i \in I \setminus M}$ is the solution of an optimization problem that determines how to divide the amount $(\delta^M(\theta) - \delta^*(\theta))$ between no merging firms (Solution in the appendix).

Note that in any of these cases, we have $\Delta\pi_M(\theta, \delta) = 0$. Thus all the gains from the merger are extracted by the AA in order to increase consumer surplus as much as possible.

4. INCOMPLETE INFORMATION CASE

In this section we study the case when the AA does not know θ . Note that we are assuming that the amount of capital k is observable and moreover the AA can use it as a screening tool. Thus, AA will consider the expected value (with respect to θ) of the difference in

consumer surplus, given the available information. We are also assuming that each firm only observes his own parameter θ_i , and the firms in the merger observe all the merger firms parameters. The equilibrium concept used is ex post Nash equilibrium, then each firm will report truthfully independently of the beliefs about other firms parameters. Again, here the merger will report all the merger firms parameters truthfully independently of any belief of no merger firms parameters.¹⁶

To solve this problem we follow a mechanism design approach. In particular, the model proposed is a non-quasi-linear payoff model with interdependent values, since transfers of capital are not linear in firms payoff and the gains due a merger depend in all firm's types. As usual, we use the revelation principle to focus only in direct mechanism. The equilibrium notion used is ex-post Nash Equilibrium, meaning that given others firms are reporting truthfully, each firm optimal report is to do it truthfully. In the particular case of the merger partners, we assume they jointly report their types, giving space to coordinate reports.

We first characterize an incentive-compatible merger rule, and secondly, solve the AA problem.

4.1. Characterization of Incentive Compatibility. We first characterise the incentive-compatible condition. Note that θ_M corresponds to a vector with the merger partner's types. Thus, we are ruling out jointly misreports from the merger partners.

Given a merger rule (x, δ) , define the acceptance sets: $A = \{\theta : x(\theta) = 1\}$, and for every $i \in I'$, $A_i(\theta_{-i}) = \{\theta_i : x(\theta) = 1\}$. Consider the following definitions.

Definition 4.1. We say a firm i is *decisive* given θ_{-i} if $A_i(\theta_{-i}) \neq \emptyset$, and $[0, 1] \setminus A_i(\theta_{-i}) \neq \emptyset$

A firm is decisive whenever it can changes the merger decision with his report.

The main idea to characterise the incentive compatibility condition is to try to consider the possible deviations individually. In particular, given some decision $x(\theta)$, over the acceptance set the only possible deviations comes from try to change the amount of divestitures. Thus, the possible deviations must belong only to the acceptance set. Consequently, the first step is to understand how an incentive compatible divestiture looks like for a fixed decision $x(\theta)$. And second, we will study how to design the decision $x(\theta)$ such that together with the

¹⁶In particular, when firms observe all other firms parameter, report truthfully will be an equilibrium.

divestiture $\delta(\theta)$ obtained in the first step, constitutes an incentive compatible merger rule. Finally, we will give a characterization of those rules.

We first give a necessary condition for incentive compatibility whenever $\theta_i \in A_i(\theta_{-i})$.

Proposition 4.2. *If a merger rule (x, δ) is incentive-compatible, then:*

- (i) *For every $i \in I \setminus M$, $\theta_{-i} \in \Theta_i$ and $\theta_i \in A_i(\theta_{-i})$, $\delta_i(\theta) = \delta_i(\theta_{-i})$; that is, the transfers of capital for firm i do not depend on firm i 's report.*
- (ii) *For every $\theta_{-M} \in \Theta_{-M}$ and $\theta_M \in A_M(\theta_{-M})$, $\delta_M(\theta) = \delta_M(\theta_{-M})$; that is, the transfers of capital for the merged firm do not depend on its report .*

Any vector δ that have this feature we will say it holds *own-report independence*.

Note that having transfers (divestitures) that do not depend on the own report is something observed in the quasi-linear environment. For example, in second price auction, over the set of reports that assign the object to some agent, the price to pay does not depend in that report. What this proposition suggests is that this feature is not exclusive from the quasi-linear environment.

Note that the amount of transfer taken from the merger firm M is divided between the rest of firms $i \in I \setminus M$. The previous result implies the following:

Corollary 4.3. *If a merger rule (x, δ) is incentive-compatible, feasible, and $|M| = n - 1$, then $\delta(\theta) = \delta$ (constant).*

In the particular case when the number of firms merging is just one below the total, any incentive compatible merger rule will have transfers of capital that do not depend in the report.

The next proposition helps to understand the relation between the divestitures and the merger decision.

Proposition 4.4. *If a merger rule (x, δ) is incentive-compatible, then:*

- (i) *For $i \in I'$, if $\Delta\pi_i(\theta, \delta(\theta)) < 0$ and i is decisive given θ_{-i} , then $x(\theta) = 0$.*
- (ii) *For $i \in I'$, if $\Delta\pi_i(\theta, \delta(\theta)) > 0$ and i is decisive given θ_{-i} , then $x(\theta) = 1$.*
- (iii) *$x(\theta)$ is monotone in θ_i (decreasing for $i \in I \setminus M$ and increasing for M).*

The last proposition is intuitive in the sense that any incentive-compatible merger rule must give positive gains from the merger to all the firms to make them willing to report

truthfully. This is crucial since to estimate the merger gains the AA needs all firms information. As a direct corollary of this proposition, any incentive-compatible merger rule will be individually rational.

An useful implication of the relation between the transfers and the merger decision is that it is enough to know one to totally determine the other.

Proposition 4.5.

- (i) *Given a monotone $x(\theta)$ (decreasing for $i \in I \setminus M$ and increasing for M), there is only one $\delta(\theta)$ such that (x, δ) is incentive-compatible.*
- (ii) *Given a vector $\delta(\theta)$ that holds own-report independence, there is only one $x(\theta)$ such that (x, δ) is incentive-compatible.*

This will be particularly useful later to simplify the AA maximization problem. Using the previous propositions we can state the characterization of incentive compatible merger rules.

Theorem 4.6. *A merger rule (x, δ) is incentive compatible if and only if δ satisfies own-report independence and x is the induced merger decision by δ .*

Comments

First, note that firm's report only change the outcome of the rule through the merger decision x , because the transfer of capital is a function of the others firms reports. Intuitively, given a merger decision x , the amount of transfers of capital received by each firm $i \in I \setminus M$ in case of merger is the amount that makes indifferent the pivotal type (the last type that makes the merger happen). This type is what we denote before $\hat{\theta}_i(\theta_{-i})$. We can see that this is analogue to how the generalized VCG mechanism works for interdependent values when transfers are linear in payoffs (Krishna, 2009). In that case, an agent report only affects the object allocation decision, but transfers are determined by the others firms reports. Furthermore, the transfers are such that the last type that receives the object is indifferent between receive it or not (pivotal).

Second, note that Proposition 5.5 help us to link the merger decision with the capital transfers between firms. As noted before, a merger decision also determines the capital transfers needed for implement that merger decision in an incentive compatible way. This is analogue to the revenue equivalence theorem in a quasi-linear environment, where the

necessary transfers for an incentive compatible mechanism are determined by the allocation, up to a constant (Krishna, 2009).

4.2. Regulator Problem. Let study now the AA maximization problem. In this subsection, we use the incentive compatible characterization to obtain the optimal one from the set of feasible rules.

In general, the AA problem is the following:

$$\max_{x(\theta), \delta(\theta)} \int_{\Theta} x(\theta) \Delta CS(\theta, \delta(\theta)) dF(\theta)$$

subject to: $(IC), (IR), (F)$

Note that from the previous discussion, incentive compatibility implies that $x(\theta) \Delta \pi_i(\theta, \delta(\theta)) \geq 0$ for every $i \in I \setminus M$. Thus (IC) implies (IR) .

Consider first the case $|M| = n - 1$. Using Corollary 5.3 and Theorem 5.6, this allow us to reduce the problem to just find an optimal value δ^* which is transferred from the merger to the other left firm.

Using proposition 4.5, we can focus only in δ , assuming that we use the *induced* x by δ . The AA problem can be rewritten as follows:

$$\max_{\delta \geq 0} \int_{\Theta} x(\theta) \Delta CS(\theta, \delta) dF(\theta)$$

subject to: $x(\theta)$ induced by δ .

Denote $I(\delta) = \{\theta : \Delta \pi_i(\theta, \delta) \geq 0\}$ the set of θ where all firms get positive gains from the merger. This is the set where the induced x take value one. Thus the problem becomes:

$$\max_{\delta \geq 0} \int_{\Theta} \Delta CS(\theta, \delta) \mathbf{1}_{I(\delta)}(\theta) dF(\theta)$$

Once we have this unconstrained problem, is direct to get the optimal merger rule.

Proposition 4.7. *In the case $|M| = n - 1$, the optimal merger rule (x, δ^*) consist on transfer δ^* from the merger to the firm left, with*

$$\delta^* = \arg \max_{\delta \geq 0} \int_{\Theta} \Delta CS(\theta, \delta) \mathbf{1}_{I(\delta)}(\theta) dF(\theta)$$

and merger decision

$$(1) \quad x(\theta) = \begin{cases} 1 & \text{if } \theta \in I(\delta^*) \\ 0 & \text{In the other case.} \end{cases}$$

Proposition 4.8. *In the case $|M| = n - 1$, an optimal merger rule always exists.*

Consider now the case when $|M| < n - 1$. Define analogously $I(\delta) = \{\theta : \Delta\pi_i(\theta, \delta) \geq 0\}$, now as a function of the vector δ . In this case, we can apply the same logic than before and reduce the AA problem to find $|M| - N$ functions $\delta_i(\theta_{-i-M}) : [0, 1]^{N-M-1} \rightarrow \mathbb{R}_+$, one for each no merged firm $i \in I \setminus M$. The problem is reduced to the following:

$$\max_{\delta_i \geq 0, i \in I \setminus M} \int_{\Theta} \Delta CS(\theta, \delta(\theta)) \mathbb{1}_{I(\delta)}(\theta) dF(\theta)$$

Using this, we can state the following:

Proposition 4.9. *In the case $|M| < n - 1$, the optimal merger rule (x, δ) consist on transfers $\delta_i^*(\theta_{-i-M})$ from the merger to the firms left, with*

$$\delta_i^*(\theta_{-i-M}) = \arg \max_{\delta_i \geq 0} \int_{\Theta_i} \int_{\Theta_M} \Delta CS(\theta, \delta(\theta)) \mathbb{1}_{I(\delta)}(\theta) dF_M(\theta_M) dF_i(\theta_i) \text{ For every } i \in I \setminus M$$

$$(2) \quad x(\theta) = \begin{cases} 1 & \text{if } \theta \in I(\delta^*) \\ 0 & \text{In the other case.} \end{cases}$$

Proposition 4.10. *In the case $|M| < n - 1$, an optimal merger rule always exists.*

4.3. Discussion. In this section we compare the complete and incomplete information optimal merger rule. In particular, we will study the distortions due to the incomplete information. Consider the case $|M| = n - 1$ and the acceptance set A from the optimal merger rule in the incomplete information case. Consider δ^* , the amount of divestiture required in the optimal merger rule. Note that $A = \{\theta : \Delta\pi_i(\theta, \delta^*) \geq 0, \text{ every } i \in I \setminus M\}$. This set can be splitted in two, A^+ and A^- , with $A^+ = \{\theta : \Delta\pi_i(\theta, \delta^*) \geq 0, \text{ every } i \in I \setminus M \text{ and } \Delta CS(\theta, \delta^*) \geq 0\}$ and A^- defined analogously. Both sets are not empty. We can partition the set A^+ in three according the distortion compared with the full information case:

- (i) $\{\theta \in A^+ : \Delta\pi_M(\theta, \delta^*) = 0\}$. There will be **no distortions**, since the amount of divestitures required will be the same in both cases.

- (ii) $\{\theta \in A^+ : \Delta\pi_M(\theta, \delta^*) > 0\}$. There will be an downward distortion. If this is the case, we say there is an **under-fixing problem**, because the AA could have been able to require more divestitures from the merger. This will affect negatively the possibility to increase the difference in consumer surplus because $\Delta CS(\theta, \delta^*) < \Delta CS(\theta, \bar{\delta})$. In particular, there are two problems. The first, for mergers that divestitures were not needed to change the difference in consumer surplus from negative to positive, the AA could have improved more $\Delta CS(\theta, \bar{\delta})$. In the second case, when divestitures were needed, the AA could not *fix* the merger, because $\Delta CS(\theta, \delta^*) < 0$.

Distortion on A^- . Thus some mergers are accepted even though there should be blocked.

- (iii) $\{\theta \in A^- : \Delta\pi_M(\theta, \delta^*) = 0\}$. No distortions on the amount of divestitures, but distortion on the merger decision because $\Delta CS(\theta, \delta^*) < 0$.
- (iv) $\{\theta \in A^- : \Delta\pi_M(\theta, \delta^*) > 0\}$. Double distortion. Under-fixing problem and wrong merger decision.

For mergers that are not in the acceptance set, there will be the following distortions:

- (v) $\{\theta \in \Theta \setminus A : \Delta\pi_M(\theta, \delta^*) > 0\}$. In this case, it must be that $\Delta\pi_i(\theta, \delta^*) < 0$ and $\Delta CS(\theta, \delta^*) > 0$. Thus, the merger should be approved but the under-fixing problem does not allow to compensate the other firm.
- (vi) $\{\theta \in \Theta \setminus A : \Delta\pi_M(\theta, \delta^*) = 0\}$. Similar than the previous case, $\Delta\pi_i(\theta, \delta^*) < 0$ and $\Delta CS(\theta, \delta^*) > 0$. Thus, the merger should be approved.
- (vii) $\{\theta \in \Theta \setminus A : \Delta\pi_M(\theta, \delta^*) < 0\}$. There will be an upward distortion. In this case we say there is an **over-fixing problem**. Since the AA required to much divestitures, some increasing consumer surplus mergers (that occurs when $\Delta\pi_i(\theta, \delta^*) < 0$) will be blocked.

5. CONCLUSION

APPENDIX A. PROOFS OF SECTION 3

Lemma 3.1 Consider a feasible merger rule (x, δ) . Then:

- (i) Merger profit is decreasing in any divestiture.
- (ii) Firm i profit is increasing in divestitures to itself, but decreasing in divestitures to other firms.

- (iii) Consumer surplus is increasing in any divestiture. The increment is higher, the higher is the receiver firm parameter θ_i .¹⁷

Proof.

- (i) Denote c_i the marginal cost of firm i . Consider the first order conditions that determine the competition outcome:

$$P(Q) - q_i = c_i, \text{ for every } i \in I$$

Summing over firms in I ,

$$NP(Q) - Q = \sum_{i \in I} c_i$$

Then the total output:

$$Q = \frac{N - \sum_{i \in I} c_i}{N + 1}$$

Firm i production is $q_i = 1 - Q - c_i$ and profit $\pi_i = q_i^2 = (1 - Q - c_i)^2$. In the merger case, considering divestitures, we have that $c_M = \theta_M(\bar{k} - M(\bar{k} - 1) + \delta_M)$ and $c_i = \theta_i(1 - \delta_i)$. Note that firm i profit depend negatively on $Q + c_i = \frac{N - \sum_{j \neq i} c_j + Nc_i}{N + 1}$.

Thus, in the merger case,

$$Q + c_M = \frac{(N - M + 1) - \sum_{j \neq i} \theta_j(1 - \delta_j) + (N - M + 1)\theta_M(\bar{k} - M(\bar{k} - 1) + \delta_M)}{(N - M + 1) + 1}$$

Which is increasing in δ_M and δ_i . Thus merger profit π_M is decreasing in δ_M and δ_i and then decreasing in any divestiture.

- (ii) By the same argument we have that

$$Q + c_i = \frac{(N - M + 1) - \sum_{j \neq i, M} \theta_j(1 - \delta_j) - \theta_M(\bar{k} - M(\bar{k} - 1) + \delta_M) + (N - M + 1)\theta_i(1 - \delta_i)}{(N - M + 1) + 1}$$

This is decreasing in δ_i and δ_M , and increasing in δ_j with $j \neq i, M$. Thus firm i profit π_i is increasing in δ_i and δ_M , and decreasing in δ_j with $j \neq i, M$, and then increasing in any divestiture to firm i . Moreover, since $\theta_M < \theta_j$ by assumption, then it is decreasing in any divestiture to firm j .

¹⁷In the case that $c(\theta_i, k_i) = \frac{\theta_i}{k + k_i}$ this is also true until a certain value of divestiture determined by the parameters. But we can always consider functions μ such that the amount of divestitures that gives zero gains to the merger is smaller than that value. Thus, the same result applies.

(iii) From previous calculation we have that:

$$Q = \frac{(N - M + 1) - \sum_{i \in I} c_i}{(N - M + 1) + 1} = \frac{(N - M + 1) - \sum_{i \neq M} \theta_i (1 - \delta_i) - \theta_M (\bar{k} - M(\bar{k} - 1) + \delta_M)}{(N - M + 1) + 1}$$

This is decreasing in δ_M and increasing in δ_i . Thus, since $\theta_M < \theta_j$ by assumption, then it is increasing in any divestiture. □

Proposition 3.3 A merger M such that $\theta \in \Delta^-$ can be fixed if and only if $\theta \in \Phi$.

Proof.

Suppose the merger M can be fixed. Then there exists $\delta > 0$ such that $\Delta CS(\theta, \delta) \geq 0$ and $\Delta \pi_i(\theta, \delta) \geq 0, i \in I'$. In particular, since $\Delta \pi_M(\theta, \delta) \geq 0$, we have that $\bar{\delta}(\theta) \geq \delta$. But, since $\theta \in \Delta^-$, $\Delta CS(\theta, 0) < 0$, and then $\delta \geq \delta^*(\theta) > 0$ using the continuity of $\Delta CS(\theta, \cdot)$. Thus $\bar{\delta}(\theta) \geq \delta^*(\theta)$, and then $\theta \in \Phi$.

Suppose now $\theta \in \Phi$. We know that $\bar{\delta}(\theta) \geq \delta^*(\theta)$. Take $\delta = \delta^*(\theta)$. By definition, we have that $\Delta CS(\theta, \delta) = 0$. Moreover, since $\Delta \pi_M(\theta, \cdot)$ is decreasing and $\bar{\delta}(\theta) \geq \delta^*(\theta)$, $\Delta \pi_M(\theta, \delta) \geq 0$. We need to show that for any other firm it is the case that $\Delta \pi_i(\theta, \delta) \geq 0$. Note that $\Delta \pi_i(\theta, \delta) = (1 - Q' - c'_i)^2 - (1 - Q - c_i)^2$. Note that for $\delta = \delta^*(\theta)$, it must be that $Q' = Q$. Any firm receiving divestitures will have $c'_i \leq c_i$. Thus, $\Delta \pi_i(\theta, \delta^*(\theta)) \geq 0$. In case a firm won't receive, $c'_i \leq c_i$ and then $\Delta \pi_i(\theta, \delta^*(\theta)) = 0$ □

Proposition 3.4 In the case $|M| = n - 1$, the optimal merger rule (x, δ) among feasible and individually rational is:

$$x(\theta) = \begin{cases} 1 & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

$$\delta_i(\theta) = \begin{cases} \bar{\delta}(\theta) & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

Proof. We have to show that

$$x(\theta) = \begin{cases} 1 & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

$$\delta_i(\theta) = \begin{cases} \bar{\delta}(\theta) & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

is the solution of the problem

$$\max_{x(\theta), \delta(\theta)} x(\theta) \Delta CS(\theta, \delta(\theta))$$

$$\text{subject to: } \Delta \pi_M(\theta, \delta(\theta)) \geq 0, \Delta \pi_{-M}(\theta, \delta(\theta)) \geq 0, \text{ (F)}$$

Note that a feasible divestitures consists only in one function $\delta : \Theta \rightarrow \mathbb{R}_+$. Using the previous lemma, since $\Delta CS(\theta, \delta(\theta))$ is increasing and $\Delta \pi_M(\theta, \delta(\theta))$ is decreasing in the value $\delta(\theta)$, whenever $x(\theta) = 1$, the first restriction will be binding in the optimum $\delta(\theta)$. Thus $\delta(\theta) = \bar{\delta}(\theta)$. Now, in an optimal merger rule, $x(\theta) = 1$ if and only if $\theta \in \{\theta \in \Theta : \Delta CS(\theta, \bar{\delta}(\theta)) \geq 0\}$. Using that $\Delta CS(\theta, \delta(\theta))$ is increasing, this will corresponds to the set of $\Delta^+ \cup [\Delta^- \cap \Phi]$. Thus we have the result. \square

Proposition 3.5 In the case $|M| < n - 1$, the optimal merger rule (x, δ) among feasible and individually rational is:

$$x(\theta) = \begin{cases} 1 & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

$$\delta_i(\theta) = \begin{cases} \delta^*(\theta) + z_h(\theta) & \text{if } i = h \text{ and } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ z_i(\theta) & \text{if } i \neq H \text{ and } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi]. \\ 0 & \text{in other case.} \end{cases}$$

Where $(z(\theta))_{i \in I \setminus M}$ is the solution of an optimization problem that determines how to divide the amount $(\delta^M(\theta) - \delta^*(\theta))$ between no merging firms (Solution in the appendix).

Proof. We have to show that

$$x(\theta) = \begin{cases} 1 & \text{if } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ 0 & \text{In the other case.} \end{cases}$$

$$\delta_i(\theta) = \begin{cases} \delta^*(\theta) + z_h(\theta) & \text{if } i = h \text{ and } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi] \\ z_i(\theta) & \text{if } i \neq h \text{ and } \theta \in \Delta^+ \cup [\Delta^- \cap \Phi]. \\ 0 & \text{in other case.} \end{cases}$$

is the solution of the problem

$$\max_{x(\theta), \delta(\theta)} x(\theta) \Delta CS(\theta, \delta(\theta))$$

subject to: $\Delta \pi_i(\theta, \delta(\theta)) \geq 0, i \in I', (F)$

A feasible divestitures consists on $I \setminus M$ functions $\delta_i : \Theta \rightarrow \mathbb{R}_+$. Using the previous lemma, since $\Delta CS(\theta, \delta(\theta))$ is increasing and $\Delta \pi_M(\theta, \delta(\theta))$ is decreasing in the value $\delta(\theta)$, whenever $x(\theta) = 1$, this restriction will be binding in the optimum $\delta(\theta)$. Thus $\sum_{i \in I \setminus M} \delta_i(\theta) = \bar{\delta}(\theta)$. Note that from the previous proposition, a merger can be fixed if and only if $\theta \in \Phi$. Since $\sum_{i \in I \setminus M} \delta_i(\theta) = \bar{\delta}(\theta)$, it is always the case that $\Delta CS(\theta, \delta(\theta)) \geq \Delta CS(\theta, \delta^*(\theta)) = 0$, thus $x(\theta) = 1$ if and only if $\theta \in \Delta^+ \cup [\Delta^- \cap \Phi]$. Now, an optimal divestiture will always give units to most unproductive firms first. Suppose you only have $\delta^*(\theta)$ units available. Thus $\delta_h(\theta) = \delta^*(\theta), \delta_i(\theta) = 0$ will maximize $\Delta CS(\theta, \delta(\theta))$, but will have the restrictions $\Delta \pi_i(\theta, \delta(\theta))$, with $i \neq h$ binding. Then, since $\pi_i(\theta, \delta(\theta))$ is decreasing in any other firm divestiture at this point, we will have to split the remaining units available $(\bar{\delta}(\theta) - \delta^*(\theta))$. Thus, fixed θ , the extra units divested must solve:

$$\text{Max}_{(z_i)_{i \in I \setminus M}} \Delta CS(\theta, \delta^*(\theta) + z)$$

subject to:

$$\Delta \pi_i(\theta, \delta^*(\theta) + z) \geq 0, i \in I \setminus M, i \neq h$$

$$\Delta \pi_h(\theta, \delta^*(\theta) + z) \geq 0$$

$$\sum_{i \in I \setminus M} z_i \leq \bar{\delta}(\theta) - \delta^*(\theta)$$

$$z_i \geq 0$$

In the optimum, the restrictions $\Delta \pi_i(\theta, \delta^*(\theta) + z) \geq 0, i \in I \setminus M, i \neq h$ are binding. Suppose not. There exists j such that $\Delta \pi_j(\theta, \delta^*(\theta) + z) > 0$. There exist an $\varepsilon > 0$ such that still $\Delta \pi_j(\theta, \delta^*(\theta) + z, z_j - \varepsilon) > 0$ and $\Delta \pi_i(\theta, \delta^*(\theta) + z, z_j - \varepsilon) > 0$ for the rest $i \neq j$ (Since all these

functions are decreasing in z_j). Now, we can give some fraction of ε to firm h , and strictly increase ΔCS and still have $\Delta\pi_i \geq 0$, every $i \in I \setminus M$. Thus, it will not be an optimum.

Which can be rewritten as:

$$\text{Max}_{(z_i)_{i \in I \setminus M}} \Delta CS(\theta, \delta^*(\theta) + z)$$

subject to:

$$\theta_j z_j = \Delta Q(\theta, \delta^*(\theta) + z)$$

$$\theta_h(z_h + \delta^*(\theta)) \geq \Delta Q(\theta, \delta^*(\theta) + z)$$

$$\sum_{i \in I \setminus M} z_i = \bar{\delta}(\theta) - \delta^*(\theta)$$

$$z_i \geq 0$$

Note that $\theta_j z_j = \theta_k z_k$, every $j, k \neq h$. From this we have that for a fixed z_h , $z_j = \frac{\bar{\delta}(\theta) - \delta^*(\theta) - z_h}{\theta_j \sum_{i \neq h} \frac{1}{\theta_i}}$. The restriction for h can be rewritten $z_h \geq \left(\frac{\bar{\delta}(\theta)}{\theta_h \sum_{i \neq h} \frac{1}{\theta_i} + 1} - \delta^*(\theta) \right)$. Thus the solution will be $z_h = 0, z_j = \frac{\bar{\delta}(\theta) - \delta^*(\theta)}{\theta_j \sum_{i \neq h} \frac{1}{\theta_i}}$ whenever $\left(\frac{\bar{\delta}(\theta)}{\theta_h \sum_{i \neq h} \frac{1}{\theta_i} + 1} - \delta^*(\theta) \right) \leq 0$, and $z_h = \left(\frac{\bar{\delta}(\theta)}{\theta_h \sum_{i \neq h} \frac{1}{\theta_i} + 1} - \delta^*(\theta) \right), z_j = \frac{\bar{\delta}(\theta) - \delta^*(\theta) - z_h}{\theta_j \sum_{i \neq h} \frac{1}{\theta_i}}$ whenever $\left(\frac{\bar{\delta}(\theta)}{\theta_h \sum_{i \neq h} \frac{1}{\theta_i} + 1} - \delta^*(\theta) \right) > 0$

□

APPENDIX B. PROOFS OF SECTION 4

Proposition 4.2 If a merger rule (x, δ) is *incentive-compatible*, then:

- (i) For every $i \in I \setminus M$, $\theta_{-i} \in \Theta_i$ and $\theta_i \in A_i(\theta_{-i})$, $\delta_i(\theta) = \delta_i(\theta_{-i})$; that is, the transfers of capital for firm i do not depend on firm i 's report.
- (ii) For every $\theta_{-M} \in \Theta_{-M}$ and $\theta_M \in A_M(\theta_{-M})$, $\delta_M(\theta) = \delta_M(\theta_{-M})$; that is, the transfers of capital for the merged firm do not depend on its report .

Proof.

- (i) Consider $i \in I \setminus M$. Fix θ_{-i} and consider a type θ_i . The proof is divided in the following steps:

Step 1. There exist a region $R \subset [0, 1] \times \mathbb{R}_+$ such that $f(\theta_i) = \delta(\theta_i, \theta_{-i}) : [0, 1] \rightarrow \mathbb{R}_+$ must be no decreasing in R and no increasing in $R^c = [0, 1] \times \mathbb{R}_+ \setminus R$ (and thus, almost everywhere differentiable).

Step 2. The function $f(\theta_i)$ is either increasing or constant function.

Step 3. If $f(\theta_i)$ is increasing, it must belong to R^c , having a contradiction. Thus, $f(\theta_i)$ must be a constant.

Step 1. Suppose first i is decisive. Consider type $\theta_i \in A(\theta_{-i})$. Any report $\theta'_i \in [0, 1] \setminus A(\theta_{-i})$ will give him payoff zero. In the other side any report $\theta'_i \in A(\theta_{-i})$ must satisfy the following condition:

$$\Delta\pi_i(\theta, \delta(\theta)) \geq \Delta\pi_i(\theta, \delta(\theta'_i, \theta_{-i}))$$

We can rewrite it only in terms of profit in the merger case.

$$\pi_i(\theta, \delta(\theta)) \geq \pi_i(\theta, \delta(\theta'_i, \theta_{-i}))$$

We can rewrite this condition as the following:

$$\pi_i(\theta, \delta(\theta)) = \max_{\theta'_i \in A_i(\theta_{-i})} \pi_i(\theta, \delta(\theta'_i, \theta_{-i}))$$

Using envelope theorem we have that:

$$\pi_i(\theta, \delta(\theta)) = \pi_i(\theta'_i, \theta_{-i}, \delta(\theta', \theta_{-i})) + \int_{\theta'_i}^{\theta_i} \pi_{i,(1)}(\tilde{\theta}_i, \theta_{-i}, \delta(\tilde{\theta}_i, \theta_{-i})) d\tilde{\theta}_i$$

Where $\pi_{i,(1)}(\theta_i, \theta_{-i}, \delta) = \frac{d\pi_i}{d\theta_i}(\theta_i, \theta_{-i}, \delta)$. Replacing this in the IC condition:

$$\pi_i(\theta'_i, \theta_{-i}, \delta(\theta', \theta_{-i})) + \int_{\theta'_i}^{\theta_i} \pi_{i,(1)}(\tilde{\theta}_i, \theta_{-i}, \delta(\tilde{\theta}_i, \theta_{-i})) d\tilde{\theta}_i \geq \pi_i(\theta, \delta(\theta'_i, \theta_{-i}))$$

Rearranging terms, we have that this is equivalent to:

$$\int_{\theta'_i}^{\theta_i} \int_{\delta(\theta'_i, \theta_{-i})}^{\delta(\tilde{\theta}_i, \theta_{-i})} \pi_{i,(1,2)}(\tilde{\theta}_i, \theta_{-i}, \tilde{\delta}) d\tilde{\delta} d\tilde{\theta}_i \geq 0$$

Where $\pi_{i,(1,2)}(\theta_i, \theta_{-i}, \delta) = \frac{d^2\pi_i}{d\delta d\theta_i}(\theta_i, \theta_{-i}, \delta)$. Thus, $\delta(\theta_i, \theta_{-i})$ must be no decreasing in θ_i whenever $\pi_{i,(1,2)}(\theta_i, \theta_{-i}, \delta) \geq 0$ and no increasing whenever $\pi_{i,(1,2)}(\theta_i, \theta_{-i}, \delta) \leq 0$.

Note that given this particular environment, $\pi_{i,(1,2)}$ can take both signs for a particular (θ_i, δ) . In other words, the usual single crossing property between type and the decision does not hold. This suggest that different shapes for the function $f(\theta_i) = \delta(\theta_i, \theta_{-i})$ may satisfy the above condition (in [Araujo and Moreira \(2010\)](#) a particular shape is discussed). Denote

$$R = \{(\theta_i, \delta) \in [\underline{\theta}, \bar{\theta}] \times \mathbb{R}: \text{ such that } \pi_{i,(1,2)}(\theta_i, \theta_{-i}, \delta) > 0\}$$

and

$$R^c = \{(\theta_i, \delta) \in [\underline{\theta}, \bar{\theta}] \times \mathbb{R}: \text{such that } \pi_{i,(1,2)}(\theta_i, \theta_{-i}, \delta) < 0\}$$

Step 2. Define $\delta^*(\theta_i, \theta_{-i})$ as the value δ^* such that $\pi_{i,(1,2)}(\theta_i, \theta_{-i}, \delta^*) = 0$.

Let consider now first order conditions. The IC problem can be rewritten as:

$$\theta_i = \arg \max_{\theta'_i \in A_i(\theta_{-i})} \Delta\pi_i(\theta, \delta(\theta'_i, \theta_{-i}))$$

For θ'_i in the interior of $A_i(\theta_{-i})$ we must have that:

$$\left. \frac{\partial \Delta\pi_i(\theta_i, \theta_{-i}, \delta)}{\partial \delta} \frac{d\delta(\theta'_i, \theta_{-i})}{d\theta'_i} \right|_{\theta'_i = \theta_i} = 0$$

Denote $\delta'(\theta)$ as the function that satisfies the condition before. Thus either $\delta'(\theta) = \delta'(\theta_{-i})$ or $\delta'(\theta)$ solves $\left. \frac{\partial \Delta\pi_i(\theta_i, \theta_{-i}, \delta(\theta'_i, \theta_{-i}))}{\partial \delta} \right|_{\theta'_i = \theta_i} = 0$. In the last case, for this particular environment, $\delta'(\theta_i, \theta_{-i})$ will be increasing in θ_i .

Step 3.

It is direct to check that the last increasing function $\delta'(\theta_i, \theta_{-i})$ will belong entirely to R^c which is a contradiction with the previous conditions. Thus, it must be that $\delta'(\theta) = \delta'(\theta_{-i})$.

- (ii) Note that an analogue analysis from part (i) can be made for an individual firm from the merger $i \in M$, resulting in that $\delta_M(\theta) = \delta_M(\theta_{-i})$. This will rule out cases of individual misreport from the firm i . Thus, the requirement to $\delta_M(\theta)$ to rule out any individual misreport is to have $\delta_M(\theta) = \delta_M(\theta_{-M})$. But, this will also rule out any jointly misreport of types, since the decision of transfer does not depend in any report from the merger firms θ_M . Thus, it must be that $\delta_M(\theta) = \delta_M(\theta_{-M})$.

□

Corollary 4.3 If a merger rule (x, δ) is incentive-compatible, feasible, and $|M| = n - 1$, then $\delta(\theta) = \delta$ (constant).

Proof. Direct from previous proposition.

□

Proposition 4.4 If a merger rule (x, δ) is *incentive-compatible*, then:

- (i) For $i \in I'$, if $\Delta\pi_i(\theta, \delta(\theta)) < 0$ and i is decisive given θ_{-i} , then $x(\theta) = 0$.
- (ii) For $i \in I'$, if $\Delta\pi_i(\theta, \delta(\theta)) > 0$ and i is decisive given θ_{-i} , then $x(\theta) = 1$.
- (iii) $x(\theta)$ is monotone in θ_i (decreasing for $i \in I \setminus M$ and increasing for M).

Proof.

- (i) Consider type θ_i such that $\Delta\pi_i(\theta_i, \theta_{-i}, \delta(\theta_{-i})) < 0$. Suppose by contradiction that $x(\theta_i, \theta_{-i}) = 1$. Since $[\underline{\theta}, \bar{\theta}] \setminus A(\theta_{-i}) \neq \emptyset$, reporting any $\theta'_i \in [\underline{\theta}, \bar{\theta}] \setminus A(\theta_{-i})$ gives a strictly higher payoff, which contradicts IC condition. Thus it must be that $x(\theta_i, \theta_{-i}) = 0$.
- (ii) Consider type θ_i such that $\Delta\pi_i(\theta_i, \theta_{-i}, \delta(\theta_{-i})) > 0$. Suppose by contradiction that $x(\theta_i, \theta_{-i}) = 0$. Thus, firm i get zero payoff. Pick $\theta'_i \in A(\theta_{-i})$. Using proposition (NUMBERRRRRR), it must be that $\Delta\pi_i(\theta'_i, \theta_{-i}, \delta(\theta_{-i})) = \Delta\pi_i(\theta, \delta(\theta_{-i})) > 0$. Thus type θ_i can report θ'_i and get a strictly positive payoff, which contradicts IC condition. Thus it must be that $x(\theta_i, \theta_{-i}) = 1$.
- (iii) Since $\Delta\pi_i(\theta, \delta(\theta_{-i}))$ is monotone in θ_i , using (i) and (ii) we get the monotonicity in θ_i .

□

Proposition 4.5

- (i) Given a monotone $x(\theta)$ (decreasing for $i \in I \setminus M$ and increasing for M), there is only one $\delta(\theta)$ such that (x, δ) is incentive-compatible.
- (ii) Given a vector $\delta(\theta)$ that holds own-report independence, there is only one $x(\theta)$ such that (x, δ) is incentive-compatible.

Proof.

- (i) Pick any monotone $x(\theta)$ (in the sense defined). Pick any $i \in I \setminus M$. For any θ_{-i} , there is a cut off type $\hat{\theta}_i(\theta_{-i})$ such that $x(\theta) = 1$ if and only if $\theta_i \leq \hat{\theta}_i(\theta_{-i})$. We know that any IC rule must have $\delta_i(\theta_{-i})$. Define $\delta_i(\theta_{-i})$ as the value δ_i that solve the following equation $\Delta\pi_i(\hat{\theta}_i(\theta_{-i}), \theta_{-i}, \delta_i) = 0$. Since δ_i does not depend on θ_i there is no possibility to manipulate. Moreover, it is the case that $\Delta\pi_i(\theta_i, \theta_{-i}, \delta_i(\theta_{-i})) \geq 0$ if and only if $x(\theta) = 1$. Thus, there is no incentives to misreport. Suppose that we define $\delta_i(\theta_{-i}) > \delta_i$. Thus a type $\hat{\theta}_i(\theta_{-i}) - \epsilon$ will report $\hat{\theta}_i(\theta_{-i}) + \epsilon$, with ϵ small enough, and get positive payoffs instead of zero. The case when $\delta_i(\theta_{-i}) < \delta_i$ is analogue. Thus, we have a unique $\delta_i(\theta_{-i})$ such that (x, δ) is IC for agent i . Repeating the argument for all the agents we have the required.
- (ii) Pick a vector $\delta(\theta)$ that satisfies own-report independence. Thus $\delta_i(\theta) = \delta(\theta_{-i})$. For every i , define $\hat{\theta}_i(\theta_{-i})$ as the value $\hat{\theta}_i$ that solve the following equation

$\Delta\pi_i(\hat{\theta}_i, \theta_{-i}, \delta_i(\theta_{-i})) = 0$. Set $x(\theta) = 1$ whenever $\theta_i \leq \hat{\theta}_i$ with $i \in I \setminus M$, and $\theta_i \geq \hat{\theta}_i$ with $i \in M$. Similar than the case before, since $\delta_i(\theta_{-i})$ does not depend on θ_i there is no possibility to manipulate. Moreover, it is the case that $\Delta\pi_i(\theta_i, \theta_{-i}, \delta_i(\theta_{-i})) \geq 0$ if and only if $x(\theta) = 1$. Thus there is no space to misreport. Thus (x, δ) is IC. Let suppose another x' with the same δ as a merger rule. There must exist some θ'_i such that either $\Delta\pi_i(\theta'_i, \theta_{-i}, \delta_i(\theta_{-i})) > 0$ whenever $x(\theta) = 0$, or $\Delta\pi_i(\theta'_i, \theta_{-i}, \delta_i(\theta_{-i})) < 0$ whenever $x(\theta) = 1$. In both cases, there are incentives to misreport, thus (x', δ) is not IC and the only IC is the (x, δ) with x defined before. □

Theorem 4.6 A merger rule (x, δ) is incentive compatible if and only if δ satisfies own-report independence and x is the *induced* merger decision by δ .

Proof. First, suppose (x, δ) is IC. From the previous propositions (NUMBER) δ satisfies own-report independence. Since that, given δ there is only one x such that (x, δ) is IC, it must be the induced one defined before.

In the other way, pick any merger rule (x, δ) such that δ satisfies own-report independence and x is the induced merger decision by δ . Since δ satisfies own-report independence, there is no possible deviation that change the amount of transfer received (given). Moreover, since x is induced by δ , $x(\theta) = 1$ if and only if $\Delta\pi_i(\theta, \delta(\theta)) \geq 0$. Thus there is no incentives to misreport from the merger decision. □

Proposition 4.7 In the case $|M| = n - 1$, the optimal merger rule (x, δ^*) consist on transfer δ^* from the merger to the firm left, with

$$\delta^* = \underset{\delta \geq 0}{\operatorname{arg\,max}} \int_{\Theta} \Delta CS(\theta, \delta) \mathbf{1}_{I(\delta)}(\theta) dF(\theta)$$

and merger decision

$$(3) \quad x(\theta) = \begin{cases} 1 & \text{if } \theta \in I(\delta^*) \\ 0 & \text{In the other case.} \end{cases}$$

Proof. Direct from previous propositions. □

Proposition 4.8 In the case $|M| = n - 1$, an optimal merger rule always exists.

Proof.

First we show that the function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by $f(\delta) = \int_{\Theta} \Delta CS(\theta, \delta) \mathbb{1}_{I(\delta)}(\theta) dF(\theta)$ is upper semi-continuous. The correspondance $I(\delta) : \mathbb{R}_+ \rightarrow \Theta$ is continuous and has closed values. Thus for a given θ , $\mathbb{1}_{I(\delta)}(\theta)$ is upper semi-continuous and $\Delta CS(\theta, \delta)$ continuous on δ , and the product will be upper semi-continuous. Since integration preserves upper semi-continuity we have the result.

Note that $\delta \in [0, \bar{\delta}]$, a compact set. By Weierstrass theorem, we have the existence of the maximizer. \square

Proposition 4.9 In the case $|M| < n - 1$, the optimal merger rule (x, δ) consist on transfers $\delta_i^*(\theta_{-i-M})$ from the merger to the firms left, with

$$\delta_i^*(\theta_{-i-M}) = \arg \max_{\delta_i \geq 0} \int_{\Theta_i} \int_{\Theta_M} \Delta CS(\theta, \delta(\theta)) \mathbb{1}_{I(\delta)}(\theta) dF_M(\theta_M) dF_i(\theta_i) \text{ For every } i \in I \setminus M$$

$$(4) \quad x(\theta) = \begin{cases} 1 & \text{if } \theta \in I(\delta^*) \\ 0 & \text{In the other case.} \end{cases}$$

Proof. Direct from previous propositions. \square

Proposition 4.10 In the case $|M| < n - 1$, an optimal merger rule always exists.

Proof.

We need to find $I \setminus M$ functions $\delta_i(\theta_{-i-M}), i \in I \setminus M$, one for each firm outside the merger. Each function is the solution of the following problem:

$$\delta_i(\theta_{-i-M}) = \arg \max_{\delta_i \geq 0} \int_{\Theta_i} \int_{\Theta_M} \Delta CS(\theta, \delta_i, \delta_{-i}(\theta)) \mathbb{1}_{I(\delta_i, \delta_{-i})}(\theta) dF_M(\theta_M) dF_i(\theta_i)$$

In order to make easier the proof, we will change the notation and adopt a general one from bayesian games. Define $T_0 = \Theta_M$ and for each $i \in I \setminus M$ define the sets $T_i = \Theta_{-M-i}$, $X_i = [0, \bar{\delta}]$, and $T = \times_{i \in \{0\} \cup I \setminus M} T_i$, $X = \times_{i \in I \setminus M} X_i$ and $u(t, x) : T \times X \rightarrow \mathbb{R}_+$ defined by $u(t, x) = \Delta CS(t, x) \mathbb{1}_{I(x)}(t)$. Define also $G_{-i}(t_{-i}/t_i) = F(\theta_i)$. Thus, the problem before can be rewritten as the following:

$$x_i(t_i) = \arg \max_{x_i \geq 0} \int_{T_{-i}} u(t, x) dG_{-i}(t_{-i}/t_i)$$

We can think the solution of the original problem as the bayesian nash equilibrium of the game $(I \setminus M, (X_i)_{i \in I \setminus M}, (T_i)_{i \in I \setminus M}, (g_i)_{i \in I \setminus M}, (u_i)_{i \in I \setminus M})$. The difficult part is that the ex-post utility function is not continuous in the decision variable. Thus, we need to apply discontinuity nash existence methods.

Claim B.1. *The Bayesian game $((T_i)X_i, u_i, p)$ satisfies condition 1 from [Carbonell-Nicolau and McLean \(2014\)](#)*

Proof. Condition 1: For each i and $\varepsilon > 0$, there exists a measurable map $\Phi : X_i \rightarrow X_i$ such that the following holds: for each (t, x) , there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, \phi(x_i), y_{-i}) > u_i(t, x_i, x_{-i}) - \varepsilon \text{ for all } y_{-i} \in V_{x_{-i}}$$

Note first that since the space of types and strategies is compact, $u_i(t, x)$ is bounded. Let denote this bound as \bar{U} .

Consider agent i and fix $\varepsilon > 0$. Define $\Phi(x_i) = x_i - \frac{\varepsilon(N+1)}{2\sqrt{\bar{U}}}$.

For each (t, x_i) , there are at most two point of discontinuity. In the case $u(t, 0) > 0$, the first discontinuity corresponds to x_{-i} such that $\Delta\pi_i(t, x_i, x_{-i}) = 0$. The second one is the x_{-i} such that $\Delta\pi_M(t, x_i, x_{-i}) = 0$. In this case $u(t, x)$ is increasing and convex between the discontinuities. In the other case $u(t, 0) < 0$, there is only one that corresponds to x_{-i} such that $\Delta\pi_M(t, x_i, x_{-i}) = 0$. In this case $u(t, x)$ is increasing and convex between zero and the discontinuities. Denote these as $\underline{x}(t, x_i)$ (whenever exists) and $\bar{x}(t, x_i)$.

- (i) $\underline{x}(t, x_i)$ is increasing in x_i : This value is defined by $\Delta\pi_{-i}(t, x) = 0$. Note that $\Delta\pi_{-i}(t, x_{-i}, x_i)$ is increasing in x_{-i} . Since $\Delta\pi_{-i}(t, x)$ is decreasing in the quantity produced in the merger case, and this quantity is increasing in x_i under the ASSUMPTION, then $\Delta\pi_{-i}(t, x)$ is decreasing in x_i . This proves that $\underline{x}(t, x_i)$ is increasing in x_i .
- (ii) $\bar{x}(t, x_i)$ is decreasing in x_i : This value is defined by $\Delta\pi_M(t, x) = 0$. Note that $\Delta\pi_M(t, x_{-i}, x_i)$ is decreasing in x_{-i} and x_i . This proves that $\bar{x}(t, x_i)$ is decreasing in x_i .

We have the following cases:

- (1) $0 \leq x_{-i} < \underline{x}(t, \Phi(x_i))$: Pick $0 < y_{-i} < \underline{x}(t, \Phi(x_i))$. Then $u_i(t, x) = u_i(t, \Phi(x_i), x_{-i}) = u_i(t, \Phi(x_i), y_{-i}) = 0$.

- (2) $\underline{x}(t, \Phi(x_i)) < x_{-i} < \underline{x}(t, x_i)$: Pick $\underline{x}(t, \Phi(x_i)) < y_{-i} < \underline{x}(t, x_i)$. Then $u_i(t, \Phi(x_i), y_{-i}) > u_i(t, x) = 0$.
- (3) $\underline{x}(t, x_i) < x_{-i} < \bar{x}(t, x_i)$: Pick $\underline{x}(t, x_i) < y_{-i} < \bar{x}(t, x_i)$. Then $\min_{y_{-i} \in V_{x_{-i}}} \{u_i(t, x_i, y_{-i}) - u_i(t, \Phi(x_i), y_{-i})\} = 2C\sqrt{\bar{U}} \frac{\varepsilon}{2C\sqrt{\bar{U}}} - (\frac{\varepsilon}{2C\sqrt{\bar{U}}})^2 < \varepsilon$. Since $u_i(t, x_i, x_{-i}) - u_i(t, \Phi(x_i), y_{-i}) < u_i(t, x_i, y_{-i}) - u_i(t, \Phi(x_i), y_{-i})$, then $u_i(t, x_i, x_{-i}) - u_i(t, \Phi(x_i), y_{-i}) < \varepsilon$.
- (4) $\bar{x}(t, x_i) < x_{-i} < \bar{x}(t, \Phi(x_i))$: Pick $\bar{x}(t, x_i) < y_{-i} < \bar{x}(t, \Phi(x_i))$. Then $u_i(t, \Phi(x_i), y_{-i}) > u_i(t, x) = 0$.
- (5) $\bar{x}(t, \Phi(x_i)) < x_{-i} < \bar{x}$: Pick $\bar{x}(t, \Phi(x_i)) < y_{-i} < \bar{x}$. Then $u_i(t, x) = u_i(t, \Phi(x_i), x_{-i}) = u_i(t, \Phi(x_i), y_{-i}) = 0$.

In the cases when x_{-i} is a discontinuity point, there always exists $V_{x_{-i}}$ such that $y_{-i} \in V_{x_{-i}}$ and belongs to some previous cases. Then the proof is analogue. \square

By *Proposition 1* in [Carbonell-Nicolau and McLean \(2014\)](#), *Condition 1* implies that the Bayesian game is uniformly payoff secure. Furthermore, by *Lemma 2* in [Carbonell-Nicolau and McLean \(2014\)](#), since initial types are independent, then the normal form game induced by the Bayesian game is **payoff secure**.

Claim B.2. For each $t \in T$, the map $\sum_{i=1}^N u_i(t, \cdot) : X \rightarrow \Re$ is upper semicontinuous.

Proof. Note that $u(t, x) = u_i(t, x) = \Delta CS(t, x) \mathbb{1}_{I(x)}(t)$. For a fixed t , $\Delta CS(t, x)$ is a continuous function in x . There are two cases. For some t , $\Delta CS(t, 0) \geq 0$ and since $\mathbb{1}_{I(x)}(t)$ is upper semi continuous in x , $u(t, x)$ will be upper semi continuous. In the other case, $\mathbb{1}_{I(x)}(t)$ equals one whenever $\Delta CS(t, x)$ is negative and upper semi continuous whenever is positive. Thus, $u(t, x)$ is upper semi continuous. \square

By the previous claim and *Lemma 3* from [Carbonell-Nicolau and McLean \(2014\)](#), and denoting \mathcal{B}_i the space of behavioral strategies for agent i , we know that **the map** $\sum_{i=1}^N U_i(\cdot) : \times_{i=1}^N \mathcal{B}_i \rightarrow \Re$ **is upper semicontinuous**.

With all the previous results, we can use *Theorem 1* from [Carbonell-Nicolau and McLean \(2014\)](#) to conclude that the induced normal form game has a Nash Equilibrium, i.e., a Bayes Nash equilibrium of Γ

\square

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