Manipulations-resistant facility location mechanisms for 
ZV-line graphs

Work in progress

Ilan Nehama*, Taiki Todo† Makoto Yokoo‡
Kyushu University
Fukuoka, JAPAN

Abstract

In many real-life scenarios, a group of agents needs to agree on a common action, e.g., on the location for a public facility, while there is some consistency between their preferences, e.g., all preferences are derived from a common metric space. The Facility Location problem models such scenarios and it is a well-studied problem in social choice. We study mechanisms for facility location on graphs, which are resistant to manipulations (strategy-proof, abstention-proof, and false-name-proof) by both individuals and coalitions and are efficient (Pareto optimal). We present a family of graphs, ZV-line graphs, which we claim includes many of the basic graphs and graph families that were studied for this problem. We show a general facility location mechanism for this family which satisfies all these desired properties. Moreover, we show that this mechanism can be computed in polynomial time, it is anonymous, and it can be equivalently defined as the first Pareto optimal location, according to some predefined order. Finally, we discuss some generalizations and limitations of the characterization.

*Corresponding author: ilan.nehama@mail.huji.ac.il
†todo@inf.kyushu-u.ac.jp
‡yokoo@inf.kyushu-u.ac.jp

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1 Introduction

Reaching to an agreement could be hard. The seminal works of Gibbard [6] and Satterthwaite [17] show that one cannot come up with a general procedure for aggregating the preferences of strategic agents to a single outcome, other than trivial procedures that a-priori ignore all agents except one (that is, the outcome is based on the preference of a predefined agent) or a-priori rule out all outcomes except two (that is, regardless of the agents’ preferences, the outcome is one of these two predefined outcomes).

The problem is that strategic agents will act strategically aiming to get an outcome they prefer. It is important to note that while we refer to a procedure, and later to a mechanism, this impossibility is not technical but a conceptual one. A procedure could be very abstract or include a deliberation process between the agents. In this work, we consider the conceptual mapping induced by such general procedure from the opinions of the agents to an agreement. For simplicity of terms, we refer to the direct mechanism which implements this mapping. That is, we think of an exogenous entity, the designer, who receives as input the opinions of the agents and returns as output the aggregated decision.

But in many natural scenarios, one does not look for a general mechanism, a mechanism that must be defined for any profile of preferences, but instead one can assume the preferences satisfy some exogenous rationality property. Two prominent examples are VCG mechanisms and median mechanisms. VCG mechanisms [21, 7, 2, 16] are the SP mechanisms for scenarios in which the agents’ preferences are quasi-linear with respect to money [10], Def. 3.b.7, and monetary transfers are allowed (i.e., the outcome space is closed under monetary exchanges between the agents or between the agents and the designer).

The second example, Median mechanisms, do not include monetary transfers and have more of an ordinal flavor. Median mechanisms [11] are the SP mechanisms when it is known the preferences are single-peaked w.r.t. the real line [1]. That is, the outcomes are locations on the real line, each agent has a unique optimal location, ℓ*, and her preference over the locations to the right of ℓ* is derived by the distance to ℓ*, and similarly for the locations to the left of ℓ*. For example, Euclidean preferences dictate preferences according to the distance to the optimal location.

A natural generalization of the latter is the facility location problem. In this problem, we assume the existence of a metric space and the preferences of the different agents are defined by their optimal location. An agent with an optimal location ℓ* prefers location a over location b if and only if the distance between a and ℓ* is lower than the distance between b and ℓ*. This family of problems arises in many real-life scenarios, such as locating a common good, e.g., a school, and in more general agreement scenarios with a common metric, e.g. partition of a common budget to different tasks.

A natural way to represent this common metric space is using a weighted undirected graph: Having a vertex for each location and weighted edges between them s.t. the distance between two locations is the distance (i.e., length of shortest path) between the respective vertices.

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1 For the properties we study in this work, this assumption does hurt the generality, as according to a simple revelation principle [13] any general procedure is equivalent (w.r.t. the properties we study) to such a direct mechanism.

2 I.e., a set of locations and the distance between any two s.t. the triangle inequality holds.
Roughly speaking, given such a graph, one seeks to find a mechanism that on one hand will not (a-priori) ignore some of the voters or rule-out some of the locations, and on the other hand will be resistant to manipulations of the agents. Since we would also like to consider manipulations of coalitions we define the *preference of a coalition* as the unanimous preference of its members. That is, we say that a coalition $C$ weakly prefers an outcome $a$ over an outcome $b$ if all the members of $C$ weakly prefer $a$ over $b$.\(^3\) In this work, we seek mechanisms which satisfy the following desired properties:

**Anonymity**  The mechanism should, not only not a-priori ignore agents, but treat them equally in the following strong sense. The mechanism should be a function of the agents’ votes (which we also refer to as *ballots*) but not their identities. Notice that most voting systems satisfy this property by first accumulating the different ballots, by that losing the voters’ identities, and applying the mechanism on the identity-less ballots.

**Citizen-sovereignty**  The mechanism should not a-priori rule-out a location, i.e., any location should be an outcome of some profile. Moreover, we require it to respect the preferences of the agents in the following way:

**Pareto-optimality**  The mechanism should not return a location $\ell$ for which the grand coalition could find a location it strictly prefers over $\ell$. In particular, if there is a unique location which is *unanimously* most-preferred by all agents, this must the outcome. This property could also be justified on efficiency grounds since in such scenarios the grand coalition could ex-post ignore the outcome and decide on a new location. Note that it is unreasonable to require that all locations are treated equally (i.e., neutrality of the mechanism) since the graph induces asymmetry of the locations.

**Strategy-proofness**  An agent should not be able to change the outcome to a location she strictly prefers by reporting a location different than her true location.

**Abstention-proofness\(^4\)**  An agent should not be able to change the outcome to a location she strictly prefers by not casting a vote.

**False-name-proofness**  An agent should not be able to change the outcome to a location she strictly prefers by casting more than one ballot. This property received less attention in the social choice literature, since in most voting scenarios there exists some central authority that can enforce the ‘one person, one vote’ principle (but cannot enforce participation or sincere voting). In contrast, many of the

\(^3\)Notice that this preference is not complete. Also note that we get that $C$ strictly prefers $a$ over $b$ if (i) all the members of $C$ weakly prefer $a$ over $b$ ($C$ weakly prefers $a$ over $b$), and (ii) at least one member of $C$ strictly prefers $a$ over $b$ ($C$ does not weakly prefer $b$ over $a$).

\(^4\)In the voting literature (e.g., \([3][12][5]\)) this property is also referred to as voluntary participation and the no-show paradox.
voting and aggregation scenarios nowadays are run in a distributed manner on some network and include virtual identities or avatars, which can be easily generated, so a manipulation of an agent pretending to represent many voters is eminent.

**Resistance to group manipulations**  We also consider the generalizations of the above three properties to require that also a coalition should not be able to change the outcome to a location it strictly prefers, by its members casting insincere ballots, abstaining, or voting more than once.

**Related work**

The characterization of manipulation-resistant mechanisms in scenarios without monetary transfers is highly related to problems in *Approximate mechanism design without money* [15]. In these problems, a specific cardinal utility of the agents is assumed, and the designer seeks to find an outcome maximizing a desired target function (e.g., sum of utilities, product of utilities, or minimal utility). These works bound the trade-off between the target function and manipulation-resistance. I.e., they bound the loss to the target function due to manipulation-resistance constraints.

Several other variants of the facility location problem were also considered in the literature: For instance, Schummer and Vohra [18] considered the case of continuous graphs, Lu et al. [9,8] studied variants in which several facilities need to be located and scenarios in which an agent is located on several locations, and Feldman et al. [4] studied the impact of constraining the input language of the agents.

False-name-proofness was first introduced by Yokoo et al. [22] in the framework of combinatorial auctions. In this work, the authors showed that the VCG mechanism does not satisfy false-name-proofness in the general case, and they proposed a condition of the preferences under which this mechanism becomes false-name-proof. Later, Conitzer and Sobel [3] analyzed false-name-proof mechanisms in voting scenarios, Todo et al. [20] characterized other false-name-proof mechanisms for combinatorial auctions, and Todo et al. [19] characterized the false-name-proof mechanisms for facility location on the continuous line and on continuous trees, and analyzed the implications for the design of mechanisms for social-welfare maximization (both for sum-of-costs and for maximal-cost).

**Our contribution**

In this paper we present a family of unweighted graphs, ZV-line graphs, and show a general mechanism for facility location over these graphs that is on one hand resistant to all the above manipulations, while on the other hand it is anonymous, so in particular no agent is ignored, and it is Pareto optimal. Roughly speaking, in ZV-line graphs there are two types of locations V and Z, and the facility will be ‘commonly’ (except if all agents agree differently as follows) located on a Z-vertex (e.g., commercial locations for locating a public mall, or a status-quo set of outcomes). In addition, we also require that the V-vertices

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Note that this property combined with Citizen-sovereignty entails Pareto optimality. Nevertheless, we prefer to think of Pareto optimality apart from this property due to the different motivation.
are divided into $k$ disjoint subgraphs (e.g., neighborhoods) and that the agents agree on some predefined order of the $Z$-vertices (similarly to the single-peaked case \textsuperscript{[II]} s.t. in any such neighborhood there is a unique vertex which is connected to $Z$-vertices, and it is connected to a sequence of $Z$-vertices. Essentially, the mechanism we present in this work locates the facility in the first (according to the predefined order) Pareto optimal $Z$-vertex, unless all agents voted unanimously for one of the subgraphs. To the best of our knowledge, this is the first work to show a general false-name-proof mechanism for a general family.

For example, for bi-cliques (full bipartite graphs), in which all the subgraphs are singletons,

\[
\begin{array}{cccccc}
\ldots & \circ & \circ & \circ & \circ & \ldots \\
\ldots & \circ & \circ & \circ & \circ & \ldots \\
\ldots & \circ & \circ & \circ & \circ & \ldots \\
\end{array}
\]

our mechanism is equivalent to returning a (predefined) status-quo location unless all agents voted unanimously to one of the other locations. A second example is the line graph, which can be represented as a $ZV$-line graph in the following way (and we use here $\bigcirc$ for $Z$-vertices and $\circ$ for $V$-vertices)

\[
\begin{array}{cccccc}
\ldots & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \ldots \\
\end{array}
\]

In particular, note that we show a strategy-proof, false-name-proof, Pareto optimal mechanism which is far from median mechanisms (for instance, in the common case the mechanism output is in a subset consisting of only half the locations), in contrary to the characterization of these mechanisms for the continuous line \textsuperscript{[19] Thm. 2}.

Two generalizations for the line graph are the following two graphs\textsuperscript{[6]}

\[
\begin{array}{cccccc}
\ldots & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \ldots \\
\end{array}
\]

\textsuperscript{6}Actually, finding mechanisms for these two graphs ignited this research.
In all these examples the $V_i$-subgraphs are singletons. From our result, one could also construct mechanisms of generalizations of these graphs when the $V$-vertices are replaced by a tree, a clique, or concatenation of these graphs.\footnote{For example,}

A common property to the above examples is their regularity: All the $V$-vertices have the same degree (and naturally also all the $Z$-vertices). Last, we encountered this graph

\begin{center}
\includegraphics[width=0.2\textwidth]{example_graph1.png}
\end{center}

which can be represted as a $ZV$-line graph in the following way:

\begin{center}
\includegraphics[width=0.5\textwidth]{example_graph2.png}
\end{center}

Notice that this graph cannot be represented as a regular $ZV$-line graph. We suspect this might be the reason that we could not find mechanisms (which satisfy the desiderata) for families we constructed.
as generalizations of the graph, e.g., the planar grid and

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
  \draw (0,1) -- (0,2) -- (1,2) -- (1,1) -- cycle;
  \draw (1,0) -- (2,0) -- (2,1) -- (1,1) -- cycle;
  \draw (1,1) -- (1,2) -- (2,2) -- (2,1) -- cycle;
  \draw (2,0) -- (3,0) -- (3,1) -- (2,1) -- cycle;
  \draw (2,1) -- (2,2) -- (3,2) -- (3,1) -- cycle;
  \draw (3,0) -- (4,0) -- (4,1) -- (3,1) -- cycle;
  \draw (3,1) -- (3,2) -- (4,2) -- (4,1) -- cycle;
  \draw (4,0) -- (5,0) -- (5,1) -- (4,1) -- cycle;
  \draw (4,1) -- (4,2) -- (5,2) -- (5,1) -- cycle;
  \draw (5,0) -- (6,0) -- (6,1) -- (5,1) -- cycle;
  \draw (5,1) -- (5,2) -- (6,2) -- (6,1) -- cycle;
\end{tikzpicture}
\end{center}

2 Model

Consider a graph \( G = \langle V, E \rangle \) with a set of vertices \( V \) and a set of (neither weighted nor directed) edges \( E \subseteq \binom{V}{2} \), and we refer to the vertices \( v \in V \) also as locations and use the two terms interchangeably. The distance between two vertices \( v, u \in V \), notated \( d(v, u) \), is the length of the shortest path connecting \( v \) and \( u \). We define \( B(v, d) \), the ball of radius \( d \geq 0 \) around a vertex \( v \in V \), to be the set of vertices of distance at most \( d \) from \( v \), i.e.,

\[
B(v, d) = \{ u \in V \mid d(v, u) \leq d \}.
\]

We say that two vertices are neighbors if the distance between them is 1 (i.e., there is an edge connecting them). We notate by \( N(v) \) the set of neighbors of a vertex \( v \), and by \( N(S) \) the set of neighbors of a set of vertices \( S \), that is, \( N(S) = \bigcup_{v \in S} N(v) \).

An instance of the facility location problem over \( G \) is comprised of \( n \) agents who are located on vertices of \( V \); Formally, we represent it by a location profile \( x \in V^n \) where \( x_i \) is the location of Agent \( i \). Given an instance \( x \), we would like to locate a facility on a vertex of the graph while taking into account the preferences of the agents over this location. In this work we assume the preference of an agent is defined solely by her distance to the facility, that is, an agent located on \( x \in V \) strictly prefers the facility to be located on \( v \in V \) over it being located on \( u \in V \) iff \( d(x, v) < d(x, u) \), and she is indifferent between the two locations in case of equality.

A general facility location mechanism (or shortly a mechanism) defines for any profile of locations a location for the facility, i.e., a function \( F: \bigcup_{n \geq 0} V^n \rightarrow V \). Note we assume a mechanism assigns a location for the facility for any profile and any number of agents. We also think on \( F \) as a voting procedure: Each agent votes (and we also refer to his vote as a ballot) for a location, and based on the ballots \( F \) returns a location for the facility. We say that a mechanism is anonymous if the outcome \( F(x) \) does not depend on the identities of the agents, i.e., it can be defined as a function of the ballot tally, the number of votes for each of locations.

\[\text{Footnote: For simplicity, we assume the graph is connected.}\]
Manipulations-resistance

A strategic agent might act untruthfully if she thinks it might cause the mechanism to return a location she prefers (i.e., closer to her). In this work we consider the following manipulations:

- **Misreport**: An agent might report to the mechanism a location different from her real location;
- **False-name-report**: An agent might pretend to be several agents and submit several (not necessarily identical) ballots;
- **Abstention**: An agent might choose not to participate in the mechanism at all. A mechanism in which no agent benefits by these manipulation, regardless to the ballots of the other agents, is said to be strategy-proof, false-name-proof, and abstention-proof, respectively. We also consider a generalization of these manipulations to manipulations of a coalition, and say a mechanism is group-manipulation-proof if no coalition can change the outcome, by misreporting, false-name-reporting, or abstaining, to a different location which they unanimously agree is no worse than the original outcome (i.e., if they vote sincerely) and at least one of its members strictly prefers the new location. Notice that this is a very strong definition of a resistance to manipulations - We do not require that all the members of the coalition will use the same (insincere) deviation and we do not require all the them to strictly prefer to deviate.

**Definition 1** (Group-manipulation-proof)[10] A mechanism \( F \) is not group-manipulation-proof if there exists a vector of locations \( x \in V^n \), a coalition of agents \( C \subseteq \{1, \ldots, n\} \), and a set of ballots \( A \in \bigcup_{n \geq 0} V^n \) s.t. (i) all the members of \( C \) weakly prefer \( F(A, x_{\neg C}) \), that is the outcome when the agents outside of \( C \) do not change their vote and the agents of \( C \) replace their ballot by \( A \), over \( F(x) \) and (ii) at least one of \( C \)'s members strictly prefers \( F(A, x_{\neg C}) \) over \( F(x) \).

Notice that for \( C = \{i\} \) being a singleton, this general manipulation coincides with misreport for \( |A| = 1 \), with false-name-report for \( |A| > 1 \), and with abstention for \( A = \emptyset \).

The revelation principle

One could also consider more general mechanisms in which the agents vote using more abstract ballots, and define similar a manipulation-resistance terms for the the general framework. Applying a simple direct revelation principle [13] shows that any such general manipulation-resistant mechanism is equivalent to a manipulation-resistant mechanism in our framework: The two mechanism implement the same mapping of the agents private preferences to a location of the facility.

Efficiency

So far we defined the desired manipulation-resistance properties for a mechanism. On the other hand, we would also like the mechanism to respect the preferences of the agents, e.g., we would like to avoid a

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9A special case of false-name-voting which is considered in the literature is double-voting - Casting the same ballot several times to increase its impact.

10For simplicity of notations, we give the formal definition for anonymous mechanisms.
scenario in which, after the mechanism have been used, the agents can agree that a different location is preferable.

Given a location profile \( x \in V^n \), the set of Pareto optimal locations, \( PO(x) \), is the set of all locations which the agents cannot agree to rule out. Formally, a location \( v \in V \) is Pareto optimal if there does not exist another location \( u \in V \) s.t. (i) all agents weakly prefer \( u \) over \( v \) and (ii) at least one agent strictly prefers \( u \) over \( v \). We say a mechanism is Pareto optimal if for any location profile \( x \), \( F(x) \in PO(x) \). In particular, notice that Pareto optimality entails unanimity, if all the agents unanimously vote for the same location then the mechanism must put the facility on that location, and citizen sovereignty, the mechanism is onto and does not rule out a-priori any location.

3 Main Result

In this work we define a family of graphs, \( ZV\text{-line graphs} \), and present a general mechanism for this family.

Definition 2 (\( ZV\text{-line graph} \)).

An unweighted undirected graph \( G = (V, E) \) is a \( ZV\text{-line graph} \) if

- There exists a partition of \( V \) to \((k + 1)\) disjoint sets \( V = V_1 \cup \ldots \cup V_k \cup Z \) s.t.

\[
E = E_1 \cup \ldots \cup E_k \cup E_Z,
\]

for \( E_i \subseteq \binom{V_i}{2} \) and \( E_Z \subseteq Z \times (V \setminus Z) \), i.e., there are no edges between vertices of different \( V_i \)-subgraphs or between vertices in \( Z \).

- \( Z \) is equipped with a line topology. That is, there exists an order of \( Z \) from left to right (an injective mapping of \( Z \) to \( \mathbb{R} \)), and we refer to a sequence of vertices according to the order as an interval.

- For \( i = 1, \ldots, k \) there exists a vertex \( \mathcal{R}(V_i) \in V_i \) which we refer to as the root of \( V_i \), s.t. \( N(\mathcal{R}(V)) \cap Z \) is an interval in \( Z \) and for any other vertex \( v \in V_i \setminus \{\mathcal{R}(V_i)\} \), \( N(\mathcal{R}(v)) \subseteq V_i \).

and we say that \( G \) is a \( ZV\text{-line graph} \) w.r.t. the partition \( V = V_1 \cup \ldots \cup V_k \cup Z \).

Two interesting special cases of \( ZV\)-line graphs are when all the \( V_i \)-subgraphs are singletons and the regular graphs of this family (for example, the graphs in the introduction).

Definition 3 (\( ZV\text{-line graph with singleton } V_i \text{-subgraphs} \)).

An unweighted undirected graph \( G = (V, E) \) is a \( ZV\text{-line graph with singleton } V_i \text{-subgraphs} \) if

- \( G \) is a bipartite graph, i.e., there exists a partition of \( V \) to two disjoint sets \( V = V' \cup Z \) s.t.

\[
E \cap \binom{V'}{2} = E \cap \binom{Z}{2} = \emptyset \text{ (there are edges only between vertices in } V' \text{ and vertices of } Z \).
• Z is equipped with a line topology.

• ∀v ∈ V', N(v) is an interval in Z.

Definition 4 ((k, ℓ)-uniform ZV-line graph with singleton V_i-subgraphs).

A graph G is (k, ℓ)-uniform ZV-line graph with singleton V_i-subgraphs, if it is a a ZV-line graph with singleton V_i-subgraphs and every interval of ℓ vertices of Z is connected to k vertices in V' (except maybe boundary vertices).

Theorem 5 (Main result)[11]

Let G = (V, E) be a ZV-line graph w.r.t. a partition V = V_1 ∪ ... ∪ V_k ∪ Z and let F_i: ∪_{n≥0} (V_i)^n → V_i be k anonymous Pareto optimal mechanisms for the V_i-subgraphs s.t. R(V_i) equals to the outcome of F_i for profiles in which each of the locations in V_i received n ballots, for an infinite number of n’s[12]

Define F to be the following mechanism: Given a vector of reports x ∈ ∪_{n≥0} V^n

- If all the ballots belong to the same V_i-subgraph, return F_i(x).
- Otherwise, return the leftmost Pareto optimal location in Z.

Then, F is an anonymous and Pareto optimal mechanism, and if for i = 1, ..., k:

For any vector of locations x ∈ (V_i)^n, a coalition of agents C, and a set of ballots A ∈ ∪_{n≥0} (V_i)^n[13] A is not a beneficial deviation for C (That is, C does not strictly prefer F_i(A, x_C) over F_i(x)).

then also for F:

For any vector of locations x ∈ V^n, a coalition of agents C, and a set of ballots A ∈ ∪_{n≥0} V^n, A is not a beneficial deviation for C.

Before proving the theorem we note the following:

• F is well defined: If x is not included in any of the V_i-subgraphs, then there are two locations, x_i and x_j and a short path between them s.t. all its vertices are in PO(x) and at least one of its vertices is in Z. Hence, PO(x) ∩ Z ≠ ∅ and in particular the leftmost location in PO(x) ∩ Z is well-defined.

• F runs in polynomial time: Checking whether an agent weakly prefers location a over location b can be done in polynomial time. Hence, also checking whether all the agents weakly prefer one location over the other, and checking for each location whether no other location is strongly preferred over it by the grand coalition, can be done in a polynomial time.

[11]We describe the strong version of the theorem, deriving from resistance of F_i to any manipulation, the same resistance for F. The same proof shows that also weaker manipulation resistance properties of F_i (e.g., against individual agents, against misreporting, or against abstentions) result in the same manipulation resistance for F.

[12]Notice that if V_i is a singleton then F_i is trivial.

[13]Since F_i (and F) are anonymous we can define it A as a set ignoring identities.
• By considering singleton coalitions, we see that in particular $F$ is strategy-proof, false-name-proof, and abstention-proof.

Moreover, no coalition can find a beneficial deviation, by assigning misreporting, false-name-reporting, or abstaining among its members\(^\text{14}\).

• If $F_1,\ldots,F_k$ can be defined as the ‘first Pareto optimal location in some order,’ then an equivalent way to define $F$ is as the first Pareto optimal location in the following order:

  First go over the vertices of the $V_i$-subgraphs in some order s.t. for each subgraph the order over its vertices matches the order of $F_i$, and then on the vertices of $Z$ from left to right.

• As a corollary for the case of singleton $V_i$-subgraphs we get

**Corollary 6 (\([\text{14}]\)).**

Let $G = (V,E)$ be a $ZV$-line graph with singleton $V_i$-subgraphs w.r.t. a partition $V = V' \cup Z$, and define $F$ to be the following mechanism

  \begin{itemize}
    \item If all ballots are identical, return this location as the outcome.
    \item Otherwise, return the leftmost Pareto optimal location in $Z$.
  \end{itemize}

Then, $F$ is an anonymous and Pareto optimal mechanism, and the following property holds: For any vector of locations $\mathbf{x} \in V^n$, a coalition of agents $C$, and a set of ballots $A \in \bigcup_{n \geq 0} V^n$, $A$ is not a beneficial deviation for $C$.

Note that the partition $V = V_1 \cup \ldots \cup V_k \cup Z$ (and the implied constraints on the edges) is neither necessary nor sufficient for the result. Even in the case of $ZV$-line graph with singleton $V_i$-subgraphs, bipartiteness by itself is neither necessary (E.g., given a clique, a mechanism which returns the first occupied location according to some predefined order satisfies the properties of the theorem) nor sufficient (E.g., $C_6$, the cycle of size 6, is a bipartite graph, but we found that there exists no mechanism for $C_6$ that satisfies the properties of the theorem\(^\text{15}\)).

Last, we note that the theorem does not hold for weighted graphs. Consider the following weighted graph

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\(^{14}\)Notice this is a strong notion of group manipulations resistance. A coalition cannot even find a deviation which is beneficial for one of its members, while not hurting the other members.

\(^{15}\)Proof sketch. We notate the vertices of $C_6$ by $\{0,1,2,3,4,5\}$ and w.l.o.g. assume that for the profile of 6 agents who vote for all 6 locations the result is 0.

• For the profile $\langle 1,2,4 \rangle$
  \begin{itemize}
    \item From false-name-resistance to manipulations for the first and last agents, the outcome must be either 0 or 2 (Any agent can change the result to be 0).
    \item From Pareto optimality, the outcome must be 2.
  \end{itemize}

Hence, from false-name-resistance, the outcome for the profile $\langle 2,4 \rangle$ must also be 2.

• Similarly, for the profile $\langle 2,4,5 \rangle$ the outcome must be 4, and hence the outcome for the profile $\langle 2,4 \rangle$ must also be 4.

So we get a contradiction.
and a profile in which Alice is located on \( z_\ell \) and Bob on \( v \). Then, the outcome is \( z_r \), but Bob can move the facility to a preferred location \( z_\ell \) both (i) by misreporting \( z_\ell \), hence \( F \) is not strategy-proof, and (ii) by false-name-reporting \( z_\ell \) in addition to his sincere report, hence \( F \) is not false-name-proof.

3.1 Implication: Mechanisms for recursive graph families

By applying the main result to recursive families of graphs, we can generate a recursive (and hence commonly simple) mechanisms which satisfy our desiderata. For instance, a corollary of our result is that for any tree \( G \) the mechanism that returns the lowest common ancestor of the ballots (with regard to some root) is a manipulations-resistant mechanism. We show that by induction over the height of \( G \), \( h(G) \).

If \( h(G) = 0 \), i.e., \( G \) consists of a single vertex and the trivial mechanism satisfies all the desired properties.

If \( h(G) \geq 1 \), then the root of \( G \), which will be a single 1-vertex, is connected to disjoint subtrees of height at most \( h(G) - 1 \), which will be the \( V_i \)-subgraphs. Hence, our recursive mechanism returns the lowest common ancestor in case that all the ballots lie in the same subtree, and the root otherwise. That is, the outcome of the mechanism is always the lowest common ancestor of the ballots.

4 Proof of Main Theorem (Thm. 5)

The anonymity and Pareto optimality of \( F \) are clear by the \( F \) is defined (Notice that if all agents are in the same \( V_i \)-subgraph, then all of them strictly prefer \( R(V_i) \) over any location outside of \( V_i \), so \( PO(x) \subseteq V_i \)). In order to prove the main part of the theorem we first prove the following two auxiliary lemmas. Notice that since \( G \) is a \( ZV \)-line graph, \( N(z) = \bigcup_{z \in Z} N(z) = \{ R(V_1), \ldots, R(V_k) \} \).

Abusing notation we notate by \( R(v) \) for \( v \notin Z \), the root of the \( V_i \)-subgraph \( v \) belongs to.

Lemma (i). For any \( v \in V \) and \( d \geq 0 \), \( B(v, d) \cap Z \) is an interval in \( Z \).

Proof of Lemma (i).

For \( d = 0 \), \( B(v, 0) \cap Z \) equals to \( \{ v \} \) if \( v \in Z \) and to the empty set if \( v \notin Z \).

We’ll prove the lemma for \( v \in N(Z) \) and odd \( d \geq 1 \). The correctness for \( v \in Z \) and even \( d \geq 2 \) follows since \( B(v, d) = \bigcup_{u \in N(v)} B(u, d - 1) \) so \( B(v, d) \cap Z \) is an interval as the union of intersecting intervals \( (N(v) \subseteq N(Z) \) and \( v \in B(u, d - 1) \cap Z \) for all \( u \in N(v) \)). The correctness for \( v \in Z \) and odd \( d \geq 1 \) (and similarly for \( v \in N(Z) \) and even \( d \geq 2 \)) follows since \( B(v, d) \cap Z = B(v, d - 1) \cap Z \). The
correctness for all other locations (that is, locations of distance at least 2 from \( Z \)) follows since there exists a location \( u = \mathcal{R}(v) \) s.t. \( u \in N(Z) \) and all shortest paths from \( v \) to locations in \( Z \) pass through \( u \), hence \( B(v,d) \cap Z = B(u,d - d(v,u)) \cap Z \).

Now, we prove the lemma for \( v \in N(Z) \) and odd radius \( d = 2k + 1 \) by induction over \( k \):

For \( k = 0 \), \( B(v,1) \cap Z = N(v) \cap Z \) which is an interval since \( G \) is a ZV-line graph.

For \( k \geq 1 \) (i.e., \( d \geq 3 \)), we notice that

\[
B(v,d) = \bigcup_{u \in B(v,2)} B(u,d - 2) \quad \text{and} \quad B(v,d) \cap Z = \bigcup_{u \in B(v,2) \cap N(Z)} B(u,d - 2) \cap Z,
\]

so \( B(v,d) \cap Z \) is an interval as the union of intersecting intervals (If \( u \in B(v,2) \cap N(Z) \) then there exists

\[
z \in N(v) \cap N(u) \cap Z,
\]

and in particular \( z \in B(u,d - 2) \cap B(v,d - 2) \). \( \Box \)

**Lemma (ii).** Let \( x \) be a vector of locations s.t \( F(x) \in Z \) and let \( v \in Z \) be a location s.t. Agent \( i \) strictly prefers \( v \) over \( F(x) \). Then \( F(x) \) is to the left of \( v \).

**Proof of Lemma (ii).**

If \( x_i \in Z \) then \( x_i \in PO(x) \cap Z \) and by the definition of \( F \), \( F(x) \) is to the left of \( x_i \). Since \( F(x) \notin B(x_i,d(x_i,v)) \cap Z \) and since this set is an interval which includes \( x_i \), we get that \( F(x) \) is to the left of the interval and in particular to the left of \( v \).

Otherwise, \( x_i \notin Z \) and there exists an Agent \( j \) for which \( x_j \) is not in the same \( V_i \)-subgraph as \( x_i \) (either in a different \( V_i \)-subgraph or in \( Z \)). Hence, there exists a location \( u \in Z \) which satisfies

- \((i)\) \( u \) is on a shortest-path from \( x_i \) to \( x_j \),
- \((ii)\) \( u \in \arg\min_{z \in Z} d(x_i,z) = N(\mathcal{R}(x_i)) \cap Z \), and
- \((iii)\) \( u \in PO(x) \).

Hence,

\[
d(x_i,u) \leq d(x_i,v) < d(x_i,F(x)),
\]

and for \( d_u, d_v \), and \( d_{F(x)} \) being the three distances, respectively,

\[
u \in B(x_i,d_u) \cap Z \subseteq B(x_i,d_v) \cap Z \subseteq B(x_i,d_{F(x)}) \cap Z.
\]

The three sets are intervals in \( Z \), \( F(x) \) is to the left of \( u \), and

\[
F(x) \in \left(B(x_i,d_{F(x)}) \cap Z\right) \setminus \left(B(x_i,d_v) \cap Z\right),
\]

so \( F(x) \) is to the left of \( v \). \( \Box \)
Now, assume for contradiction that there exists a vector of locations \( \mathbf{x} \in V^n \), a coalition of agents \( C \), and a set of ballots \( \mathbf{A} \in \bigcup_{n \geq 0} V_n \), s.t. \( C \) can, by voting \( \mathbf{A} \), get an outcome \( F(\mathbf{A}, \mathbf{x}_C) \) which is strictly prefers, that is, all of its members weakly prefer \( F(\mathbf{A}, \mathbf{x}_C) \) over \( F(\mathbf{x}) = F(\mathbf{x}_C, \mathbf{x}_C) \), and at least one of \( C \)'s members, Agent \( i \) for \( i \in C \), strictly prefers \( F(\mathbf{A}, \mathbf{x}_C) \) over \( F(\mathbf{x}) \). Since \( F(\mathbf{x}) \in PO(\mathbf{x}) \), there exists an Agent \( j \), for \( j \notin C \), who strictly prefers \( F(\mathbf{x}) \) over \( F(\mathbf{A}, \mathbf{x}_C) \).

If both \( F(\mathbf{x}) \) and \( F(\mathbf{A}, \mathbf{x}_C) \) are in \( Z \): By applying Lemma [ii] for the profile \( \mathbf{x} \) and Agent \( i \), we get that \( F(\mathbf{x}) \) is to the left of \( F(\mathbf{A}, \mathbf{x}_C) \); and by applying Lemma [ii] for the profile \( (\mathbf{A}, \mathbf{x}_C) \) and Agent \( j \), we get that \( F(\mathbf{A}, \mathbf{x}_C) \) is to the left of \( F(\mathbf{x}) \). Hence, we get a contradiction.

If \( F(\mathbf{x}) \) is not in \( Z \): Then necessarily, all the locations in \( \mathbf{x} \) and \( F(\mathbf{x}) \) belong to the same \( V_i \)-subgraph, w.l.o.g., \( V_1 \), so \( F(\mathbf{x}) = F_1(\mathbf{x}) \). Since \( F_1 \) is resistant to FNP-manipulations of Agent \( i \), we get that Agent \( i \) weakly prefers \( F_1(\mathbf{x}) = F(\mathbf{x}) \) over \( R(V_1) \), which we assumed Agent \( i \) can achieve by casting enough false ballots. Since for any \( u \) outside of \( V_1 \) it holds that \( d(x_i, R(V_1)) < d(x_i, u) \), we get that \( F(\mathbf{A}, \mathbf{x}_C) \in V_1 \), and hence \( \mathbf{A} \subseteq V_1 \) and \( F(\mathbf{A}, \mathbf{x}_C) = F_1(\mathbf{A}, \mathbf{x}_C) \). Hence, we get a contradiction to the FNP resistance of \( F_1 \).

Similarly, if \( F(\mathbf{A}, \mathbf{x}_C) \) is not in \( Z \): Then necessarily, \( F(\mathbf{A}, \mathbf{x}_C) \) and all the locations in \( \mathbf{A} \) and \( \mathbf{x}_C \) belong to the same \( V_i \)-subgraph, w.l.o.g., \( V_1 \), so \( F(\mathbf{A}, \mathbf{x}_C) = F_1(\mathbf{A}, \mathbf{x}_C) \). Since \( F_1 \) is resistant to FNP-manipulations of Agent \( j \), we get that Agent \( j \) weakly prefers \( F(\mathbf{A}, \mathbf{x}_C) \) over \( R(V_1) \). Since for any \( u \) outside of \( V_1 \) it holds that \( d(x_j, R(V_1)) < d(x_j, u) \), we get that \( F(\mathbf{x}) \in V_1 \), and hence \( \mathbf{x} \subseteq V_1 \) and \( F(\mathbf{x}) = F_1(\mathbf{x}) \). Hence, we get a contradiction to the FNP resistance of \( F_1 \). □

## 5 Summary & Extensions

In this work we presented a family of graphs, ZV-line graphs, and a general anonymous Pareto optimal manipulations-resistant mechanism for the facility location problem on these graphs. To the best of our knowledge, this is the first work to show a general false-name-proof mechanism for a large family. The construction we presented is recursive, and derives a mechanism for a given ZV-line graph from mechanisms for its subgraphs (which might not be ZV-line graphs). Hence, it is straightforward to derive from the construction general mechanisms for recursive families of graphs.

We assumed that our graphs are connected, but it is not hard to see that the following immediate extension for unconnected graphs will satisfy the same desired properties.

- At the first stage, choose the first connected component according to some predefined order s.t. at least one agent voted for a location in this component.

- At the second stage, run our mechanism taking into account only agents who voted for locations in the chosen component.

Note that, just like the original mechanism, also this mechanism can be equivalently defined as the first Pareto optimal location according to some order of the vertices (which is the concatenation on the respective orders for the different components).
Clearly, the mechanism we presented is not the only mechanism satisfying the desired properties. A mechanism that will take at the second stage the rightmost Pareto optimal $Z$-vertex will also satisfy them. Moreover, a generalization of our proof shows that a mechanism which outputs at the second stage the Pareto optimal $Z$-vertex closest (in the topology of $Z$) to a predefined vertex $\alpha \in Z$ will also satisfy the same desiderata. Notice that by taking $\alpha$ to be the leftmost vertex in $Z$ we get the mechanism described in the main result. We conjecture that essentially these are the only anonymous Pareto optimal manipulations-resistant mechanisms.

A second simple generalization of our result, omitted due to space considerations, is assuming a more complex topology, e.g. topology of a tree, over $Z$ instead of the line topology we assumed in this work.

Last, an important continuation of this work is analyzing the implications for Approximate mechanism design without money. That is, assuming the agents are accurately represented by a cost function (e.g., the distance to the facility or a monotone function of the distance) and analyzing the implications of manipulation-resistance on the approximability of natural social welfare functions, e.g., the average cost (Harsanyi’s social welfare), the geometric mean of the costs (Nash’s social welfare), or the maximal cost (Rawls’ criterion).

\footnote{We omit here some details regarding how to choose a location in case there are Pareto optimal locations on both sides of $\alpha$.}
References


