Stochastic Dominance and Decomposable Measures of Inequality and Poverty

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January 18, 2018

Abstract: In this note we show a closer link between second-degree stochastic dominance (SD) and measures of inequality and poverty than previously recognized. For a normalized second-degree SD curve, the weighted area between the curve and that of the equal distribution characterizes the class of decomposable transfer-sensitive inequality measures. For a censored second-degree SD curve, the weighted area between the curve and that of the zero-poverty distribution characterizes the class of decomposable distribution-sensitive poverty measures. Some extensions are considered and the properties of the weighting functions are also explored.

JEL Classifications: I32

Key Words: Stochastic Dominance, Normalized Stochastic Dominance, Censored Stochastic Dominance, Decomposable Inequality Measure, Decomposable Poverty Measure.
1. Introduction

A monument in the history of income inequality measurement is the establishment of the close connection between Lorenz curve, summary measures of inequality such as the Gini coefficient and the Pigou-Dalton principle of transfers. Atkinson (1970), drawing upon Rothchild and Stiglitz’s (1970) characterization of second-degree stochastic dominance (SSD), proved that the dominance of Lorenz curves between two income distributions implies and is implied by the unanimous rankings for all relative inequality measures that satisfy the Pigou-Dalton principle of transfers. The equivalence between Lorenz dominance and second-degree stochastic dominance (for equal-mean distributions) also makes SSD a dual inequality ordering condition. By normalizing all distributions (e.g., dividing all incomes in each distribution by its respective mean income) to have the same mean income, the application of SSD would yield the same inequality rankings as the Lorenz dominance. Although not as commonly used as Lorenz dominance, this dual inequality ordering condition has been first noted in Foster and Sen (1997) and is generally known as “normalized stochastic dominance” (Zheng et al., 2000) to distinguish it from stochastic dominance which does not require an equal mean condition.

The close connection that Atkinson uncovered also allows us to define and characterize Lorenz-consistent inequality measures by extracting and summarizing the information contained in the Lorenz curve. The best known example is the Gini coefficient which is twice the area enclosed by the Lorenz curve and the 45 degree line (which represents the Lorenz curve for the equal distribution). Another well-known example is the Schutz coefficient which is the maximum distance between the Lorenz curve and the 45 degree line (though the measure does not satisfy Pigou-Dalton’s principle of transfers). Generalizing the Gini coefficient which is the un-weighted (or equally weighted) area, Shorrocks and Slottje (2002) defined a class of “Gini type inequality measures” by weighting the area between a Lorenz curve and the 45 degree line. The Gini type measures, as noted by Shorrocks and Slottje (2002), are closely related to the “generalized Gini” measures examined by Donaldson and Weymark (1980), Weymark (1981) and Bossert (1990), and the linear measures by Mehran (1976). Aaberge (2000), drawing upon the similarity between the Lorenz curve and a cumulative probability distribution function, characterized the moments of the Lorenz curve as inequality measures. Since Aaberge’s “Lorenz family of inequality measures” essentially employs a specific function (the power function) in weighting the area, the family is also contained in the class that Shorrocks and Slottje (2002) defined.

None of the inequality measures characterized via the Lorenz curve is decomposable. In fact, all Gini type measures are the so-called rank-based measures since the relative position of an income matters in determining the level of inequality for a society. For a decomposable inequality measure, the relative rank does not matter and the overall inequality can be decomposed as a sum of “within-group inequality” and “between-group inequality.” Given the importance of decomposable inequality
measures in inequality measurement, it is useful to establish a similar connection between decomposable inequality measures and a partial ordering condition. In this note, we show that the dual inequality ordering condition - the normalized stochastic dominance (NSD) - is able to characterize the entire class of decomposable inequality measures. Specifically, we show that the weighted area between the second-degree NSD curve of a given distribution and that of the equal distribution is a decomposable inequality measure; the different weighting functions lead to different measures. The use of the area between the NSD curves to characterize an inequality measure is not new: Formby et al. (1999) demonstrated that the area between the two second-degree NSD curves is one-half of the squared coefficient of variation which is a member the Generalized Entropy family (Shorrocks, 1980, 1984). This characterization of decomposable inequality measures also provides a direct proof that the decomposable class is sufficient to characterize inequality orderings. Thus the unanimous rankings by all decomposable inequality measures and by all Gini type inequality measures provide two alternative but equivalent necessary and sufficient conditions for inequality orderings.

The note then extends the characterizations for inequality measurement to poverty measurement. In poverty measurement, income distributions are no longer normalized. Instead, all income distributions are censored at the poverty line (assume a fixed poverty line). That is, all incomes above the poverty line are replaced by the poverty line. The censored stochastic dominance (CSD) results when stochastic dominance is applied to the censored income distributions. We show that the weighted area between the two CSD curves (an arbitrary distribution and the distribution with no poverty) is a decomposable poverty measure. We also show that all known decomposable poverty measures are members of this CSD decomposable class with appropriate weighting functions. Again, the decomposable class is sufficient to characterize partial poverty orderings.

2. The Characterizations

Let $x \in [a, b]$ be an income variable with $0 < a < b < \infty$. The distribution of income is summarized by a right continuous and strictly increasing cumulative distribution function (cdf) $F$ and the set of all such pdfs is $\Omega$. Denote the mean income of $F$ as $\mu_F$, i.e., $\mu_F = \int_a^b xdF(x)$. Also define $F_1 \equiv F$ and $F_k(x) = \int_x^b F_{k-1}(t)dt$ for any integer $k \geq 2$. For two cdfs $F$ and $G \in \Omega$, $F$ (k-1)th degree stochastic dominates $G$ if and only if $F_{k-1}(x) \leq G_{k-1}(x)$ for all $x \in [a, b]$ with the inequality holding strictly for some $x$.

2.1. Decomposable inequality measures

An inequality measure $I(F)$ indicates the inequality level associated with distribution $F$; $I(F)$ is assumed to be differentiable in income and satisfies Pigou-Dalton’s principle of transfers that a progressive (rich-to-poor) transfer of income reduces inequality. A decomposable inequality measure is the one that the overall inequality
value of the population “can be expressed as a weighted sum of the inequality value calculated for population subgroups plus the contribution arising from differences between subgroup means” (Shorrocks, 1980). The best known measures of decomposable inequality measures are the members of the generalized entropy (GE) class characterized by Cowell (1978), Bourguignon (1979), Foster (1983) and Shorrocks (1980, 1984). Although these previous characterizations may require working with discrete distributions, the GE class can nevertheless be expressed for a continuous distribution $F$ as

$$I^r(F) = \begin{cases} 
\frac{1}{\alpha(\alpha-1)} \int_a^b \left( \left( \frac{x}{\mu_F} \right)^\alpha - 1 \right) dF(x), & \alpha \neq 0, 1, \\
\int_a^b \ln \frac{\mu_x}{x} dF(x), & \alpha = 0, \\
\int_a^b x \ln \frac{x}{\mu_F} dF(x), & \alpha = 1.
\end{cases}$$

The GE class contains some well-known measures of inequality: the (squared) coefficient of variation ($\alpha = 2$) and the two Theil measures ($\alpha = 0, 1$).

The GE measures are the so-called relative measures in the sense that only relative income matters (doubling all incomes leaves the inequality value unchanged). The two other types of inequality measures are absolute measures (adding the same dollar amount to all incomes leaves the inequality value unchanged) and intermediate measures (doubling all incomes increases inequality and adding the same dollar amount reduces inequality). Chakravarty and Tyagarupananda (1998) showed that the only decomposable absolute inequality measures are the measures introduced by Kolm (1976)

$$I^l(F) = \int_a^b \left[ e^{\alpha(x-\mu_F)} - 1 \right] dF(x), \alpha \neq 0.$$

and the variance

$$I^l(F) = \int_a^b (x-\mu_F)^2 dF(x).$$

The inequality measures characterized in Zheng (2007) are examples of intermediate decomposable inequality measures (for any real value $\beta$):

$$I^m(F) = \begin{cases} 
\frac{1}{c(c-1)\mu_F} \int_a^b \left[ \left( \frac{x}{\mu_F} \right)^c - 1 \right] dF(x), & \alpha \neq 0, 1, \\
\frac{1}{\mu_F} \int_a^b \ln \frac{\mu_x}{x} dF(x), & \alpha = 0, \\
\frac{1}{\mu_F} \int_a^b x \ln \frac{x}{\mu_F} dF(x), & \alpha = 1.
\end{cases}$$

There are at least two other alternative definitions for “decomposability.” One definition is by Blackorby, Donaldson and Auersperg (1981) who suggested to use a “generalized mean” such as the geometric mean instead of the arithmetic mean in the decomposition. The other definition is by Ebert (2010) who suggested to define the “between-group” term as “the inequality between all pairs of individuals belonging to the groups involved” (p. 94). In this note, we characterize decomposition in its traditional sense as defined by Shorrocks (1980, 1984).
All these measures are derived using Shorrocks’s fundamental theorem about the structure of a decomposable inequality measure. Shorrocks (1980) proved that, under certain regularity conditions, an inequality measure $I(F)$, in its continuous form, is decomposable if and only if there exist functions $\lambda(\cdot)$ and $\phi(\cdot)$ such that

$$I(F) = \frac{1}{\lambda(\mu_F)} \int_a^b [\phi(x) - \phi(\mu_F)] dF(x)$$

(2.1)

where $\lambda(\cdot)$ is positive and differentiable, $\phi'(\cdot)$ is continuous and $\phi(\cdot)$ is strictly convex.

2.2. Normalized stochastic dominance and decomposable inequality measures

When stochastic dominance (SD) is applied to income distributions that are normalized to have equal means, the resulting dominance is referred to as normalized stochastic dominance (NSD). To achieve equal means, one can use the relative approach (e.g., dividing all incomes in a distribution by the mean of the distribution), the absolute approach (e.g., subtracting all incomes by the mean income and then adding the same income so all incomes are positive) or the intermediate approach (a mixture between the relative and absolute approaches). The different approaches will characterize different types of decomposable inequality measures. In our characterizations below, we do not specify which approach is used in the normalization process; we only assume that the mean incomes are the same across all distributions.

For any income distribution $F$, let $\tilde{F}$ be its equalized version of the distribution. That is, $\mu_{\tilde{F}} = \mu_F$, and

$$\tilde{F}(x) = \begin{cases} 0 & \text{if } x < \mu_F \\ 1 & \text{if } x \geq \mu_F. \end{cases}$$

It follows that

$$\tilde{F}_2(x) = \int_a^x \tilde{F}(t) dt = \begin{cases} 0 & \text{if } x < \mu_F \\ x - \mu_F & \text{if } x \geq \mu_F. \end{cases}$$

Clearly, $\tilde{F}$ second-degree stochastic dominates $F$ (i.e., $\tilde{F}_2(x) \leq F_2(x)$ for all $x \in [a, b]$) and we are interested in the area between $\tilde{F}_2$ and $F_2$.

Suppose the distance between $\tilde{F}_2$ and $F_2$, $F_2(x) - \tilde{F}_2(x)$, is weighted by a non-negative function $w(x)$. The weighted area between $\tilde{F}_2$ and $F_2$ is

$$\int_a^b w(x) [F_2(x) - \tilde{F}_2(x)] dx. \quad (2.2)$$

The first main result of the note is:

**Proposition 2.1.** For each weighting function $w(x) \geq 0$, the weighted area (2.2) is a decomposable inequality measure

$$\int_a^b [\tilde{w}(x) - \tilde{w}(\mu_F)] dF(x)$$

(2.3)

where $\tilde{w}(x) = \int_a^x \int_a^t w(s) ds dt$. Assume $\tilde{w}(b) < \infty$. 

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Proof. Denote $\Delta F_2(x) = F_2(x) - \bar{F}_2(x)$ and $\tilde{w}(x) = \int_a^x w(t)dt$. We first have

$$\int_a^b w(x) [F_2(x) - \bar{F}_2(x)] \, dx = \int_a^b \int_a^x w(t)dt \, d\int_a^b \Delta F_2(x) d\tilde{w}(x)$$

$$= \Delta F_2(x)\tilde{w}(x)|^b_a - \int_a^b \tilde{w}(x)d\{\Delta F_2(x)\}$$

$$= \Delta F_2(a)\tilde{w}(a) - \Delta F_2(b)\tilde{w}(b) - \int_a^b \tilde{w}(x)d\{\Delta F_2(x)\}$$

$$= -\int_a^b \tilde{w}(x)d\{\Delta F_2(x)\}$$

where we have used the fact that

$$\Delta F_2(a) = 0$$

and

$$\Delta F_2(b) = \int_a^b [F(x) - \bar{F}(x)] \, dx$$

$$= [F(x) - \bar{F}(x)]|_a^b - \int_a^b xd[F(x) - \bar{F}(x)]$$

$$= \int_a^b xd\bar{F}(x) - \int_a^b xdF(x) = 0$$

since the means of $F$ and $\bar{F}$ are equal. Further denote $\Delta F = F - \bar{F}$ and $\tilde{w}(x) = \int_a^x \tilde{w}(t)dt = \int_a^x \int_a^t w(s)dsdt$, then (using $\Delta F(a) = \Delta F(b) = 0$)

$$\int_a^b w(x) [F_2(x) - \bar{F}_2(x)] \, dx = -\int_a^b \tilde{w}(x)d\{\Delta F_2(x)\}$$

$$= -\int_a^b \tilde{w}(x)\Delta F(x)dx$$

$$= -\int_a^b \Delta F(x)d\tilde{w}(x)$$

$$= -\Delta F(x)\tilde{w}(x)|_a^b + \int_a^b \tilde{w}(x)d\{\Delta F(x)\}$$

$$= \int_a^b \tilde{w}(x)dF(x) - \int_a^b \tilde{w}(x)d\bar{F}(x)$$

$$= \int_a^b \tilde{w}(x)dF(x) - \tilde{w}(\mu_F)$$

$$= \int_a^b [\tilde{w}(x) - \tilde{w}(\mu_F)] \, d\bar{F}(x).$$

The proposition is proved by invoking Shorrocks's result in (2.1). Note that since we do not require $w(x)$ to satisfy any additional conditions (such as $\int_a^b w(x)dx = 6$).
1) beyond \( w(x) \geq 0 \), we do not further normalize inequality measures by a factor such as \( \lambda(\mu_F) \) and, thus, Shorrocks’s decomposable measure is simply defined as 
\[
\int_a^b [\phi(x) - \phi(\mu_F)]dF(x).
\]
□

A special case of Proposition 2.1 is when the area is weighted equally or unweighted, i.e., \( w(x) = 1 \) for all \( x \in [a,b] \). In this case, \( \hat{w}(x) = \int_a^x w(s)ds = \frac{1}{2}(x-a)^2 \), then
\[
\int_a^b [\hat{w}(x) - \hat{w}(\mu_F)]dF(x) = \frac{1}{2} \int_a^b [(x-a)^2 - (\mu_F - a)^2] dF(x)
\]
\[
= \frac{1}{2} \int_a^b [(x - \mu_F)^2] dF(x)
\]
which is one-half of the variance (Formby et al., 1999, demonstrated this via direct calculation of area).

In general, for a decomposable inequality measure (2.1), the weighting function used in (2.2) is \( w(x) = \frac{\partial^2 \phi(x)}{\partial x^2} \). For example, for a GE measure with \( \alpha \neq 0,1 \), \( \hat{w}(x) = \frac{x^\alpha}{\alpha(\alpha-1)} \), \( w(x) = x^{\alpha-2} \); for the two Theil measures, \( w(x) = x^{-2} \) and \( x^{-1} \), respectively; for the Kolm measure, \( \hat{w}(x) = e^{\alpha x} \) and thus \( w(x) = \alpha^2 e^{\alpha x} \). These results are summarized in the following corollary.

Corollary 2.1. The weighted NSD-area class of inequality measures (2.2) contains all known decomposable inequality measures.

Surely new decomposable inequality measures can be generated by using some new weighting functions. An interesting thing to note is that decomposable inequality measures together are sufficient to characterize stochastic dominance. By choosing appropriate weighting functions, it is easy to see that

Corollary 2.2. For two distributions \( F \) and \( G \in \Omega \) with equal mean, if all decomposable inequality measures in (2.1) indicate that \( F \) has less inequality than \( G \), then \( F \) second-degree stochastic dominates \( G \).

Given the equivalence between second-degree SD and Lorenz dominance, we know that \( F \) would Lorenz dominate \( G \) as well. Thus the class of decomposable inequality measures can also characterize Lorenz dominance.

A final consideration concerns the satisfaction of Kolm’s “principle of diminishing transfers” by a decomposable inequality measure (2.2). Kolm’s principle states that the inequality-reducing effect of a progressive transfer gets stronger as it happens lower down in the income distribution. Since the principle amounts to requiring \( \hat{w}'''(x) < 0 \) which implies \( w'(x) < 0 \) if \( w(x) \) is also assumed to be differentiable, we have

Corollary 2.3. A weighted-area decomposable inequality measure satisfies Kolm’s principle of diminishing transfers if and only if the weighting function is a decreasing of income.
Note that Pigou-Dalton’s principle of transfers requires \( \dot{w}'(x) > 0 \) which is readily satisfied by Shorrocks’s specification.

2.3. Decomposable poverty measures

For a given poverty line \( z \in [a, b] \), the poverty level associated with distribution \( F \) is indicated by a poverty measure \( P(F; z) \). A decomposable poverty measure is defined as\(^2\)

\[
P(F; z) = \int_a^b p(x, z) dF(x)
\]

where \( p(x, z) \) is an individual deprivation function defined over \([a, z] \) that satisfies \( p(x, z) \geq 0, p(z, z) = \lim_{x \to z} p(x, z) = 0, p_z(x, z) < 0 \) and \( p_{xz}(x, z) > 0 \). Examples of decomposable poverty measures include the Watts measure (Watts, 1968)

\[
P(F; z) = \int_a^z (\ln z - \ln x) dF(x),
\]

the Chakravarty measure (Chakravarty, 1981)

\[
P(F; z) = \int_a^z \left[ 1 - \left( \frac{x}{z} \right)^\beta \right] dF(x), 0 < \beta < 1
\]

the FGT measure (Foster et al, 1984)

\[
P(F; z) = \int_a^z \left( 1 - \frac{x}{z} \right) ^\alpha dF(x), \alpha \geq 0
\]

and the CDS measure (Zheng, 2000a)

\[
P(F; z) = \int_a^z \left[ e^{\gamma(z-x)} - 1 \right] dF(x), \gamma > 0.
\]

The FGT measure includes the commonly (and officially) used headcount ratio (\( \alpha = 0 \)) and the income gap ratio (\( \alpha = 1 \)) as special cases.

2.4. Censored stochastic dominance and decomposable poverty measures

For a given poverty line \( z \), income distributional information above the poverty line is not used in calculating the poverty value of the distribution. This property enables the use of censored income distributions in poverty measurement. A distribution is censored at \( z \) when all incomes above \( z \) are set to \( z \). A celebrated result in the poverty measurement literature is that a unanimous ranking between two income distributions for all poverty measures amounts to requiring stochastic dominance between the two censored distributions (Atkinson, 1987; Foster and Shorrocks, 1988; Zheng, 2000).

\(^2\)For a survey of poverty measurement including decomposable poverty measures and rank-based poverty measures such as those proposed by Sen (1976) and Thon (1979), see, for example, Zheng (1997).
1999). For instance, second-degree stochastic dominance between two censored distributions is equivalent to requiring a unanimous ranking by all distribution-sensitive poverty measures. For ease of reference, we may refer to stochastic dominance applied to censored income distributions as censored stochastic dominance (CSD). In this section, we establish an even closer link between CSD and decomposable poverty measures.

For distribution $F$, its censored distribution by the poverty line is

$$F^z(x) = \begin{cases} F(x) & \text{if } x \leq z \\ 1 & \text{if } x > z \end{cases}$$

and the censored distribution of zero-poverty (i.e., all incomes are above $z$) is

$$F^z(x) = \begin{cases} 0 & \text{if } x \leq z \\ 1 & \text{if } x > z \end{cases}.$$

It follows that the difference between the two censored distributions is

$$\Delta F^z(x) = F^z(x) - F^z(x) = \begin{cases} F(x) & \text{if } x \leq z \\ 0 & \text{if } x > z \end{cases}$$

and the difference between the cumulated cdfs is

$$\Delta F^z(x) = \begin{cases} F_2(x) & \text{if } x \leq z \\ F_2(z) & \text{if } x > z \end{cases}.$$

As we did in inequality measurement, we are interested in weighting the area between the two censored distributions. Considering the second-degree, the weighted area with a weighting function $v(x,z)$ is

$$\int_a^b v(x,z) \left[ F^z_2(x) - F^z_2(x) \right] dx. \quad (2.5)$$

The second main result of the note is:

**Proposition 2.2.** For each weighting function $v(x,z) > 0$, the weighted area (2.5) is a decomposable poverty measure

$$\int_a^b q(x,z)dx \quad (2.6)$$

where $q(x,z) = \tilde{v}(b,z)(z - x) - \hat{v}(z,z) + \hat{v}(x,z)$ with $\tilde{v}(x,z) = \int_a^x v(t,z)dt$ and $\hat{v}(x,z) = \int_a^z \tilde{v}(t,z)dt$.

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$^3$Similar results (albeit much less complete) between censored (generalized) Lorenz curve and some rank-based poverty measures such as those defined by Sen (1976) and Thon (1979) are sketched in Zheng (2000b) which surveys the literature of poverty orderings.
Proof. By the definition of $\Delta F^z_2(x)$ above, we have for a given $z$, 

$$\int_a^b v(x, z) \left[ F^z_2(x) - \bar{F}^z_2(x) \right] dx = \int_a^z v(x, z) \Delta F^z_2(x) dx + \int_z^b v(x, z) \Delta F^z_2(x) dx$$

$$= \int_a^z v(x, z) F_2(x) dx + \int_a^b v(x, z) F_2(z) dx$$

$$= \int_a^z F_2(x) d\tilde{v}(x, z) + \int_z^b F_2(z) d\tilde{v}(x, z).$$

The second part of the rhs of the above equation is simply $F_2(z)[\tilde{v}(b, z) - \tilde{v}(z, z)]$. For the first part, we further have 

$$\int_a^z F_2(x) d\tilde{v}(x, z) = \tilde{v}(x, z) F_2(x)\bigg|_a^z - \int_a^z \tilde{v}(x, z) F(x) dx$$

$$= \tilde{v}(z, z) F_2(z) - \int_a^z F(x) d\tilde{v}(x, z)$$

$$= \tilde{v}(z, z) F_2(z) - \tilde{v}(z, z) F(z) + \int_a^z \tilde{v}(x, z) dF(x).$$

Combining the two parts and use the fact $F_2(z) = \int_0^z (z - x) dF(x)$ (e.g., Fishburn, 1976), we arrive at (2.6). □

If we consider only the area over the poverty domain $[a, z]$ instead of the entire range $[a, b]$ as in the proposition, then an alternative and perhaps more constructive way to look at the issue is from the opposite direction of Proposition 2.2. That is, to construct the weighting function from a given decomposable measure. For a given poverty measure (2.1), we have

$$\int_a^z p(x, z) dF(x) = \left. p(x, z) F(x) \right|_a^z - \int_a^z p(x, z) F(x) dx$$

$$= - \int_a^z p_x(x, z) dF_2(x)$$

$$= - p_x(z, z) F_2(z)\bigg|_a^z + \int_a^z p_{xx}(x, z) F_2(x) dx$$

$$= - p_x(z, z) F_2(z) + \int_a^z p_{xx}(x, z) F_2(x) dx.$$

Thus, if $p_x(z, z) = 0$, i.e., $p_x(x, z)$ is continuous at $z$, then $\int_a^z p(x, z) dF(x) = \int_a^z p_{xx}(x, z) F_2(x) dx$ and the weighting function is simply $v(x, z) = p_{xx}(x, z)$ for $x \leq z$ and $v(x, z) = 0$ for $x > z$. But if $p_x(z, z) \neq 0$, then the weighting function is a bit more complicated:

$$v(x, z) = \begin{cases} p_{xx}(x, z) & \text{if } x < z \\ p_{xx}(z, z) - p_x(z, z) & \text{if } x = z \\ 0 & \text{if } x > z \end{cases}.$$
That is, the weighting function may have a “bump” at \( x = z \).

From equation (2.7), we can verify that all known decomposable poverty measures can be characterized as weighted area between the censored income distributions. For the FGT measure with \( \alpha \geq 2 \), the weighting function is \( v(x, z) = \alpha(\alpha - 1)(z - x)^{\alpha - 2} \) for \( x \leq z \) and \( v(x, z) = 0 \) for \( x > z \); for the Watts measure, \( v(x, z) = \frac{1}{z^2} \) for \( x < z \), \( v(z, z) = \frac{1}{z^2} + \frac{1}{z} \) and \( v(x, z) = 0 \) for \( x > z \); for the Chakravarty measure, \( v(x, z) = \beta(1 - \beta)z^{\beta - 2} \) for \( x < z \), \( v(z, z) = \beta z^{\beta - 2}(1 - \beta + z) \) and \( v(x, z) = 0 \) for \( x > z \); for the CDS measure, \( v(x, z) = \gamma^2 e^{\gamma(z-x)} \) for \( x < z \), \( v(z, z) = \gamma(1 + \gamma) \) and \( v(x, z) = 0 \) for \( x > z \).

These results are formally summarized in a corollary.

**Corollary 2.4.** The weighted CSD-area class of poverty measures (2.5) contains all known decomposable poverty measures as special cases.

Other two corollaries that follow Proposition 2.1 also find their counterparts in poverty measurement. For completeness, we include both as the final two results of the note. Note that the first result is well-known since the seminal work of Atkinson (1987), but our characterization provides a more direct proof.

**Corollary 2.5.** For two distributions \( F \) and \( G \) and a given poverty line \( z \), if all decomposable poverty measures in (2.4) agree that \( F \) has less poverty than \( G \), then \( F \) second-degree censored stochastic dominates \( G \).

**Corollary 2.6.** A weighted-area decomposable poverty measure satisfies Kolm’s principle of diminishing transfers if and only if the weighting function is a decreasing function of income.

3. Conclusion

The link between stochastic dominance (Lorenz dominance) and measures of inequality and poverty has been well established in the literature. In this note we establish a stronger link than what is already known. We show that each decomposable inequality measure is a weighted area between the NSD curves for some weighting function. The class of decomposable transfer-sensitive inequality measures alone is sufficient to characterize second-degree normalized stochastic dominance. Similarly, each decomposable poverty measure is a weighted area between the CSD curves for some weighting function. The class of decomposable distribution-sensitive poverty measures alone is also sufficient to characterize second-degree censored stochastic dominance.

These characterizations provide graphical interpretations for decomposable inequality and poverty measures, much like the graphical interpretations that the
Lorenz curve renders for the Gini and the generalized Gini inequality measures. We believe that our characterizations illustrate the differences among the different measures - it is the different ways of weighting the SD curves that produces the different measures. Much like that the class of generalized Gini inequality measures uniquely determines the Lorenz curve, the class of decomposable inequality/poverty measures also uniquely determines the SD curve. Given the equivalence between second-degree SD and Lorenz dominance, the generalized Gini class and the decomposable class provide alternative and yet equivalent ways to completely characterize both SD and Lorenz dominance.

The results of this can also be used to further enrich inequality and poverty measurement. For example, we can extend the link between stochastic dominance and decomposable measures to higher degrees of stochastic dominance. In this note, we have focused on second-degree stochastic dominance which amounts to requiring inequality and poverty measures to satisfy Pigou-Dalton’s principle of transfers. We have also derived a condition to enable a subset of the measures to further satisfy Kolm’s principle of diminishing transfers. One could certainly require Kolm’s principle for all decomposable measures and that amounts to using third-degree stochastic dominance. We believe the discussion and derivations for third-degree SD (and any higher degrees for that matter) should largely resemble those for second-degree.

Another direction of extension is to establish “almost normalized stochastic dominance” and “almost censored stochastic dominance” that are akin to Leshno and Levy’s (2002) “almost stochastic dominance” and Zheng’s (2016) “almost Lorenz dominance.” The notion of “almost dominance” addresses the situation where a clear dominance is absent but the violation is minor and a unanimous ranking can still be reached by “almost all” summary measures of utility (poverty or inequality). For inequality rankings, for example, while Zheng’s “almost Lorenz dominance” establishes conditions for all Gini type inequality measures, “almost normalized stochastic dominance” would establish condition for all decomposable inequality measures. Thus “almost Lorenz dominance” and “almost normalized stochastic dominance” together would establish conditions for all inequality measures.
References


