Resource Allocation by Frugal Majority Rule: An Arrovian Possibility Result

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Abstract

In the context of a simple resource allocation problem, we propose a model of 'frugal aggregation' in which the information about an agent's type is restricted to her top choice under a general background assumption of separable and concave individual utility functions (without further eliciting individual preferences). We show that a suitable frugal version of majority rule is consistent (i.e. acyclic and decisive) and amounts to minimizing the sum of the natural resource distances to the individually proposed allocations, that is, it coincides in this model with the 'median rule.' We show that the set of 'frugal majority winners' represents a well-behaved and easily computable solution concept, and we provide a normative foundation in an Arrovian spirit of the corresponding choice function based on a 'frugal' independence condition.

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1 Introduction

Many political decisions involve alternative ways to collectively allocate resources under a budget constraint. Examples are the allocation of public goods, redistribution across classes of beneficiaries, the allocation of tax burden, the choice of intertemporal expenditure streams, or the macro-allocation between expenditure, tax receipts and net debt.

The applicability of standard preference aggregation theory in this context is impeded by the following features of such problems. First, the inputs of a social choice mechanism can become quite complex if the space of alternatives is high dimensional; this is true in particular if the aggregation mechanism requires an entire preference ordering from each agent. Secondly, the operation of many standard aggregation rules such as the Borda count, the Copeland or Kemeny rules is unclear; on an infinite domain, these rules are not even well-defined. Third, except for the one-dimensional case with two public goods (Black, 1948; Arrow, 1953/62), one obtains generic impossibility results even under reasonable domain restrictions (Kalai, Muller and Satterthwaite, 1979; Le Breton and Weymark, 2004) just as in spatial voting models (Plott, 1968; McKelvey, 1986). In particular, there is no hope to find a Condorcet winner in the usual sense and the indeterminacy of sequential majority voting is generically maximal, i.e. *every* alternative can be the outcome of a dynamic majority vote for an appropriate agenda. Finally, there is an open issue about the existence of strategy-proof mechanisms, again with the exception of the one-dimensional case (Moulin 1980, 2016).

Starting with the problem of practicability, we explore in this paper the potential of *frugal aggregation* in the context of collective resource allocation. By 'frugal' aggregation we mean aggregation procedures that (i) elicit only the top choice of each agent's preference ordering, and (ii) are assessed under a background restriction of the domain of true (but not individually elicited) complete preferences. In the present context of multi-dimensional resource allocation, an appropriate and economically meaningful background restriction is given by the assumption that each agent's preferences can be represented by a separable and concave utility function.

The problem addressed in the present paper is to determine a suitable social welfare ordering, and in particular, suitable social welfare optima in the frugal aggregation model applied to the resource allocation context. We show that under the assumption of separable and concave individual utility functions, a frugal version of (pairwise) majority voting gives rise to a natural notion of social welfare optimum, and in fact to an entire ordering of allocations in terms of social welfare. An allocation is called *frugal majority winner* given a profile of individual tops if it wins every binary frugal majority comparison with any of its neighbors. The crucial property underlying our 'frugal' approach is the *(ex ante) indifference principle* which treats pairs of alternatives as indifferent for all agents whose tops are compatible with either ranking of the two alternatives vis-á-vis each other.

The following are our three main results. First, we show that, for all profiles of tops, the set of frugal majority winners is non-empty and coincides with the allocations selected by the well-known *median rule* (Young and Levenglick, 1978; Nehring and Pivato, 2017a,b) applied to the present context, i.e. with the allocations that minimize the sum of the distances in the natural resource metric to the individual tops (Theorem 1). In particular, the sum of the negative distances to the individuals' tops represents an appropriate measure of 'frugal' social welfare. Secondly, we provide a normative foundation via an axiomatic characterization of the choice rule induced by frugal majority rule (Theorem 2). The two key axioms are a *symmetry* axiom invoking an appropriate version of the principle of insufficient reason, and a novel axiom called *frugal independence*; the latter axiom requires the choice from a feasible set to depend only on the *discernible* properties of a profile of tops vis-á-vis the given feasible set under the general background assumption of preference admissibility. Third, we provide a simple characterization in terms of a profile-dependent ('endogenous') quota that allows one to efficiently compute the frugal majority winners in the resource allocation context (Theorem 3).

Frugal majority rule has further attractive properties.

(i) While the induced set of social welfare optima is typically set-valued, the set of frugal majority winners shrinks as the distances between the individual tops gets smaller. In this spirit, we prove that the frugal majority winners are 'essentially unique' if the support of the individual tops is 'coordinate-wise connected.' Moreover, we propose a natural selection from the set of frugal majority winners that yields a unique optimum in the continuum.

(ii) Since frugal majority rule admits a 'scoring rule representation' (with the negative of the natural resource distance as scores) it escapes the central paradoxes that characterize majoritarian choice rules within the standard voting framework based on complete preference orderings; in particular, it does not suffer from the *no-show paradox*.

(iii) The interpretation of the set of frugal majority winners as social welfare optima is enhanced by the demonstration that they maximize the sum of appropriately imputed utilities, referred to as individual 'goal satisfaction' functions. The goal satisfaction functions have an economically meaningful interpretation as frugal money metric utility functions.

The remainder of the paper is organized as follows. In the next Section 2, we introduce the frugal aggregation model for resource allocation problems. Section 3 contains the first main result demonstrating the well-behavedness of the set of frugal majority winners as a solution to the collective choice problem under the background assumption of separable and concave preferences. In Section 3.2, we show that the frugal majority winners coincide with an appropriately defined version of 'frugal Borda winners.' We also show there that frugal majority rule avoids the no-show paradox in the strong sense that, for each agent, the frugal majority set under own participation dominates the corresponding set without own participation. Finally, we show that the frugal majority winners admit an interpretation as utilitarian maximizers with respect to a natural class of imputed utility functions. Section 4 contains the normative foundation and axiomatic characterization of frugal majority rule. Section 5 provides a simple characterization of the set of frugal majority winners in terms of an endogenous quota. Besides the efficient computability of the set of frugal majority winners, this has a number of other corollaries. Most importantly, it gives a tight upper bound on the diameter (and the number of elements) of the set of frugal majority winners; the issue of uniqueness is addressed in Section 5.1. Section 6 concludes.

2 Frugal majority rule

In this section, we formally introduce the collective resource allocation problem and propose a general framework, the frugal aggregation model, to address it and to assess possible solutions.

2.1 The resource agenda

A group of agents (a 'society') has to collectively decide on how to allocate a fixed budget $Q \geq 0$ to a number L of public goods. Throughout, we assume fixed prices, thus the problem is fully determined by specifying the expenditure shares. Furthermore, we assume that expenditure shares are measured in discrete amounts of money and that all individuals have monotone preferences. Expenditure x^ℓ on public good ℓ may be bounded from below and above, so that feasibility requires $x^\ell \in [q_-^\ell, q_+^\ell]$ for some integers q_-^ℓ, q_+^ℓ where we allow that $q_-^\ell = -\infty$ and/or $q_+^\ell = \infty$. Together, these assumptions allow us to model the allocation problem as the choice of an element of the following $(L-1)$-dimensional polytope

$$X := \left\{ x \in \mathbb{Z}^L \ : \ \sum_{\ell=1}^{L} x^\ell = Q \text{ and } x^\ell \in [q_-^\ell, q_+^\ell] \text{ for all } \ell = 1, ..., L \right\}, \tag{2.1}$$

where $\mathbb{Z}$ is the set of integers and $x = (x^1, ..., x^L)$. The space X is referred to as the set of *feasible allocations*.

2.2 Frugal aggregation

In this paper, we study collective choice mechanisms on sets of alternatives of the form described in (2.1). Closely related problems have been addressed by numerous contributions in the literature, see e.g., Kalai, Muller and Satterthwaite (1979), and Le Breton and Weymark (2004, Sect. 4 & 5) for an extensive survey on Arrovian aggregation on economic domains.

The *frugal voting model* promoted here assumes that individual preferences are only partially elicited. Specifically, we assume a maximally sparse message space of one single alternative for each individual (their respective top alternative). Moreover, the collective choice mechanisms are assessed under the background assumption that the non-elicited preferences of individuals come from a common and known domain $\mathcal{D}$ of *admissible* complete preference orderings on the set X. Henceforth, a *frugal resource allocation problem* is given by a pair $(X, \mathcal{D})$, where X is as specified in (2.1) and $\mathcal{D}$ is the associated domain of admissible preference orderings on X. Throughout, we assume that all admissible preference orderings $\succcurlyeq \in \mathcal{D}$ have a unique top element, denoted by $\tau(\succcurlyeq) \in X$. Observe that, obviously, the discernible information of an individual message also depends on $\mathcal{D}$, since by submitting a specific allocation $\theta \in X$ an individual 'reveals' that her complete preference ordering must come from the set $\mathcal{D}_\theta$, where, for all $\theta \in X$,

$$\mathcal{D}_\theta := \{\succcurlyeq \in \mathcal{D} \ : \ \tau(\succcurlyeq) = \theta\}.$$

For all $\theta \in X$, and all distinct $x, y \in X$, let

$$x >_\theta^{\mathcal{D}} y \ :\Leftrightarrow \ \text{for all } \succcurlyeq \in \mathcal{D}_\theta, \ x \succ y.$$

Thus, $>^{\mathcal{D}}_{\theta}$ is the (strict) partial order that describes the discernible strict preference judgements of a voter with top alternative θ.[1]

We will also assume that the preference domain is *minimally rich* in the sense that it contains, for every allocation $\theta \in X$, at least one preference ordering $\succcurlyeq$ with $\tau(\succcurlyeq) = \theta$. This guarantees that the partial order $>^{\mathcal{D}}_{\theta}$ is well-defined for all $\theta \in X$.

2.3 The indifference principle

Our goal in this paper is to determine one or more *welfare optima* under the informational restriction imposed by the frugal aggregation model. To achieve this, we will take a truly 'Arrovian' route, that is, we will first determine a 'social betterness' relation among pairs of alternatives and then infer from this optimality in more general feasible sets. Unlike the general Arrovian program (on an unrestricted domain), we will demonstrate the possibility of non-trivial solutions to this problem in the present context of resource allocation.

Thus, suppose that each individual i submits an alternative $\theta_i \in X$ representing his or her most preferred alternative from X. We will call θ_i the *top* of individual i, and denote by $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ the profile of tops, allowing for variable population size $n \in \mathbb{N}$. For every $x \in X$ denote by $\#_{\boldsymbol{\theta}}(x)$ the *mass of x under $\boldsymbol{\theta}$*. i.e. the number of individuals i such that $\theta_i = x$. Moreover, denote by $\operatorname{supp} \boldsymbol{\theta} := \{x \in X : \#_{\boldsymbol{\theta}}(x) > 0\}$ the *support* of $\boldsymbol{\theta}$. Finally, for all subsets $Y \subseteq X$, denote by $\#_{\boldsymbol{\theta}}(Y) := \#\{i : \theta_i \in Y\} = \sum_{x \in Y} \#\boldsymbol{\theta}(x)$ the popular support of Y.

Consider any pair $x, y \in X$ of alternatives, and denote by $\mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}}$ the *social betterness* relation given the profile $\boldsymbol{\theta}$ in the frugal aggregation problem $(X, \mathcal{D})$. In the special case in which the support of $\boldsymbol{\theta}$ is concentrated on the pair x, y, i.e. if $\operatorname{supp} \boldsymbol{\theta} = \{x, y\}$, then by standard arguments (using the conditions of anonymity, neutrality and positive responsiveness, cf. May, 1952) one obtains

$$x \, \mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}} \, y \;\text{ if and only if a (weak) majority votes for } x.$$

But $\operatorname{supp} \boldsymbol{\theta} = \{x, y\}$ is clearly a very special case, and the key question becomes how to count other votes in the comparison between x and y. The fundamental idea underlying the frugal aggregation model is to treat missing information *symmetrically*, i.e. to treat the absence of unambiguous comparison as indifference. This is embodied in the fundamental *indifference principle* to which we turn now.

For all pairs of distinct alternatives $x, y \in X$, let

$$\rangle x, y \rangle^{\mathcal{D}} \; := \; \{\theta \in X \; : \; x >^{\mathcal{D}}_{\theta} y\}.$$

Thus, $\rangle x, y \rangle^{\mathcal{D}}$ the set of alternatives such that all agents with top in $\rangle x, y \rangle^{\mathcal{D}}$ are known to strictly prefer x to y, and we refer to this set as the *set supporting x over y*. If the popular support of $\rangle x, y \rangle^{\mathcal{D}}$ is larger than the popular support of $\rangle y, x \rangle^{\mathcal{D}}$, we say that x is the *net majority winner against* y under $\boldsymbol{\theta}$, and denote this by $x \, \mathrm{NM}_{\boldsymbol{\theta}} \, y$. Formally, for all $x, y \in X$,

$$x \, \mathrm{NM}^{\mathcal{D}}_{\boldsymbol{\theta}} \, y \;:\Leftrightarrow\; \#_{\boldsymbol{\theta}} \left(\rangle x, y \rangle^{\mathcal{D}} \right) \geq \#_{\boldsymbol{\theta}} \left(\rangle y, x \rangle^{\mathcal{D}} \right).$$

[1] Note that, as a binary relation on X, $>^{\mathcal{D}}_{\theta}$ is a (generally proper) subset of the partial order $\bigcap \mathcal{D}_{\theta}$.

The binary relation $\mathrm{NM}^{\mathcal{D}}_{\boldsymbol{\theta}}$ will also be referred to as the *net majority tournament* (given $\boldsymbol{\theta}$). Note that, in contrast to the agents with top alternative in the set $\rangle x, y\rangle^{\mathcal{D}}$ who strictly prefer x to y, and the agents with top alternative in $\rangle y, x\rangle^{\mathcal{D}}$ who strictly prefer y to x, the background assumption of preference admissibility is compatible with every possible ranking between x and y (indifference included) for agents with top outside the union of the sets $\rangle x, y\rangle^{\mathcal{D}}$ and $\rangle y, x\rangle^{\mathcal{D}}$. For future reference, denote for all distinct $x, y \in X$,

$$x \bowtie^{\mathcal{D}} y \ := \ X \setminus \left(\rangle x, y\rangle^{\mathcal{D}} \cup \rangle y, x\rangle^{\mathcal{D}} \right).$$

The frugal aggregation approach refrains from eliciting individual preferences beyond the information about the tops and admissibility. By consequence, an appeal to the principle of insufficient reason suggests to treat the agents with top alternative in $x \bowtie^{\mathcal{D}} y$ symmetrically, i.e. as indifferent between x and y. Our provisional answer to the question of social betterness among pairs of alternatives within the frugal aggregation model is thus to identify the relation $\mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}}$ with the net majority tournament, i.e. for all distinct $x, y \in X$,

$$x \, \mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}} \, y \ \Leftrightarrow \ x \, \mathrm{NM}^{\mathcal{D}}_{\boldsymbol{\theta}} \, y,$$

and to define the set $C(X; \boldsymbol{\theta})$ of 'social welfare optima' given the profile $\boldsymbol{\theta}$ as the maximal elements in X with respect to $\mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}}$, i.e.

$$C(X; \boldsymbol{\theta}) \ := \ \left\{ x \in X \ : \ \text{ for no } y \in X, \ y \, \mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}} \, x \right\}. \tag{2.2}$$

This proposal of course raises the immediate question whether $C(X; \boldsymbol{\theta})$ is always non-empty. Before we further analyze this problem, let us consider two simple but instructive special cases that will help to understand the underlying motivation.

Example 1 (Plurality voting) Suppose that admissibility does not convey any additional information, i.e. consider the special case $\mathcal{D} = \mathcal{U}$ where $\mathcal{U}$ is the unrestricted domain of all weak preference orderings with a unique top. Evidently, in this case we have for all $x, y \in X$, $\rangle x, y\rangle^{\mathcal{U}} = \{x\}$, and hence

$$x \, \mathrm{NM}^{\mathcal{U}}_{\boldsymbol{\theta}} \, y \ \Leftrightarrow \ \#_{\boldsymbol{\theta}}(x) \geq \#_{\boldsymbol{\theta}}(y),$$

i.e. the maximal elements of the net majority tournament are simply the plurality winners: the alternatives that are named by most agents as their respective top choice. The indifference principle is justified since, from an impartial welfare perspective, there is simply no basis to favor an alternative x over any other alternative y given a vote for $\theta \notin \{x, y\}$. Clearly, the net majority tournament $\mathrm{NM}^{\mathcal{U}}_{\boldsymbol{\theta}}$ and hence also the social betterness relation $\mathrm{B}^{\mathcal{U}}_{\boldsymbol{\theta}}$ are transitive in this case; consequently, the set $C(X; \boldsymbol{\theta})$ of social welfare optima is non-empty for every profile (and coincides with the plurality winners, as noted).

As another special case, consider $L = 2$, i.e. suppose that the money allocation is between two public goods only. In this case, the space X in (2.1) is one-dimensional, and a natural assumption on agents' preferences is that they are single-peaked. Then, net majority voting amounts to the choice of the median allocation, as follows.

Example 2 (Median voting on the line) Suppose that $L = 2$ and, for simplicity, that $X = \{x \in \mathbb{Z}^2 \ : \ x^1 + x^2 = Q$ and $x^1, x^2 \geq 0\}$; moreover, let $\mathcal{D}^*$ denote the set of all weak orderings on X that are single-peaked with a unique top alternative.[2] Consider two distinct alternatives $x, y \in X$, say such that $x < y$. Any top θ not in the interior of the interval $[x, y]$ induces a strict preference either for x (if $\theta \leq x$), or for y (if $\theta \geq y$); on the other hand, a top strictly between x and y is compatible with any preference between x and y, i.e. $x \bowtie^{\mathcal{D}^*} y = \{w \in X : x < w < y\}$. In particular, we obtain

$$x \, \mathrm{NM}_{\boldsymbol{\theta}}^{\mathcal{D}^*} y \ \Leftrightarrow \ \#_{\boldsymbol{\theta}}(\{w \in X : w \leq x\}) \geq \#_{\boldsymbol{\theta}}(\{w \in X : w \geq y\}).$$

As is easily verified the net majority tournament is transitive in this case. Moreover, the social welfare optima $C(X; \boldsymbol{\theta})$ according to (2.2) are given by the median alternative(s).[3]

The indifference principle may seem less compelling in this case, because one could argue in favor of using information about the distances between the top θ and the alternatives x and y, respectively; for instance, θ could be an element of $x \bowtie^{\mathcal{D}^*} y$ but still closer to x than to y. However, it is in fact sufficient to consider *adjacent* pairs x and y, i.e. to restrict the social betterness relation to *neighbors*. For adjacent pairs x, y, one has $x \bowtie^{\mathcal{D}^*} y = \emptyset$, i.e. the indifference principle is vacuously satisfied for such pairs. Nevertheless, one obtains the same set of social welfare optima if one restricts $\mathrm{B}_{\boldsymbol{\theta}}^{\mathcal{D}^*}$ to all adjacent pairs; indeed, if an alternative is dominated by y then it is also dominated by its neighbor in direction of y, i.e. the median alternative(s) remain the only undominated alternatives. The restriction to adjacent pairs will also play an important role in our subsequent analysis.

3 Well-behavedness under separable and convex preferences

In the remainder of this paper we will explore the frugal voting model of resource allocation under the assumption that each agent's preferences can be represented by an additively separable and strictly concave utility function. This represents an economically meaningful and plausible restriction, and we will show that it yields a remarkably tractable model. In fact, the frugal resource allocation model with this particular notion of preference admissibility arguably represents a distinguished compromise between a priori structure and neglect of unavailable information.

Specifically, we will consider the frugal resource allocation problem $(X, \mathcal{D}^*)$, where $\mathcal{D}^*$ is given by the class of weak preference orderings that (i) have a unique top element among all allocations in X and (ii) can be represented by a separable utility function of the form

$$u(x) \ = \ u(x^1, ..., x^L) \ = \sum_{\ell=1}^{L} u^\ell(x^\ell), \tag{3.1}$$

where the $u^\ell : \mathbb{Z} \to \mathbb{R}$ are strictly increasing and concave for all $\ell = 1, ..., L$. The class $\mathcal{D}^*$ contains many economically meaningful preferences such as Cobb-Douglas or perfect comple-

[2] A preference ordering $\succcurlyeq$ is *single-peaked* on a one-dimensional space X if all upper contour sets are connected in X.

[3] More precisely, the unique median top if the number of agents is odd, and all alternatives between the two middle tops if the number agents is even.

ments. Note that, on the other hand, perfect substitutes (of expenditure shares) are ruled out by the assumption that there be a unique maximizer on X. Also observe that the utility functions of the form (3.1) are defined over the underlying economic domain $\mathbb{Z}^L$; and clearly, monotonicity is reasonable as an assumption on individual preferences only on this underlying space of alternatives, not on the restrictions to feasible resource agendas. Finally, note that either separability and concavity are invariant properties with respect to a rescaling of the coordinates; in particular, they are well-defined properties of preferences both over physical units and over expenditure shares.

Examples 1 and 2 above are special cases: the universal domain (on L alternatives) is obtained by considering $(X, \mathcal{D}^*)$ with the agenda $X = \{x \in \mathbb{Z}_+^L : \sum_{\ell=1}^L x^\ell = 1\} = \{(1,0,...,0),(0,1,0,...,0),...,(0,...,0,1)\}$; the line considered in Example 2 is obtained by setting $L = 2$.[4]

The following result characterizes, for $(X, \mathcal{D}^*)$ and every pair of allocations $x, y \in X$, the set supporting x over y, and provides thus a key technical fact for the subsequent analysis. For all $x, y \in X$, let

$$[x,y] \ := \ \left\{ w \in X \ : \ \text{for all } \ell = 1,...,L, \ x^\ell \leq w^\ell \leq y^\ell \text{ or } y^\ell \leq w^\ell \leq x^\ell \right\}.$$

We will refer to $[x, y]$ as the *interval spanned by x and y*, and to its elements as the allocations *between x and y*. It turns out that, for the domain $\mathcal{D}^*$, a top θ supports x over y if and only if x is between θ and y in this natural sense.

Lemma 3.1 *For all distinct $x, y \in X$,*

$$\rangle x, y \rangle^{\mathcal{D}^*} \ = \ \{\theta \in X \ : \ x \in [\theta, y]\}. \tag{3.2}$$

(Proof in Appendix.)

Henceforth, we will fix the domain $\mathcal{D}^*$ and often simplify notation by writing "$\rangle x, y\rangle$" instead of "$\rangle x, y\rangle^{\mathcal{D}^*}$," "$\mathrm{B}_{\boldsymbol{\theta}}$" instead of "$\mathrm{B}_{\boldsymbol{\theta}}^{\mathcal{D}^*}$" and "$\mathrm{NM}_{\boldsymbol{\theta}}$" instead of "$\mathrm{NM}_{\boldsymbol{\theta}}^{\mathcal{D}^*}$" etc., whenever no confusion can arise.

It follows from Example 2 above that the set of maximal elements of the net majority tournament in the frugal resource allocation problem $(X, \mathcal{D}^*)$ is non-empty for any profile $\boldsymbol{\theta}$ if $L = 2$. The following example shows that this does not generalize to the case $L > 2$.

Example 3 (Non-existence of an unrestricted majority winner) Suppose that X is given as in (2.1) above with $L = 3$ and $Q = 3$. Consider the following profile $\boldsymbol{\theta}$ with seven agents (see Figure 1): $\theta_1 = (1,1,1)$, $\theta_2 = \theta_3 = (3,0,0)$, $\theta_4 = \theta_5 = (0,3,0)$, and $\theta_6 = \theta_7 = (0,0,3)$. Using Lemma 3.1, it is easily verified that $(1,1,1) >_{\theta_i} (0,1,2)$ for $i = 1,2,3$ and $(0,1,2) >_{\theta_i} (1,1,1)$ for $i = 6,7$, while any ranking between $(1,1,1)$ and $(0,1,2)$ is compatible with admissibility for agents $i = 4,5$. Thus,

$$(1,1,1)\,\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}\,(0,1,2), \tag{3.3}$$

[4] In our discrete setting, any single-peaked preference without indifferences can be represented by a concave utility function. If we allow for non-trivial indifferences, the class of all preferences representable by a concave utility function is somewhat smaller than the class of all single-peaked weak preferences, but this does not affect any of our results.

where $\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}$ is the asymmetric ('strict') part of $\mathrm{NM}_{\boldsymbol{\theta}}$. Moreover, by Lemma 3.1, we have $(0,1,2) >_{\theta_i} (0,0,3)$ for $i = 1,4,5$ while any ranking between $(0,1,2)$ and $(0,0,3)$ is compatible with admissibility for agents $i = 2.3$; hence, notwithstanding the fact $(0,0,3) >_{\theta_i} (0,1,2)$ for $i = 6,7$, we obtain

$$(0,1,2)\,\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}\,(0,0,3). \tag{3.4}$$

Finally, again using Lemma 3.1, we obtain

$$(0,0,3)\,\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}\,(1,1,1) \tag{3.5}$$

since agents $i = 6,7$ have their top at $(0,0,3)$ while only agent $i = 1$ has her top at $(1,1,1)$ and the ranking between these two allocations is not determined by admissibility for the other agents $i = 2,3,4,5$. Combining (3.3), (3.4) and (3.5) we thus obtain that both $(1,1,1)$ and $(0,0,3)$ are contained in a $\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}$-cycle. By a completely symmetric argument, also the allocations $(3,0,0)$ and $(0,3,0)$ are part of a $\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}$-cycle. This easily implies that the set of (unrestricted) net majority winners is in fact empty for the profile $\boldsymbol{\theta}$.

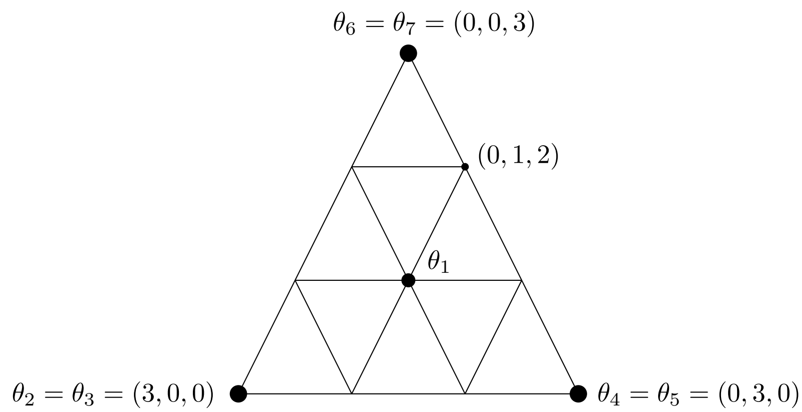


Fig. 1: Non-existence of a global net majority winner

The last example demonstrates that an naive application of the indifference principle may lead to a cyclic net majority tournament, and hence does not deliver a coherent account of 'frugal social betterness.' To provide the diagnosis of the underlying difficulty, consider the following binary comparisons between x and y of agents with top θ.

(a) $x = (2,1,0)$, $y = (1,2,0)$, $\theta = (1,1,1)$.
(b) $x = (3,0,0)$, $y = (1,1,1)$, $\theta = (0,0,3)$.
(c) $x = (2,1,0)$, $y = (1,2,0)$, $\theta = (0,1,2)$.

In each case, the top θ neither supports x nor y since any preference between x and y is compatible with a top θ under separability and concavity of the utility function. However, the indifference principle is not equally plausible in all cases. Consider case (a) first; here we have full symmetry of x and y vis-á-vis the top θ, say under permutation of coordinates, and the appeal to the indifference principle appears to be safe. On the other hand, in case (b) the indifference between x and y given θ is more problematic; in the pairwise comparison an impartial arbitrator could come up with favoring y because it is closer to θ (in the natural 'resource metric' defined below). We will not settle this and avoid an appeal to the indifference principle in this case. Finally, in (c) we still do not have full symmetry as in (a), but less of an asymmetry as compared to (b); in particular, x and y are equidistant from θ. We will appeal to the indifference principle in this case. For now, the distinction between cases (b) and (c) in terms of their qualitative distance properties is only meant heuristically. Below, we will provide an axiomatic characterization that will justify the use of the indifference principle in case (c) but not in (b).

One important difference between cases (a) and (c) on the one hand and case (b) on the other is that in two former cases x and y are adjacent, i.e. they differ in the allocation of one unit of expenditure only. This suggests to refine the indifference principle and to restrict judgements of social betterness to a subset of *well-decidable* binary comparisons via a *comparison graph* Γ, i.e. to consider $\mathrm{B}_{\boldsymbol{\theta}} = \mathrm{NM}_{\boldsymbol{\theta}} \cap \Gamma$ for an appropriate graph Γ on X. In our present context, the natural choice of the comparison graph is given by the set of all neighbors, i.e. all pairs of allocations that result from each other by transferring one unit from one coordinate to another.

Formally, say that two allocations $x, y \in X$ are *neighbors* if they differ only by the allocation of one monetary unit, i.e. if $\sum_{\ell=1}^{L} |x^\ell - y^\ell| = 2$. Denote by Γ_{res} the graph that results from connecting all neighbors in X by an edge. (Observe that all our figures in fact depict this graph.) As is easily seen, two distinct allocations x and y are neighbors if and only if $[x, y] = \{x, y\}$. Moreover, neighbors are characterized by the following property which we state here for future reference.

Fact 3.1 *Two distinct allocations $x, y \in X$ are neighbors if and only if there exist two preference orderings $\succcurlyeq$ and $\succcurlyeq'$ in $\mathcal{D}^*$ such that x is top and y is second in $\succcurlyeq$ while y is top and x is second in $\succcurlyeq'$.*

(Proof in Appendix.)

For all $x, y \in X$, denote by

$$d(x, y) := \frac{1}{2} \sum_{\ell=1}^{L} |x^\ell - y^\ell|.$$

the natural 'resource' metric on X. The normalization ensures that neighbors have distance one, i.e. that a transfer of one unit of expenditure from one public good to another yields an allocation with unit distance from the original allocation. Also observe that, for all $x, y \in X$,

$$[x, y] \;=\; \{w \in X \;:\; d(x, y) = d(x, w) + d(w, y)\}, \tag{3.6}$$

i.e. the allocations between two other allocations are precisely those that are geodesically between them with respect to the natural 'resource' metric d; in other words, the allocation between x and y are precisely those that lie on some shortest path connecting x and y in the graph Γ_{res}.

The following figure depicts the sets $\rangle x, y\rangle$, $\rangle y, x\rangle$ and $x \bowtie y$ for two neighbors x and y. It shows that, for fixed $x \in X$, the 'agnostic' region $x \bowtie y$ is inclusion minimal if and only if y is a neighbor of x. Moreover, it illustrates the following *equidistance property* that justifies the appeal to the indifference principle for adjacent alternatives in our present context. For all $x, y, \theta \in X$,

$$[d(x, y) = 1 \text{ and } \theta \in x \bowtie y] \;\Rightarrow\; d(\theta, x) = d(\theta, y), \tag{3.7}$$

which follows easily from Lemma 3.1 using (3.6).

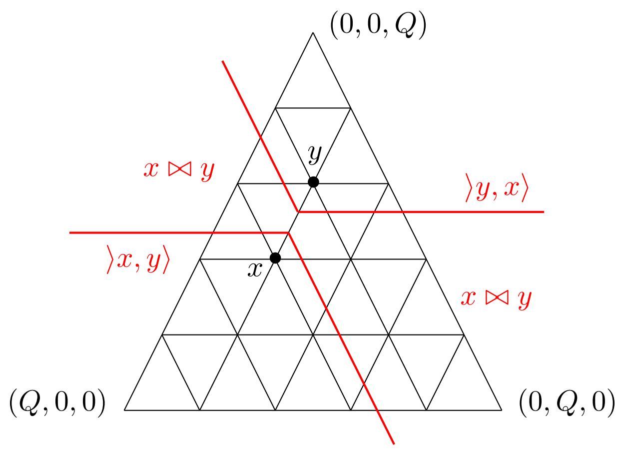


Fig. 2: The sets $\rangle x, y\rangle$, $\rangle y, x\rangle$ *and* $x \bowtie y$

By 'localizing' the net majority tournament, i.e. by restricting it to the graph Γ_{res}, we obtain the central solution concept advocated here, the *set of frugal majority winners*, henceforth simply the *frugal majority set*, formally defined by

$$\text{Maj}(\boldsymbol{\theta}) \;:=\; \{x \in X \;:\; \text{for no } \; w \in X \text{ with } \; w\Gamma_{\text{res}}x,\; w\,\text{NM}_{\boldsymbol{\theta}}\,x\}.$$

Thus, our claim is that the indifference principle is justified if applied to adjacent alternatives due to (3.7), and that therefore a coherent account of frugal betterness is obtained by restricting the net majority tournament to neighbors, i.e. by letting

$$\text{B}_{\boldsymbol{\theta}} \;=\; \text{NM}_{\boldsymbol{\theta}} \cap \Gamma_{\text{res}}.$$

In Example 3, the allocation $(1, 1, 1)$ is the unique frugal majority winner. In the following, we will show that the frugal majority set is always non-empty; indeed, the net majority tournament restricted to Γ_{res} is acyclic and decisive, as follows.

Consider a weak tournament (i.e. a complete binary relation) R' on X and its restriction R to a connected comparison graph Γ, i.e. $R = R' \cap \Gamma$. Denote by P the asymmetric part of R, and by R^* and P^* the transitive closures of R and P, respectively. The relation R is *acyclic* if P displays no cycles, i.e. if P^* is irreflexive. Say that $x \in X$ is a *local optimum (with* *respect to* Γ*)* if xRy for all $y \in X$ such that $x\Gamma y$, and say that R is *decisive (on* X*)*, if for some $x \in X$, xR^*y for all $y \in X$. Finally, say that a subset $Y \subseteq X$ is *connected* if any pair of elements of Y are connected by a Γ-path that stays in Y. The following fact is easily verified.

Fact 3.2 *If* R *is acyclic, there exists a local optimum. Moreover,* R *is decisive if and only if the set of local optima is non-empty and connected.*

Intuitively, the smaller Γ, the easier it is to achieve the existence of local optima but the harder it is to keep their set connected. As our first main result, we will now show that, for all profiles $\boldsymbol{\theta}$, the restricted tournament $\mathrm{NM}_{\boldsymbol{\theta}} \cap \Gamma_{\mathrm{res}}$ is acyclic and decisive;[5] in particular, the frugal majority set is non-empty and connected. Moreover, the frugal majority set will be shown to consist exactly of the allocations that minimize the aggregate graph distance with respect to the graph Γ_{res}, as follows.

For all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and all $x \in X$, denote by

$$\Delta_{\boldsymbol{\theta}}(x) := \sum_{i=1}^{n} d(x, \theta_i)$$

the *aggregate distance* of the feasible allocation x given the profile $\boldsymbol{\theta}$. The aggregate distance is a natural way to quantify the 'overall remoteness' of an allocation to the set of individual top alternatives. The collective choice rule that selects, for any profile, the allocations that minimize the aggregate distance represents the well-known *median rule* (see Barthélemy and Monjardet, 1981; Barthélemy and Janowitz, 1991; McMorris, Mulder and Powers, 2000) applied to the present context of resource allocation. Accordingly, we will refer to an allocation as a *median allocation* if it solves

$$\arg\min_{x \in X} \Delta_{\boldsymbol{\theta}}(x) \;\; = \;\; \arg\min_{x \in X} \sum_{i=1}^{n} d(x, \theta_i),$$

and we denote the set of median allocations given the profile θ by $\mathrm{Med}(\boldsymbol{\theta})$. Evidently, $\mathrm{Med}(\boldsymbol{\theta})$ need not be a singleton but it is always non-empty since the median allocations are obtained as the solution of a maximization problem on a finite set. Note that, if $L = 2$, the set of median allocations coincides with the standard medians. In particular, if there is an odd number of individuals, the set $\mathrm{Med}(\boldsymbol{\theta})$ consists of the unique median top.

Say that a subset $Y \subseteq X$ is *box-convex* if it contains with any two allocations x, y all allocations between them, i.e. if $x, y \in Y$ implies $[x, y] \subseteq Y$. Evidently, the box-convex subsets of X are exactly the intersections with X of 'boxes' of the form $\prod_{\ell=1}^{L} [w^\ell, z^\ell] \subseteq \mathbb{Z}^L$ for any $w^\ell, z^\ell \in \mathbb{Z}$ with $w^\ell \leq z^\ell$ for all $j = 1, ..., L$.

Theorem 1 *For all profiles* $\boldsymbol{\theta}$*, the restricted net majority tournament is acyclic, in particular* $\mathrm{Maj}(\boldsymbol{\theta})$ *is non-empty. Moreover,* $\mathrm{Maj}(\boldsymbol{\theta})$ *is box-convex and coincides with* $\mathrm{Med}(\boldsymbol{\theta})$*.*

(Proof in Appendix.)

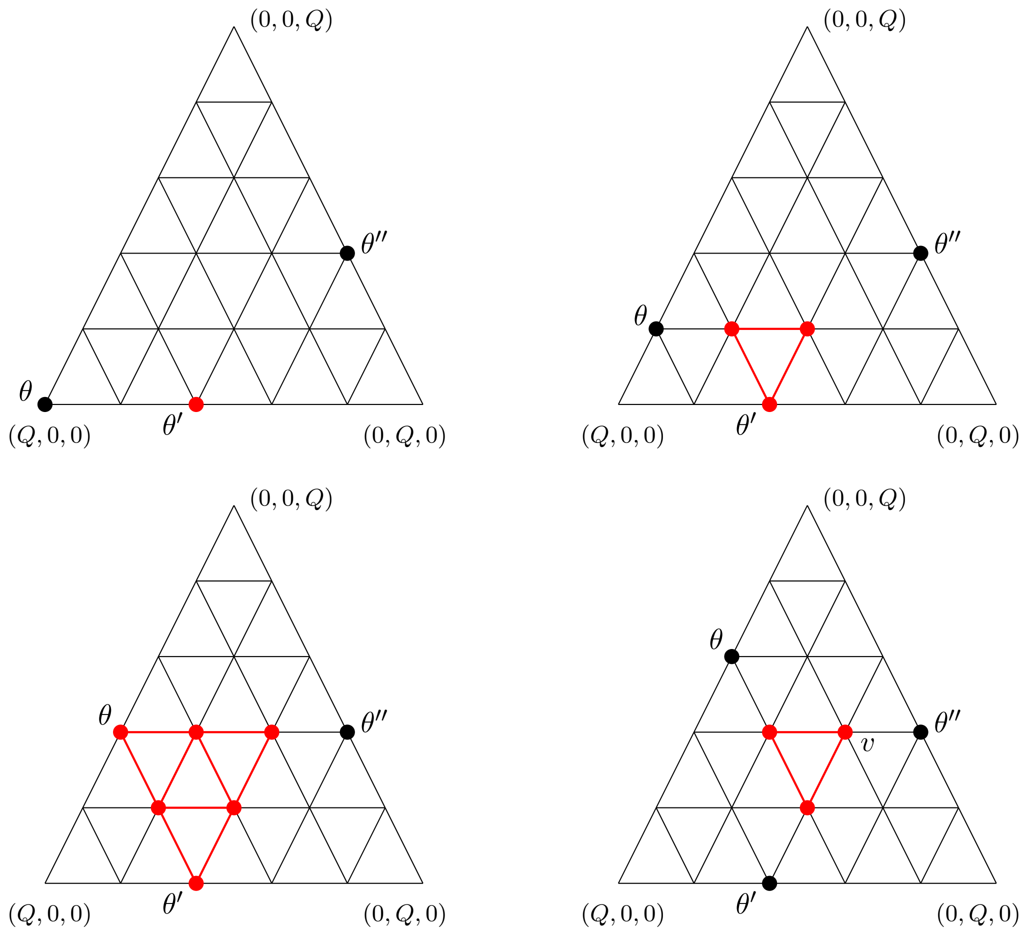
[5] It follows from Fact 3.1 above that decisiveness of $\mathrm{NM}_{\boldsymbol{\theta}} \cap \Gamma$ in fact *requires* that $\Gamma \supseteq \Gamma_{\mathrm{res}}$.

Illustration: The net majority set with three voters

The following figure illustrates the frugal majority set in the case of three voters. First note that, evidently, for two voters with tops θ and θ', respectively, the frugal majority set is given by the interval $[\theta, \theta']$. Indeed, every allocation $x \in [\theta, \theta']$ lies on a shortest path between θ and θ', and therefore has aggregate distance equal to $d(\theta, \theta')$ by (3.6), while every allocation outside $[\theta, \theta']$ has strictly larger distance.

Now consider the case of three agents with distinct tops θ, θ' and θ'', respectively. In Figure 3, we fix the two tops θ' and θ'' in generic position, and describe how the frugal majority set changes when θ moves clockwise 'around' the interval $[\theta', \theta'']$ (with the frugal majority winners marked in red in each case; the depicted shapes of $\mathrm{Maj}(\boldsymbol{\theta})$ can be verified by computing aggregate distances and using Theorem 1).

We note that while the frugal majority set with three voters is always a triangle (possibly consisting of a single allocation, cf. Fig. 3), it can take on more general shapes if there are more agents.



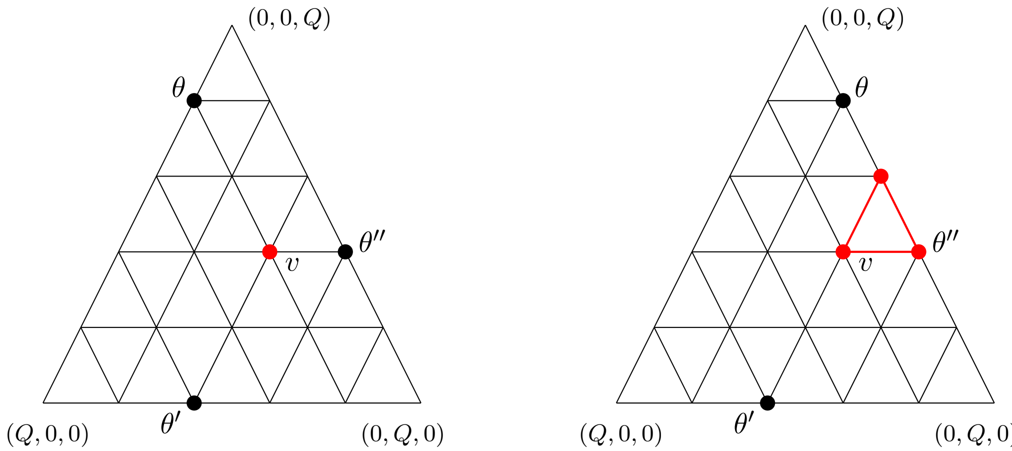


Fig. 3: The frugal majority set with three agents

3.1 On the scope of Theorem 1: When is it valid (and why)?

In this section we observe that while Theorem 1 crucially hinges on the assumptions of separability and convexity, it does in fact not depend on *additive* separability, nor on the concavity of the component utility functions in each coordinate. We also show that the net majority tournament $\text{NM}_{\boldsymbol{\theta}}^{\mathcal{D}^*}$ restricted to Γ_{res} is decisive on a feasible subset if and only if that subset is box-convex, i.e. itself a (sub-)resource agenda.

3.1.1 General validity for separably convex preferences

An inspection of the proof of Theorem 1 provided in the Appendix shows that the result holds for all preference domains $\mathcal{D}$ that satisfy the identity (3.2) stated in Lemma 3.1.[6] Say that $\succcurlyeq$ is *generalized single-peaked on* X if $\succcurlyeq$ has a unique top alternative, $\tau(\succcurlyeq) = \theta$, such that for all distinct $x, y \in X$,

$$x \in [\theta, y] \;\Rightarrow\; x \succ y.$$

(cf. Nehring and Puppe, 2007). Denote by $\mathcal{D}_{\text{sp}}$ the domain of all generalized single-peaked preferences, and say that a domain $\mathcal{D} \subseteq \mathcal{D}_{\text{sp}}$ is *rich* if, for each triple $x, y, \theta \in X$ with $x \notin [\theta, y]$ there exists an ordering $\succcurlyeq$ with $\tau(\succcurlyeq) = \theta$ and $y \succ x$. Evidently, this richness condition implies, but is strictly stronger than, our minimal richness condition above. Every subdomain $\mathcal{D} \subseteq \mathcal{D}_{\text{sp}}$ of generalized single-peaked preferences that is rich in this sense verifies Lemma 3.1. Indeed, the generalized single-peakedness says that, for any given pair $x, y \in X$ of distinct elements, all preferences with top in the set $\{\theta \in X : x \in [\theta, y]\}$ support x over y; and the richness condition says that, for all $\tilde{\theta}$ outside this set there exists an element $\succcurlyeq \in \mathcal{D}$ with top $\tilde{\theta}$ such that $y \succ x$. Together, this yields precisely the identity (3.2), thus verifying Lemma 3.1 and hence Theorem 1.

[6] In fact, it is sufficient that a domain verifies the statement of Lemma A.1 in the Appendix, i.e. the local coincidence of the net majority tournament with the median rule; this is implied but weaker than (3.2).

A special case of a rich generalized single-peaked domain is the set $\mathcal{D}_{\text{qlin}} \subsetneq \mathcal{D}^*$ of all preferences that have a unique top and that can be represented by a quasi-linear utility function of the form

$$u(x) = u(x^1, ..., x^L) = x^1 + \sum_{\ell=2}^{L} u^\ell(x^\ell),$$

with each $u^\ell(\cdot)$ strictly increasing and concave.

3.1.2 Non-validity for general convex preferences

Now consider the domain of all *convex* (but not necessarily separable) preferences on X, denoted by $\mathcal{D}_{\text{conv}}$. As is easily seen, for each pair of neighbors $x, y \in X$ the supporting set $\rangle x, y\langle^{\mathcal{D}_{\text{conv}}}$ is given by the set of all tops θ such that x is on the (Euclidean) straight line connecting θ and y (see Figure 4 which depicts the sets $\rangle x, y\langle^{\mathcal{D}_{\text{conv}}}$ and $\rangle y, x\langle^{\mathcal{D}_{\text{conv}}}$ in red).

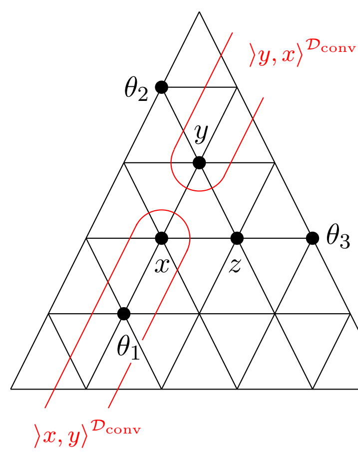


Fig. 4: Cyclic frugal majorities with convex preferences

Evidently, frugal majority rule under the domain assumption $\mathcal{D}_{\text{conv}}$ does not coincide with the median rule, i.e. Theorem 1 fails to hold for this domain. In fact, the restricted net majority tournament is in general not even acyclic. To see this, consider the binary comparisons among the alternatives x, y, z under the three-agent profile $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ depicted in Fig. 4. Agent 1 with top θ_1 supports x against y but not against z, and no other alternative against any other alternative; similarly, the only discernible binary support of agent 2 is for y against z, and the only discernible binary support of agent 3 is for z against x. Thus, frugal majority rule yields the cycle $x \, \widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{conv}}} y$, $y \, \widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{conv}}} z$, $z \, \widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{conv}}} x$.

3.1.3 Non-validity for Euclidean preferences

The previous example of all convex but not necessarily separable preferences shows that Theorem 1 may fail on 'too large' domains. But the result may also fail for 'small' domains. Specifically, consider the domain $\mathcal{D}_{\text{Euclid}}$ of all Euclidean preferences, i.e. the class of preferences that can be represented by the negative Euclidean distance to some allocation $\theta \in X$. Evidently, the domain $\mathcal{D}_{\text{Euclid}}$ is minimally rich, but the top reveals the entire preference ordering; in particular, for each $\theta \in X$, the order $>_\theta^{\mathcal{D}}$ coincides with the asymmetric part of the unique preference ordering in $\mathcal{D}_{\text{Euclid}}$ with top θ. For all pairs $x, y \in X$ of neighbors, the set $x \bowtie^{\mathcal{D}_{\text{Euclid}}} y$ is given by the straight line normal to the (Euclidean) segment connecting x and y, and the sets $\rangle x, y\rangle^{\mathcal{D}_{\text{Euclid}}}$ and $\rangle y, x\rangle^{\mathcal{D}_{\text{Euclid}}}$ are on the two respective sides of this line (see Figure 5). Local net majority rule under the domain of Euclidean preferences is in general not acyclic. This can be verified using the same example as in Fig. 4 above. Interestingly, however, under the domain $\mathcal{D}_{\text{Euclid}}$, we obtain the reverse cycle $y \,\widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{Euclid}}} x$, $z \,\widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{Euclid}}} y$, $x \,\widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{Euclid}}} z$. Indeed, now both θ_2 and θ_3 support y over x, both θ_1 and θ_3 support z over y, and both θ_1 and θ_2 support x over z (see Fig. 5).

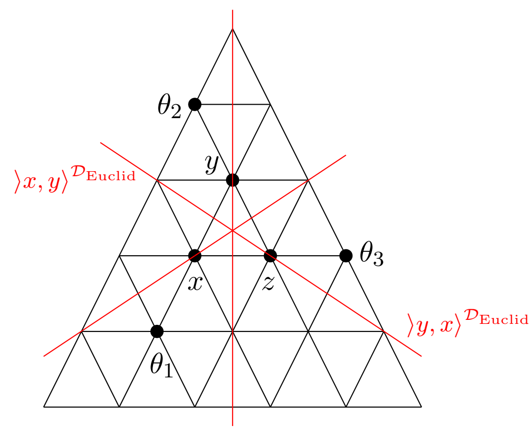


Fig. 5: Cyclic frugal majorities with Euclidean preferences

3.1.4 Decisiveness on resource sub-agendas

Theorem 1 entails that the restricted net majority tournament is decisive on all resource agendas. Evidently, a subset $Y \subseteq X$ of a resource agenda is itself a resource agenda if and only if it is box-convex. Hence, for all profiles, frugal majority rule when restricted to box-convex subsets is decisive on that subset. The following result states that it is in fact *only* in this case decisive for all profiles.

Fact 3.3 *Let $Y \subseteq X$, and consider the restricted net majority tournament $\text{NM}_{\boldsymbol{\theta}}|_Y$ restricted to Y. Then, $\text{NM}_{\boldsymbol{\theta}}|_Y$ is decisive for all profiles $\boldsymbol{\theta}$ on X if and only if Y is box-convex, i.e. a resource sub-agenda.*

3.2 On the interpretation of frugal majority rule: Condorcet meets Borda

By relying on an appropriate notion of (local) majorities, our approach to frugal social optimality may appear to have a strong 'Condorcetian' flavor. However, as we shall presently argue, the claim of the frugal majority set as the appropriate concept of social welfare optimum does not rest on a specific Condorcetian philosophy. Indeed, as shown in Subsection 3.2.1 below, the frugal majority set arises naturally also from the perspective of formulating an appropriate frugal version of Borda rule. Moreover, the frugal majority set does not suffer from the typical deficiencies that afflict Condorcetian aggregation methods in general, such as the well-known no-show paradox; this is shown in the subsequent Subsection 3.2.2. Finally, one 'philosophy' driving Borda's approach to collective choice is to determine the utilitarian optimum with respect to appropriately imputed utility functions. In Subsection 3.2.3, we show that the frugal majority set allows for this interpretation as well; in fact, it represents the utilitarian optimum with respect to a particularly natural class of imputed utility functions, the 'goal satisfaction' functions.

3.2.1 Frugal majority rule coincides with frugal Borda rule

How can one apply the Borda rule under the informational constraints of the frugal aggregation model? One immediate problem is that each agent is characterized by an *incomplete* ordering, namely by the partial order $>_{\boldsymbol{\theta}}^{\mathcal{D}^*}$, and that the Borda rule is defined for complete orderings only. In the following, we propose a natural general way to apply the Borda rule for partial orders, and we show that it coincides in our setting with frugal majority rule, i.e. with the ranking of allocations according to the median rule according to Theorem 1.

The Borda rule adds, for each alternative, the 'score' it receives from each individual, it is thus fully specified once the individual scores are determined. Suppose that an individual is characterized by the (strict) partial order $>_\theta$ on a finite set X with unique top alternative $\theta \in X$. We define the (non-positive) *score* $s_\theta(x) \leq 0$ of any alternative $x \in X$ as follows. A *chain (with respect to* $>_\theta$*)* is any subset of X that is totally ordered by the partial order $>_\theta$. A chain $Y \subset X$ has $x \in Y$ *at its bottom* if $y >_\theta x$ for all $y \in Y \setminus \{x\}$. For each $x \in X$, let $\tilde{s}_\theta(x)$ be the maximal cardinality of a chain that has x at its bottom. Then, define $s_\theta(x) := \tilde{s}_\theta(x) - 1$, so that θ itself uniquely receives the highest score $s_\theta(\theta) = 0$.

For every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every alternative x, let

$$\text{Borda}(\boldsymbol{\theta}) := \arg\max_{x \in X} \sum_{i=1}^{n} s_{\theta_i}(x)$$

denote the set of *frugal Borda winners.*

Proposition 1 *Consider the frugal aggregation problem $(X, \mathcal{D}^*)$. For all $\theta \in X$ and all $x \in X$, we have $s_\theta(x) = -d(x, \theta)$ for the scores derived from the partial order $>_\theta^{\mathcal{D}^*}$. Thus, in particular, for all profiles $\boldsymbol{\theta}$,*

$$\text{Borda}(\boldsymbol{\theta}) \;=\; \text{Med}(\boldsymbol{\theta}) \;=\; \text{Maj}(\boldsymbol{\theta})$$

Proof. The fact that $s_\theta(x) = -d(x,\theta)$ for the scores derived from the partial order $>_\theta^{\mathcal{D}^*}$ follows easily using Lemma 3.1; indeed, any shortest path from x to θ forms a chain with x at its bottom, and conversely, any chain with respect to $>_\theta^{\mathcal{D}^*}$ which has x at its bottom is contained in a shortest path connecting x and θ. From this, all other statements are immediate. □

The frugal Borda rule thus gives the same answer in the frugal aggregation problem $(X, \mathcal{D}^*)$ as both, the median rule and frugal majority rule. As further support for the chosen definition of the frugal version of Borda rule, let us note that it coincides with the frugal majority rule also in any frugal aggregation problem $(X, \mathcal{U})$ with an unrestricted domain; indeed, both rules coincide in this case with plurality rule. This follows for the frugal Borda rule from noting that $x >_\theta^{\mathcal{U}} y \Leftrightarrow (x = \theta \ \& \ y \neq \theta)$; thus, the top alternative receives individual score 0 while all other allocations receive an individual score of -1. But this, of course, means that the frugal Borda score ranks alternatives according to the number of agents who have the respective alternative at the top, i.e. according to their plurality score.

3.2.2 Frugal majority rule avoids the no-show paradox

The scoring rule representation of the frugal majority winners as the set of allocations that minimize the aggregate distance function implies that the corresponding choice rule avoids the famous 'no-show' paradox (Brams and Fishburn, 1983; Moulin 1988). For instance, by voting for the most preferred allocation among the frugal majority winners that would result without her vote, an individual can guarantee this allocation to be the unique new frugal majority winner.

More generally, we have the following result which shows that it can never be harmful for an agent to participate in collective decision according to frugal majority rule. For every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every agent $h \notin \{1, ..., n\}$, denote by $\boldsymbol{\theta} \sqcup \theta_h$ the profile $(\theta_1, ..., \theta_n, \theta_h)$. Moreover, denote by $\text{Maj}(\boldsymbol{\theta})^h$ the set of all allocations $x \in X$ such that $\text{Maj}(\boldsymbol{\theta}) \cap [x, \theta_h] = \{x\}$; thus, $\text{Maj}(\boldsymbol{\theta})^h$ is the subset of those allocations $x \in \text{Maj}(\boldsymbol{\theta})$ for which no other allocation of $\text{Maj}(\boldsymbol{\theta})$ lies on any shortest path connecting x and θ_h.

Proposition 2 *Consider any profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and any agent $h \notin \{1, ..., n\}$ with top θ_h and preference $\succcurlyeq_h$. Then,*

$$\text{Maj}(\boldsymbol{\theta})^h \subseteq \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \subseteq \bigcup_{x \in \text{Maj}(\boldsymbol{\theta})^h} [x, \theta_h]. \tag{3.8}$$

In particular, (i) for every $y \in \text{Maj}(\boldsymbol{\theta}) \setminus \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$ there exists $x \in \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \cap \text{Maj}(\boldsymbol{\theta})$ with $x \succ_h y$, and (ii) for every $z \in \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \setminus \text{Maj}(\boldsymbol{\theta})$ there exists $x \in \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \cap \text{Maj}(\boldsymbol{\theta})$ such that $z \succ_h x$.

The proof of (3.8) is provided in the appendix, but let us here verify that (3.8) implies the other statements in Proposition 2. First observe that by (3.8), we have $\text{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \cap \text{Maj}(\boldsymbol{\theta}) = \text{Maj}(\boldsymbol{\theta})^h$, that is, some frugal majority winners without agent h remain frugal majority winners also under h's participation. If $\text{Maj}(\boldsymbol{\theta}) \setminus \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$ is empty, statement (i) is trivially satisfied.

Thus, let $\mathrm{Maj}(\boldsymbol{\theta}) \setminus \mathrm{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \neq \emptyset$ and consider any $y \in \mathrm{Maj}(\boldsymbol{\theta}) \setminus \mathrm{Maj}(\boldsymbol{\theta} \sqcup \theta_h) = \mathrm{Maj}(\boldsymbol{\theta}) \setminus \mathrm{Maj}(\boldsymbol{\theta})^h$. By construction, there exists $x \in \mathrm{Maj}(\boldsymbol{\theta})^h$ on a shortest path connecting y and θ_h, hence $x \succ_h y$ by Lemma 3.1; this shows (i). Next, suppose that $\mathrm{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \setminus \mathrm{Maj}(\boldsymbol{\theta}) \neq \emptyset$ and let $z \in \mathrm{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \setminus \mathrm{Maj}(\boldsymbol{\theta}) = \mathrm{Maj}(\boldsymbol{\theta} \sqcup \theta_h) \setminus \mathrm{Maj}(\boldsymbol{\theta})^h$. By the second inclusion in (3.8), z is on some shortest path connecting θ_h with some $x \in \mathrm{Maj}(\boldsymbol{\theta})^h$; this implies $z \succ_h x$ by Lemma 3.1, and hence statement (ii).

Proposition 2 says that, from the viewpoint of agent h, the frugal majority set if h participates 'dominates' the frugal majority set if h does not participate. Indeed, statement (i) in Proposition 2 says that every frugal majority winner that is discarded by h's participation is strictly less preferred by h to some remaining frugal majority winner; and by statement (ii), every gained frugal majority winner is strictly preferred by h to some remaining frugal majority winner.

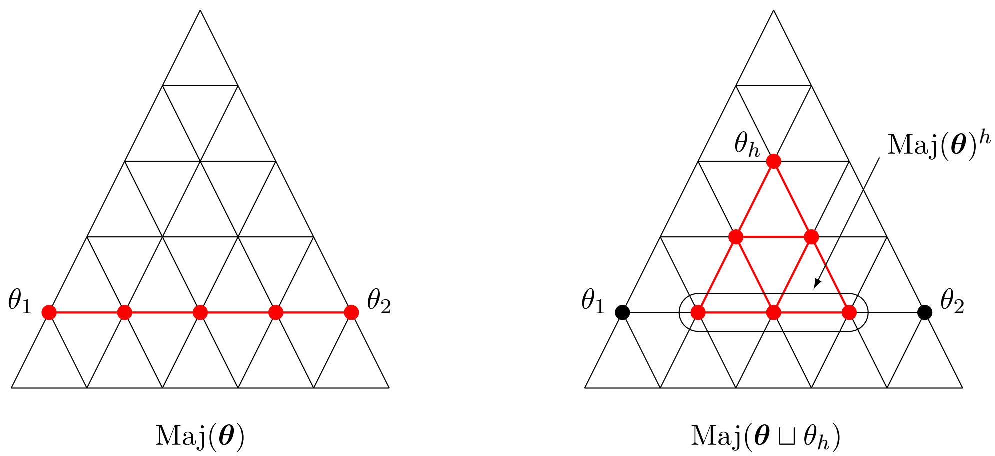


Fig. 6: Additional participation of agent h

Figure 6 illustrates Proposition 2. On the left-hand side, the frugal majority set $\mathrm{Maj}(\boldsymbol{\theta})$ (without agent h's participation) is marked in red. The right-hand side depicts the frugal majority set with participation of agent h whose top is at the upper vertex of the red triangle representing $\mathrm{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$; the set $\mathrm{Maj}(\boldsymbol{\theta})^h$ is encircled by the black oval. From h's perspective, the two allocations discarded by h's participation, θ_1 and θ_2, are strictly worse than their right and left neighbor, respectively; and each of the three frugal majority winners gained by h's participation are strictly preferred by h to at least one element (actually, here: at least two elements) of $\mathrm{Maj}(\boldsymbol{\theta})^h$.

3.2.3 The frugal majority set as 'imputed' utilitarian solution

In this subsection, we show that the frugal majority set coincides with the utilitarian maxima with respect to a natural class of 'imputed' utility functions. Specifically, for each agent with top θ consider the *goal satisfaction* function $v_\theta : \mathbb{Z}^L \to \mathbb{R}$ defined by

$$v_\theta(x) := \sum_{\ell=1}^{L} \min\{x^\ell, \theta^\ell\},$$

for all $x \in \mathbb{Z}^L$. Evidently, for every $\theta \in X$, $v_\theta(\cdot)$ is an additively separable function which is (weakly) increasing and concave in each component. The term $\min\{x^\ell, \theta^\ell\}$ measures the extent to which the 'goal' θ^ℓ is satisfied in coordinate ℓ by the allocation x. Note that with monotone preferences oversatisfaction of the goal in the sense that $x^\ell > \theta^\ell$ does not hurt *per se*; but for a feasible allocation $x \in X$, oversatisfaction in one coordinate is necessarily accompanied by undersatisfaction of the goal in some other coordinate due to the budget constraint incorporated in the resource agenda X. Thus, the term

$$\sum_{\ell=1}^{L} |x^\ell - \theta^\ell|_+,$$

where $|x^\ell - \theta^\ell|_+ := \max\{x^\ell - \theta^\ell, 0\}$, can be interpreted as the potential 'waste' of resources at allocation x from the point of view of an agent with top allocation θ. Noting that, for all $\theta, x \in X$,

$$\sum_{\ell=1}^{L} |x^\ell - \theta^\ell|_+ \;=\; \sum_{\ell=1}^{L} |\theta^\ell - x^\ell|_+ \;=\; d(x,\theta)/2,$$

we obtain for all $x \in X$ and all $\theta_i \in X$,

$$\begin{aligned} v_{\theta_i}(x) &= \sum_{\ell=1}^{L} x^\ell - \sum_{\ell=1}^{L} |x^\ell - \theta_i^\ell|_+ \;=\; Q - \sum_{\ell=1}^{L} |x^\ell - \theta_i^\ell|_+ \qquad (3.9)\\ &= Q - d(x,\theta_i)/2. \qquad (3.10) \end{aligned}$$

Equation (3.9) states that, up to a constant, goal satisfaction simply measures aggregate (potential) waste, and (3.10) implies that minimizing aggregate distance of $x \in X$ for a profile $\boldsymbol{\theta}$ of tops in X amounts to maximizing the aggregate goal satisfaction $v_{\boldsymbol{\theta}}(\cdot)$ defined by

$$v_{\boldsymbol{\theta}}(x) \;:=\; \sum_{i=1}^{n} v_{\theta_i}(x).$$

Thus, for all profiles $\boldsymbol{\theta}$ in X, the frugal majority set coincides with the utilitarian maximizers of the individual goal satisfaction functions, i.e. we have the following result.

Proposition 3 *Consider the frugal aggregation problem* $(X, \mathcal{D}^*)$. *Then,*

$$\mathrm{Maj}(\boldsymbol{\theta}) = \mathrm{Med}(\boldsymbol{\theta}) = \arg\max_{x \in X} v_{\boldsymbol{\theta}}(x),$$

for all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$.

Finally, observe that, for every agent with top $\theta_i \in X$ and each $x \in X$, one needs exactly the money amount $d(x, \theta_i)$ in addition to Q in order to finance an allocation $y \in \mathbb{Z}^L$ such that $y^\ell \geq x^\ell$ for all $\ell = 1, ..., L$, and $y \succcurlyeq_i x$ for all $\succcurlyeq \in \mathcal{D}^*_{\theta_i}$. By (3.10), the individual goal satisfaction function v_{θ_i} can thus be viewed as a *frugal money metric* utility function as it measures the money equivalent of the individual loss due to the collective compromise.

4 Normative foundation

In this section, we provide an axiomatic characterization of frugal majority rule aka median rule as a choice rule from a fixed resource agenda X and all its resource sub-agendas $S \subseteq X$. Recall from Section 3.1.4 above that a subset $S \subseteq X$ of a resource agenda represents itself a resource agenda if and only if it is a box-convex subset of X. Arguably, the class of all (non-empty) resource sub-agendas of a given agenda X, denoted by $\mathcal{S}_{\text{res}}$ in the following, is a natural choice of feasible sets. Moreover, it turns out that these always represent 'well-decidable' feasible sets. In particular, observe that a pair $\{x, y\} \subseteq X$ represents a resource agenda if and only if x and y are neighbors, i.e if $\{x, y\}$ is an element of the comparison graph Γ_{res}.

In general, a *(collective) choice rule* C assigns, for every profile $\boldsymbol{\theta}$ and every element $S \in \mathcal{S}$ of some family $\mathcal{S} \subseteq 2^X \setminus \{\emptyset\}$ of non-empty subsets of X, a subset

$$C(S; \boldsymbol{\theta}) \subseteq S$$

of *eligible* alternatives. A particular choice rule is given by the *median choice rule*

$$\text{Med}(S; \boldsymbol{\theta}) := \arg\min_{x \in S} \sum_{\theta \in \boldsymbol{\theta}} d(x, \theta), \tag{4.1}$$

for all profiles $\boldsymbol{\theta}$ and all non-empty subsets $S \subseteq X$; evidently, we have $\text{Med}(X; \boldsymbol{\theta}) = \text{Med}(\boldsymbol{\theta})$.

Our first axiom is an uncontroversial unanimity condition.

Unanimity. For all $S \in \mathcal{S}_{\text{res}}$ and $x \in S$, if $\text{supp}(\boldsymbol{\theta}) = \{x\}$, then

$$C(S; \boldsymbol{\theta}) \; = \; \{x\}.$$

For any two profiles $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ denote by $\boldsymbol{\theta} \sqcup \boldsymbol{\theta}'$ the profile that results from merging the two profiles, i.e. from 'adding' the two underlying populations. The following condition states that the choice from a set is unaffected by adding an 'indifferent population.'

Cancellation. For all $S \in \mathcal{S}_{\text{res}}$, and all profiles $\boldsymbol{\theta}, \boldsymbol{\theta}'$,

$$C(S; \boldsymbol{\theta}') = S \;\Rightarrow\; C(S; \boldsymbol{\theta} \sqcup \boldsymbol{\theta}') = C(S; \boldsymbol{\theta}).$$

The next condition is a symmetry condition which plays an important rule in the following analysis and can be justified by an appeal to the principle of insufficient reason.

Symmetry (with respect to $\mathcal{D}$). For all $S \in \mathcal{S}_{\text{res}}$ such that $\mathcal{D}|_S = \mathcal{U}$, if $\boldsymbol{\theta}$ is a profile such that each $x \in S$ is the top of exactly one voter, then

$$C(S; \boldsymbol{\theta}) \; = \; S.$$

Observe that when applied to pairs $S = \{x, y\} \in \mathcal{S}_{\text{res}}$, the Symmetry axiom becomes a standard condition of 'majority ties,' stating that a tie in the comparison of two (neighboring) allocations should result in the choice of either. The strength of the symmetry axiom derives in particular from its requirement of symmetric treatment of alternatives that are indistinguishable by information from within the feasible set S. One could try to argue that information about preferences *outside* the set S could be relevant as well, but this is ruled out by the symmetry axiom.

Together with the above symmetry condition, the next condition is the key axiom in the subsequent characterization result. For a given domain $\mathcal{D}$, consider any $S \in \mathcal{S}_{\text{res}}$ and any top $\theta \in X$, and denote by $>_\theta^\mathcal{D}|_S$ the restriction to S of the partial order $>_\theta^\mathcal{D}$. Note that while every order $>_\theta^\mathcal{D}$ has a unique top on X (namely, θ) it is not true that $>_\theta^\mathcal{D}|_S$ has a unique top for every $S \in \mathcal{S}_{\text{res}}$. For instance, if $d(x,\theta) = d(y,\theta)$, then $>_\theta^\mathcal{D}|_S$ does not admit a unique top on the adjacent pair $S = \{x, y\}$ since x and y are incomparable with respect to $>_\theta^\mathcal{D}|_S$. On the other hand, if $\theta \in \rangle x, y\langle$, then the unique top of the restriction of $>_\theta^\mathcal{D}$ to the adjacent pair $\{x, y\}$ is clearly x.

Frugal Independence (with respect to $\mathcal{D}$). For all $S \in \mathcal{S}_{\text{res}}$, and all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, $\boldsymbol{\theta}' = (\theta'_1, ..., \theta'_n)$, if, for all $i = 1, ..., n$, $>_{\theta_i}^\mathcal{D} |_S = >_{\theta'_i}^\mathcal{D} |_S$ and if each $>_{\theta_i}^\mathcal{D} |_S$ has a unique top on S, then

$$C(S; \boldsymbol{\theta}) \;=\; C(S; \boldsymbol{\theta}').$$

Our final two axioms are familiar choice consistency properties (see, e,g, Sen, 1971).[7]

Property $\boldsymbol{\alpha}$. For all $\boldsymbol{\theta}$ and all $T, S \in \mathcal{S}_{\text{res}}$ such that $T \subseteq S$,

$$C(S; \boldsymbol{\theta}) \cap T \subseteq C(T; \boldsymbol{\theta})$$

Property $\boldsymbol{\beta}$. For all $\boldsymbol{\theta}$ and all $T, S \in \mathcal{S}_{\text{res}}$ such that $T \subseteq S$,

$$[\{x, y\} \subseteq C(T; \boldsymbol{\theta}) \text{ and } x \in C(S; \boldsymbol{\theta})] \;\Rightarrow\; y \in C(S; \boldsymbol{\theta}).$$

Theorem 2 *A collective choice rule on the domain $\mathcal{S}_{\text{res}}$ of all resource sub-agendas satisfies Unanimity, Cancellation, Symmetry with respect to $\mathcal{D}^*$, Frugal Independence with respect to $\mathcal{D}^*$, as well as Properties $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, if and only if it is the median choice rule.*

Proof. Necessity of the stated conditions are proved in the Appendix. The proof of their sufficiency works as follows. Consider a collective choice rule C satisfying the stated conditions (with respect to the domain $\mathcal{D}^*$). First, we show that C then satisfies the following *indifference property.* For all $\{x, y\} \in \mathcal{S}_{\text{res}}$, and all profiles $\boldsymbol{\theta}$,

$$(z \in x \bowtie y \text{ and } \operatorname{supp}(\boldsymbol{\theta}) = \{z\}) \;\Rightarrow\; C(\{x, y\}; \boldsymbol{\theta}) \;=\; \{x, y\}. \tag{4.2}$$

This will allow us to show that the stated conditions imply that all choices from adjacent pairs coincide with the median choice rule. Finally, we use Properties $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to extend this conclusion to all box-convex subsets, using Theorem 1 above.

[7] It is well known that, if the class of feasible menus contains all subsets of cardinality two and three, then Properties $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both necessary and jointly sufficient for a choice rule C to be rationalizable by a weak order (Sen, 1971). Since $\mathcal{S}_{\text{res}}$ does not contain all two- and three-element subsets, Properties $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have much weaker implications on the domain $\mathcal{S}_{\text{res}}$ of feasible menus.

To prove the implication (4.2), consider an adjacent pair x, y and one agent with top θ in $x \bowtie y$. Let z be the common neighbor of x and y such that z is the unique top of the restriction $>_\theta^{\mathcal{D}^*}$ to the adjacent triple $\{x, y, z\}$. By Frugal Independence the choice from $\{x, y, z\}$ does not change by moving the top of this agent to z. By Cancellation and Symmetry, the choice from $\{x, y, z\}$ does not change by adding one agent with top x and one agent with top y; call this profile $\boldsymbol{\theta}'$. Evidently, we have $\mathcal{D}^*|_{\{x,y,z\}} = \mathcal{U}$,[8] hence by Symmetry, $C(\{x, y, z\}; \boldsymbol{\theta}') = \{x, y, z\}$, and by Property $\boldsymbol{\alpha}$, $C(\{x, y\}; \boldsymbol{\theta}') = \{x, y\}$. This shows that the indifference property (4.2) holds whenever $\boldsymbol{\theta}$ contains only one voter; from this the general case with an arbitrary $\boldsymbol{\theta}$ such that $\text{supp}(\boldsymbol{\theta}) = \{z\}$ follows at once using Cancellation.

Now consider an arbitrary pair $\{x, y\} \in \mathcal{S}_{\text{res}}$, and any profile $\boldsymbol{\theta}$. By Cancellation and the indifference property (4.2) we may modify $\boldsymbol{\theta}$ without altering the choice from the pair $\{x, y\}$ by deleting all but one of the agents with top in $x \bowtie y$ (if any). If $\text{supp}(\boldsymbol{\theta}) \subseteq x \bowtie y$, we obtain $C(\{x, y\}; \boldsymbol{\theta}) = \{x, y\}$ by (4.2). In this case, we clearly have $C(\{x, y\}; \boldsymbol{\theta}) = C^{\text{Med}}(\{x, y\}; \boldsymbol{\theta})$. If $\text{supp}(\boldsymbol{\theta}) \not\subseteq x \bowtie y$, we may in fact by Cancellation and (4.2) delete all agents with top in $x \bowtie y$ and assume henceforth that $\text{supp}(\boldsymbol{\theta}) \subseteq \rangle x, y\rangle \cup \rangle y, x\rangle$. Observe that for all $\theta \in \rangle x, y\rangle$ the induced top on the pair $\{x, y\}$ is $\theta_{\{x,y\}} = x$, and for all $\theta \in \rangle y, x\rangle$ the induced top on the pair $\{x, y\}$ is $\theta_{\{x,y\}} = y$; by Frugal Independence we may thus modify $\boldsymbol{\theta}$ to $\boldsymbol{\theta}'$ such that the support of $\boldsymbol{\theta}'$ is concentrated on $\{x, y\}$ without changing the choice from $\{x, y\}$. There are now three cases: (i) if there is an equal number of agents on x and on y under $\boldsymbol{\theta}''$, we obtain $C(\{x, y\}; \boldsymbol{\theta}) = C(\{x, y\}; \boldsymbol{\theta}'') = \{x, y\}$ using Cancellation and Symmetry; (ii) if there are more agents on x than on y, we obtain $C(\{x, y\}; \boldsymbol{\theta}) = C(\{x, y\}; \boldsymbol{\theta}'') = \{x\}$ using Cancellation, Symmetry and Unanimity; (iii) and, finally, if there are more agents on y than on x, we obtain $C(\{x, y\}; \boldsymbol{\theta}) = C(\{x, y\}; \boldsymbol{\theta}'') = \{y\}$ again using Cancellation, Symmetry and Unanimity. Evidently, in each case the choice under C coincides with the median choice rule. Thus, we have

$$C(\{x, y\}; \boldsymbol{\theta}) = C^{\text{Med}}(\{x, y\}; \boldsymbol{\theta}) \tag{4.3}$$

for all pairs $\{x, y\} \in \mathcal{S}_{\text{res}}$.

Now consider any resource sub-agenda $S \subseteq X$ with more than two elements. We have to show that $C(S; \boldsymbol{\theta}) = C^{\text{Med}}(S; \boldsymbol{\theta})$. Let $x \in C(S; \boldsymbol{\theta})$; by Property $\boldsymbol{\alpha}$, $x \in C(\{x, y\}; \boldsymbol{\theta})$ for all $y \in S$ that are adjacent to x. By (4.3), x is the frugal majority winner against all its neighbors in S, hence by Theorem 1 applied to the resource agenda $S \subseteq X$, we have

$$x \in \arg\min_{z \in S} \sum_{\theta \in \boldsymbol{\theta}} d(z, \theta),$$

i.e. $x \in C^{\text{Med}}(S; \boldsymbol{\theta})$, hence $C(S; \boldsymbol{\theta}) \subseteq C^{\text{Med}}(S; \boldsymbol{\theta})$. Conversely, let $x \in C^{\text{Med}}(S; \boldsymbol{\theta})$, and consider $w \in C(S; \boldsymbol{\theta})$. By the first part we have $w \in C^{\text{Med}}(S; \boldsymbol{\theta})$. Consider a shortest path $(y_1, ..., y_m) \subseteq S$ with $y_1 = w$ and $y_m = x$. By Observation 1 in the Appendix, the aggregate distance of $\boldsymbol{\theta}$ is constant along the path $(y_1, ..., y_m)$, i.e. $\sum_{\theta \in \boldsymbol{\theta}} d(y_j, \theta) = \sum_{\theta \in \boldsymbol{\theta}} d(y_k, \theta)$ for all $j, k \in \{1, ..., m\}$, hence $C^{\text{Med}}(\{y_j, y_{j+1}\}; \boldsymbol{\theta}) = \{y_j, y_{j+1}\}$ for all $j \in \{1, ..., m-1\}$. By equation (4.3), this implies $C(\{y_j, y_{j+1}\}; \boldsymbol{\theta}) = \{y_j, y_{j+1}\}$ for all $j \in \{1, ..., m-1\}$. By repeated application of Property $\boldsymbol{\beta}$, we obtain $y_j \in C(S; \boldsymbol{\theta})$ for all $j \in \{1, ..., m\}$, in particular $x \in C(S; \boldsymbol{\theta})$ and hence

[8] More generally, it can be verified that, for all $S \in \mathcal{S}_{\text{res}}$, $\mathcal{D}^*|_S = \mathcal{U}$ if and only if $\{x, y\} \in \mathcal{S}_{\text{res}}$ for all $x, y \in S$.

$C^{\text{Med}}(S;\boldsymbol{\theta}) \subseteq C(S;\boldsymbol{\theta})$. This completes the sufficiency proof of Theorem 2. $\square$

The main thrust of the axiomatization and normative foundation of the frugal majority (aka median) rule provided by Theorem 2 lies in the interplay between the particular use of the information contained in the agents' tops and the background domain assumption (separability and concavity of preference) on the one hand, and the treatment of the *lack* of information on the other hand. This is exemplified in the two key conditions, Frugal Independence and Symmetry: the first condition describes how known information has to be used on resource sub-agendas, while the second condition requires symmetric treatment of missing information on certain resource sub-agendas that are symmetric with respect to both the profile and background information.

Note that in the special case of the resource agenda $X^{1,L}$ defined by

$$X^{1,L} := \left\{ x \in \mathbb{Z}^L \ : \ \sum_{\ell=1}^{L} x^\ell = 1 \text{ and } x^\ell \in \{0,1\} \text{ for all } \ell = 1,...,L \right\},$$

every non-empty subset is a sub-resource agenda; moreover, we have $\mathcal{D}^*|_S = \mathcal{U}$ for all $S \subseteq X^{1,L}$. Thus in this case, the Frugal Independence axiom is vacuous, while the Symmetry condition gains full force. Theorem 2 provides an axiomatization of plurality rule alternative to the one given by Goodin and List (2006). The main conclusion of either characterization, however, is similar: in the absence of any preference information beyond the agents' tops, plurality rule is the only collective choice rule satisfying a small set of reasonable and basic conditions.

Finally, consider the somewhat more general case in which the only feasible values of x^ℓ in each coordinate are still either 0 or 1, but in which $1 \leq Q \leq L$, i.e. consider the agenda

$$X_{\text{com}} \ = \ \left\{ x \in \mathbb{Z} \ : \ \sum_{\ell=1}^{L} x^\ell = Q \text{ and } x^\ell \in \{0,1\} \text{ for all } \ell = 1,...,L \right\}.$$

A possible interpretation is that each coordinate corresponds to a candidate and the feasible allocations are committees of exactly $Q \leq L$ candidates. A natural domain assumption on X_{com} is to restrict individual preferences to the set $\mathcal{D}_{\text{add}}$ of all additively separable preferences on X_{com}. Note that, since there are only two possible values of x^ℓ in each coordinate, concavity imposes no restriction here; the only difference between $\mathcal{D}^*$ and $\mathcal{D}_{\text{add}}$ on the agenda X_{com} is that $\mathcal{D}_{\text{add}}$ does not assume monotonicity.

A top allocation $\theta \in X_{\text{com}}$ can be interpreted as *approving* exactly the candidates $\ell = 1,...,L$ with $\theta^\ell = 1$. It is easily verified that, under the background assumption that individual preferences belong to the domain $\mathcal{D}_{\text{add}}$, a committee is a frugal majority winner if and only if it consists of the Q candidates who receive the greatest number of individual approvals. In other words, frugal majority rule (and hence, by Theorem 1, also the median rule) amounts to *Q-approval voting.* By consequence, Theorem 2 provides a novel axiomatization of Q-approval voting.

5 Simple Characterization

In this subsection, we provide a simple and powerful characterization of the frugal majority set that allows one to compute it very efficiently and to immediately derive a number of interesting further properties.

In the following fix a profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ with n voters and denote, for every $\ell = 1, ..., L$ and every $k = 1, ..., n$, by $\theta^\ell_{[k]} \in X$ the k-th smallest vote in coordinate ℓ, i.e. the vector $(\theta^\ell_{[1]}, \theta^\ell_{[2]}, ..., \theta^\ell_{[n]})$ results from the values $\theta^\ell_1, \theta^\ell_2, ..., \theta^\ell_n$ simply by re-arranging the latter in ascending order so that $\theta^\ell_{[1]} \leq \theta^\ell_{[2]} \leq ... \leq \theta^\ell_{[n]}$ (possibly with some equalities). Denote by $Q_{[k]} := \sum_{\ell=1}^{L} \theta^\ell_{[k]}$, and let $k^*(\boldsymbol{\theta})$ be the largest $k = 1, ..., n$ such that $Q_{[k]} \leq Q$. Finally, say that the profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is *unanimous* if $\theta_1 = \theta_2 = ... = \theta_n$. Note that for a unanimous profile one has $k^*(\boldsymbol{\theta}) = n$ since, evidently, $\theta^\ell_{[1]} = \theta^\ell_{[2]} = ... = \theta^\ell_{[n]} = \theta^\ell_i$ for all $i = 1, ..., n$ and all $\ell = 1, ..., L$. Also observe that $k^*(\boldsymbol{\theta}) < n$ for all non-unanimous profiles.

Theorem 3 *Consider the frugal aggregation problem $(X, \mathcal{D}^*)$. For every non-unanimous profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every $x \in X$ the following are equivalent.*

a) $x \in \mathrm{Maj}(\boldsymbol{\theta})$,

b) x maximizes aggregate goal satisfaction $v_{\boldsymbol{\theta}}(\cdot)$,

c) for all $\ell = 1, ..., L$,

$$\theta^\ell_{[k^*(\boldsymbol{\theta})]} \leq x^\ell \leq \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}. \tag{5.1}$$

(Proof in Appendix.)

Condition (5.1) means that $k^*(\boldsymbol{\theta})/n$ is the *'endogenous' (i.e. profile-dependent) quota* of voters who can be satisfied in all coordinates (of course, these have to be different sets of voters in different coordinates).

Example 4 As a simple example illustrating the endogenous quota interpretation of the characterization in Theorem 3c), consider the case $L = 3$, $Q = 10$, and a profile $\boldsymbol{\theta}$ with four voters such that $\theta_1 = (5, 0, 5)$, $\theta_2 = (0, 2, 8)$, $\theta_3 = (2, 6, 2)$ and $\theta_4 = (4, 3, 3)$, say. For the corresponding matrices (θ^ℓ_i) and $(\theta^\ell_{[k]} | \, Q_{[k]})$ with $\ell = 1, ..., L$ and $i, k = 1, ..., n$ we thus obtain

$$(\theta^\ell_i) \;=\; \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 8 \\ 2 & 6 & 2 \\ 4 & 3 & 3 \end{pmatrix} \quad \text{and} \quad (\theta^\ell_{[k]} | \, Q_{[k]}) \;=\; \begin{pmatrix} 0 & 0 & 2 & | & 2 \\ 2 & 2 & 3 & | & 7 \\ 4 & 3 & 5 & | & 12 \\ 5 & 6 & 8 & | & 19 \end{pmatrix}.$$

Since $Q_{[2]} = 7 < 10 \; (= Q) < 12 = Q_{[3]}$, we obtain $k^*(\boldsymbol{\theta}) = 2$, and thus an endogenous quota of 0.5; in accordance with (5.1), the frugal majority set is given by

$$\mathrm{Maj}(\boldsymbol{\theta}) = \{(2, 3, 5), (3, 2, 5), (3, 3, 4), (4, 2, 4), (4, 3, 3)\}.$$

Now suppose that voter 4 changes her vote to $\tilde{\theta}_4 = (3, 2, 5)$ while the other voters keep their position. If we denote the resulting profile by $\tilde{\boldsymbol{\theta}}$, we obtain

$$(\tilde{\theta}_i^\ell) \;=\; \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 8 \\ 2 & 6 & 2 \\ 3 & 2 & 5 \end{pmatrix} \quad \text{and} \quad (\tilde{\theta}_{[k]}^\ell |\, Q_{[k]}) \;=\; \begin{pmatrix} 0 & 0 & 2 & | & 2 \\ 2 & 2 & 5 & | & 9 \\ 3 & 2 & 5 & | & 10 \\ 5 & 6 & 8 & | & 19 \end{pmatrix}.$$

Now, since $Q_{[3]} = 3 + 2 + 5 = 10$ $(= Q)$, we obtain $k^*(\tilde{\boldsymbol{\theta}}) = 3$, hence an endogenous quota of 0.75. Moreover, since $Q_{[k^*(\tilde{\boldsymbol{\theta}})]} = Q$ there is a unique net majority winner, and indeed $\text{Maj}(\tilde{\boldsymbol{\theta}}) = \{(3, 2, 5)\}$.

The following is a simple but important corollary of the characterization in Theorem 3c). For any subset $Y \subseteq X$, denote by $hull(Y)$ the box-convex hull of Y, i.e. the smallest box-convex set containing Y. Furthermore, for any $s \in \mathbb{R}$, denote by $\lfloor s \rfloor$ the largest integer $\leq s$, and by $\lceil s \rceil$ the smallest integer $\geq s$.

Corollary 1 *For all $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, we have*

$$\text{Maj}(\boldsymbol{\theta}) \;\subseteq\; [\theta_{[\lfloor \frac{n}{L} \rfloor]}, \theta_{[\lceil n - \frac{n}{L} \rceil]}] \;\subseteq\; hull(\{\theta_1, ..., \theta_n\}).$$

5.1 On the uniqueness of frugal majority winners

In contrast to solution concepts that may be refined by using additional information, the frugal majority set is characterized by the indifference principle, i.e. by a deliberate neglect of unavailable information. Therefore, ties seem to be unavoidable. However, two important qualifications are in order. First, the 'more dense' the support of a profile becomes, the smaller is the frugal majority set; and secondly, if the unit of measuring expenditure becomes sufficiently small, there exists a natural and unique selection from the frugal majority set. This is described in the next two subsections, respectively.

5.1.1 Essential uniqueness for connected supports

Say that a subset $Y \subseteq X$ is *essentially unique* if

$$\max_{x,y \in Y,\ \ell=1,...,L} |x^\ell - y^\ell| \leq 1.$$

Thus, a subset of X is essentially unique if every two of its elements differ in each coordinate by at most one unit. Also, say that the support of a profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is *coordinate-wise connected* if, for each $\ell = 1, ..., L$, the set $\{\theta_i^\ell\}_{i=1,...,n}$ forms an interval in $\mathbb{Z}$, i.e. $\{\theta_1^\ell, ..., \theta_n^\ell\} = [\min_i \theta_i^\ell, \max_i \theta_i^\ell]$. The following is an immediate corollary of the characterization in Theorem 3c).

Corollary 2 *The frugal majority set* $\text{Maj}(\boldsymbol{\theta})$ *is essentially unique whenever* $\boldsymbol{\theta}$ *is coordinate-wise connected.*

The following figure depicts the set Maj($\boldsymbol{\theta}$) for a profile of five agents. Observe that while the support of $\boldsymbol{\theta}$ is not connected in the usual sense (because different agents' tops cannot be connected by a path within the support of $\boldsymbol{\theta}$), the depicted profile is nevertheless coordinate-wise connected in the sense defined above.

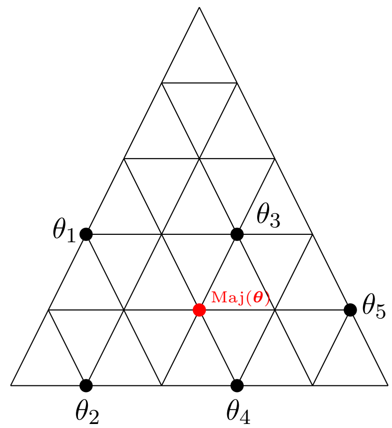


Fig. 7: Essential uniqueness under coordinate-wise connectedness

Denoting by $\delta_Y := \max_{x,y\in Y} d(x,y)$ the *diameter* of a subset Y, we obtain that, for any essentially unique set $Y \subseteq X$, $\delta_Y \leq L/2$ if L is even and $\delta_Y \leq (L-1)/2$ if L is odd. Moreover, it can be verified that the maximal number of elements of an essentially unique set is $\binom{L}{L/2}$ if L is even, and $\binom{L}{(L-1)/2}$ if L is odd.[9] For $L = 3$ this gives the upper bound $\#\text{Maj}(\boldsymbol{\theta}) \leq 3$ whenever the profile $\boldsymbol{\theta}$ is coordinate-wise connected. Note that this upper bound is attained if the support of $\boldsymbol{\theta}$ is itself symmetric and essentially unique, e.g. if each of the following feasible allocations receives exactly one third of the popular support: (x^1, x^2, x^3), $(x^1 - 1, x^2 + 1, x^3)$, $(x^1 - 1, x^2, x^3 + 1)$. By contrast, we have full uniqueness in Fig. 7.

Remark. Evidently, in the one-dimensional case with an odd number of agents, the frugal majority winner coincides with the median top no matter whether the support is connected or not. More generally, the frugal majority set does not change by moving agents at the tails of the support. This shows that coordinate-wise connectedness of supp($\boldsymbol{\theta}$) is a *sufficient* but certainly not a necessary condition for essential uniqueness of Maj($\boldsymbol{\theta}$). As an illustration of this point, consider again the example depicted in Fig. 7. The unique frugal majority winner depicted there does not change if one replaces θ_2 by its left neighbor on the same horizontal line, say; however, the resulting support is no longer coordinate-wise connected.

This suggests that the 'size' of the frugal majority set is bounded (in an appropriate way) by the size of the 'median gap' in each coordinate (no matter how large the distance of agents' tops at the tails of the support are). A more detailed analysis of this is left to future work.

[9] This follows from the fact that, for any two elements x, y of an essentially unique set $Y \subseteq X$, one has $\#\{\ell : x^\ell > y^\ell\} \leq L/2$ and $\#\{\ell : x^\ell < y^\ell\} \leq L/2$.

5.1.2 A unique selection in the continuum

The characterization given in Theorem 3 suggests a natural selection method from the frugal majority set. Specifically, for any resource agenda $X \subseteq \mathbb{Z}^L$ consider the Euclidean convex hull $co_2(X) \subseteq \mathbb{R}^L$, i.e. the set of all real-valued convex combinations of the elements of X. For every non-unanimous profile $\boldsymbol{\theta}$, consider the box

$$\prod_{\ell=1}^{L} \left[\theta^{\ell}_{[k^*(\boldsymbol{\theta})]}, \theta^{\ell}_{[k^*(\boldsymbol{\theta})+1]}\right]$$

and the point x^* of intersection with $co_2(X)$ of the diagonal in this box, i.e. the straight line connecting $(\theta^1_{[k^*(\boldsymbol{\theta})]}, ..., \theta^L_{[k^*(\boldsymbol{\theta})]})$ and $(\theta^1_{[k^*(\boldsymbol{\theta})+1]}, ..., \theta^L_{[k^*(\boldsymbol{\theta})+1]})$. Concretely, for every $\ell = 1, ..., L$, $(x^*)^\ell$ is given by

$$(x^*)^\ell \;=\; \alpha \cdot \theta^{\ell}_{[k^*(\boldsymbol{\theta})]} + (1-\alpha) \cdot \theta^{\ell}_{[k^*(\boldsymbol{\theta})+1]},$$

where

$$\alpha \;:=\; \frac{Q_{[k^*(\boldsymbol{\theta})+1]} - Q}{Q_{[k^*(\boldsymbol{\theta})+1]} - Q_{[k^*(\boldsymbol{\theta})]}}.$$

By Theorem 3, we have

$$x^* \in co_2(\mathrm{Maj}(\boldsymbol{\theta})),$$

and we will refer to x^* as the *resolute net majority winner*. While x^* need not be an element of $X \subseteq \mathbb{Z}^L$ itself due to integer problems, it will be close to an element of X if the 'grid' is fine enough (i.e. if Q is sufficiently large). Moreover, one can show that, for a finite set of agents, the frugal majority winners can be defined in the continuum in an analogous way and the characterization entailed by Theorem 3 remains unchanged. In this case, we have $x^* \in \mathrm{Maj}(\boldsymbol{\theta})$ for all profiles $\boldsymbol{\theta}$ (with finite support).

6 Conclusion

Instead of summarizing our findings, we close with a remark on what appears to be an obvious contender of the frugal majority (aka median) rule: the *mean rule*, i.e. the coordinate-wise average of the agents' tops. Evidently, however, this rule violates the Frugal Independence condition even in the simplest cases of the choice between two adjacent alternatives in the line. But in this case, the Frugal Independence axiom reduces to the Arrovian condition of independence of irrelevant alternatives. Due to the violation of this basic independence conditions, the mean rule is in particular highly manipulable through misrepresentation of preferences. By contrast, the frugal majority vote is much more robust against manipulation, as we show in a companion paper.

Appendix: Remaining proofs

(To be completed)

We start with the following key technical observation that highlights the role of separability and convexity in our present context (observe that it does not require monotonicity of preferences). For any allocation $x \in X$ denote by $x_{(kj)}$ the allocation that results from x by transferring one unit of money from good j to good k, i.e. $x^k_{(kj)} = x^k + 1$, $x^j_{(kj)} = x^j - 1$ and $x^\ell_{(kj)} = x^\ell$ for all $\ell \neq k, j$.

Observation 1 *Let $f : X \to \mathbb{R}$ be a separable function with $f(x) = \sum_{\ell=1}^{L} f^\ell(x^\ell)$ such that all functions $f^\ell(\cdot)$ are concave. Then,*

$$f(x) > f(x_{(kj)}) \;\Rightarrow\; f(y) > f(y_{(kj)})$$

for all k, j, x, y such that $y^k \geq x^k$ and $y^j \leq x^j$ (see Figure A.1 in which $L = 3$, $k = 3$ and $j = 2$).

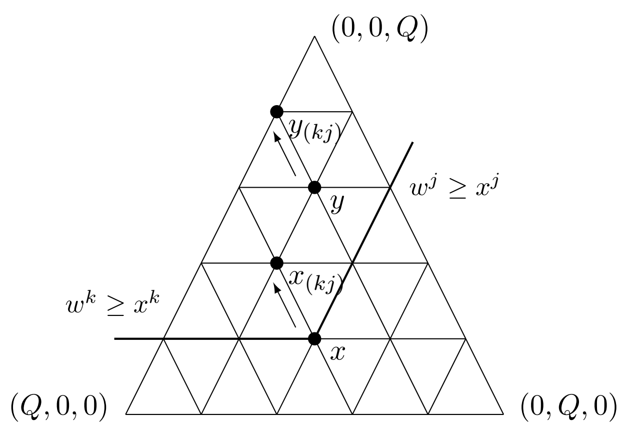


Fig. A.1: An implication of separability and concavity

Proof. Since x and $x_{(kj)}$ differ only in coordinates k and j, we have $f(x) > f(x_{(kj)})$ if and only if $f^j(x^j) - f^j(x^j - 1) > f^k(x^k + 1) - f^k(x^k)$. By the concavity of $f^k(\cdot)$ and $f^j(\cdot)$ we obtain

$$\left[f^j(y^j) - f^j(y^j - 1)\right] \geq \left[f^j(x^j) - f^j(x^j - 1)\right] > \left[(f^k(x^k + 1) - f^k(x^k)\right] \geq \left[f^k(y^k + 1) - f^k(y^k)\right],$$

and hence $f(y) > f(y_{(kj)})$ as desired. □

Proof of Lemma 3.1 We have to show that, for all distinct $x, y \in X$,

$$\rangle x, y \langle^{\mathcal{D}^*} \;=\; \{\theta \in X \;:\; x \in [\theta, y]\}\,. \tag{A.1}$$

First part: "$\subseteq$" Consider $x, y, \theta \in X$ such that $x \notin [\theta, y]$. We will show that there exists $\succcurlyeq \in \mathcal{D}^*_\theta$ such that $y \succ x$. This is trivial if $\theta = y$; thus, assume henceforth that $\theta \neq y$. In the following, we explicitly construct appropriate strictly increasing and strictly concave functions $u^\ell : X^\ell \to \mathbb{R}$ for $\ell = 1, ..., L$, where X^ℓ is the projection of X to coordinate ℓ. First observe that it is clearly possible, for any given $\theta^\ell \in X^\ell$ and any $\epsilon > 0$, to slightly 'perturb' the identity function $f(x^\ell) = x^\ell$ to a strictly concave and strictly increasing function $\tilde{f}$ such that the difference $\tilde{f}(\theta^\ell) - \theta^\ell$ is strictly larger that $\tilde{f}(w^\ell) - w^\ell$ for all $w^\ell \in X^\ell \setminus \{\theta^\ell\}$, and such that the absolute values $|\tilde{f}(w^\ell) - w^\ell| < \epsilon$ for all $w^\ell \in X^\ell$. Note that if all utility functions u^ℓ arise from such perturbations, we obtain in particular

$$\sum_{\ell=1}^{L} (u^\ell(\theta^\ell) - \theta^\ell) > \sum_{\ell=1}^{L} (u^\ell(w^\ell) - w^\ell) \tag{A.2}$$

for all $w \in X \setminus \{\theta\}$ (note that every $w \in X \setminus \{\theta\}$ differs from θ in at least one coordinate, hence the inequality in (A.2) is indeed strict). Since $\sum_{\ell=1}^{L} \theta^\ell = \sum_{\ell=1}^{L} w^\ell = Q$, this implies $\sum_{\ell=1}^{L} u^\ell(\theta^\ell) > \sum_{\ell=1}^{L} u^\ell(w^\ell)$ for all $w \in X \setminus \{\theta\}$, i.e. θ is the unique top of the preference ordering represented by the utility function $u = \sum_\ell u^\ell$.

Now let $x \notin [\theta, y]$ and assume with loss of generality that x, y, θ are pairwise distinct. Since $x \notin [\theta, y]$ there exists a coordinate $j = 1, ..., L$ such that $x^j \notin [\theta^j, y^j]$. Thus, either ($x^j < \theta^j$ & $x^j < y^j$) or ($x^j > \theta^j$ & $x^j > y^j$). Consider the first case. It is possible to choose, for any position of θ^j and y^j, a strictly increasing and strictly concave function $u^j : X^l \to \mathbb{R}$ such that

$$u^j(\theta^j) - \theta^j \geq u^j(y^j) - y^j \geq \delta > 0 \geq u^j(x^j) - x^j, \tag{A.3}$$

where the first inequality in (A.3) is strict whenever $\theta^j \neq y^j$, and such that the difference $u^j(\theta^j) - \theta^j$ is in fact strictly larger than $u^j(w^j) - w^j$ for all $w^j \in X^j \setminus \{\theta^j\}$. Figure A.2 depicts the two cases $\theta^j < y^j$ (left) and $y^j < \theta^j$ (right).

Now choose all other functions u^ℓ strictly increasing and strictly concave such that $u^\ell(\theta^\ell) - \theta^\ell$ is strictly larger than $u^\ell(w^\ell) - w^\ell$ for all $w^\ell \in X^\ell \setminus \{\theta^\ell\}$, and such that $|u^\ell(w^\ell) - w^\ell| < \delta/[2(L-1)]$ for all $w^\ell \in X^\ell$, as described above. Let $\succcurlyeq$ be the preference order represented by $u = \sum_{\ell=1}^{L} u^\ell$. As argued above, θ is the top alternative of $\succcurlyeq$, i.e. $\succcurlyeq \in \mathcal{D}^*_\theta$. Moreover, we have

$$u^j(y^j) - y^j \ + \ \sum_{\ell \neq j} (u^\ell(y^\ell) - y^\ell) \ > \ \delta/2 \ > \ u^j(x^j) - x^j \ + \ \sum_{\ell \neq j} (u^\ell(x^\ell) - x^\ell),$$

i.e. $u(y) > u(x)$, and hence $y \succ x$ as desired. The argument in the case $x^j > \theta^j$ & $x^j > y^j$ is completely symmetric. By contraposition, we have thus shown that, if $\theta \in X$ is such that for every preference ordering $\succcurlyeq \in \mathcal{D}^*_\theta$ one has $x \succ y$, then $x \in [\theta, y]$, i.e. that the set on the left side of equation (A.1) is contained in the set on the right side of (A.1).

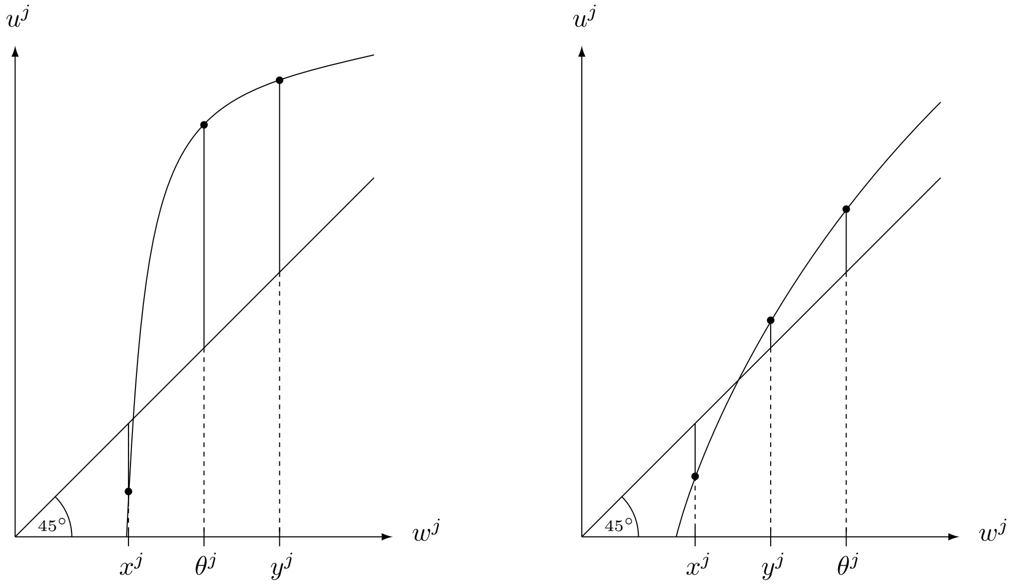


Figure A.2: Construction of u^j if $x^j < \theta^j$ and $x^j < y^j$

Second part: "$\supseteq$" Now consider $\succcurlyeq \in \mathcal{D}^*_\theta$ and $x \in [\theta, y]$. Let u be a separable and concave utility representation of $\succcurlyeq$. By Observation 1, u is strictly decreasing along any shortest path from θ to y. Since $[\theta, y]$ is the union of all shortest path, cf.(3.6), the conclusion follows, i.e. the set on the right side of equation (A.1) is contained in the set on the left side of (A.1). □

Proof of Fact 3.1 First assume that x and y are not neighbors, i.e. $[x, y] \neq \{x, y\}$. Then, any shortest path connecting x and y contains an allocation $z \notin \{x, y\}$; hence, by Observation 1, if x is the top of a preference ordering $\succcurlyeq \in \mathcal{D}^*$, then $z \succ y$; and if y is the top of $\succcurlyeq \in \mathcal{D}^*$, then $z \succ x$.

Now, let x and y be two neighbors. It follows as in the proof of Lemma 3.1 above that one can construct, for any $\ell = 1, ..., L$ two strictly increasing and strictly concave functions u^ℓ_x and u^ℓ_y such that, for all $w \in X \setminus \{x, y\}$,

$$[u^\ell_x(x^\ell) - x^\ell] > [u^\ell_x(y^\ell) - y^\ell] > [u^\ell_x(w^\ell) - w^\ell],$$

and

$$[u^\ell_y(x^\ell) - x^\ell] > [u^\ell_y(y^\ell) - y^\ell] > [u^\ell_y(w^\ell) - w^\ell].$$

Let $u_x := \sum_{\ell=1}^L u^\ell_x$ and $u_y := \sum_{\ell=1}^L u^\ell_y$; then, for all $w \in X \setminus \{x, y\}$, $u_x(x) > u_x(y) > u_x(w)$ and $u_y(y) > u_y(x) > u_y(w)$, as desired. □

The next result shows that the restricted net majority tournament coincides with the ranking induced by the median rule, i.e. with the ranking of neighbors according to their aggregate distance.

Lemma A.1 *For any profile $\boldsymbol{\theta}$ and any two neighbors x and y,*

$$\#_{\boldsymbol{\theta}}(\rangle x, y\rangle) - \#_{\boldsymbol{\theta}}(\rangle y, x\rangle) = \Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(y).$$

In particular, $x \, \mathrm{NM}_{\boldsymbol{\theta}} \, y$ if and only if $\Delta_{\boldsymbol{\theta}}(x) < \Delta_{\boldsymbol{\theta}}(y)$.

Proof. Combining Lemma 3.1 and (3.6), we obtain that, for all $w \in \rangle x, y\rangle$, $d(w,x) - d(w,y) = -1$, for all $w \in \rangle y, x\rangle$, $d(w,x) - d(w,y) = 1$, and for all other $w \in X$, $d(w,x) - d(w,y) = 0$. From this the result immediate. □

As an immediate corollary of Lemma A.1, we obtain the acyclicity of the restricted net majority tournament and the inclusion $\text{Med}(\boldsymbol{\theta}) \subseteq \text{Maj}(\boldsymbol{\theta})$. Note moreover, that a neighbor y of a median allocation x is itself a median allocation if and only if $x \,\text{NI}_{\boldsymbol{\theta}}\, y$, where for all neighbors x, y, $x \,\text{NI}_{\boldsymbol{\theta}}\, y :\Leftrightarrow$ (not $x \,\text{NM}_{\boldsymbol{\theta}}\, y$ and not $y \,\text{NM}_{\boldsymbol{\theta}}\, x$).

Observation 2 *Let $f : X \to \mathbb{R}$ be a separable function with $f(x) = \sum_{\ell=1}^{L} f^\ell(x^\ell)$ such that all functions $f^\ell(\cdot)$ are concave. Then, any local optimum of f on X is also a global optimum of f on X, i.e. if $f(x) \geq f(w)$ for all neighbors $w \in X$ of x, then $f(x) \geq f(w)$ for all $w \in X$. Moreover, the set of optima is box-convex, i.e. every point on a shortest path between two optima is also an optimum.*

Proof. By Observation 1, f must be constant along any shortest path connecting two local optima. From this, all assertions follow at once. □

Proof of Theorem 1. Since the negative of the aggregate distance function $-\Delta_{\boldsymbol{\theta}}(\cdot)$ is the sum of the separable and concave functions $-d(\cdot, \theta_i)$ it is itself separable and concave. Hence, by Observation 2, each of its local optima is a global optimum. This implies $\text{Maj}(\boldsymbol{\theta}) \subseteq \text{Med}(\boldsymbol{\theta})$; using Lemma A.1, we thus obtain $\text{Maj}(\boldsymbol{\theta}) = \text{Med}(\boldsymbol{\theta})$ for all profiles $\boldsymbol{\theta}$. Finally, from this the box-convexity of $\text{Maj}(\boldsymbol{\theta})$ follows using Observation 2 again. This completes the proof of Theorem 1.

Remark. Observations 1 and 2 show that every separable and concave function f is *single-plateaued* on X in the sense that there exist a box-convex set of optima (the 'plateau') outside of which f is strictly increasing on every shortest path to the plateau. Single-plateaued preferences in this sense on a one-dimensional space have been introduced by Moulin (1984).

Proof of Proposition 2 We first show the first inclusion in (3.8), i.e. that $\text{Maj}(\boldsymbol{\theta})^h \subseteq \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$. Let $x \in \text{Maj}(\boldsymbol{\theta})^h$ and consider any neighbor $w \in X$ of x. There are three cases. (i) If $w \in \rangle x, \theta_h\rangle$, then $\Delta_{\boldsymbol{\theta} \sqcup \theta_h}(x) - \Delta_{\boldsymbol{\theta} \sqcup \theta_h}(w) = \Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(w) - 1$. Since $x \in \text{Maj}(\boldsymbol{\theta})$ we have $\Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(w) \leq 0$ by Lemma A.1. Hence, $\Delta_{\boldsymbol{\theta} \sqcup \theta_h}(x) - \Delta_{\boldsymbol{\theta} \sqcup \theta_h}(w) < 0$, and thus $x \,\text{NM}_{\boldsymbol{\theta} \sqcup \theta_h}\, w$ again by Lemma A.1. (ii) If $w \in x \bowtie \theta_h$, then $\Delta_{\boldsymbol{\theta} \sqcup \theta_h}(x) - \Delta_{\boldsymbol{\theta} \sqcup \theta_h}(w) = \Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(w)$, and since $x \in \text{Maj}(\boldsymbol{\theta})$ the last term is non-positive; hence $x \,\text{NM}_{\boldsymbol{\theta} \sqcup \theta_\text{h}}\, w$ or $x \,\text{NI}_{\boldsymbol{\theta} \sqcup \theta_\text{h}}\, w$. (iii) if $w \in \rangle \theta_h, x\rangle$, then $\Delta_{\boldsymbol{\theta} \sqcup \theta_h}(x) - \Delta_{\boldsymbol{\theta} \sqcup \theta_h}(w) = \Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(w) + 1$; since $x \in \text{Maj}(\boldsymbol{\theta})$ and $w \notin \text{Maj}(\boldsymbol{\theta})$, we have $\Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(w) < 0$ and therefore $\Delta_{\boldsymbol{\theta} \sqcup \theta_h}(x) - \Delta_{\boldsymbol{\theta} \sqcup \theta_h}(w) \leq 0$; hence, $x \,\text{NM}_{\boldsymbol{\theta} \sqcup \theta_\text{h}}\, w$ or $x \,\text{NI}_{\boldsymbol{\theta} \sqcup \theta_\text{h}}\, w$ by Lemma A.1. Summarizing, we obtain that x wins against all its neighbors, hence $x \in \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$ by Theorem 1. Note that the argument also shows that $(\text{Maj}(\boldsymbol{\theta}) \setminus \text{Maj}(\boldsymbol{\theta})^h) \cap \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h) = \emptyset$.

Now consider the second inclusion in (3.8). By the preceding remark, it suffices to show that $y \notin \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$ for every $y \notin \text{Maj}(\boldsymbol{\theta}) \cup \bigcup_{x \in \text{Maj}(\boldsymbol{\theta})^h} [x, \theta_h]$. If $y \notin \text{Maj}(\boldsymbol{\theta}) \cup \bigcup_{x \in \text{Maj}(\boldsymbol{\theta})^h} [x, \theta_h]$ then y cannot be on any shortest path connecting any element $w \in \text{Maj}(\boldsymbol{\theta})$ and θ_h. Consider a path $\{w_1, ..., w_m\}$ of minimal length connecting y and $\text{Maj}(\boldsymbol{\theta})$ such that $w_1 = y$

and $\{w_1, ..., w_m\} \cap \text{Maj}(\boldsymbol{\theta}) = \{w_m\}$. Since $w_1 = y$ is not on a shortest path connecting θ_h and w_m, there must exist two neighbors w_k and w_{k+1} with $k \in \{1, ..., m-1\}$ such that $d(w_k, \theta_h) \geq d(w_{k+1}, \theta_h)$. Since $w_k \notin \text{Maj}(\boldsymbol{\theta})$, we have $\Delta_{\boldsymbol{\theta}}(w_k) > \Delta_{\boldsymbol{\theta}}(w_{k+1})$ by Observation 1 and Theorem 1. Hence, we also have $\Delta_{\boldsymbol{\theta} \sqcup \theta_h}(w_k) > \Delta_{\boldsymbol{\theta} \sqcup \theta_h}(w_{k+1})$. In particular, $w_k \notin \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$, and hence $y \notin \text{Maj}(\boldsymbol{\theta} \sqcup \theta_h)$ by Observation 1. □

Proof of Theorem 3. The equivalence of (i) and (ii) follows from Proposition 3. Thus, consider statement (iii). The idea of the following proof is to show that (5.1) is equivalent to x being a local maximum of aggregate goal satisfaction; this implies the equivalence of (ii) and (iii) by Observation 2.

We first introduce some notation. For a fixed profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in X^n$, each $\ell = 1, ..., L$ and $r \in \mathbb{Z}$, denote by $v^\ell(r) := \sum_{i=1}^n \min\{r, \theta_i^\ell\}$ so that for the aggregate goal satisfaction function $v(\cdot)$ we have $v(x) = \sum_\ell v^\ell(x^\ell)$. Moreover, let

$$\begin{aligned} \nabla_- v^\ell(r) &:= v^\ell(r) - v^\ell(r-1), \\ \nabla_+ v^\ell(r) &:= v^\ell(r+1) - v^\ell(r). \end{aligned}$$

By construction, we obtain

$$\begin{aligned} \nabla_- v^\ell(r) &= \#\{i : \theta_i^\ell \geq r\}, \\ \nabla_+ v^\ell(r) &= \#\{i : \theta_i^\ell \geq r+1\}. \end{aligned} \tag{A.4}$$

By definition of $\theta_{[k]}^\ell$, we have $\#\{i : \theta_i^\ell \geq r\} \geq (n-k+1)$ whenever $r \leq \theta_{[k]}^\ell$, and hence by (A.4),

$$r \leq \theta_{[k]}^\ell \;\Rightarrow\; \nabla_- v^\ell(r) \geq (n-k+1). \tag{A.5}$$

Similarly, we have $\#\{i : \theta_i^\ell \geq r+1\} \leq (n-k)$ whenever $r \geq \theta_{[k]}^\ell$, hence, again by (A.4),

$$r \geq \theta_{[k]}^\ell \;\Rightarrow\; \nabla_+ v^\ell(r) \leq (n-k). \tag{A.6}$$

Now consider any $x \in X$ satisfying (5.1), i.e. for all $\ell = 1, ..., L$, $\theta_{[k^*(\boldsymbol{\theta})]}^\ell \leq x^\ell \leq \theta_{[k^*(\boldsymbol{\theta})+1]}^\ell$. We will show that x is a local maximizer of aggregate goal satisfaction v. By Observation 2, x is then also a global optimum, hence a frugal majority winner by Proposition 3. Thus, consider any neighbor y of x. Without loss of generality, assume that $y = x_{(21)}$ in the notation of Observation 1 above, i.e.$y^1 = x^1 - 1$, $y^2 = x^2 + 1$, and $y^\ell = x^\ell$ for all $\ell = 3, ..., L$. We have $x^1 \leq \theta_{[k^*(\boldsymbol{\theta})+1]}^\ell$ and $x^2 \geq \theta_{[k^*(\boldsymbol{\theta})]}^\ell$, therefore, using (A.5) and (A.6),

$$\begin{aligned} v(x) - v(y) &= \nabla_- v^1(x^1) - \nabla_+ v^2(x^2) \\ &\geq n - (k^*(\boldsymbol{\theta}) + 1) + 1 - (n - k^*(\boldsymbol{\theta})) \\ &= 0. \end{aligned}$$

This proves that every $x \in X$ satisfying (5.1) is indeed a maximizer of aggregate goal satisfaction.

Conversely, consider $x \in X$ that violates (5.1). There are two (not mutually exclusive) cases.

Case 1. For some coordinate h, $x^h < \theta^h_{[k^*(\boldsymbol{\theta})]}$. In this case, there must exist some other coordinate j such that $x^j > \theta^j_{[k^*(\boldsymbol{\theta})]}$. Consider the neighbor y of x such that $y^h = x^h + 1$, $y^j = x^j - 1$, and $y^\ell = x^\ell$ for all coordinates $\ell \neq h, \ell$, i.e. $y = x_{(hj)}$. By the same arguments as above, we obtain using (A.4),

$$r < \theta^\ell_{[k]} \ \Rightarrow\ \nabla_+ v^\ell(r) \geq (n - k + 1) \tag{A.7}$$

and

$$r > \theta^\ell_{[k]} \ \Rightarrow\ \nabla_- v^\ell(r) \leq (n - k). \tag{A.8}$$

Therefore,

$$\begin{aligned} v(y) - v(x) &= \nabla_+ v^h(x^h) - \nabla_- v^j(x^j) \\ &\geq n - k^*(\boldsymbol{\theta}) + 1 - (n - k^*(\boldsymbol{\theta})) \\ &= 1, \end{aligned}$$

hence x is not a maximizer of aggregate goal satisfaction.

Case 2. For some coordinate h, $x^h > \theta^h_{[k^*(\boldsymbol{\theta})+1]}$. In this case, there must exist some other coordinate ℓ such that $x^\ell < \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}$. Consider the neighbor y of x such that $y^h = x^h - 1$, $y^\ell = x^\ell + 1$, and $y^\ell = x^\ell$ for all coordinates $\ell \neq h, \ell$, i.e. $y = x_{(jh)}$. By (A.7) and (A.8), we obtain

$$\begin{aligned} v(y) - v(x) &= \nabla_+ v^j(x^j) - \nabla_- v^h(x^h) \\ &\geq n - (k^*(\boldsymbol{\theta}) + 1) + 1 - (n - (k^*(\boldsymbol{\theta}) + 1)) \\ &= 1, \end{aligned}$$

hence x is not a maximizer of aggregate goal satisfaction in this case either. This completes the proof of Theorem 3. □

(References to be added)