Dichotomous multi-type Games with a Coalition structure

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December 3, 2015

Abstract

This work focuses on the evaluation of the voting power in dichotomous multi-type games endowed with a coalition structure (DMGCs). Dichotomous multi-type games have been introduced by Courtin et al. [2015] in order to generalize simple game. They modelize game in which there are a number of non-ordered types of support in the input, while the output is dichotomous, i.e. the proposal is accepted or rejected. In game with a coalition structure it is supposed that players organize themselves into a priori disjoint coalitions. We extend the well known Owen-Shapley index (Owen [1977]) and Owen-Banzhaf index (Owen [1981]) to DMGCs. A full characterization of these power indices are provided.

KEYWORDS: Multi-type games, Simple games, Coalition structure, Owen-Shapley index, Owen-Banzhaf index

JEL Classification Numbers: C71 ;D71.

*The authors would like to thank Fabian Gouret, Mathieu Martin, Matias Nunez and Issofa Moyouwou for their useful comments and encouragement. This work has also benefited from comments by a number of conference and seminar participants. This research has been developed within the center of excellence MME-DII (ANR-11-LBX-0023-01), and the CoCoRICO-CoDEC research program (ANR-14-CE24-0007-02).

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1 Introduction

How a voter can affect the outcome of an election? This is one of the main questions that the literature on voting theory has been concerned with, both in political science and in game theory. The analysis of this question can be conducted within the theoretical framework of simple games and power indices. Power indices are quantitative measures of the a priori power of a voter in a Committee (Parliament). Although there are different approaches of how a voter can influence an election, most of the literature concerns a limited scenario. Especially, two main criticism can be adress to simple games. The fist one concerns the alternatives and the second one is about the players.

Multiples Alternatives

First of all, in simple game, a proposal is approve only on the basis of the votes cast by those who are in favor. In other words, voting “yes” and “no” are the only feasible alternatives. Consequently, the possibility of other alternatives, like “abstention” and “non-participation”, is not taken into account. It has been pointed out by many scholars that a number of interesting questions from economics, politics, and the social sciences more generally, cannot be described by a classical simple game. For example, in some real voting systems such as the United Nations Security Council, the United States federal system and the Council of Ministers of the European Union, “abstention” plays a key role. However, these voting systems cannot be modeled by such games. This explains the introduction these recent years, of many alternative generalization of simple games.

In this paper we follow the work of Courtin et al. [2015], who introduce multi-type games. They model games for which the following conditions are met: (i) Several, say \( r \) non-ordered types (not levels) of support are allowed in the input. Each player chooses one of the \( r \) types of support. Note that a player can also make no choice. The players’ choices then lead to a choice configuration. (ii) The characteristic function maps each choice configuration to a real number, the value of the configuration. When considering a dichotomous output, Courtin et al. [2015] introduced Dichotomous multi-type games (DMG). In such a game the characteristic

\[ \text{References} \]

See Andjiga et al. [2003] and Laruelle and Valenciano [2008] for a detailed description of the different notions of power indices.
function can be seen as a voting rule that maps each choice configuration to either a collective approval or disapproval. In this context [Courtin et al., 2015] introduced the following example.

**Example 1. Junior professor selection**

Consider a committee that must decide on the promotion of a certain junior colleague in an economics department. There are three full professors (namely 1, 4 and 5) and two associate professors (namely 2, 3) on this committee, and each of the professors may express a Research Support (RS), a Teaching Support (TS) or an Administrative Support (AS). The candidate will be selected when both of the following conditions are met:

1. At least two full professor shows a Research Support.
2. One or more associate professors expresses at least one of the supports.

The multi-type dichotomous game associates with this situation is such that: i) the input consists of three non-ordered types: RS, TS, AS; ii) the output is dichotomous, i.e. the junior colleague have the promotion or not.


**Relation between players**

The second problem of the traditional measure of voting power, is that they do not take into consideration a priori relation between different players. Indeed in many negotiations some players prefer to cooperate with each other rather than with other players. For example, in the European Union, some States are considered as "Euro-enthusiastic" (pro-European) and others as "Euro-skeptic" (opposed to Europe). We can imagine that some States are closer to one another, and this for their proximity with the European ideas. Therefore ignoring the "policy positions" of the European governments is problematic. For example, if we consider two major European States, France and the United Kingdom (UK), it is known that the first one is rather "pro-European", while the UK is more "Euro-skeptic". So when it comes to deal with questions about the evolution of the EU and these institutions, there will be a problem to agree. On the contrary, France and Germany act together about this subject, since they agree about the evolution of the EU.
One of the sophisticated models which take into account these situation are those of the game endowed with a priori unions, that is, a partition of the player set which describes a pre-defined (exogenously given) coalition structure. This strand of the literature was pioneered by Owen [1977] and Owen [1981] who propose and characterize a modification of the Shapley-Shubik and Banzhaf indices with respect to a coalition structure.

Our main contribution is to extend and fully characterize in a related model (à la Owen [1977] and Owen [1981]), voting power indices with a coalition structure when dichotomous multi-type games are considered. In other words, we extend the notion of DMG to dichotomous multi-type games endowed with a coalition structure(DMGs).

This work is structured as follows. After briefly reviewing the literature on simple (and cooperative) game with multiple alternatives, Section 2 introduces the general framework. Section 3 presents the main results, while Section 4 concludes the paper.

Related Literature

In the context of simple games with multiple alternatives, different theories of power have been proposed. The Shapley-Shubik and Banzhaf-Coleman indices were defined for \((j,k)\) simple games by Freixas [2005a] and Freixas [2005b]. In the \((j,k)\) simple games, each player expresses one of \(j\) ordered possible levels of input support, and the output consists not of a real value but of one of \(k\) possible levels of collective support. \((j,k)\) games are direct extensions of ternary voting (Felsenthal and Machover [1997]), and bicooperative game (Bilboa et al. [2000]). In ternary game “abstention” is permitted as a distinct third option of a voter, whereas in bicooperative game, the third one is “no participation”. Clearly, these three input supports are totally ordered in the sense that a “yes” vote is more favorable to the collective acceptance than an “abstention” vote (or a “not participating” vote), which in turn is more favorable than a “no” vote. An other model of game with ordered inputs is the so-called multichoice game (Hsiao and Raghavan [1993]). In these games, each player is allowed to have a given number of effort levels, each of which is assigned a nonnegative weight. The weight assigned to an effort level leads to an ordering on the set of effort levels. Any choice configuration is then associated with a real value (in a context of a cooperative game).
An example of a model in which alternatives in the inputs are not totally ordered is the one formalized by Laruelle and Valenciano [2012]. They study quaternary voting rules in which the four possible alternatives are “yes”, “no”, “abstention” and “non-participation”. The collective decision is dichotomous, i.e., either the proposal is accepted or rejected. Levels of support here are not totally ordered, since the “non-participation” and the “abstention” options are not ranked. Indeed, in some situations, the “abstention” option may be more favorable to the rejection of the proposal than the “non-participation” option, while in other situations the converse is observed. However, the “yes” alternatives is always more favorable than any other alternative while the “no” alternative is less favorable than any alternatives. This model and the other models above are particular cases of the more general framework of games on lattices developed by Grabisch and Lange [2007]. In this model, each player has a set of possible actions and this set is endowed with a lattice, i.e., a partial order such that any pair of actions possesses a least upper bound and a greatest lower bound.

A model of a cooperative game (not only a simple game) which is not based on lattice is the one of $r$-games, initially developed by Bolger [1986], Bolger [1993], and later extended by Amer et al. [1998] and Magana [1996]. In such games, there are $r$ possible input alternatives that are not ordered. Each alternative $j$ attractions its own coalition of supporting voters. A configuration, which is a partition of the set of players into $r$ subsets (some of which might be empty), is then associated with an $r$-tuple of cardinal values. The component $j$ represents the value of the coalition of the configuration that has chosen the input $j$. This model is related to dichotomous multi-type games in the sense that the set of inputs is not ordered. No alternative is a priori more favorable than another. However, both models differ in their outputs. Indeed in a DMG, the output consists of a single value.

Finally, note that Albizuri and Zarzuelo [2000] generalize the values considered by Bolger [1993] when a coalition structure is considered. Indeed, they consider games with $n$ players and $r$ alternatives, in which the value of a coalition depends not only on that coalition, but also on the organization of the other players in the game. They propose coalitional values that are direct extensions of those of Owen [1977].
2 General framework

Firstly, we remind the notion of (dichotomous) multi-type cooperative game introduced by Courtin et al. [2015]; secondly, we introduce games with coalition structures; and finally we present different notions of power indices.

Multi-type games

A finite set of players is denoted by $N = \{1, 2, ..., n\}$, $\varphi(N)$ is the set of all subsets of $N$ and $2^N$ is the set of all nonempty subsets of $N$: $2^N = \varphi(N) \setminus \{\emptyset\}$. We refer to any subset $S$ of $N$ as a coalition. A classical cooperative $n$-person game in characteristic form is defined by a function $v : \varphi(N) \rightarrow \mathbb{R}$, such that $v(\emptyset) = 0$. In this paper we consider monotonic cooperative $n$-person games, i.e. cooperative $n$-person games satisfying $v(S) \leq v(T)$ if $S \subseteq T \subseteq N$.

Let $R = \{a_1, ..., a_k, ..., a_r\}$, where $r$ is a non-null integer, be the set of types of support the players can choose. Denote $a_0$ the option which means no support at all. Each player can then choose between $r + 1$ possible actions: no support, or one of the $r$ types of support. Let $\overline{R} = \{a_0, a_1, ..., a_k, ..., a_r\}$ be the set of all possible actions. A configuration is a sequence $F = (F_k)_{k \in R}$, such that for all $k \in \overline{R}$, $F_k \subseteq N$ and for all $k, j \in \overline{R}$, $k \neq j \implies F_k \cap F_j = \emptyset$. $F$ can be seen as a division of the voters according to their action, while $F_k$ is the set of voters who choose the action $k \in \overline{R}$. We denote by $\overline{R}^N$ the set of all configurations and by $\emptyset^r$ the configuration defined by: $\emptyset_0^r = (N, \emptyset, ..., \emptyset)$. In words, $\emptyset^r$ represents the situation where no player chooses one of the support.

A multi-type game is a pair $(N, V)$ where $V : \overline{R}^N \rightarrow \mathbb{R}$, such that $V(\emptyset^r) = 0$.

Let $F = (F_0, F_1, ..., F_r)$ be a configuration. Then it is clear that $F_0 = N - \bigcup_{k=1}^{r} F_k$. This means that $F$ is completely described by the given of $(F_k)_{1 \leq k \leq r}$. This allows us to consider in the sequel that a configuration is a sequence $(F_k)_{1 \leq k \leq r}$ and we let $N_F = \bigcup_{k=1}^{r} F_k = \bigcup_{k \in R} F_k$. $N_F$ will be referred to as the support of the configuration $F$, which is the set of voters who chooses one support action.

Clearly a multi-type game reduces to a classical cooperative game when there is only two possible actions, one type of support i.e. when $|R| = 1$.

For the sake of simplicity and when there is no ambiguity, we write $k \in R$ for an element $a_k \in R$. 

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Dichotomous multi-type games

In the classical framework, dichotomous games are represented by simple games. A simple (voting) game (SG) consists of a pair \((N,v)\) such that: (i) \(v(S) = \{0,1\}\) for all \(S \in \varnothing(N)\); (ii) \(v(N) = 1\); and (iii) \(v\) is monotonic. A coalition \(S\) is said to be winning in \(v \in \varnothing(N)\) if \(v(S) = 1\) and losing otherwise.

As a natural extension of SG, a dichotomous multi-type game (DMG), denoted \(V\), is given by a map \(V : R^N \rightarrow \mathbb{R}\) such that: (i) \(V(F) \in \{0,1\}\) for all \(F \in R^N\); (ii) \(V(F) = 1\) for at least one \(F \in R^N\) and (iii) \(V\) is monotonic. A configuration \(F\) is a winning configuration for \(V(F) = 1\) while it is a losing configuration for \(V(F) = 0\).

Dichotomous multi-type games with coalition structure

A coalition structure on \(N\) is a finite partition \(P = \{P_l : l \in M = \{1,...,m\}\}\) of \(m\) non-empty and disjoint subsets of \(N\), i.e. \(\cup_{l=1}^m P_l = N\) and \(P_h \cap P_l = \emptyset\) for all \(h, l \in \{1,...,m\}, h \neq l\). A coalition structure is assumed to be given exogenously. In the following, an element \(P_l\) of the partition \(P\) is called a structural coalition ([Hamiache 1999]).

A simple game with coalition structure (SGCs), denoted \((v,P)\), is simply a SG which take into account a given partition of the voters.

Let \((v,P)\) be a SGCs and let \(l \in M\). The game \((v,P)\) is \(l\)-anonymous if there exist two positives integers \(\mu_l\) and \(q_l\) such that for all \(S \subseteq N\), \(v(S) = 1\) if and only if \([|M(S\setminus P_l)| \geq \mu_l]\) or \([|M(S\setminus P_l)| = \mu_l - 1\) and \([S \cap P_l]| \geq q_l]\), where for all \(T \in 2^N\), \(M(T) = \{h \in M : P_h \subseteq T\}\).

A dichotomous multi-type game with coalition structure (DMGCs), denoted \((V,P)\), is simply a DMG which take into account a given partition of the voters.

For all \(F \in R^N\), we denoted by \(F^{-l} = (F_k \setminus P_l)_{k \in R}\) and we will refer to \(M(F) = \{h \in M : P_h \subseteq N_F\}\) as the set of indices \(h \in M\) such that the support of \(F\), \(N_F\) contains \(P_h\). Moreover, in the following, \(R^N \setminus B^l\) will be the set of configurations without \(P_l\), i.e. such that for all \(F \in R^N\), \(P_l \cap N_F = \emptyset\).

Let \((V,P)\) be a DMGCs and let \(l \in M\). \((V,P)\) is said to be \(l\)-anonymous if there exist integers \(\mu_l \geq 1\) and \((q_k^l)_{k=1,...,r}\) such that for all \(F \in R^N\), \(V(F) = 1\) if and only if \([|M(F^{-l})| \geq \mu_l]\) or \([|M(F^{-l})| = \mu_l - 1\) and \(\forall k \in R, |F_k \cap P_l| \geq q_k^l]\).

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3For the sake of simplicity and when there is no ambiguity, we denoted \(l \in M\) for a structural coalition \(P_l \in P\).
Power index

A power index, for a simple game endowed with an exogenous coalition structure, is a mapping \( \varphi \) assigning to each SGCs an \( n \)-dimensional real-valued vector \( \varphi(v,P) = (\varphi_1(v,P), ..., \varphi_n(v,P)) \). A general formula for a power index for a game \((v,P)\) and a given player \( i \in P_l \) can be summarized by:

\[
\varphi_i(v,P) = \sum_{T \in 2_{N \setminus P_l}} \sum_{S \subseteq P_l, i \in S} f_i(T,S) \left[ v(S \cup T) - v((S \setminus i) \cup T) \right]
\]

with \( 2_{N \setminus P_l} = \{ T \subseteq N \setminus P_l : \forall h \in M, P_h \cap T \neq \emptyset \Rightarrow P_h \subseteq T \} \).

The factor \( f_i(T,S) \) depends a priori on \( T \) and \( S \) and not on \( i \in S \) even though the problem is the evaluation of the power of \( i \). If \( S \) is a subset of \( P_l \), then \( T \) is such that \( M(T) \) is any subset of \( M \) not containing \( l \).

Many power indices for games endowed with coalition structure can be constructed in this form. For \( f_i(T,S) = \frac{|M(T)||m-|M(T)|-1||S|-|S||!}{|P_l||P_l|!} \), we obtain the well-known Owen-Shapley index (OS), introduced by [Owen 1977]. OS is a direct generalization of the famous Shapley-Shubik index. Analogously, [Owen 1981] presented the Owen-Banzhaf index (OB), an extension of the Banzhaf-Coleman index to SGCs. In that case, \( f_i(T,S) = \frac{1}{2^{m-1}} \frac{1}{2^{|P_l|-1}} \).

In the following, we call this class of power indices, the **coalition structure weight-dependant** power indices (CSWD).

We propose now an extension of CSWD power index, in the context of DMGCs. A CSWD power index for a player \( i \in P_l, l \in M \) is in the form:

\[
\Phi_i(V, P) = \sum_{L \in R^*_{N \setminus P_l}} \left( \sum_{G \in R_h} f_i(L,G) \left[ V(L \cup G) - \bar{V}(L \cup (G \setminus i)) \right] \right)
\]

where:

- \( L \cup G = (L_k \cup G_k) \)
- \( R^*_{N \setminus P_l} = \{ F \in R^{N \setminus P_l} : \forall h \in M, l \neq h, P_h \cap N_F \neq \emptyset \Rightarrow P_h \subseteq N_F \} \)
- \( \bar{V}(F) = \max_{Z \in R^N} \{ V(Z), N_Z = N_F \} \)
- \( F \setminus i = (F_k \setminus i)_{k \in R} \).

Note that for \( |R| = 1, \Phi_i(V, P) = \varphi_i(v,P) \) for a player \( i \).
3 Power index for DMGCs

In this section, we first outline an axiomatic approach of the Owen-Shapley power index both for SGCs and DMGCs. Following the same approach, we present an alternative characterization of the Owen-Banzhaf power index.

Owen-Shapley for SGCs

Axiom 1. Equal Share for SGCs

A power index $\varphi$ satisfies Equal Share axiom if for all $l$-anonymous SGCs, for all $l \in M$, and for all $i \in P_l$, $\varphi_i(v, P) = \frac{1}{|M|}$.

The Equal Share axiom for SGCs implies that in an $l$-anonymous SGCs, the power of each voter in $P_l$ is the same and moreover is proportional to the number of structural coalition and to the number of voters in $P_l$.

We introduce below another property for CSWD power indices.

Axiom 2. Equal Size for SGCs

A power index $\varphi$ satisfies the Equal Size axiom if for all $l \in M$, for all $L_1, L_2 \in 2^{N\setminus B_l}$ and for all $S_1, S_2 \subseteq P_l$,

$$[|M(L_1)| = |M(L_2)| \text{ and } |S_1| = |S_2|] \implies f_i(L_1, S_1) = f_i(L_2, S_2).$$

Consider two situations in which: i) the union of the structural coalition different from $P_l$ have the same size and, ii) two coalitions $S$ of $P_l$ have the same size. Hence, according to the Equal Size axiom for SGCs, the factor $f_i(T, S)$ must be the same in both situations.

We can now characterize OS in the class of CSWD power indices.

Theorem 1. There exists one and only one CSWD power index satisfying Equal Share and Equal Size axioms, which is the Owen-Shapley power index.

This theorem presents an alternative characterization of OS, which is based on the characterisation of Shapley-Shubik given by [Courtin et al., 2015].

Owen-Shapley for DMGCs

We now focus on $DMG$ endowed with a coalition structure.
We first extend the Equal Share axiom to DMGCs. Once again, in a DMGCs which is $l$-anonymous, the power of each voters in $P_i$ is the same and is proportional both to the number of coalitional structures and to the number of voters in $P_i$.

**Axiom 3. Equal Share for DMGCs**

A power index $\Phi$ satisfies Equal Share if for all $l$-anonymous DMGCs, for all $l \in M$, and for all $i \in P_l$, $\Phi_i (V, P) = \frac{1}{m|P_i|}$

We extend the Equal Size axiom as follows.

**Axiom 4. Equal Size for DMGCs**

$\Phi$ satisfies Equal Size axiom if for all $l \in M$, for all $L_1, L_2 \in R^*_{N \setminus P_l}$, and for all $G^1, G^2 \in R^P_l$,

$$\left| M(L_1) \right| = \left| M(L_2) \right| \quad \text{and} \quad \forall k \in R, \left| G^1_k \right| = \left| G^2_k \right| \implies f_i(L_1, G^1) = f_i(L_2, G^2).$$

According to the Equal Size axiom, two couples $(L_1, G^1)$ and $(L_2, G^2)$ that contain the same number of elements must be equally weighted.

Before to present a generalization of Theorem 1, let us denote by $[L]$ the set of family of indices $I$ for which both the cardinalities of $M(L)$ and $I$ coincide, formally, $[L] = \{ I \subseteq M : |M(L)| = |I| \}$.

**Theorem 2.** There exists one and only one CSWD power index for DMGCs satisfying the Equal Share and the Equal Size axioms, the factor of which is such that: for all DMGCs $(V, P)$, all $l \in M$, all $L \in R^N_{N \setminus P_l}$ and all $G \in R^P_l$,

$$f_i(L, G) = \left[ m \sum_{I \in [L]} r(I) \right]^{-1} \left[ \left| \prod_{k \in R} |G_k| \right| \left| \prod_{k \in R} |G_k| \right| \right]$$

**Phrase theoreme 2**

We can now define the Owen-Shapley power index for DMGCs. Let $(V, P)$ be a multi-type game endowed with a coalition structure $P = \{ P_1, ..., P_m \}$ and let $i \in N$. The Owen-Shapley power index of player $i$ in $(V, P)$ is given by:

$$OS_i^f (V, P) = \sum_{L \in R^N_{N \setminus P_l}} \left[ \sum_{G \in R^P_l} \left[ m \sum_{I \in [L]} r(I) \right]^{-1} \left[ \left| \prod_{k \in R} |G_k| \right| \left| \prod_{k \in R} |G_k| \right| \right] \right] \times \left[ V(L \cup G) - V(L \cup (G \setminus i)) \right]$$
Corollary 1. If $|R| = 1$, then the Owen-Shapley power index for DMGCs is equivalent to the Owen-Shapley power index for SGCs.

Owen-Banzhaf power index for SGCs and DMGCs

We provide now a characterization of the class of CSWD power indices based on the famous Banzhaf score (Banzhaf [1965]). Given a SG, the Banzhaf score of a player $i$ in this game, is the number of coalitions in which he is decisive ($\sum_{S \in 2^N} |v(S) - v(S - i)|$).

Hence, for all $i \in P_l$, the generalization to a SGCs of the Banzhaf score of a player $i$, is the number of $(T, S) \in 2^{N\setminus P_l} \times P_l$ such that $[v(S \cup T) - v((S \setminus i) \cup T)] = 1$. More formally, for all $i \in P_l$, $\eta_i(v, P) = \sum_{T \in 2^{N\setminus P_l} \cap P_l} \sum_{S \subseteq P_l, i \in S} [v(S \cup T) - v((S \setminus i) \cup T)]$.

Axiom 5. Constant-weight axiom for SGCs

A coalition structure weight-dependant power index satisfies the Constant Weight axiom if for all $l \in M$, there exists $\lambda^l > 0$ such that for all $T \in 2^{N\setminus P_l}$ and for all $S \subseteq P_l$, $f_l(T, S) = \lambda^l$.

Note that, if this is the case then the CSWD power index is said to be a $(\lambda^l)$-constant-weight. We obtain the following proposition.

Proposition 1. A CSWD power index for SGCs is $(\lambda^l)$-constant-weight if and only if for all SGCs $(v, P)$ and for all $i \in P_l$, $\psi_i(v, P) = \lambda^l \eta_i(v, P)$.

Let us remark that for $\lambda^l = \frac{1}{2^{m-1} \times 2^{|P_l|-1}}$, we obtain a new characterization of Owen-Banzhaf $OB$.

We can also extend the Banzhaf scores to a DMGCs. Let $(V, P)$ be a DMGCs, $i \in P_l$, the Banzhaf score of $i$ is given by $\eta_i^l(V, P) = \sum_{L \in R_{N\setminus P_l}^N} \sum_{G \in R_P^G} [V(L \cup G) - \overline{V}(L \cup (G \setminus i))]$.

Axiom 6. Constant-weight axiom for DMGCs

A CSWD power index for DMGCs satisfies the Constant Weight axiom if for all $l \in M$, there exists $\delta^l > 0$ such that for all $L \in R_{N\setminus P_l}^N$ and for all $G \in R_P^G$, $f_l(L, G) = \delta^l$.

The following theorem is a direct extension of Proposition [1].

Proposition 2. A CSWD power index for DMGCs is $(\delta^l)$-constant-weight if and only if for all DMGCs $(V, P)$, and for all $i \in P_l$, $\Phi_i(V, P) = \delta^l \eta_i^l(V, P)$.
Consequently, for $\delta^l = \frac{1}{(r+1)^{m-1} \times (r+1)^{|P_i|-1}}$, we obtain a generalization of $OB$. More formally, for a player $i \in P_i$, we have

$$OB_i^r(V, P) = \frac{\sum_{L \in R_N \setminus P_i} \sum_{G \in R_{Pl}|} [V(L \cup G) - \overline{V}(L \cup (G \setminus i))]}{(r + 1)^{m-1} \times (r + 1)^{|P_i|-1}}$$

To conclude, let us consider the the extension of Example 1.

**Example 2.** (example 2 continued)

Consider that among the three full Professors, two of them organize themselves to defend their interests. Likewise, the two associate Professors will form an a priori coalition. The last full Professor stay alone since he is the Chairman of the department. Hence $P = \{(1, 4), (2, 3), (5)\}$.

i) In an SGCs context they only have to say if they are “for” or “against” the promotion. The candidate will be selected when at least two full professor vote ”yes” and at least one associate professor vote “yes”. Therefore $OB(v, P) = OS(v, P) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$.

ii) In a DMGCs context, we can compute the power of each professors, and we find that $OS^r(V, P) = \left(\frac{3}{12}, \frac{4}{12}, \frac{4}{12}, \frac{3}{12}, 0\right)$; while $OB^r(V, P) = \left(\frac{9}{32}, \frac{12}{32}, \frac{12}{32}, \frac{9}{32}, 0\right)$.

4 Conclusion

This work deals with dichotomous multi-type supports games in the context of a priori coalition between the players. It is worth noting that simple games with coalition structure do not take into account the possibility for voters to express different types of support. The extension of the classical notions of coalition structure voting power to such a games was the main objective of this paper. Thanks to alternatives characterizations of the Owen-Shapley and Owen-Banzhaf power indices, we provide full characterizations of these indices in our framework. Note that when considering the trivial partition, $P = \{\{i\}_{i \in N}\}$, it is easily to check that the Owen-Shapley and the Owen-Banzhaf indices for DMGCs reduce respectively to the Shapley and Banzahf indices for dichotomous multi-type supports games, introduced by Courtin et al. [2015].

There are several ways in which dichotomous multi-type games could be explored in further research. Firstly, we can extend the notion of desirability relation,
to qualitatively compare the a priori influence of voters in DMGCs. In the context of SGCs, Courtin and Tchantcho [2015] already shows that OS and OB are not ordinally equivalent. Secondly, following the works of Albizuri et al. [2006] and Albizuri and Aurrekoetxea [2006], we can also provide power indices in the context of dichotomous multi-type game with coalition configuration. In games with coalition configuration, the players are organized themselves into coalitions that are not necessarily disjoint.

References


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A Appendix

A.1 Proof of Theorem [1]

The following straightforward lemma will be useful in order to prove the theorem.

**Lemma 1.** Let \((v, P)\) be a SGCs, \(l \in M, T \in 2^{N \setminus P}_l\) and \(S \subseteq P_l\).

If \((v, P)\) is \(l\)-anonymous with two positives integers \(\mu_l\) and \(q_l\) then, for all \(i \in P_l\), \(v(S \cup T) - v((S \setminus i) \cup T) = 1 \iff i \in S, |S| = q_l\) and \(|M(T)| = \mu_l - 1\).

**Proof.** Theorem 1

\(\implies\) Let \(\varphi\) be a CSWD power index that satisfies Equal Share and Equal Size axioms. We shall show that for all \(l \in M\), all \(T \in 2^{N \setminus P}_l\) and all \(S \subseteq P_l\), \(f_l(T, S) = \frac{|M(T)|!|S|!(m-|M(T)|-1)!}{|P|!|S|!(|P|-|S|)!}\).

Let \(l \in M, T \in 2^{N \setminus P}_l\) and \(S \subseteq P_l\), consider the SGCs \((v, P)\) defined by:

for all \(\Lambda \subseteq N\), \(v(\Lambda) = 1 \iff \begin{cases} |M(\Lambda \setminus P_l)| \geq |M(T)| + 1 \ or \\ |M(\Lambda \setminus P_l)| = |M(T)| \ and \ |\Lambda \cap P_l| \geq |S| \end{cases}\).

It follows that \((v, P)\) is \(l\)-anonymous with the coefficients factors \(|M(T)| + 1\) and \(|S|\).

Let \(i \in P_l\): set \(W^i_l = \{(Z, \Lambda) : Z \in 2^{N \setminus P}_l, \Lambda \subseteq P_l, v(\Lambda \cup Z) - v((\Lambda \setminus i) \cup Z) = 1\}\).

Then, by lemma [1] we have:

\[W^i_l = \{(Z, \Lambda) : Z \in 2^{N \setminus P}_l, \Lambda \subseteq P_l, i \in \Lambda, |M(Z)| = |M(T)| \ and \ |\Lambda| = |S|\} \times \{\Lambda \subseteq P_l, i \in T, |\Lambda| = |S|\}\]
Moreover,

\[ q_i(v, P) = \sum_{Z \in 2^N \setminus \Lambda} \sum_{i \in \Lambda} f_i(Z, \Lambda)[v(\Lambda \cup Z) - v((\Lambda \setminus i) \cup Z)] \]

\[ = \sum_{(Z, \Lambda) \in W_i} f_i(T, S), \]

Since for all \((Z, \Lambda) \in W_i\), we have \(|M(Z)| = |M(T)|\) and \(|\Lambda| = |S|\) and by Equal Size, it holds that \(f_i(Z, \Lambda) = f_i(T, S)\). Thus,

\[ q_i(v, P) = \sum_{(Z, \Lambda) \in W_i} f_i(T, S) \]
\[ = f_i(T, S) \left| W_i \right| \]
\[ = f_i(T, S) \left| \{ Z \in 2^N \setminus \Lambda, |M(Z)| = |M(T)| \times \{ \Lambda \subseteq P_l, i \in T, |\Lambda| = |S| \} \right| \]
\[ = f_i(T, S) \left| \{ Z \in 2^N \setminus \Lambda, |M(Z)| = |M(T)| \times |\Lambda \subseteq P_l, i \in T, |\Lambda| = |S| \} \right| \]
\[ = f_i(T, S) \sum_{m-1}^{M(T)} \left( \binom{m-1}{1} \right) \binom{|S|-1}{|P|} \frac{(m-1)!}{|P|-1!} \frac{|P|-1!}{(m-1)! (|S|-1)! (|P|-1)!} \]

Finally, by Equal Share, \(\forall i \in P_l\), \(q_i(v, P) = \frac{1}{m[|P|]}\).

It follows that \(f_i(T, S) \frac{(m-1)!}{|M(T)|! (m-1)! (|S|-1)! (|P|-1)!} = \frac{1}{m[|P|]}\), which implies that

\[ f_i(T, S) = \frac{|M(T)|! (m-1)! (|S|-1)! (|P|-1)!}{m! \frac{|M(T)|! (m-1)! (|S|-1)! (|P|-1)!}{m!}} \cdot \]

\(\Leftarrow\) Conversely, the CSWD power index for which the factor of a given \((T, S)\) is given by \(f_i(T, S) = \frac{|M(T)|! (m-1)! (|S|-1)! (|P|-1)!}{m! \frac{|M(T)|! (m-1)! (|S|-1)! (|P|-1)!}{m!}}\) is clearly the Owen-Shapley power index, which obviously satisfies Equal Share and Equal Size axioms. \(\square\)

### A.2 Proof of Theorem 2

We need two lemmas in order to prove this theorem. The first one is taken from Courtin et al. [2015] and is recalled below.

**Lemma 2.** [Courtin et al. 2015]

Let \(N\) be a set of \(n\) players and \((n_t)\) a sequence of \(r\) integers such that \(\sum_{t=1}^{r} n_t = n\). The number of partitions of \(N\) into \(r\) coalitions \(S_1, ..., S_r\) such that for all \(t = 1, 2, ..., r, |S_t| = n_t,\)
is given by the formula $\frac{n!}{n_1! \times n_2! \times \cdots \times n_l!}$.

**Lemma 3.** Let $(V, P)$ be an DMGCs, $l \in M$, $F \in R_s^{N \setminus P_l}$ and $G \in R^p_l$.

If $(V, P)$ is l-anonymous, with the coefficients $\mu_i, \left( q^l_k \right)_{k \in R}$ then: for all $i \in N_G$

$V(F \cup G) - \overline{V}(F \cup (G \setminus i)) = 1 \iff |M| = \mu_i - 1$ and $\forall k \in R, |G_k| = q^l_k$

**Proof.** Let $(V, P)$ be an DMGCs, $l \in M$, $i \in P_l$, $F \in R_s^{N \setminus P_l}$ and $G \in R^p_l$. Assume that $(V, P)$ is l-anonymous, with the coefficients $\mu_i, \left( q^l_k \right)_{k \in R}$

$\Rightarrow$) Assume that $V(F \cup G) - \overline{V}(F \cup (G \setminus i)) = 1$, we shall show that $|M| = \mu_i - 1$ and $\forall k \in R, |G_k| = q^l_k$.

We know that $V(F \cup G) - \overline{V}(F \cup (G \setminus i)) = 1$ if and only if $V(F \cup G) = 1$ and $\overline{V}(F \cup (G \setminus i)) = 0$ (1).

Moreover, $\overline{V}(F \cup (G \setminus i)) = 0$ means that for all $H \in R^N$, if $N_H = N_{(F \cup (G \setminus i))}$ then $V(H) = 0$ (2).

By hypothesis, $(V, P)$ is l-anonymous, with the coefficients $\mu_i, \left( q^l_k \right)_{k \in R}$, thus:

$V(F \cup G) = 1 \Rightarrow \left\{ \begin{array}{l} |M|(F \cup G) \geq \mu_i \\ |M|(F \cup G) \geq \mu_i - 1 \text{ and } \forall k \in R, |(F_k \cup G_k) \cap P_l| \geq q^l_k \end{array} \right. \Rightarrow \left\{ \begin{array}{l} |M(F)| \geq \mu_i \\ |M(F)| \geq \mu_i - 1 \text{ and } \forall k \in R, |G_k| \geq q^l_k \end{array} \right. (3)$

We claim that $|M(F)| \geq \mu_i$ is impossible. Indeed, since $F^{-l} = F$, $|M(F)| \geq \mu_i$ implies that $|M(F^{-l})| \geq \mu_i$ and that $V(F) = 1$ by $(V, P)$ l-anonymous. Furthermore, $(F \subseteq F \cup (G \setminus i))$. Therefore $V(F \cup (G \setminus i)) = 1$ and hence $\overline{V}(F \cup (G \setminus i)) = 1$. This is a contradiction with (1).

It then follows from (3) that $|M(F)| = \mu_i - 1$ and $\forall k \in R, |G_k| \geq q^l_k$. We prove now that $\forall k \in R, |G_k| = q^l_k$. Assume on the contrary that there exists $e \in R$ such that $|G_e| > q^l_e$. Consider the integer $j \in R$ such that $i \in G_j$:

Let $i' \in G_e$ and $L \in R^p_l$ defined by $\forall k \in R, L_k = \left\{ \begin{array}{ll} G_k - \{i, i'\} & \text{if } k \neq j \\ (F_k - \{i\}) \cup \{i'\} & \text{if } k = j \end{array} \right.$

It is easy to show that $\forall k \in R, |L_k| \geq q^l_k$, then $V(F \cup L) = 1$.

On the other hand, $N_{F \cup L} = N_{(F \cup (G \setminus i))}$ and thanks to (2) it follows that $V(F \cup L) = 0$, which is a contradiction.

We deduce that $|M(F)| = \mu_i - 1$ and $\forall k \in R, |G_k| = q^l_k$, which concludes the first part of the proof.

$\Leftarrow$) Assume that $|M(F)| = \mu_i - 1$ and $\forall k \in R, |G_k| = q^l_k$. Let us prove that $V(F \cup G) - \overline{V}(F \cup (G \setminus i)) = 1$. 

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We have \(|M(F)| = \mu_l - 1\) and \(\forall k \in R, |G_k| = q_k^l\) thus, by \((V, P)\) \(l\)-anonymous, it follows that \(V(F \cup G) = 1\).

Let \(H \in R^N\) such that \(N_H = N_{(F \cup (G \setminus i))}\).

Then, \(|M(H^{-1})| = |M((F \cup (G \setminus i))^{-1})| = |M(F)| = \mu_l - 1\).

Furthermore, \(\cup_k (H_k \cap P_l) = G \setminus i\) since \(N_H = N_{(F \cup (G \setminus i))}\).

Hence, 
\[
\sum_{k \in R} |H_k \cap P_l| = |(G \setminus i)| < |G| = \sum_{k \in R} |G_k| = \sum_{k \in R} q_k^l.
\]

Therefore, \(\sum_{k \in R} (|H_k \cap P_l| - q_k^l) < 0\) which means that there exists \(j \in R : |H_j \cap P_l| < q_j^l\).

To summarize, we have: \(|M(F)| = \mu_l - 1\), \(|H_j \cap P_l| < q_j^l\) and \((V, P)\) is \(l\)-anonymous with coefficients \(\mu_l, (q_k^l)_{k \in R}\).

It follows that \(V(H) = 0\) for all \(H \in R^N\) such that \(N_H = N_{(F \cup (G \setminus i))}\) and consequently \(V(F \cup (G \setminus i)) = 0\). Finally, \(V(F \cup G) - \overline{V}(F \cup (G \setminus i)) = 1\).

We can now present the proof of Theorem 2.

Proof. \(\Rightarrow\) Let \(\Phi\) a CSWD power index which verifies Equal Share and Equal Size property, with \((f_l(L, G))_{L,G}\). Let \(l \in M, L \in R^{N\setminus B}\) and \(G \in R^B\), we will show that:

\[
f_l(L, G) = \left( m \sum_{I \in [L]} r(I)^{-1} \left( \frac{\left( |F| - \prod_{k \in R} |G_k| \right)}{|P| \prod_{k \in R} |G_k|} \right) \right)
\]

Consider the DMGCs, \((V, P)\) such that:

\[
\forall F \in R^N, V(F) = 1 \iff \left\{ \begin{array}{l} |M(F^{-1})| \geq |M(L)| + 1 \text{ or} \\ |M(F^{-1})| = |M(L)| \text{ and } \forall k \in R, |F_k \cap P_l| \geq |G_k| \end{array} \right\}.
\]

It follows that \((V, P)\) is \(l\)-anonymous with the coefficients \(|M(L)| + 1\), \((|G_k|)_{k \in R}\).

Let \(i \in N\) and set:

\[
\overline{W}_i = \left\{ (F, H) : F \in R^{N\setminus B}, H \in R^B \text{ and } V(F \cup H) - \overline{V}(F \cup (H \setminus i)) = 1 \right\}
\]

\[
= \left\{ (F, H) : F \in R^{N\setminus B}, H \in R^B, i \in N_H, |M(F)| = |M(L)| \text{ and } \forall k \in R, |H_k| = |G_k| \right\}
\]

\[
= \left\{ (F, H) : F \in R^{N\setminus B}, H \in R^B, i \in N_H, M(F) \in [L] \text{ and } \forall k \in R, |H_k| = |G_k| \right\}
\]

\[
= \left\{ (F, H) : F \in R^{N\setminus B}, H \in R^B, i \in \bigcup_{j \in R} H_j, M(F) \in [L] \text{ and } \forall k \in R, |H_k| = |G_k| \right\}
\]

\[
= \bigcup_{j \in R} \left\{ (F, H) : F \in R^{N\setminus B}, H \in R^B, i \in H_j, M(F) \in [L] \text{ and } \forall k \in R, |H_k| = |G_k| \right\}
\]

\[
= \bigcup_{j \in R} \overline{W}^i_{j}
\]

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where:

\[
\overline{W}_{i,j}^l = \begin{cases} 
(F, H) : F \in R^N \setminus P_l, H \in R^P_l, i \in H_j, M(F) \in [L] \text{ and } \forall k \in R, |H_k| = |G_k| \end{cases}
\]

\[
= \left\{ F \in R^N \setminus P_l : M(F) \in [L] \right\} \times \left\{ H \in R^P_l, i \in H_j, \forall k \in R, |H_k| = |G_k| \right\}
\]

\[
= \left( \bigcup_{I \in [L]} \left\{ F \in R^N \setminus P_l : M(F) = I \right\} \right) \times \left\{ H \in R^P_l : i \in H_j, \forall k \in R, |H_k| = |G_k| \right\}
\]

Moreover

\[
\Phi_i(V, P) = \sum_{F \in R^N \setminus P_l} \left( \sum_{H \in R^P_l} f_i(F, H) \left[ V(F \cup H) - \overline{V}(F \cup (H \setminus i)) \right] \right)
\]

\[
= \sum_{(L, G) \in \overline{W}_{i,j}^l} f_i(L, G)
\]

because for all \((F, H) \in \overline{W}_{i,j}^l\), we have \(|M(F)| = |M(L)|\) and \(\forall k \in R, |H_k| = |G_k|\) and \(f_i(F, H) = f_i(L, G)\) holds thanks to equal weight axiom.

Thus,

\[
\Phi_i(V, P) = \sum_{(L, G) \in \overline{W}_{i,j}^l} f_i(L, G)
\]

\[
= f_i(L, G) \left| \overline{W}_{i,j}^l \right|
\]

\[
= f_i(L, G) \bigg| \bigcup_{j \in R} \left| \overline{W}_{i,j}^l \right|igg|
\]

\[
= f_i(L, G) \sum_{j \in R} \left| \overline{W}_{i,j}^l \right|
\]

\[
= f_i(L, G) \sum_{j \in R} \left| \bigcup_{I \in [L]} \left\{ F \in R^N \setminus P_l : M(F) = I \right\} \right| \times \left\{ H \in R^P_l : i \in H_j, \forall k \in R, |H_k| = |G_k| \right\}
\]

\[
= f_i(L, G) \left[ \sum_{I \in [L]} \left| \left\{ F \in R^N \setminus P_l : M(F) = I \right\} \right| \right] \times \left[ \sum_{j \in R} \left| \left\{ H \in R^P_l : i \in H_j, \forall k \in R, |H_k| = |G_k| \right\} \right| \right]
\]

However,

\[
\forall I \in [L], \left| \left\{ F \in R^N \setminus P_l : M(F) = I \right\} \right| = r_{|P_l|}^{\sum \left\{ P_I \right\}}
\]
and thanks to Lemma 2 we have: for all $j \in R$,

$$\left| \{ H \in R^i, i \in H_j : \forall k \in R, |H_k| = |G_k| \} \right| = \frac{|P|-1!}{(|P|-\sum_{k \in R} |G_k|)! \prod_{k \in R \setminus k_j} |G_k|! (|G_j|-1)!}$$

Thus,

$$\Phi_1(V, P) = f_1(L, G) \left( \sum_{l \in [L]} \left( \sum_{h \in h_l} |P_h| \right) \right) \left( \sum_{j \in R} \frac{|P|-1! |G_j|}{(|P|-\sum_{k \in R} |G_k|)! \prod_{k \in R \setminus k_j} |G_k|!} \right)$$

$$= f_1(L, G) \left( \sum_{l \in [L]} \left( \sum_{h \in h_l} |P_h| \right) \right) \frac{|P|-1! \sum_{j \in R} |G_j|}{(|P|-\sum_{k \in R} |G_k|)! \prod_{k \in R \setminus k_j} |G_k|!}$$

Finally, by Equal share, $\forall i \in P_1, \Phi_1(V, P) = \frac{1}{m|P|}$. It then follows that

$$f_1(L, G) \left( \sum_{l \in [L]} \left( \sum_{h \in h_l} |P_h| \right) \right) \frac{|P|-1! \sum_{j \in R} |G_j|}{(|P|-\sum_{k \in R} |G_k|)! \prod_{k \in R \setminus k_j} |G_k|!} = \frac{1}{m|P|}$$

which implies that

$$f_1(L, G) = \frac{1}{m|P|} \times \left[ \sum_{l \in [L]} \left( \sum_{h \in h_l} |P_h| \right) \right]^{-1} \left[ \frac{|P|-\sum_{k \in R} |G_k|! \prod_{k \in R} |G_k|!}{|P|-1! \sum_{j \in R} |G_j|} \right]$$

$$= \left[ m \sum_{l \in [L]} \left( \sum_{h \in h_l} |P_h| \right) \right]^{-1} \left[ \frac{|P|-\sum_{k \in R} |G_k|! \prod_{k \in R} |G_k|!}{|P|! \sum_{k \in R} |G_k|} \right]$$

$\Leftarrow$) Conversely, let $\Phi$ be a CSWD power index such that

$$f_1(L, G) = \left[ m \sum_{l \in [L]} \left( \sum_{h \in h_l} |P_h| \right) \right]^{-1} \left[ \frac{|P|-\sum_{k \in R} |G_k|! \prod_{k \in R} |G_k|!}{|P|! \sum_{k \in R} |G_k|} \right]$$

• It is obvious that $\Phi$ satisfies the Equal Size condition.
Let us now prove that \( \Phi \) satisfies the Equal share axiom.

Consider a DMGCs which is \( l \)-anonymous with

\[
\forall F \in R^N, V(F) = 1 \iff \left\{ \begin{array}{l}
\left| M(F^l) \right| \geq \mu_i \\
\left| M(F^l) \right| = \mu_i - 1 \text{ and } \forall k \in R, |F_k \cap P| \geq q_k^l
\end{array} \right.
\]

We have:

\[
\Phi_i(V,P) = \sum_{L \in R^N} \left( \sum_{G \in R^P} f_i(L,G) \left[ V(L \cup G) - \overline{V} (L \cup (G \setminus i)) \right] \right)
\]

\[
= \sum_{(L,G) \in \overline{W}_i^j} f_i(L,G)
\]

where \( \overline{W}_i^j = \{(L,G) : L \in R^N \setminus R^P, G \in R^P \text{ and } V(L \cup G) - \overline{V} (L \cup (G \setminus i)) = 1 \} \)

We know from Lemma 3 that if \( (L,G) \in W_i^j \), then \( |M(L)| = \mu_i - 1 \) and \( \forall k \in R, |G_k| = q_k^l \), which implies that \( f_i(L,G) \) does not depend on \( (L,G) \) and is given by \( f_i(L,G) = \left( m \sum_{|P| \in [L]} \left( \frac{\sum_{|P| \in [L]} |P_i| - \sum_{k \in R} |q_k^l|}{\prod_{k \in R} |q_k^l|} \right) \right)^{-1} \)

Hence, \( \Phi_i(V,P) = f_i(L,G)|\overline{W}_i^j| \).

As we did in the first part of the proof, let us remark that \( \overline{W}_i^j = \bigcup_{j \in R} X_{i,j}^j \), where

\[
X_{i,j}^j = \{(F,H) \in \overline{W}_i^j : i \in H_j \}
\]

\[
= \{(F,H) : F \in R^N \setminus R^P, H \in R^P, i \in G_j, M(F) = \mu_i - 1 \text{ and } \forall k \in R, |H_k| = q_k^l \}
\]

\[
= \{F \in R^N \setminus R^P : M(F) = \mu_i - 1 \} \times \{H \in R^P : i \in H_j, \forall k \in R, |H_k| = q_k^l \}
\]

\[
= \{F \in R^N \setminus R^P : M(F) = M(L) \} \times \{H \in R^P : i \in H_j, \forall k \in R, |H_k| = q_k^l \}
\]

\[
= \{F \in R^N \setminus R^P : F \in [L] \} \times \{H \in R^P : i \in H_j, \forall k \in R, |H_k| = q_k^l \}
\]

\[
= \left( \bigcup_{i \in [L]} \{F \in R^N \setminus R^P : M(L) = 1 \} \right) \times \{H \in R^P, i \in H_j, \forall k \in R, |H_k| = q_k^l \}
\]
Thus,
\[
\overline{W}_i^j = \left| \bigcup_{j \in R} X^i_{j} \right| \\
= \sum_{j \in R} X^i_{j} \\
= \sum_{j \in R} \left| \left\{ F \in R^N \setminus B : M(L) = I \right\} \times \left\{ G \in R^P : i \in G_j, \forall k \in R, |G_k| = q_k^j \right\} \right| \\
= \left[ \sum_{j \in R} \left| \left\{ F \in R^N \setminus B : M(L) = I \right\} \times \left\{ G \in R^P : i \in G_j, \forall k \in R, |G_k| = q_k^j \right\} \right| \right] \\
\]

For all \( j \in R, \left| \left\{ G \in R^P, i \in G_j : \forall k \in R, |G_k| = q_k^j \right\} \right| \) is the number of partitions of \( N - i \) into \( r + 1 \) coalitions \( G_1, G_2, ..., G_r \) and \( G_{r+1} \) such that \( |G_1| = q_1^j, ..., |G_{j-1}| = q_{j-1}^j, |G_j| = q_j^j - 1 \), (by assuming that \( i \in G_j \), \( |G_{j+1}| = q_{j+1}^j, ..., |G_r| = q_r \) and \( |G_{r+1}| = (n - \sum_{k \in R} |G_k|) \).

As above, we obtain:
\[
\left| \left\{ G \in R^P, i \in G_j : \forall k \in R, |G_k| = q_k^j \right\} \right| = \frac{|P_j|!|q_j^j|}{\left( |P_j| - \sum_{k \in R} |q_k^j| \right) \prod_{k \in R} |q_k^j|} \\
\]

Further, for all \( I \in [L], \left| \left\{ F \in R^N \setminus B : M(L) = I \right\} \right| = r^{\left( \sum_{i \in I} |P_i| \right)} \\
\]
hence,
\[
\overline{W}_i^j = \left( \sum_{I \in [L]} r^{\left( \sum_{i \in I} |P_i| \right)} \right) \sum_{j \in R} \frac{|P_j|!|q_j^j|}{\left( |P_j| - \sum_{k \in R} |q_k^j| \right) \prod_{k \in R} |q_k^j|} \\
\]

And finally,
\[
\Phi_i(V, P) = f_i(L, G)\overline{W}_i^j \\
= m \sum_{I \in [L]} r^{\left( \sum_{i \in I} |P_i| \right)} \left( \frac{|P_j|!\left( \sum_{k \in R} |q_k^j| \right) \prod_{k \in R} |q_k^j|}{|P_j| - \sum_{k \in R} |q_k^j|} \right) \\
\times \left( \sum_{I \in [L]} r^{\left( \sum_{i \in I} |P_i| \right)} \right) \sum_{j \in R} \frac{|P_j|!|q_j^j|}{\left( |P_j| - \sum_{k \in R} |q_k^j| \right) \prod_{k \in R} |q_k^j|} \\
= \frac{1}{m|P|} \\
\]

This concludes the proof.
A.3 Proof of corollary 1

Proof. Let |R| = 1, l ∈ M, L ∈ R_{N/P}^i and G ∈ R_P, then there exist T ∈ 2_{N/P} and S ⊆ P_l such that: L = T and G = S.

We have,

\[ f_1(L, G) = f_1(T, S) = \left[ m \sum_{I \in [L]} \left( \sum_{h \in [L]} |P_h| \right)^{-1} \left[ \frac{(|P| - \sum_{k \in [1]} |G_k|)!}{|P|! \left( \sum_{k \in [1]} |G_k| \right)!} \right] \right] \]

\[ = \left[ m \sum_{I \in [L]} \left( \sum_{h \in [L]} |P_h| \right)^{-1} \frac{(|P| - |S|)!|S|!}{|P|!|S|!} \right] \]

\[ = \left[ m \frac{|I \subseteq M - l : |I| = |M(L)||}{|M(L)|} \left[ \frac{(|P| - |S|)!}{|P|!} \right] \left[ \frac{|S|!}{|S|!} \right] \right] \]

\[ = \left[ m \frac{|I \subseteq M - l : |I| = |M(T)||}{|M(T)|} \left[ \frac{(|P| - |S|)!}{|P|!} \right] \left[ \frac{|S|!}{|S|!} \right] \right] \]

\[ = \left[ m \frac{(m-1)!}{m!} \frac{|M(T)|}{|M(T)|!} \left[ \frac{(|P| - |S|)!}{|P|!} \right] \left[ \frac{|S|!}{|S|!} \right] \right] \]

\[ = OS_1(V, P) \]

It follows that:

\[ \Phi_i(V, P) = \sum_{L \in R_{N/P}^i} \sum_{G \in R_P^i} f_1(L, G) \left[ V(L \cup G) - \bar{V}(L \cup (G \setminus i)) \right] \]

\[ = \sum_{T \in 2_{N/P}^i} \sum_{S \subseteq P_l, i \in S} f_1(T, S) \left[ V(S \cup T) - V((S \setminus i) \cup T) \right] \]

\[ = \sum_{T \in 2_{N/P}^i} \sum_{S \subseteq P_l, i \in S} \frac{|M(T)|!(m-|M(T)|-1)!}{m!} \frac{|S|!(|P| - |S|)!}{|P|!} \left[ V(S \cup T) - V((S \setminus i) \cup T) \right] \]

This concludes the proof. \[ \square \]

A.4 Proof of proposition 1

Proof. \[ \Rightarrow \] Let \( \varphi_i(v, P) \) be a CSWD. Assume further that \( \varphi \) is \( (\lambda^l)^i \)-constant-weight.

Then, for all \( T \in 2_{N/P}^i \) and for all \( S \subseteq P_l, f_1(T, S) = \lambda^l \)

Let \( (v, P) \) be SGCs and \( i \in P_l \), we have:
\[\varphi_i(v, P) = \sum_{T \in \mathcal{P}} \sum_{S \subseteq P, i \in S} f_i(T, S)[v(S \cup T) - v((S \setminus i) \cup T)]\]
\[= \sum_{T \in \mathcal{P}} \sum_{S \subseteq P, i \in S} \lambda^l \cdot [v(S \cup T) - v((S \setminus i) \cup T)]\]
\[= \lambda^l \sum_{T \in \mathcal{P}} \sum_{S \subseteq P, i \in S} \eta_i(v, P)\]

\(\Leftarrow\) The converse is straigntforward.

\[\square\]

### A.5 Proof of Proposition 2

**Proof.** Let \(\Phi\) be a CSWD power index. If \(\Phi\) is \((\sigma^l)\)-constant-weight, then for all \(i \in N\),

\[\Phi_i(V, P) = \sum_{L \in \mathcal{P}} \sum_{G \in \mathcal{P}} f_i(L, G)[V(L \cup G) - \overline{V}(L \cup (G \setminus i))]\]
\[= \sum_{L \in \mathcal{P}} \sum_{G \in \mathcal{P}} \delta^l \cdot [V(L \cup G) - \overline{V}(L \cup (G \setminus i))]\]
\[= \delta^l \sum_{L \in \mathcal{P}} \sum_{G \in \mathcal{P}} \eta_i(V, P)\]

\(\Leftarrow\) The converse is straightforward. \(\square\)