Fair Intergenerational Decision Making*

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Abstract

Most analyses of climate policy rely on an expected discounted utilitarian criterion, ignoring the questions of welfare measurement, preference diversity and preference change. The paper develops a set of alternative methods for policy evaluation, named fair utilitarian, fair ex ante and fair ex post approaches, that explicitly address these issues. Following the fair social choice approach, we rely on Pigou-Dalton transfer principles to embody fairness ideals. In combination with Pareto principles and social rationality requirements, they help delineating a set of criteria that can be used when we are uncertain about the size and the preferences of future populations, as is typically the case for climate policy.

Keywords: Social choice, fairness, social risk, intergenerational equity.

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1 Introduction

Recent debate about climate change policy have addressed the problem of the correct welfare framework to be used. The debate has focused primarily on the choice of the social discount rate, which is a key parameter to compute the present values of future impacts and is therefore crucial because of the long term impacts of most greenhouse gases. Most of the debate following the publication of the Stern review (Stern, 2006) has been about the appropriate value of the discount rate. Some authors (Schelling, 1995; Stern, 2006) endorsed an 'ethical' approach supporting a low value of the discount rate, while others (Nordhaus, 2007, 2008; Weitzman, 2007) called for larger values based on the preferences revealed on financial markets.

All these papers address the problem within the framework of the workhorse model of welfare economics, namely the expected discounted utility (EDU) model. Prospects $\mathbf{x}$ are assessed using the social welfare function:

$$\sum_{t=0}^{\infty} e^{-\delta t} \sum_{i} \mathbb{E}\big(u_i(\mathbf{x}_i)\big). \tag{1}$$

This model has several drawbacks. First, the EDU criterion has been criticized for using utility discounting embodied in the factor $e^{-\delta t}$ (Dasgupta and Heal, 1979; Stern, 2006), inducing an unfair treatment of future generations. Second, it is unable to disentangle risk aversion, inequality aversion and the intertemporal elasticity of substitution (Atkinson et al., 2009; Dasgupta, 2008; Anthoff, Tol and Yohe, 2009; Gollier, 2002; Traeger, 2014; Bommier and LeGrand, 2013). This is a source of difficulty for applied work, because it is not clear how the model should be parametrized. and and the inability to disentangle risk aversion and inequality aversion. We do not discuss these issues in details here (see Fleurbaey and Zuber, 2015a, for a more detailed discussions of these issues, as well as the population ethics issue).

Formula 1 also assumes that there exists no risk on the composition of generations: all individuals belonging to a generation always exist. A risk on the existence of generations is sometimes said to be embodied in the term $e^{-\delta t}$ (see Dasgupta and Heal, 1979; Stern, 2006, for classical references), but it is independent of the risk on consumption, and generations sizes are taken as given. The difficult issues with the comparison of population of different sizes are completely ignored by the formula (these issues are discussed for instance in Millner, 2013). There is a significant literature on population ethics discussing how to compare populations of different sizes (classical references in that field include Broome, 1991, 2004; Ng, 1989; Blackorby, Bossert and Donaldson, 2005). However, the applied literature on climate change has hardly taken stock of these contributions.

Other problems that have seldom been discussed. One of this problem is the intricate problem of welfare measurement. Most applied papers use a common utility function $u(c)$, which is the same for all individuals and all generations, and which only depends on consumption. Usually, the utility function is iso-elastic, $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$. This parsimonious model ignores the possibility of preference diversity and preference change. Also, Formula 1 implicitly assumes that the cardinally meaningful utility representations are the Von Neumann and Morgenstern (VNM) utilities, which means that risk preferences are key in welfare measurement. This Bernoulli assumption that risk preferences should be used in welfare measurement is part of a venerable tradition in welfare economics,[1] but it is not undisputed.

From the viewpoint of a resource-based approach to welfare measurement that we endorse in the present paper, an issue with the expected utilitarian criterion when one acknowledges preference diversity is that progressive income transfers may not be welfare enhancing, even when there is no risk. Indeed, if two individuals have different utility functions u_i, they in particular have different marginal utilities at a some income level, which may justify regressive transfers.

Preference diversity is a real challenge when analyzing the impact of public policies. How should the welfare individuals with different preferences be com-

[1]John Harsanyi is a well-known representative and advocate of this tradition. For a more recent defense, see Adler (2012).

pared when allocating resources? This has been one of the central issues of social choice theory ever since the birth of welfare economics. In that context, it has been showed that we face a real dilemma: when there is more than one good, it is impossible to design welfare evaluation criteria satisfying both the Pareto principle and a natural principle of fairness, expressed in terms of multidimensional Pigou-Dalton principle of transfer of resources between individuals (Fleurbaey and Trannoy, 2003; Brun and Tungodden, 2004). In problems involving risks, such as climate policy, this dilemma gives rise to the distinction between an ex ante approach that respect individuals' ex ante preferences (ex ante Pareto), and an ex post approach that satisfies fairness principles ex post (Gajdos and Tallon, 2002; Fleurbaey and Zuber, 2015c)

In the context of climate policy, the question of preference diversity is particularly important. Indeed, environmental questions involve long-lasting phenomena, and current decisions will have long-lasting consequences. So it seems necessary to take into account the possibility of changes in preferences in the future. The attitude of future generations towards environmental assets may well be different from ours. The formation of preferences is however a very complex issue so that we cannot be completely sure what will be future generations' preferences. A few papers have highlighted the issue of the uncertainty about future preference and its consequences for climate policy (Beltratti, Chichilnisky and Heal, 1998; Ayong Le Kama, 2001; Ayong Le Kama and Schubert, 2004). But they do not discuss in details the welfare framework to be used for evaluating such changes. In this paper, we clearly spell out the assumptions behind different welfare models and highlight their implication for a key parameter of climate policy, namely the social discount rate.

The aim of the present paper is to introduce welfare frameworks that can be used to explicitly address the issue of welfare measurement in situations involving preference diversity, preference change and uncertainty about the identity of future people. Following the fair social choice approach (Fleurbaey and Maniquet, 2011), we put forward fairness principles embodied in Pigou-Dalton transfer axioms. We introduce and characterize fair ex ante and fair ex post approaches.

We also show how weakened fairness principle suggest specific normalizations of the VNM utility functions when using a utilitarian approach.

In Section 2, we present a framework to analyze intergenerational policies in the context of uncertainty about the existence of future generations, and uncertainty about there preferences. This involves setting up a variable population framework like in the literature on populations ethics (Blackorby, Bossert and Donaldson, 2005). The main difference is that we do not take utilities or welfare as given, like in the existing literature, but we only use information on ordinal preferences and resource.

In Section 3, we first consider the case where the same population exists in all risky prospects. We highlight a basic dilemma between three principles: social rationality, the respect of individual preferences, fairness. This suggests that three routes can be followed. First, we can abandon fairness, which is what the EDU approach does. However, we prove that the utilitarian approach can still satisfy a weakened fairness principle, which delivers what we name the fair utilitarian approach. Second, we can use a weakened principle of respect of individual preferences, which yields to the fair ex post approach. Third, we can abandon social rationality, which is what the fair ex ante approach does. These three approaches are axiomatically characterized. Doing so, we contribute to social welfare theory. Although there already exist methods to deal with risk within the fair social choice approach (see Fleurbaey and Maniquet, 2011 – Chap. 6.2, for an ex ante approach and Fleurbaey and Zuber, 2015c, for an ex post approach), we obtain more general results. And we propose what we believe is a novel version of utilitarianism.

In Section 4, we extend the previous results to the case of population of varying size and composition. To do so, we need to adapt the Pareto principles, because people may not live in all states of the world. We hence only rely on individual preferences for the comparison of prospects where people live in the same states of the world (although they may be different for different people). In that respect, the paper also offer a new methodological contribution. We are able to compare populations with different size by introducing a critical-level principle expressed in terms of resources (while it is usually expressed in terms of utility in the literature on population ethics). An to compare populations where people have different risk preferences by relying on their certainty equivalents.

2 The framework

We let $\mathbb{N}$ denote the set of positive integers, $\mathfrak{N}$ the set of non-empty finite subsets of $\mathbb{N}$, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+$ the set of non-negative real numbers. For a set D and any $n \in \mathbb{N}$, D^n is the n-fold Cartesian product of D. Also, for two sets D and E, D^E denotes the set of mappings from E into D.

The set of *potential* individuals who may or may not exist is $\mathbb{N}$. In the definition of a person, we include all her relevant characteristics and in particular the generation she belongs to. Hence there exists a mapping $T : \mathbb{N} \to \mathbb{N}$ that associates to each individual i the period she will exist provided she comes to life, $T(i)$. Individuals live for one period only, so that we call this period a generation.

We consider the allocation of resources between the individuals composing the economy. To simplify, we assume that a single good, labelled as income, has to be distributed. It can be a summary statistics of all the resources available (for instance an equivalent income).[2]

We let $\mathcal{X} = \bigcup_{\mathcal{N}\in\mathfrak{N}} (\mathbb{R}_+)^{\mathcal{N}}$ denote the set of possible alternatives when at least one individual exists. An alternative then is a function assigning a non-negative quantity of good to each individual living in that particular alternative, where $\mathcal{N}$ is the subset of such individuals living. Hence the size of the population may vary from one alternative to another and it is important to keep track of the population in an alternative. For any $x \in \mathcal{X}$, we let $\mathcal{N}(x) \in \mathfrak{N}$ be the set of individuals in the alternative and $n(x) = |\mathcal{N}(x)|$ be the number of individuals in the alternative. We assume that there always are finitely many people in any alternative.

We also assume that it is not known for sure what will be the final allocation of income nor the set of individuals who will eventually exist. To model this,

[2]See **?**for a history of the use of equivalent incomes in economics.

we assume that there exists an infinite set of states of the world $\mathcal{S}$, with typical element $s \in \mathcal{S}$. We denote Σ a σ-algebra over $\mathcal{S}$, so that $(\mathcal{S}, \Sigma)$ is a measurable space. There is a probability measure P on the measurable space $(\mathcal{S}, \Sigma)$. For any random variable K, that is any $\Sigma-$measurable function $K : \mathcal{S} \to \mathbb{R}$, we denote $\mathbb{E}\big(K\big) = \int_{\mathcal{S}} K(s) dP(s)$. We assume that we are in a situation of risk and that all individuals consider P as the correct belief function. We also make the following assumption of convex-rangedness:

Assumption 1 *For any $A \in \Sigma$ and $\kappa \in [0,1]$, there exists $A' \in \Sigma$ such that $A' \subset A$ and $P(A') = \kappa P(A)$.*

A prospect $\mathbf{x}$ is a function $\mathbf{x}$ from $\mathcal{S}$ to $\mathcal{X}$, which is supposed to be $\Sigma-$measurable. For $s \in \mathcal{S}$, $\mathbf{x}(s)$ is therefore the alternative induced by the prospect $\mathbf{x}$ in state s. We assume that prospects are such that there exists a finite partition $(A_1, \cdots, A_m)$ for which, whatever $k = 1, \cdots, m$, $\mathbf{x}(s) = \mathbf{x}(s')$ for all $s, s' \in A_k$. Hence prospects only induce a finite number of final allocations. We denote $\mathbf{X}$ the set of all such prospects.

For an alternative $x \in \mathcal{X}$, whenever $i \in \mathcal{N}(x)$, $x_i \in \mathbb{R}_+$ denotes the income of individual i. For a prospect $\mathbf{x} \in \mathbf{X}$, whenever $i \in \mathcal{N}\big(\mathbf{x}(s)\big)$, $\mathbf{x}_i(s)$ denotes the income of individual i in state of the world $s \in \mathcal{S}$. We denote $A_i(\mathbf{x}) = \big\{s \in \mathcal{S} \mid i \in \mathcal{N}\big(\mathbf{x}(s)\big)\big\}$ the event where individual i exists and $p_i(\mathbf{x}) = P\big(A_i(\mathbf{x})\big)$ the probability that he exists. We denote $\mathbf{x}_i$ the mapping $\mathbf{x}_i \in \mathbb{R}_+^{A_i(\mathbf{x})}$ assigning in each state of the world to individual i her allocation induced by the social prospect $\mathbf{x}$. The mapping $\mathbf{x}_i$ represents the prospects of individual i.

For a subpopulation $\mathcal{N} \in \mathfrak{N}$, we denote by $\mathbf{X}_{\mathcal{N}}$ the set of prospects such that, for every $\mathbf{x} \in \mathbf{X}_{\mathcal{N}}$, and every $s \in \mathcal{S}$, $\mathcal{N}\big(\mathbf{x}(s)\big) = \mathcal{N}$. These are the prospects such that the same individuals are present in all states of the world. Without loss of generality, for any $x \in \mathcal{X}$, we denote by x the prospect $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x}(s) = x$ for all $s \in \mathcal{S}$. These are sure prospects yielding the same allocation in all states of the world. Similarly, we abuse notation and denote $\mathcal{X} \subset \mathbf{X}$ the set of sure prospects and $\mathcal{X}_{\mathcal{N}} \subset \mathbf{X}_{\mathcal{N}}$ the set of sure prospects such that the population is $\mathcal{N}$.

Last, we want to define egalitarian alternatives and prospects. An alternative x is egalitarian if, for all $i, j \in \mathcal{N}(x)$, $x_i = x_j$. We denote $\mathcal{X}^e$ the set of egalitarian allocations. Similarly, a prospect $\mathbf{x}$ is egalitarian if for all $s \in \mathcal{S}$ $\mathbf{x}(s) \in \mathcal{X}^e$. We denote $\mathbf{X}^e$ the set of egalitarian prospects, $\mathcal{X}^e_{\mathcal{N}}$ the set of egalitarian alternatives such that the population is $\mathcal{N}$, and $\mathbf{X}^e_{\mathcal{N}}$ the set of egalitarian prospects such that the population is always $\mathcal{N}$.

In this setting, we want to define how individuals rank different prospects and alternatives. One difficulty is that, for a given social prospect, an individual $i \in \mathbb{N}$ may not exist in all states of the world. We hence need to define individual preferences conditional on their existence in an event $A \in \Sigma$. To do so, for any $A \in \Sigma$ we let $\Sigma_A = \{B \in \Sigma, B \subset A\}$ be a sigma-algebra over A. For any Σ_A-measurable function $K : A \to \mathbb{R}$, we denote $\mathbb{E}_A\big(K\big) = \frac{1}{P(A)} \int_A K(s)dP(s)$. And we denote $\mathbf{X}_i(A)$ the set of mappings $\mathbb{R}_+^A$ that are Σ_A-measurable. We assume that individual preferences are conditional expected utilities.

Assumption 2 *For each $i \in \mathbb{N}$ and $A \in \Sigma$, individual i's preferences are represented by complete reflexive and transitive relation $\succsim_i^A$ over $\mathbf{X}_i(A)$. Furthermore, there exists an increasing continuously differentiable and concave function $u_i : \mathbb{R}_+ \to \mathbb{R}$ such that, for all $\mathbf{x}_i, \tilde{\mathbf{x}}_i \in \mathbf{X}_i(A)$,*

$$\mathbf{x}_i \succsim_i^A \tilde{\mathbf{x}}_i \Longleftrightarrow \mathbb{E}_A\big(u_i(\mathbf{x}_i)\big) \geq \mathbb{E}_A\big(u_i(\tilde{\mathbf{x}}_i)\big).$$

Note that, if expected utilities are a representation of preferences, it is only an ordinal one. An alternative representation, that we use in this paper as a metric to assess individual welfare is the the certainty equivalent. It is defined as follows. For any $\mathbf{x}_i \in \mathbf{X}_i(A)$, and given the preferences $\succsim_i^A$ of individual i, the certainty equivalent of $\mathbf{x}_i$, denoted $ce_i(\mathbf{x}_i)$ is

$$ce_i(\mathbf{x}_i) = u_i^{-1}\left(\mathbb{E}_A\big(u_i(\mathbf{x}_i)\big)\right)$$

so that $u_i\big(ce_i(\mathbf{x}_i)\big) = \mathbb{E}_A\big(u_i(\mathbf{x}_i)\big)$. The certainty equivalent is a money-metric of utility in the sense that it is expressed as an equivalent income level.

We also assume that there exists a complete social ranking of prospects.

Assumption 3 *There exist a social ordering $\succsim$, which is a complete reflexive and transitive relation over $\mathbf{X}$. Furthermore, for all $\mathcal{N} \in \mathfrak{N}$, the relation $\succsim$ restricted to $\mathcal{X}_{\mathcal{N}}$ is continuous and increasing.*

The fact that the social ordering is increasing for allocations is an expression of the Pareto principle in the absence of risk: if all individuals have more (and thus are better-off according to Assumption 2, the situation is socially better.

Assumptions 1, 2 and 3 will be maintained throughout the paper. We call them the *core assumptions.* For some results, we will also make the following assumptions regarding individuals preferences.

Assumption 4 *For any $\mathcal{N} \in \mathfrak{N}$, the functions $(u_i)_{i \in \mathcal{N}}$ are linearly independent.*

Assumption 5 *There exists an individual $i \in \mathbb{N}$ such that for all $j \in (\mathbb{N} \setminus \{i\})$ there exists a continuous, increasing and concave function ψ_j such that either $u_i = \psi_j \circ u_j$ or $u_j = \psi_j \circ u_i$.*

Assumption 4 is an assumption of preference diversity, implying that no too individuals have exactly the same preferences. This may seem rather strong, but differences in preferences for small intervals in $\mathbb{R}_+$ are sufficient to obtain it, which may be acceptable. Assumption 5 states that there exists an individual which can be ambiguously ranked with respect to others in terms of risk aversion. It is verified if there is a completely risk neutral (potential) individual or an extremely risk averse one (focusing on the worst outcome).

3 Fair social choice under risk for fixed populations

In this section, we first highlight the difficulty to combine three requirements for fair social choice in the presence of risk, even when population is fixed. The three requirements are: 1/ social rationality (expected utility); 2/ respect of individual preferences (the Pareto principle); and 3/ fairness (the Pigou-Dalton principle).

Let us first introduce the these three requirements.

Axiom 1 (Social expected utility) *There exists a continuous function* $U : \mathcal{X} \to \mathbb{R}$ *such that, for all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$*:*

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \mathbb{E}\big(U(\mathbf{x})\big) \geq \mathbb{E}\big(U(\mathbf{y})\big)$$

Social expected utility is a requirement of social rationality. In particular, it implies a dominance property (if a prospect yield a better alternative in all states of the world, it is also better *ex ante*). It also guarantees that there is a positive value for society learn information about the true state of the world. Also the expected utility model has been disputed from the behavioral point of view, it almost unquestioned from the normative point of view, especially in the context of risk that we consider here. Given that our focus is on normative assessment, it seems natural to invoke expected utility as a principle of social rationality.

The second requirement concern the respect of individual preferences. It is the well-known Pareto principle.

Axiom 2 (Pareto) *For all* $\mathcal{N} \in \mathfrak{N}$*, for all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$*, if* $\mathbf{x}_i \succsim_i^{\mathcal{S}} \mathbf{y}_i$ *for all* $i \in \mathcal{N}$*, then* $\mathbf{x} \succsim \mathbf{y}$*. If furthermore* $\mathbf{x}_i \succ_i^{\mathcal{S}} \mathbf{y}_i$ *for all* $i \in \mathcal{N}$*, then* $\mathbf{x} \succ \mathbf{y}$*.*

Our version of the Pareto principle is a version of the so-called "Weak Pareto" principles that guarantees Pareto indifference. It allows a wide range of social welfare ranking, including utilitarian and egalitarian (maxmin) ones.

Last, we want to introduce a principle a fairness. As is standard in the literature, this principle is expressed in terms of social welfare enhancing transfers of resources (the Pigou-Dalton principle). It is however known that strong fairness principles conflict with the Pareto principle (Fleurbaey and Trannoy, 2003; Brun and Tungodden, 2004). We thus only assume that fairness holds when there is no risk.

Axiom 3 (Pigou-Dalton transfer for no risk alternatives) *For all* $\mathcal{N} \in \mathfrak{N}$*, for all* $x, y \in \mathcal{X}_{\mathcal{N}}$*, if there exists* $i, j \in \mathcal{N}$ *and* $\varepsilon \in \mathbb{R}_{++}$ *such that*

1. $y_i + \varepsilon = x_i \leq x_j = y_j - \varepsilon$;
2. $x_k = y_k$ *for all* $k \in (\mathcal{N} \setminus \{i, j\})$;

then $x \succ y$.

Denote $\succsim_{\mathcal{N}}$ the restriction of the social welfare ordering $\succsim$ to prospects in $\mathcal{N}$. The next proposition highlights the impossibility to satisfy the three requirements when people have diverse risk preferences.

Proposition 1 *Under the core assumptions, for any* $\mathcal{N} \in \mathfrak{N}$ *such that there exists* $i, j \in \mathcal{N}$ *and* $\mathbf{x}_i \in \mathbf{X}_i(\mathcal{S})$ *for which* $ce_i(\mathbf{x}_i) \neq ce_j(\mathbf{x}_i)$*, there is no social ordering* $\succsim_{\mathcal{N}}$ *satisfying the three Axioms 1, 2 and 3.*

Proof. By Prop. 2 below, we know that the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 1, 2 and 4 (which is implied by 3) if and only if there exists exists a poverty level $z_p \in \mathbb{R}_+$ such that, for all $x, y \in \mathcal{X}_{\mathcal{N}}$,

$$x \succsim_{\mathcal{N}} y \Longleftrightarrow \sum_{i \in \mathcal{N}} \frac{u_i(x_i)}{u_i'(z_p)} \geq \sum_{i \in \mathcal{N}} \frac{u_i(y_i)}{u_i'(z_p)}.$$

Because there exists $i, j \in \mathcal{N}$ and $\mathbf{x}_i \in \mathbf{X}_i(\mathcal{S})$ for which $ce_i(\mathbf{x}_i) \neq ce_j(\mathbf{x}_i)$, the must exist un interval $[\underline{z}, \bar{z}] \subset \mathbb{R}_+$ such that for all $z \in [\underline{z}, \bar{z}]$ $\frac{u_i'(z)}{u_i'(z_p)} < \frac{u_j'(z)}{u_j'(z_p)}$ (or the opposite, but we consider this case without loss of generality). For any $\varepsilon \in \mathbb{R}_+$ such $0 < \varepsilon < \frac{\bar{z} - \underline{z}}{2}$, define $x, y \in \mathcal{X}$ in the following way:

1. $y_i = \underline{z}$, $x_i = \underline{z} + \varepsilon$, $x_j = \bar{z} - \varepsilon$ and $y_j = \bar{z}$;
2. $x_k = y_k$ for all $k \in (\mathcal{N} \setminus \{i, j\})$.

By the above results, $x \succ_{\mathcal{N}} y \Longleftrightarrow \frac{u_i(\underline{z}+\varepsilon)}{u_i'(z_p)} + \frac{u_j(\bar{z}-\varepsilon)}{u_j'(z_p)} > \frac{u_i(\underline{z})}{u_i'(z_p)} + \frac{u_j(\bar{z})}{u_j'(z_p)}$. But, by concavity of the functions u_i and u_j, we have $\frac{u_i(\underline{z}+\varepsilon)}{u_i'(z_p)} - \frac{u_i(\underline{z})}{u_i'(z_p)} < \frac{u_i(\underline{z})}{u_i'(z_p)}\varepsilon$ and $\frac{u_j(\bar{z})}{u_j'(z_p)} - \frac{u_j(\bar{z}-\varepsilon)}{u_j'(z_p)} > \frac{u_j(\bar{z})}{u_j'(z_p)}\varepsilon$. We thus have

$$\frac{u_i(\underline{z}+\varepsilon)}{u_i'(z_p)} - \frac{u_i(\underline{z})}{u_i'(z_p)} < \frac{u_i(\underline{z})}{u_i'(z_p)}\varepsilon < \frac{u_j(\underline{z})}{u_j'(z_p)}\varepsilon < \frac{u_j(\bar{z})}{u_j'(z_p)} - \frac{u_j(\bar{z}-\varepsilon)}{u_j'(z_p)}.$$

This contradicts $x \succ_{\mathcal{N}} y$ and hence Axiom 3. ∎

In light of this impossibility, we will develop three alternative paths out of the impossibility by weakening one of the three requirements.

A first route is to follow Harsanyi's lead and focus on expected utility and the Pareto principle (Harsanyi, 1955). Harsanyi's theorem then implies that the social welfare ordering must be represented by a weighted sum of individual expected utilities. The key problem then is to know how the weights should be chosen. The main solution that has been proposed in the literature is *relative utilitarianism*, where individual expected utilities are normalized so that 0 corresponds to be the worst outcome and 1 to the best one. A problem however is that we consider the outcome space $\mathbb{R}_+$ where there is no well-defined best outcome. We do so to potentially cover any real problem the society may face and that would impose constraints on the value of the highest level of income people can get. More substantially, relative utilitarianism implies that the ranking of prospects may not be invariant to adding feasible options that change the best and worst outcomes for individual. We propose a way to normalize utilities that would on the contrary be invariant.

In the literature, to obtain normalizations of the expected utilities, scholars have proposed two kinds of axioms. First, some people have introduced impartiality axioms that states that it is indifferent to favor one individual or the other is the utility gains are the same (in a specific sense, see Karni, 1998; Segal, 2000). Second, other people have proposed restricted invariance axioms (Dhillon and Mertens, 1999; Sprumont, 2013). In this paper, we want to suggest an alternative principle, which is a restricted fairness principle. It states that it must always be possible to make social welfare enhancing transfers raising people above a poverty level z_p.

Axiom 4 (Pigou-Dalton transfer for the poor) *There exists a poverty level $z_p \in \mathbb{R}_+$ such that, for all $\mathcal{N} \in \mathfrak{N}$, for all $i, j \in \mathcal{N}$ and for all $z > z_p$ there exists $\varepsilon \in \mathbb{R}_+$, with $0 < \varepsilon < \frac{z-z_p}{2}$, such that, if $x, y \in \mathcal{X}$ satisfy*

1. $y_i = z_p$, $x_i = z_p + \varepsilon$, $x_j = z - \varepsilon$ *and* $y_j = z$;
2. $x_k = y_k$ *for all* $k \in (\mathcal{N} \setminus \{i, j\})$;

then $x \succ y$.

This restricted fairness principle makes it possible to characterize the following "fair utilitarian" criterion.

Proposition 2 *Under the core assumptions, for any $\mathcal{N} \in \mathfrak{N}$, the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 1, 2 and 4 if and only if there exists a poverty level $z_p \in \mathbb{R}_+$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$,*

$$\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow \mathbb{E}\left[\sum_{i \in \mathcal{N}} \frac{u_i(\mathbf{x}_i)}{u_i'(z_p)}\right] \geq \mathbb{E}\left[\sum_{i \in \mathcal{N}} \frac{u_i(\mathbf{y}_i)}{u_i'(z_p)}\right]$$

Proof. See the Appendix. ■

Fair utilitarian criteria gives equal weight to individual utilities that are normalized so that marginal utility (measured by the first derivative of the utility function) is that same at the poverty level. To the best of our knowledge, this is the first time this criterion is axiomatized.

A second route, given Prop. 1, is to completely drop the requirement of social rationality which has perhaps less ethical appealing than the other two axioms (although it has a strong appeal in terms of guaranteeing consistent social choices). In that case, it is possible to characterize a large class of fair ex ante criteria.

Proposition 3 *Under the core assumptions, for any $\mathcal{N} \in \mathfrak{N}$, the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 2 and 3 if and only if there exists a continuous, non-decreasing, Schur-concave and normalized function*[3] *$E_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|} \to \mathbb{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$,*

$$\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow E_{\mathcal{N}}\Big(\big(ce_i(\mathbf{x}_i)\big)_{i \in \mathcal{N}}\Big) \geq E_{\mathcal{N}}\Big(\big(ce_i(\mathbf{y}_i)\big)_{i \in \mathcal{N}}\Big),$$

where $\big(ce_i(\mathbf{x}_i)\big)_{i \in \mathcal{N}}$ and $\big(ce_i(\mathbf{y}_i)\big)_{i \in \mathcal{N}}$ are the vectors of certainty equivalents.

Proof. Obviously, the criterion axiomatized satisfies Axioms 2 and 3.

Assume the social ordering $\succsim_{\mathcal{N}}$ satisfies that Axioms 2 and 3. Its restriction to $\mathcal{X}_{\mathcal{N}}$ satisfies continuity (by Assumption 3), so that there exists a continuous social welfare function $W_{\mathcal{N}} : \mathcal{X}_{\mathcal{N}} \to \mathbb{R}$ such that for all $x, y \in \mathcal{X}_{\mathcal{N}}$, $x \succsim_{\mathcal{N}} y \Longleftrightarrow W_{\mathcal{N}}(x) \geq$

[3] A function $f : \mathbb{R}_+^n \to \mathbb{R}$ is Schur-concave if whenever $\tilde{z} \in \mathbb{R}_+^n$ majorizes $z \in \mathbb{R}_+^n$ we have $f(z) \geq f(\tilde{z})$. It is normalized if, for all $a \in \mathbb{R}_+$, $f(a, \cdots, a) = a$.

$W_{\mathcal{N}}(y)$. By Axiom 3, $W_{\mathcal{N}}$ must be Schur-concave (because any majorizing vector can be obtained by series of inverse Pigou-Dalton transfers). And we can define a continuous increasing function ψ such that $\psi\left(W_{\mathcal{N}}(a,\cdots,a)\right) = a$ for all $a \in \mathbb{R}_+$. Hence $E_{\mathcal{N}} := \psi \circ W_{\mathcal{N}}$ is a social welfare function representing $\succsim_{\mathcal{N}}$ on $\mathcal{X}_{\mathcal{N}}$.

Consider any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Let $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ be such that for all $i \in \mathcal{N}$ and $s \in \mathcal{S}$ $\tilde{\mathbf{x}} = ce_i\big(\mathbf{x}\big)$ and $\tilde{\mathbf{y}} = ce_i\big(\mathbf{y}\big)$. By definition of certainty equivalents, $\mathbf{x}_i \sim_i \tilde{\mathbf{x}}_i$ and $\mathbf{y}_i \sim_i \tilde{\mathbf{y}}_i$. By Assumption 3 and Axiom 2, $\mathbf{x} \succsim_{\mathbf{N}} \mathbf{y} \iff \tilde{\mathbf{x}} \succsim_{\mathbf{N}} \tilde{\mathbf{y}}$. And, by the representation of $\succsim_{\mathcal{N}}$ on $\mathcal{X}_{\mathcal{N}}$ defined above:

$$\tilde{\mathbf{x}} \succsim_{\mathcal{N}} \tilde{\mathbf{y}} \iff E_{\mathcal{N}}\Big(\big(ce_i(\mathbf{x}_i)\big)_{i\in\mathcal{N}}\Big) \geq E_{\mathcal{N}}\Big(\big(ce_i(\mathbf{y}_i)\big)_{i\in\mathcal{N}}\Big).$$

■

A possible issue with the ex ante approach is the conflict between the ex ante and ex post assessment of prospects that can imply time inconsistencies. Even in a simple case with two equally probable states of the world and two individuals with different risk preferences, the fair ex ante social welfare ordering may not satisfy the natural dominance principle that whenever a prospect is better than another in each state of the world, the former should also be preferred ex ante to the latter. If two individuals have different risk preferences it means that there exist a sure level of income a and two levels of income $\bar{a}$ and $\underline{a}$ such that individual 1 prefers the sure level a to a prospect of having each of $\bar{a}$ and $\underline{a}$ with probability 1/2, while individual 2 prefers the uncertain prospect to the sure consumption. Now the society may face the following prospects, described by matrices by matrices in which a cell gives the income of an individual in a particular state of the world (rows are for individuals and columns for states of the world):

$$\begin{pmatrix} a & a \\ \bar{a} & \underline{a} \end{pmatrix} \text{ and } \begin{pmatrix} \bar{a} & \underline{a} \\ a & a \end{pmatrix}.$$

By Pareto, the left prospect should be socially preferred. But if social decisions are fair, the two prospects have socially indifferent consequences in all states of the world, so that by dominance they should be indifferent ex ante. This simple example illustrate the potential time inconsistency of ex ante criteria. This time inconsistency issue has been discussed in the literature on social choice under risk and uncertainty following the seminal paper by Harsanyi (1955), for instance Diamond (1967) and Epstein and Segal (1992).

To avoid time inconsistency, a solution is to adopt an ex post approach instead. To do so, we need a weakened version of the Pareto principle. We follow Fleurbaey (2010) and Fleurbaey and Zuber (2015c) to propose the following principle, where unanimity is respected only when individuals share exactly the same fate in all states of the world.

Axiom 5 (Pareto for equal risk) *For all $\mathcal{N} \in \mathfrak{N}$, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^e_{\mathcal{N}}$, if $\mathbf{x}_i \succsim^{\mathcal{S}}_i \mathbf{y}_i$ for all $i \in \mathcal{N}$, then $\mathbf{x} \succsim \mathbf{y}$. If furthermore $\mathbf{x}_i \succ^{\mathcal{S}}_i \mathbf{y}_i$ for all $i \in \mathcal{N}$, then $\mathbf{x} \succ \mathbf{y}$.*

With this axiom, we obtain the following class of fair ex post criteria.

Proposition 4 *Under the core assumptions, for any $\mathcal{N} \in \mathfrak{N}$, the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 1, 3 and 5 if and only if there exist non-negative weights $(\alpha^{\mathcal{N}}_i)_{i\in\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}_+$ and a continuous, non-decreasing, Schur-concave and normalized function $E_{\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}_+ \to \mathbb{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$,*

$$\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow \sum_{i\in\mathcal{N}} \alpha^{\mathcal{N}}_i \mathbb{E}\left[u_i\Big(E_{\mathcal{N}}\big((\mathbf{x}_i)_{i\in\mathcal{N}}\big)\Big)\right] \geq \sum_{i\in\mathcal{N}} \alpha^{\mathcal{N}}_i \mathbb{E}\left[u_i\Big(E_{\mathcal{N}}\big((\mathbf{y}_i)_{i\in\mathcal{N}}\big)\Big)\right].$$

If furthermore Assumption 4 holds, then the weights $(\alpha^{\mathcal{N}}_i)_{i\in\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}_+$ are unique.

Proof. It is easily checked that the characterized class of social welfare orderings satisfy Axioms 1, 3 and 5.

Similarly to the first step of the proof of Prop. 2, it can be showed that, if the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 1 and 5, there exist non-negative weights $(\alpha^{\mathcal{N}}_i)_{i\in\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}_+$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^e_{\mathcal{N}}$,

$$\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow \sum_{i\in\mathcal{N}} \alpha^{\mathcal{N}}_i \mathbb{E}\left[u_i\Big(E_{\mathcal{N}}\big((\mathbf{x}_i)_{i\in\mathcal{N}}\big)\Big)\right] \geq \sum_{i\in\mathcal{N}} \alpha^{\mathcal{N}}_i \mathbb{E}\left[u_i\Big(E_{\mathcal{N}}\big((\mathbf{y}_i)_{i\in\mathcal{N}}\big)\Big)\right].$$

Using standard arguments about the linear independence of the utility functions (see for instance Coulhon and Mongin, 1989), Assumption 4 implies that the weights $(\alpha_i^{\mathcal{N}})_{i\in\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|}$ are unique.

Similarly to proof of Prop. 3, it can be showed that, if the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 3 and 5, there exists a continuous, non-decreasing, Schur-concave and normalized function $E_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|} \to \mathbb{R}$ such that, for all $x, y \in \mathcal{X}_{\mathcal{N}}$, $x \succsim_{\mathcal{N}} y \Longleftrightarrow E_{\mathcal{N}}(x) \geq E_{\mathcal{N}}(y)$.

For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$, define $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}_{\mathcal{N}}^e$ such that for all $s \in \mathcal{S}$ and $i \in \mathcal{N}$ $\tilde{\mathbf{x}}_i(s) = E_{\mathcal{N}}\big(\mathbf{x}(s)\big)$ and $\tilde{\mathbf{y}}_i(s) = E_{\mathcal{N}}\big(\mathbf{y}(s)\big)$. By definition, we have for all $s \in \mathcal{S}$ $\mathbf{x}(s) \sim_{\mathcal{N}} \tilde{\mathbf{x}}(s)$ and $\mathbf{y}(s) \sim_{\mathcal{N}} \tilde{\mathbf{y}}(s)$. Therefore, by Axiom 1, $\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow \tilde{\mathbf{x}} \succsim_{\mathcal{N}} \tilde{\mathbf{y}}$. But given that $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}_{\mathcal{N}}^e$:

$$\tilde{\mathbf{x}} \succsim_{\mathcal{N}} \tilde{\mathbf{y}} \Longleftrightarrow \sum_{i\in\mathcal{N}} \alpha_i^{\mathcal{N}} \mathbb{E}\left[u_i\Big(E_{\mathcal{N}}\big((\mathbf{x}_i)_{i\in\mathcal{N}}\big)\Big)\right] \geq \sum_{i\in\mathcal{N}} \alpha_i^{\mathcal{N}} \mathbb{E}\left[u_i\Big(E_{\mathcal{N}}\big((\mathbf{y}_i)_{i\in\mathcal{N}}\big)\Big)\right].$$

■

4 Fair criteria for populations of variable sizes and compositions

Until now we have assumed that the same individuals exist in all states of the world. So there is not uncertainty on the preferences or existence of people composing the population, in particular future people. In this section, we to incorporate this possibility in our model. This raises the issue of how the Pareto principle should be applied. Indeed, individuals preferences are normatively relevant only for states of the world where they exist. We thus need to distinguish two kind of risks: *individual risks* that individuals (endowed with a given fixed preference ordering) bear; and *social risks* about the composition and size of the population.

We first state the two generalized versions of the Pareto principles, where individual preferences are respected whenever individuals live in the same states of the world in the two prospects that are being compared.

Axiom 6 (Generalized Pareto) *For all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$*, if* $A_i(\mathbf{x}) = A_i(\mathbf{y}) = A_i$ *for all* $i \in \mathbb{N}$*, and if* $\mathbf{x}_i \succsim_i^{A_i} \mathbf{y}_i$ *for all* $i \in \mathbb{N}$ *such that* $P(A_i) > 0$*, then* $\mathbf{x} \succsim \mathbf{y}$*. If furthermore* $\mathbf{x}_i \succ_i^{A_i} \mathbf{y}_i$ *for all* $i \in \mathbb{N}$ *such that* $P(A_i) > 0$*, then* $\mathbf{x} \succ \mathbf{y}$*.*

Axiom 7 (Generalized Pareto for equal risk) *For all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}^e$*, if* $A_i(\mathbf{x}) = A_i(\mathbf{y}) = A_i$ *for all* $i \in \mathbb{N}$*, and if* $\mathbf{x}_i \succsim_i^{A_i} \mathbf{y}_i$ *for all* $i \in \mathbb{N}$ *such that* $P(A_i) > 0$*, then* $\mathbf{x} \succsim \mathbf{y}$*. If furthermore* $\mathbf{x}_i \succ_i^{A_i} \mathbf{y}_i$ *for all* $i \in \mathbb{N}$ *such that* $P(A_i) > 0$*, then* $\mathbf{x} \succ \mathbf{y}$*.*

A key feature of intergenerational problems, as highlighted in the introduction, is that we have compare populations with different and uncertain size and composition. The next two axioms describe how population of different sizes and composition are being compared. The first axiom introduce the notion of a critical-level of income such that adding a person with a this level is a matter of social indifference. The main novelty is that this level is expressed in terms of resource and not in term of welfare.

Axiom 8 (Critical-level) *There exists* $z_c \in \mathbb{R}_{++}$ *such that for all* $x, y \in \mathcal{X}$*, if* $\mathcal{N}(y) = \mathcal{N}(x) \cup \{j\}$ *(with* $j \in (\mathbb{N} \setminus \mathcal{N}(x))$*),* $y_i = x_i = z$ *for all* $i \in \mathcal{N}(x)$ *and* $x_j = z$ *then* $x \sim y$*.*

Remark that it is a weak critical-level assumption: we do not require the critical-level to be the same for any possible allocation, but only that there is a level a which population expansion is indifferent.

The second axiom compares populations of same size facing an egalitarian (aggregate) prospect. The axiom states that replacing an individual by a more risk averse one reduces social welfare. The reason why this should be so is that the more risk averse individual always has a lower certainty-equivalent, and is therefore willing to pay more to avoid the aggregate risk on income. The distribution of certainty-equivalents is thus worse (in terms of first order stochastic dominance), so that society may be deemed worse-off in risky situation.

Axiom 9 (Dominance of more risk averse populations) *For all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}^e$*, if there exists* $\mathcal{M} \in \mathfrak{N}$ *and* $i, j \in (\mathbb{N} \setminus \mathcal{M})$ *such that* $\mathcal{N}\big(\mathbf{x}(s)\big) = \mathcal{M} \cup \{i\}$ *and*

$\mathcal{N}\big(\mathbf{y}(s)\big) = \mathcal{N} \cup \{j\}$ *for all* $s \in \mathcal{S}$*, there exists a continuous, increasing and concave function* ψ *such that* $u_j = \psi \circ u_i$*, and* $\mathbf{x}_i(s) = \mathbf{y}_i(s)$ *for all* $i \in \mathcal{M}$ *and* $s \in \mathcal{S}$*, then* $\mathbf{x} \succ \mathbf{y}$.

It is then possible to characterize a class of fair utilitarian intergenerational criteria.

Proposition 5 *Under the core assumptions and Assumption 5, the social ordering* $\succsim$ *satisfies Axioms 1, 6, 8 and 9 if and only if there exists a critical level* $z_c \in \mathbb{R}_+$ *such that, for all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \sum_{i \in \mathbb{N}} p_i(\mathbf{x}) \mathbb{E}\left[\frac{u_i(\mathbf{x}_i) - u_i(z_c)}{u_i'(z_c)}\right] \geq \sum_{i \in \mathbb{N}} p_i(\mathbf{y}) \mathbb{E}\left[\frac{u_i(\mathbf{y}_i) - u_i(z_c)}{u_i'(z_c)}\right]$$

Proof. See the Appendix. ■

A first noticeable consequence of Prop. 5 is that fair utilitarian intergenerational criteria satisfy Axiom 4 (Pigou-Dalton transfer for the poor), although it is not an assumption of the characterization. Indeed, their restrictions to same population prospect correspond to the criteria characterized in Prop. 2. In addition, the income poverty level is shown to be exactly equal to the critical-level.

A second implication of Prop. 5 is that fair utilitarian intergenerational criteria satisfy the following Axiom of existence independence.

Axiom 10 (Existence Independence) *For all* $\mathcal{M} \in \mathfrak{N}$ *and for all* $\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{y}, \tilde{\mathbf{y}} \in \mathbf{X}$*, if*
$A_i(\mathbf{x}) = A_i(\mathbf{y})$ *and* $A_i(\tilde{\mathbf{x}}) = A_i(\tilde{\mathbf{y}}) = \emptyset$ *for all* $i \in \mathcal{M}$,
$A_i(\mathbf{x}) = A_i(\tilde{\mathbf{x}})$ *and* $A_i(\mathbf{y}) = A_i(\tilde{\mathbf{y}})$ *for all* $i \in \mathbb{N} \setminus \mathcal{M}$,
$\mathbf{x}_i(s) = \mathbf{y}_i(s)$ *for all* $i \in \mathcal{M}$ *and* $s \in A_i(\mathbf{x})$,
$\mathbf{x}_j(s) = \tilde{\mathbf{x}}_j(s)$ *for all* $j \in \mathbb{N} \setminus \mathcal{M}$ *and* $s \in A_j(\mathbf{x})$,
$\mathbf{y}_j(s) = \tilde{\mathbf{y}}_j(s)$ *for all* $j \in \mathbb{N} \setminus \mathcal{M}$ *and* $s \in A_j(\mathbf{x})$,
then $x \succsim y \Longleftrightarrow \tilde{x} \succsim \tilde{y}$.

Existence independence asserts that the existence of unconcerned people should not affect the social ranking. In particular, it implies the existence independence of the dead (Blackorby, Bossert and Donaldson, 2005), making the social judgement independent of the welfare level and number of past people. This property can be viewed as a property of informational parsimony: we do not need precise knowledge of what happen in the past to make social choices now. It also simplifies dynamic planning by making future choices independent of the welfare of people that could have existed, but did not, making time consistent choice easier.

If we drop the strong Pareto requirement, we can obtain the class of fair ex post intergenerational social orderings.

Proposition 6 *Under the core assumptions and Assumptions 4 and 5, the social ordering* $\succsim$ *satisfies Axioms 1, 3, 7, 8 and 9 if and only if there exists a critical level* $z_c \in \mathbb{R}_+$ *and, for each* $\mathcal{N} \in \mathfrak{N}$*, a continuous, non-decreasing, Schur-concave and normalized function* $E_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|} \to \mathbb{R}$ *such that, for all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$*,*

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \mathbb{E}\left[\sum_{i \in \mathcal{N}(\mathbf{x})} \frac{u_i\big(E_{\mathcal{N}}(\mathbf{x})\big) - u_i(z_c)}{u_i'(z_c)}\right] \geq \mathbb{E}\left[\sum_{i \in \mathcal{N}(\mathbf{y})} \frac{u_i\big(E_{\mathcal{N}}(\mathbf{y})\big) - u_i(z_c)}{u_i'(z_c)}\right]$$

Proof. It is easily checked that the characterized class of social welfare orderings satisfy Axioms 1, 3, 7, 8 and 9.

Using similar reasoning as in the the proof of Prop. 5, it can be showed that, if the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 1, 7, 8 and 9, there exists a critical level $z_c \in \mathbb{R}_+$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^e$,

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \sum_{i \in \mathbb{N}} p_i(\mathbf{x}) \mathbb{E}\left[\frac{u_i(\mathbf{x}_i) - u_i(z_c)}{u_i'(z_c)}\right] \geq \sum_{i \in \mathbb{N}} p_i(\mathbf{y}) \mathbb{E}\left[\frac{u_i(\mathbf{y}_i) - u_i(z_c)}{u_i'(z_c)}\right]$$

Similarly to proof of Prop. 3, it can be showed that, if the social ordering $\succsim_{\mathcal{N}}$ satisfies Axioms 3 and 5, there exists a continuous, non-decreasing, Schur-concave and normalized function $E_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|} \to \mathbb{R}$ such that, for all $x, y \in \mathcal{X}_{\mathcal{N}}$, $x \succsim_{\mathcal{N}} y \Longleftrightarrow E_{\mathcal{N}}(x) \geq E_{\mathcal{N}}(y)$.

For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, define $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}^e$ such that for all $s \in \mathcal{S}$ and $i \in \mathcal{N}(\mathbf{x}(s))$ $\tilde{\mathbf{x}}_i(s) = E_{\mathcal{N}}\big(\mathbf{x}(s)\big)$ and for all $s \in \mathcal{S}$ and $i \in \mathcal{N}(\mathbf{y}(s))$ $\tilde{\mathbf{y}}_i(s) = E_{\mathcal{N}}\big(\mathbf{y}(s)\big)$. By definition, we have for all $s \in \mathcal{S}$ $\mathbf{x}(s) \sim_{\mathcal{N}} \tilde{\mathbf{x}}(s)$ and $\mathbf{y}(s) \sim_{\mathcal{N}} \tilde{\mathbf{y}}(s)$. Therefore, by

Axiom 1, $\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \tilde{\mathbf{x}} \succsim \tilde{\mathbf{y}}$. But given that $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}^e$:

$$\tilde{\mathbf{x}} \succsim \tilde{\mathbf{y}} \Longleftrightarrow \mathbb{E}\left[\sum_{i \in \mathcal{N}(\mathbf{x})} \frac{u_i\big(E_{\mathcal{N}}(\mathbf{x})\big) - u_i(z_c)}{u_i'(z_c)}\right] \geq \mathbb{E}\left[\sum_{i \in \mathcal{N}(\mathbf{y})} \frac{u_i\big(E_{\mathcal{N}}(\mathbf{y})\big) - u_i(z_c)}{u_i'(z_c)}\right].$$

■

In addition to not satisfying the Pareto principle in full force, fair ex post intergenerational social orderings display a lot of non separability: the social assessment depends on the risk preferences of all people eventually affected, and in particular of all past generations who have already been affected. If we want to avoid these complications, a solution if to turn to a fair ex ante approach.

The problem of the fair ex ante approach in the case of prospects involving a fixed population is that it does not satisfy the expected utility hypothesis, and often not even a dominance property. We would like to avoid as much as possible deviations from the idea of social rationality. To this end, we introduce the following axiom.

Axiom 11 (Restricted social expected utility hypothesis) *There exists a continuous function $\tilde{U} : \mathcal{X} \to \mathbb{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if for all $i \in \mathbb{N}$ $\mathbf{x}_i(s) = \mathbf{x}_i(s')$ for all $s, s' \in A_i(\mathbf{x})$ and $\mathbf{y}_i(s) = \mathbf{y}_i(s')$ for all $s, s' \in A_i(\mathbf{y})$, then:*

$$\mathbf{x} R \mathbf{y} \Longleftrightarrow \mathbb{E}\big(\tilde{U}(\mathbf{x})\big) \geq \mathbb{E}\big(\tilde{U}(\mathbf{y})\big)$$

In the above Axiom, the prospects $\mathbf{x}$ and $\mathbf{y}$ are such that individuals do not bear risk (the receive the same allocation in states of the world where they exist). We are thus in a situation of social risk where the composition of the population may be uncertain, but this not individual outcomes when individuals exist. This permits to avoid a conflict with the Generalized Pareto principle as shown by the following proposition.

Proposition 7 *Under the core assumptions, the social ordering $\succsim$ satisfies Axioms 3, 6, 8, 10 and 11 if and only if there exists a critical level $z_c \in \mathbb{R}_+$ and an increasing continuous and concave function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that for all*

$\mathbf{x}, \mathbf{y} \in \mathbf{X}$:

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \sum_{i \in \mathbb{N}} p_i(\mathbf{x}) \Big[\phi\big(ce_i(\mathbf{x}_i)\big) - \phi(z_c) \Big] \geq \sum_{i \in \mathbb{N}} p_i(\mathbf{y}) \Big[\phi\big(ce_i(\mathbf{y}_i)\big) - \phi(z_c) \Big].$$

Proof. See the Appendix. ■

Note that the criteria axiomatized in Prop. 7 also satisfy the dominance of more risk averse populations (Axiom 9). Indeed, the use certainty-equivalents as a welfare metrics of individual prospects, so that more risk populations are always worse-off in risky situations.

5 Conclusion

The present paper contributes to the fields of intergenerational fairness, population ethics and social evaluation in risky situations by proposing fair ex ante and ex post methods of social choice, as well as a fair utilitarian approach. By doing so we have taken an important step towards preparing fair social choice criteria (for practical use, e.g. for evaluation of climate policies and other policy issues with long-run consequences.

With respect to most of the existing literature, the paper makes two significant contributions. A first contribution explicitly derive welfare metrics from axioms, and state how individuals' wellbeing can be measured and compared when people have different risk preferences. The literature often escapes the question by assuming that all individuals have the same preferences. Doing so, it ignores the fact that the cardinalization of the utilities is itself important, and it ignores important aspects of realistic problems of environmental and climate problems, including the issues of preference diversity and preference change.

A second contribution has been to explicitly model and account for the uncertainty about future generations preferences. Climate policy have impacts that span many generations, and it is clearly difficult to now for sure what future generations will be. Being able to include this uncertainty in the design of policies would be an improvement if we want to satisfy future generations needs.

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Appendix

Proof of Proposition 2

Step 1: The social VNM function is an affine combination of individuals ones.

We first show that social choices on $\mathbf{X}_\mathcal{N}$ can be translated as choices on the set of simple lotteries over $\mathcal{X}_\mathcal{N}$. To do so, define $\Delta(\mathcal{X}_\mathcal{N})$ the set of simple lotteries, that is mapping $p : \mathcal{X}_\mathcal{N} \to [0,1]$ such that $\sum_{x\in\mathcal{X}_\mathcal{N}} p(x) = 1$ and $p(x) > 0$ for a finite number of elements in $\mathcal{X}_\mathcal{N}$.

First remark that there is perfect correspondence between the two frameworks. By assumption, prospects in $\mathbf{X}_\mathcal{N}$ only induce a finite number of realized allocations, in the sense that there exists a finite partition $(A_1, \cdots, A_m)$ for which, whatever $k = 1, \cdots, m$, $\mathbf{x}(s) = \mathbf{x}(s') \neq \mathbf{x}(s'')$ for all $s, s' \in A_k$ and $s'' \in (\mathcal{S} \setminus A_k)$. So the mapping $p : \mathcal{X}_\mathcal{N} \to [0,1]$ such that $p(x) = P(A_k)$ whenever $\mathbf{x}(s) = x$ for any $s \in A_k$ and $p(y) = 0$ when there exists no l and $s' \in A_l$ for which $\mathbf{x}(s') = y$ is a well-defined lottery. Conversely, for any lottery $p \in \Delta(\mathcal{X}_\mathcal{N})$, there are m elements $(x_1, \cdots, x_k) \in \mathcal{X}_\mathcal{N}$ such that $p(x_k) > 0$ and $p(x_1) + \cdots + p(x_m) = 1$. By Assumption 1, we can find a measurable partition $(A_1, \cdots, A_m)$ such that $P(A_k) = p(x_k)$ for each $k = 1, \cdots, m$. So the mapping $\mathbf{x} : \mathcal{S} \to \mathcal{X}_\mathcal{N}$ such that for $k = 1, \cdots, m$ and all $s \in A_k$ $\mathbf{x}(s) = x_k$ is a well-defined prospect. For any $p \in \Delta(\mathcal{X}_\mathcal{N})$, we denote $\mathbf{x}^p \in \mathbf{X}_\mathcal{N}$ the associated prospect.

Hence, we can define the following orderings on $\Delta(\mathcal{X}_\mathcal{N})$:

- For all $i \in \mathcal{N}$, for all $p, q \in \Delta(\mathcal{X}_\mathcal{N})$,

$$p \widetilde{\succsim}_i q \iff V_i(p) = \mathbb{E}\big(u_i(\mathbf{x}_i^p)\big) \geq \mathbb{E}\big(u_i(\mathbf{x}_i^q)\big) = V_i(q).$$

- for all $p, q \in \Delta(\mathcal{X}_\mathcal{N})$,

$$p \widetilde{\succsim}_0 q \iff V_u(p) = \mathbb{E}\big(U(\mathbf{x}_i^p)\big) \geq \mathbb{E}\big(U(\mathbf{x}_i^q)\big) = V_i(q).$$

Clearly, because of Assumption 2 and Axiom 1, we have $V_i\big(\lambda p + (1-\lambda) q\big) =$

$\lambda V_i(p) + (1-\lambda)V_i(q)$. Using the Pareto principle (Axiom 2), we know by Proposition 2 in Coulhon and Mongin (1989) that there exist real numbers $\left(\alpha_i^{\mathcal{N}}\right)_{i\in N}$ and β such that

$$V_0(p) = \sum_{i\in\mathcal{N}} \alpha_i^{\mathcal{N}} V_i(p) + \beta.$$

Because we use a Weak Pareto Axiom, we must have $(\alpha_i \alpha_i^{\mathcal{N}})_{i\in\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|}$.

Given the correspondence between preferences over prospects and preferences over lotteries, this implies that:

$$U(x) = \sum_{i\in N} \alpha_i^{\mathcal{N}} u_i(x_i) + \beta,$$

where U is the function used to compute the social expected utility in Assumption 1. So, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$,

$$\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow \mathbb{E}\left[\sum_{i\in\mathcal{N}} \alpha_i^{\mathcal{N}} u_i(\mathbf{x}_i)\right] \geq \mathbb{E}\left[\sum_{i\in\mathcal{N}} \alpha_i^{\mathcal{N}} u_i(\mathbf{y}_i)\right].$$

Step 2: Characterizing the weights on utilities.

Assume now that there are two individuals $i, j \in \mathcal{N}$ such that $\alpha_i^{\mathcal{N}} u_i'(z_p) < \alpha_j^{\mathcal{N}} u_j'(z_p)$, where z_p is the poverty level defined in Axiom 4. By continuity of the derivative functions u_i' this implies that there exist $z > z_p$ such that $u_j'(z) > u_i(z_p)$. For any $\varepsilon \in \mathbb{R}_+$ such $0 < \varepsilon < \frac{z-z_p}{2}$, define $x, y \in \mathcal{X}$ in the following way:

1. $y_i = z_p$, $x_i = z_p + \varepsilon$, $x_j = z - \varepsilon$ and $y_j = z$;
2. $x_k = y_k$ for all $k \in (\mathcal{N} \setminus \{i,j\})$.

By the above results, $x \succ_{\mathcal{N}} y \Longleftrightarrow \alpha_i^{\mathcal{N}} u_i(z_p+\varepsilon) + \alpha_j^{\mathcal{N}} u_j(z-\varepsilon) > \alpha_i^{\mathcal{N}} u_i(z_p) + \alpha_j^{\mathcal{N}} u_j(z)$. But, by concavity of the functions u_i and u_j, we have $u_i(z_p+\varepsilon) - u_i(z_p) < u_i'(z_p)\varepsilon$ and $u_j(z) - u_j(z-\varepsilon) > u_j'(z)\varepsilon$. We thus have

$$u_i(z_p+\varepsilon) - u_i(z_p) < u_i'(z_p)\varepsilon < u_j'(z)\varepsilon < u_j(z) - u_j(z-\varepsilon)$$

this contradicts $x \succ_{\mathcal{N}} y$ and hence Axiom 4.

Therefore, for all $i \in \mathcal{N}$, we must have $\alpha_i^{\mathcal{N}} u_i'(z_p) = \kappa$, where κ is a strictly positive real number. This implies that $\alpha_i^{\mathcal{N}} = \frac{\kappa}{u_i'(z_p)}$ for all $i \in \mathcal{N}$.

Proof of Prop. 5

It is easily checked that the characterized class of social welfare orderings satisfy Axioms 1, 6, 8 and 9 .

Assume that the social ordering $\succsim$ satisfies Axioms 1, 6, 8 and 9. By the first step of the proof of Prop. 2, 1 and 6 imply that for any $\mathcal{N} \in \mathfrak{K}$, there exist weights $(\alpha_i \alpha_i^{\mathcal{N}})_{i\in\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|}$, such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$,

$$\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow \mathbb{E}\left[\sum_{i\in\mathcal{N}} \alpha_i^{\mathcal{N}} u_i(\mathbf{x}_i)\right] \geq \mathbb{E}\left[\sum_{i\in\mathcal{N}} \alpha_i^{\mathcal{N}} u_i(\mathbf{y}_i)\right].$$

Given that individual outcomes are independent, the functions u_i over the prospects $\mathbf{X}_{\mathcal{N}}$ are linearly independent so that weights are unique.

By Axioms 1, it is also the case that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathcal{N}}$, $\mathbf{x} \succsim_{\mathcal{N}} \mathbf{y} \Longleftrightarrow \mathbb{E}\Big[U(\mathbf{x})\Big] \geq \mathbb{E}\Big[U(\mathbf{x})\Big]$. Given that VNM utility functions are unique up to an increasing affine transformation, there exist $a_{\mathcal{N}} \in \mathbb{R}_{++}$ and $b_{\mathcal{N}} \in \mathbb{R}$ such that, for all $x \in \mathcal{X}$:

$$U(x) = b_{\mathcal{N}(x)} + \sum_{i\in\mathcal{N}(x)} \tilde{\alpha}_i^{\mathcal{N}(x)} u_i(x_i),$$

where $\tilde{\alpha}_i^{\mathcal{N}(x)} = \Gamma_{\mathcal{N}(x)} \times \alpha_i^{\mathcal{N}(x)}$.

Now consider any $i \in \mathbb{N}$ and any $\mathcal{N} \in \mathfrak{N}$ such that $i \in \mathcal{N}$. Let $A, B, C \in \Sigma$ such that A, B, C form a partition of $\mathcal{S}$ and $P(A) = P(B) = P(C) = 1/3$ (it is possible to find such sets by Assumption 1). For any $z_1, z_2, z_3, z_4 \in \mathbb{R}_+$, define $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that:[4]

For all $s \in A$, $\mathcal{N}(\mathbf{x}(s)) = \mathcal{N}(\mathbf{y}(s)) = \mathcal{N}$, $\mathbf{x}_j(s) = z_1$ and $\mathbf{y}_j(s) = z_3$ for all $j \in \mathcal{N}$;

For all $s \in B$, $\mathcal{N}(\mathbf{x}(s)) = \mathcal{N}(\mathbf{y}(s)) = \{i\}$, $\mathbf{x}_i(s) = z_2$ and $\mathbf{y}_i(s) = z_4$;

[4]Remark that $\mathbf{x}, \mathbf{y}$ are egalitarian prospects so that reasoning can be used for the Fair ex post case.

For all $s \in C$, $\mathcal{N}(\mathbf{x}(s)) = \mathcal{N}(\mathbf{y}(s)) = \mathcal{N} \setminus \{i\}$, $\mathbf{x}_j(s) = z_3$ and $\mathbf{y}_j(s) = z_1$ for all $j \in (\mathcal{N} \setminus \{i\})$.

Denote $\tilde{\alpha}_i := \tilde{\alpha}_i\{i\}$. By Axiom 1 and the results above, $\mathbf{x} \succsim \mathbf{y} \iff \frac{1}{3}\tilde{\alpha}_i u_i(z_1) + \frac{1}{3}\tilde{\alpha}_i^{\mathcal{N}} u_i(z_2) \geq \frac{1}{3}\tilde{\alpha}_i u_i(z_3) + \frac{1}{3}\tilde{\alpha}_i^{\mathcal{N}} u_i(z_4)$. But by Axiom 6, it is also the case that $\mathbf{x} \succsim \mathbf{y} \iff \frac{1}{2}u_i(z_1) + \frac{1}{2}u_i(z_2) \geq \frac{1}{2}u_i(z_3) + \frac{1}{2}u_i(z_4)$, because other individuals than $\{i\}$ are indifferent (they face a fifty-fifty chance of receiving either z_1 or z_3). This is true for any $z_1, z_2, z_3, z_4 \in \mathbb{R}_+$, so there must exist a continuous increasing function Ψ such that for all $z_1, z_2 \in \mathbb{R}_+$:

$$\frac{1}{3}\tilde{\alpha}_i u_i(z_1) + \frac{1}{3}\tilde{\alpha}_i^{\mathcal{N}} u_i(z_2) = \Psi\Big(\frac{1}{2}u_i(z_1) + \frac{1}{2}u_i(z_2)\Big).$$

For this to be the case, we necessarily need $\tilde{\alpha}_i = \tilde{\alpha}_i^{\mathcal{N}}$.

To sum up, there exist $(a_i)_{i\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ and $b_{\mathcal{N}} \in \mathbb{R}$ such that, for all $x \in \mathcal{X}$:

$$U(x) = b_{\mathcal{N}(x)} + \sum_{i \in \mathcal{N}(x)} \tilde{\alpha}_i u_i(x_i),$$

is a representation of preferences for allocations.

By Axiom 8, for any $i \in \mathbb{N}$, we have $\tilde{\alpha}_i u_i(z_c) + b_{\{i\}} = \tilde{\alpha}_i u_i(z_c) + \tilde{\alpha}_1 u_1(z_c) + b_{\{1,i\}}$, where z_c is the critical level mentioned in the axiom. But we also have $\tilde{\alpha}_1 u_1(z_c) + b_{\{1\}} = \tilde{\alpha}_i u_i(z_c) + \tilde{\alpha}_1 u_1(z_c) + b_{\{1,i\}}$, and thus $\tilde{\alpha}_i u_i(z_c) + b_{\{i\}} = \tilde{\alpha}_1 u_1(z_c) + b_{\{1\}}$. Also, for any $\mathcal{M} \in \mathfrak{N}$ and any $j \notin \mathcal{M}$, $\sum_{i\in\mathcal{M}} \tilde{\alpha}_i u_i(z_c) + b_{\mathcal{M}} = \sum_{i\in\mathcal{M}} \tilde{\alpha}_i u_i(z_c) + \tilde{\alpha}_j u_j(z_c) + b_{\mathcal{M}\cup\{j\}}$. Hence, by induction we obtain that there exists $\Lambda \in \mathbb{R}$ such that, for all $\mathcal{L} \in \mathfrak{N}$ $\sum_{i\in\mathcal{L}} \tilde{\alpha}_i u_i(z_c) + b_{\mathcal{L}} = \Lambda$. Thus for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\begin{aligned}
\mathbf{x} \succsim \mathbf{y} &\iff \mathbb{E}\Big[U(\mathbf{x})\Big] \geq \mathbb{E}\Big[U(\mathbf{x})\Big] \\
&\iff \mathbb{E}\Big[b_{\mathcal{N}(\mathbf{x})} + \sum_{i\in\mathcal{N}(\mathbf{x})} \tilde{\alpha}_i u_i(\mathbf{x}_i)\Big] \geq \mathbb{E}\Big[b_{\mathcal{N}(\mathbf{y})} + \sum_{i\in\mathcal{N}(\mathbf{y})} \tilde{\alpha}_i u_i(\mathbf{y}_i)\Big] \\
&\iff \mathbb{E}\Big[b_{\mathcal{N}(\mathbf{x})} + \sum_{i\in\mathcal{N}(\mathbf{x})} \tilde{\alpha}_i u_i(\mathbf{x}_i)\Big] - \Lambda \geq \mathbb{E}\Big[b_{\mathcal{N}(\mathbf{y})} + \sum_{i\in\mathcal{N}(\mathbf{y})} \tilde{\alpha}_i u_i(\mathbf{y}_i)\Big] - \Lambda \\
&\iff \mathbb{E}\Big[\sum_{i\in\mathcal{N}(\mathbf{x})} \tilde{\alpha}_i\big(u_i(\mathbf{x}_i) - u_i(z_c)\big)\Big] \geq \mathbb{E}\Big[\sum_{i\in\mathcal{N}(\mathbf{y})} \tilde{\alpha}_i\big(u_i(\mathbf{y}_i) - u_i(z_c)\big)\Big]
\end{aligned}$$

Let i be the individual in Assumption **??** and let $j \in \mathbb{N}$ be any other individual. Let $\mathcal{M} \in \mathfrak{N}$ be such that $i \notin \mathcal{M}$ and $j \notin \mathcal{M}$. Assume that $\alpha_i u_i'(z_c) \neq \alpha_j u_j'(z_c)$. Let $x, y \in \mathcal{X}^e$ be such $\mathcal{N}\big(x\big) = \mathcal{M} \cup \{i\}$ $\mathcal{N}\big(y\big) = \mathcal{N} \cup \{j\}$, and $x_i = y_i = z$ for all $i \in \mathcal{M}$. Then, $x \succsim y \Longleftrightarrow \tilde{\alpha}_i u_i(z) \geq \tilde{\alpha}_j u_j(z)$.

Assume that $\tilde{\alpha}_i u_i'(z_c) > \tilde{\alpha}_j u_j'(z_c)$. Then there exists $z > z_c$ (with z close enough to z_c) such that $\tilde{\alpha}_i u_i(z) > \tilde{\alpha}_j u_j(z)$ because $\tilde{\alpha}_i u_i(z_c) = \tilde{\alpha}_j u_j(z_c)$. Similarly, there exists $z < z_c$ such that $\tilde{\alpha}_i u_i(z) < \tilde{\alpha}_j u_j(z)$. Conversely, if $\tilde{\alpha}_i u_i'(z_c) < \tilde{\alpha}_j u_j'(z_c)$, then there exists $z > z_c$ such that $\tilde{\alpha}_i u_i(z) < \tilde{\alpha}_j u_j(z)$, and there exists $z < z_c$ such that $\tilde{\alpha}_i u_i(z) > \tilde{\alpha}_j u_j(z)$. But this constitutes a violation of Axiom 9 because, by definition, there exists a continuous, increasing and concave function ψ such that either $u_j = \psi \circ u_i$ or $u_i = \psi \circ u_j$.

Denote $Z = \tilde{\alpha}_i u_i'(z_c)$, we must have, for all $j \in \mathbb{N}$, $Z = \tilde{\alpha}_j u_j'(z_c)$ and thus $\tilde{\alpha}_j = \frac{Z}{u_j'(z_c)}$

Lemma 1

Lemma 1 *Under the core assumptions, the social ordering $\succsim$ satisfies Axioms 4, 8 and 10 if and only if there exists a continuous increasing and concave function $\phi : \mathbb{R}_+ \to \mathbb{R}$ and a positive real number $z_c \in \mathbb{R}_{++}$ such that, for all $x, y \in \mathcal{X}$*

$$x \succsim y \Longleftrightarrow \sum_{i \in \mathcal{N}(x)} \Big(\phi(x_i) - \phi(z_c)\Big) \geq \sum_{i \in \mathcal{N}(y)} \Big(\phi(y_i) - \phi(z_c)\Big)$$

Proof.

Step 1: The social ordering $\succsim$ restricted to $\mathcal{X}_\mathcal{N}$ admits a symmetric additively separable representation.

For any $\mathcal{N} \in \mathfrak{N}$ such that $|\mathcal{N}| \geq 3$, the restriction of the relation $\succsim$ on $\mathbf{X}_\mathcal{N}$ is transitive, reflexive, complete, increasing and continuous (Assumption 3). By Axiom 10, any subset of $\mathcal{N}$ is separable. Therefore, by Theorem 3 in Debreu (1960), as $|\mathcal{N}| \geq 3$, there exist continuous and increasing functions $\phi_\mathcal{N}^i$ such that, for all $x, y \in \mathcal{X}_\mathcal{N}$,

$$x \succsim y \Longleftrightarrow \sum_{i \in \mathcal{N}} \phi_\mathcal{N}^i(x_i) \geq \sum_{i \in \mathcal{N}} \phi_\mathcal{N}^i(x_i). \tag{2}$$

By Lemma 2 in Fleurbaey and Zuber (2013), because the social ordering $\succsim$ restricted to $\mathcal{X}_{\mathcal{N}}$ satisfies the Pigou-Dalton transfer principle (Axiom 3), it must be the case that the $\phi^i_{\mathcal{N}}$ functions in Eq. (2) are all identical and concave. Hence there exists a continuous, increasing and concave function $\phi_{\mathcal{N}}$ such that, for any $x, y \in \mathcal{X}_{\mathcal{N}}$:

$$x \succsim y \Longleftrightarrow \sum_{i\in\mathcal{N}} \phi_{\mathcal{N}}(x^1_i) \geq \sum_{i\in\mathcal{N}} \phi_{\mathcal{N}}(y^1_i). \tag{3}$$

Step 2: The function $\phi_{\mathcal{N}}$ in Eq. (3) does not depend on $\mathcal{N}$.

Denote $\mathcal{N}_0 \in \mathfrak{N}$ the set $\mathcal{N}_0 = \{1, 2, 3\}$. Consider any $\mathcal{N} \in \mathfrak{N}$ such that $|\mathcal{N}| \geq 3$ and let $\mathcal{M} \in \mathcal{N}$ be any set such that $|\mathcal{M}| = 3$. Let $\pi : \mathcal{N}_0 \to \mathcal{M}$ be a bijection. Let $x, y, x', y', \bar{x}, \bar{y}, \widetilde{x}, \widetilde{y} \in \mathcal{X}$ be as follows:

1. $\mathcal{N}(x) = \mathcal{N}(y) = \mathcal{N}(x') = \mathcal{N}(x') = \mathcal{N}_0 \cup \mathcal{N}$, $\mathcal{N}(\bar{x}) = \mathcal{N}(\bar{y}) = \mathcal{N}_0$, $\mathcal{N}(\widetilde{x}) = \mathcal{N}(\widetilde{y}) = \mathcal{N}$;

2. $x_i = \bar{x}_i$ and $y_i = \bar{y}_i$ for all $i \in \mathcal{N}_0$;

3. $x'_i = \widetilde{x}_i$ and $y'_i = \widetilde{y}_i$ for all $i \in \mathcal{N}$;

4. $x_i = y_i$ for all $i \in \mathcal{N}$ and $x'_i = y'_i$ for all $i \in (\mathcal{N} \setminus \mathcal{M}) \cup \mathcal{N}_0$.

5. $x_i = x'_j$, $y_i = y'_j$, $x'_i = x_j$ and $y'_i = y_j$ for all $i \in \mathcal{N}_0$ and $j \in \mathcal{M}$ such that $j = \pi(i)$; and $x_k = x'_k$, $y_k = y'_k$, for all $k \in \mathcal{N} \setminus \mathcal{M}$.

Hence x' is just a permutation of elements in x, y' a permutation of elements in y. Thus, by the (symmetric) representation in Eq. (3), we know that $x \succsim y \Longleftrightarrow x' \succsim y'$. In addition, $x, y, \bar{x}, \bar{y}$ satisfy the conditions of Axiom 10, so that $x \succsim y \Longleftrightarrow \bar{x} \succsim \bar{y}$. Similarly, $x', y', \widetilde{x}, \widetilde{y}$ satisfy the conditions of Axiom 10, so that $x' \succsim y' \Longleftrightarrow \widetilde{x} \succsim \widetilde{y}$.

So, in the end, we have that $\bar{x} \succsim \bar{y} \Longleftrightarrow \widetilde{x} \succsim \widetilde{y}$. Using Eq. (3) and the construction of $\bar{x}, \bar{y}, \widetilde{x}, \widetilde{y}$, this implies that:

$$\begin{aligned} \sum_{i\in\mathcal{N}_0} \phi_{\mathcal{N}_0}(x_i) \geq \sum_{i\in\mathcal{N}_0} \phi_{\mathcal{N}_0}(y_i) \quad &\Longleftrightarrow \quad \sum_{i\in\mathcal{M}} \phi_{\mathcal{N}}(x'_i) \geq \sum_{i\in\mathcal{M}} \phi_{\mathcal{N}}(y'_i) \\ &\Longleftrightarrow \quad \sum_{i\in\mathcal{N}_0} \phi_{\mathcal{N}}(x_i) \geq \sum_{i\in\mathcal{N}_0} \phi_{\mathcal{N}}(y_i). \end{aligned}$$

By Debreu (1960, Theorem 3), we know that additively separable representations of an ordering over a product of connected separable sets are unique up to positive affine transformations. Hence, there must exist $a_{\mathcal{N}} \in \mathbb{R}_+$ and $b_{\mathcal{N}} \in \mathbb{R}$ such that $\phi_{\mathcal{N}}(x) = a_{\mathcal{N}}\phi_{\mathcal{N}_0}(z) + b_{\mathcal{N}}$ for all $z \in \mathbb{R}$.

Denoting $\phi := \phi_{\mathcal{N}_0}$, by Eq. (3) we therefore obtain that, for any $\mathcal{N} \in \mathfrak{N}$ such that $|\mathcal{N}| \geq 3$, and for all $x, y \in \mathcal{X}_{\mathcal{N}}$:

$$x \succsim y \Longleftrightarrow \sum_{i \in \mathcal{N}} \phi(x_i) \geq \sum_{i \in \mathcal{N}} \phi(y_i). \tag{4}$$

If $|\mathcal{N}| < 3$, Axiom 10 yields a similar result. Indeed consider any $\mathcal{N} \in \mathfrak{N}$ such that $|\mathcal{N}| < 3$, and any $x, y \in \mathcal{X}_{\mathcal{N}}$. Then take any $\mathcal{M} \in \mathfrak{N}$ such that $\mathcal{N} \subset \mathcal{M}$ and construct $\bar{x}, \bar{y} \in \mathcal{X}_{\mathcal{M}}$ such that $\bar{x}_i = x_i$ and $\bar{y}_i = y_i$ for all $i \in \mathcal{N}$ and $\bar{x}_j = \bar{y}_j$ for all $j \in \mathcal{M} \setminus \mathcal{N}$. By Axiom 10, $x \succsim y \Longleftrightarrow \bar{x} \succsim \bar{y}$. By representation (4), $\bar{x} \succsim \bar{y} \Longleftrightarrow \sum_{i \in \mathcal{N}} \phi(\bar{x}_i) \geq \sum_{i \in \mathcal{N}} \phi(\bar{y}_i)$. Hence, using the definition of $\bar{x}$ and $\bar{y}$,

$$x \succsim y \Longleftrightarrow \sum_{i \in \mathcal{N}} \phi(x_i) \geq \sum_{i \in \mathcal{N}} \phi(y_i).$$

Step 3: Comparisons of allocations with different populations. By Th. 6.9 of Blackorby, Bossert and Donaldson (2005), the Pareto principle, existence independence and the weak existence of critical levels (which is implied by Axiom 8) imply that there exists a constant critical level $z_c \in \mathbb{R}_{++}$ such that for any $x \in \mathcal{X}$, if $y \in \mathcal{X}$ is such that there exists $j \in (\mathbb{N} \setminus \mathcal{N}(x))$ for which $y_j = z_c$ while $\mathcal{N}(y) = \mathcal{N} \cup \{j\}$ and $x_i = y_i$ for all $i \in \mathcal{N}$, then $x \sim y$.

Consider any $x, y \in \mathcal{X}$ and let $\mathcal{L} = \mathcal{N}(x) \setminus \mathcal{N}(y)$ and $\mathcal{M} = \mathcal{N}(y) \setminus \mathcal{N}(x)$ so that $\mathcal{L} \cup \mathcal{N}(x) = \mathcal{M} \cup \mathcal{N}(y) = \mathcal{K}$. Define $\tilde{x}, \tilde{y} \in \mathcal{X}_{\mathcal{K}}$ in the following way:

1. $\tilde{x}_i = x_i$ for all $i \in \mathcal{N}(x)$ and $\tilde{x}_j = z_c$ for all $j \in (\mathcal{L} \setminus \mathcal{N}(x)$;

2. $\tilde{y}_i = y_i$ for all $i \in \mathcal{N}(y)$ and $\tilde{y}_j = z_c$ for all $j \in (\mathcal{L} \setminus \mathcal{N}(y)$.

By Assumption 3 and the fact that z_c is the constant critical level, we have $x \sim \tilde{x}$, $y \sim \tilde{y}$ and thus $x \succsim y \Longleftrightarrow \tilde{x} \succsim \tilde{y}$. Given that $\tilde{x}, \tilde{y} \in \mathcal{X}_{\mathcal{K}}$, the previous results yield:

$$\tilde{x} \succsim \tilde{y} \Longleftrightarrow \sum_{i \in \mathcal{K}} \phi(\tilde{x}_i) \geq \sum_{i \in \mathcal{K}} \phi(\tilde{y}_i). \tag{5}$$

Subtracting $|\mathcal{K}| \times \phi(z_c)$ to both terms of the inequation in (5) and using the definition of $\tilde{x}$ and $\tilde{y}$ we obtain:

$$\tilde{x} \succsim \tilde{y} \Longleftrightarrow \sum_{i \in \mathcal{N}(x)} \big(\phi(x_i) - \phi(z_c)\big) \geq \sum_{i \in \mathcal{N}(y)} \big(\phi(y_i) - \phi(z_c)\big).$$

■

Proof of Prop. 6

It is easily checked that the characterized class of social welfare orderings satisfy Axioms 3, 6, 8, 10 and 11 .

Assume that the social ordering $\succsim$ satisfies Axioms 3, 6, 8, 10 and 11. By Lemma 1, there exist a continuous increasing and concave function $\phi : \mathbb{R}_+ \to \mathbb{R}$ and a positive real number $z_c \in \mathbb{R}_{++}$ such that, for all $x, y \in \mathcal{X}$

$$x \succsim y \Longleftrightarrow \sum_{i \in \mathcal{N}(x)} \Big(\phi(x_i) - \phi(z_c)\Big) \geq \sum_{i \in \mathcal{N}(y)} \Big(\phi(y_i) - \phi(z_c)\Big).$$

But, by Axiom 11, it is also the case that, for all $x, y \in \mathcal{X}$, $x \succsim y \Longleftrightarrow \tilde{U}(x) \geq \tilde{U}(y)$, where $\tilde{U}$ is the function in Axiom 11. Hence there exists a continuous and increasing function Θ such that for all $x \in \mathcal{X}$:

$$\tilde{U}(x) = \Theta\Bigg(\sum_{i \in \mathcal{N}(x)} \Big(\phi(x_i) - \phi(z_c)\Big)\Bigg).$$

Denote $h(x_i) = \phi(x_i) - \phi(z_c)$. For any $\mathcal{N} \in \mathfrak{N}$ such that $1 \notin \mathcal{N}$, Axiom 10 implies that, for all $x, y \in \mathcal{X}_\mathcal{N}$, for any $x_1 \in \mathbb{R}_+$:

$$\begin{aligned} & \Theta\Big(\sum_{i\in\mathcal{N}} h(x_i)\Big) \geq \Theta\Big(\sum_{i\in\mathcal{N}} h(y_i)\Big) \\ \iff \quad & \Theta\Big(h(x_1) + \sum_{i\in\mathcal{N}} h(x_i)\Big) \geq \Theta\Big(h(x_1) + \sum_{i\in\mathcal{N}} h(y_i)\Big) \end{aligned}$$

By the unicity of additive representation up to positive affine transformations (see Debreu, 1960, Theorem 3), there must exists $\alpha(h(x_1)\Big) \in \mathbb{R}_+$ and $\beta\Big(h(x_1)\Big) \in \mathbb{R}$ such that:

$$\Theta\Big(h(x_1) + \sum_{i\in\mathcal{N}} h(x_i)\Big) = \alpha\Big(h(x_1)\Big)\Theta\Big(\sum_{i\in\mathcal{N}} h(x_i)\Big) + \beta\Big(h(x_1)\Big). \tag{6}$$

This is true for all $\mathcal{N} \in \mathfrak{N}$ such that $1 \notin \mathcal{N}$ and for any $x_1 \in \mathbb{R}_+$. By continuity of the social ordering on allocations, there must exist continuous functions α and β such that Eq. (6) always holds. Denote $I = \{t \in \mathbb{R} \mid \exists z \in \mathbb{R}_+, t = h(z)\}$, which is an interval in $\mathbb{R}$ by continuity of function h. Eq. (6) then implies the following functional equation for all $t \in I$ and $z \in \mathbb{R}$:

$$\Theta(t + y) = \alpha(t)\Theta(y) + \beta(t).$$

By Corollary 1 (pp. 150–151) in Aczél (1966), this equation implies that either Θ is affine in y or that it is a positive affine transformation of the function $y \to \alpha e^{\alpha y}$ for some $\alpha \neq 0$.

Assume that the second case is true, and without loss of generality that for all $x \in \mathcal{X}$:

$$\tilde{U}(x) = \alpha \exp\left(\alpha \left[\sum_{i\in\mathcal{N}(x)} \Big(\phi(x_i) - \phi(z_c)\Big)\right]\right)$$

with $\alpha > 0$ (the case $\alpha < 0$ can be treated in a similar way). Consider $z_1 < z_c < z_2$ such that $\phi(z_2) - \phi(z_c) = \phi(z_c) - \phi(z_1) = \varepsilon$, for ϵ sufficiently small, and $A \in \Sigma$ such that $P(A) = 1/2$ (such a A exists by Assumption 1). Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ be such that:

- for all $s \in A$, $\mathcal{N}(\mathbf{x}(s)) = \mathcal{N}(\mathbf{y}(s)) = \{1\}$, $x_1(s) = z_1$, $y_1(s) = z_c$.
- for all $s \in (\mathcal{S} \setminus A)$, $\mathcal{N}(\mathbf{x}(s)) = \mathcal{N}(\mathbf{y}(s)) = \{2\}$, $x_2(s) = z_2$, $y_2(s) = z_c$.

By Axiom 11 and the previous results, we have $\mathbf{x} \succ \mathbf{y}$ because $\frac{\alpha}{2}e^{\alpha\varepsilon} + \frac{\alpha}{2}e^{-\alpha\varepsilon} > \alpha$.

Now construct $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ be such that:

- for all $s \in A$, $\mathcal{N}(\tilde{\mathbf{x}}(s)) = \mathcal{N}(\tilde{\mathbf{y}}(s)) = \{1\}$, $x_1(s) = z_1$, $y_1(s) = z_c$.
- for all $s \in (\mathcal{S} \setminus A)$, $\mathcal{N}(\tilde{\mathbf{x}}(s)) = \mathcal{N}(\tilde{\mathbf{y}}(s)) = \{2, 3\}$, $x_2(s) = z_2$, $y_2(s) = z_c$, $x_3(s) = y_3(s) = z_3 < z_c$.

By existence independence (Axiom 10), we should have $\tilde{\mathbf{x}} \succ \tilde{\mathbf{y}}$. But if $\phi(z_c) - \phi(z_3)$ is sufficiently large (and ε sufficiently small), it is possible that $\frac{\alpha}{2}e^{\alpha\varepsilon}e^{-\alpha(\phi(z_c)-\phi(z_3))} + \frac{\alpha}{2}e^{-\alpha\varepsilon} < \frac{\alpha}{2}e^{-\alpha(\phi(z_c)-\phi(z_3))} + \frac{\alpha}{2}$, which, by 11 and the previous results, implies that $\tilde{\mathbf{y}} \succ \tilde{\mathbf{x}}$. This is a contradiction that rules out the case where Θ is the affine transformation of an exponential function.

Hence, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if for all $i \in \mathbb{N}$ $\mathbf{x}_i(s) = \mathbf{x}_i(s')$ for all $s, s' \in A_i(\mathbf{x})$ and $\mathbf{y}_i(s) = \mathbf{y}_i(s')$ for all $s, s' \in A_i(\mathbf{y})$, then:

$$\mathbf{x} R \mathbf{y} \Longleftrightarrow \mathbb{E}\left[\sum_{i \in \mathcal{N}(\mathbf{x})} \Big(\phi(\mathbf{x}_i) - \phi(z_c)\Big)\right] \geq \mathbb{E}\left[\sum_{i \in \mathcal{N}(\mathbf{y})} \Big(\phi(\mathbf{y}_i) - \phi(z_c)\Big)\right].$$

Now consider any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Let $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ be such that, for all $i \in \mathbb{N}$, $A_i(\mathbf{x}) = A_i(\tilde{\mathbf{x}})$ and $\tilde{\mathbf{x}}_i(s) = ce_i(\tilde{\mathbf{x}}_i)$ for all $s \in A_i(\tilde{\mathbf{x}})$, $A_i(\mathbf{y}) = A_i(\tilde{\mathbf{y}})$ and $\tilde{\mathbf{y}}_i(s) = ce_i(\tilde{\mathbf{y}}_i)$ for all $s \in A_i(\tilde{\mathbf{y}})$. By the generalized Pareto axiom (Axiom 6), $\mathbf{x} \sim \tilde{\mathbf{x}}$ and $\mathbf{y} \sim \tilde{\mathbf{y}}$ so that, by Assumption 3, $\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \tilde{\mathbf{x}} \succsim \tilde{\mathbf{y}}$. But prospects $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ satisfy the conditions of Axiom 1 so that:

$$\tilde{\mathbf{x}} R \tilde{\mathbf{y}} \Longleftrightarrow \mathbb{E}\left[\sum_{i \in \mathcal{N}(\tilde{\mathbf{x}})} \Big(\phi(\tilde{\mathbf{x}}_i) - \phi(z_c)\Big)\right] \geq \mathbb{E}\left[\sum_{i \in \mathcal{N}(\tilde{\mathbf{y}})} \Big(\phi(\tilde{\mathbf{y}}_i) - \phi(z_c)\Big)\right].$$

This can be rewritten:

$$\tilde{\mathbf{x}} R \tilde{\mathbf{y}} \Longleftrightarrow \sum_{i \in \mathbb{N}} p_i(\mathbf{x}) \Big[\phi\big(ce_i(\mathbf{x}_i)\big) - \phi(z_c)\Big] \geq \sum_{i \in \mathbb{N}} p_i(\mathbf{y}) \Big[\phi\big(ce_i(\mathbf{y}_i)\big) - \phi(z_c)\Big].$$