A Study of Turkish High School Admissions
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Abstract

This paper studies the recent reforms in the Turkish High School admissions system. There are two types of high schools in Turkey: public and private schools. Public school admissions are administered in a centralized manner, whereas private school admissions are done in a decentralized manner. Up until 2013, private school assignments followed the centralized public school assignments. Starting from 2014, private school assignments are carried out in a first round along with a policy change in the terms re-entry into the second round. We study the performance of both sequential assignment systems. We show that although the current system is an improvement over its predecessor, alternative solutions that outperform both systems are available.

1 Introduction

1.1 Overview

In Turkey, more than 1.3 million students enter High School Admissions every year. The Ministry of Education (MoE) implemented some changes in the admission systems two years ago. To begin with, all public schools are merged in centralized admissions. Students take two nationwide high school entrance exams in the eighth grade of elementary school. Scores for Placement (SFP) are calculated by taking weighted average

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of the test scores (e.g., Science, Math, etc.) in two nationwide exams, and Grade Point Average in the last three years of elementary school. Each student submits ranking of 15 public schools. The public schools are assigned via a clearinghouse with respect to SFP. The students do not apply to private schools in the centralized admissions system. The private schools accept students whose score are higher than their threshold score. Students who are assigned to a public school and having a private school that they prefer to their current assignment, leave their seats and participate in the decentralized private school admissions. As a result, many public school seats became vacant, leading to multiple rounds of reassignments which in turn caused some frustration on students and their parents. In 2014, there were around 100,000 empty seats in public schools. Five rounds of reassignments followed. Yet, there were still tens of thousands empty seats in public schools. To address these problems, MoE revised the admissions system last year. Firstly, the order of the admissions has been changed. That is, private schools admission started preceding the centralized public school admissions. Secondly, any student who is assigned to a private school is not allowed to participate in the centralized public schools assignments. Basically, MoE wanted to eliminate possible wasteful outcomes and reduce number of students participating in the reassignment rounds. Even though the changes mitigate the problems to a certain degree, it gave rise to some other complications. The students are left with a risky choice: either get your best private school and opt out of public school admissions or wait and try your chance for a better public school. This created serious pressure on the students and their parents. A letter written by group of parents to the President of the Republic of Turkey manifests this concern: “The current system forces us to make a decision between public and private schools in the beginning... Hence many parents are forming a line before private schools fearing they would not be admitted to any private school...”

1.2 Related Literature

The main characteristic that distinguishes the type of problems we study here from the vast majority of the problems considered in the literature is that they involve sequential assignment of indivisible resources. Whereas the set of agents and resources are predetermined in a standard, single-round simultaneous assignment problem, in a sequential assignment problem agents and resources considered within a round may depend on decisions made in a previous round. Differently put, in a sequential assignment problem, an agent might have the ability to choose which round he is to get which assignment. Yet, the two types of problem still share similar strategic and distributional objectives.

Abdulkadireroglu and Sonmez (2003) introduce the school choice problem by investig-
ing the school choice plans in major metropolitan areas in the US and offer alternative superior solutions that are efficient or stable while maintaining good incentive properties. These proposals highlight a striking tradeoff between stability and efficiency. Kesten (2010) proposes an intermediate solution to the problems caused by the tradeoff between stability and efficiency in school choice. There is now an extant literature on school choice plans but as far as we are aware, virtually all of these models abstract away from the multi-round nature of these problems and focus exclusively on a single-round simultaneous assignment system. In another sequential project on NYC student assignment plan, Abdulkadiroglu, Pathak, and Roth (2009) argue that the current multi-round assignment plan may result in unstable student assignments.

The current problem is mathematically similar to the student placement problem (SPP) due to Balinski and Sonmez (1999) and the house allocation problem with existing tenants (HAPwET) due to Abdulkadiroglu and Sonmez (1999). Another paper that is related to ours is Ergin and Sonmez (2006), where the authors characterize the set of NE of the widely used Boston mechanism and show that this set coincides with the set of stable matchings.

The only paper (that we are aware of) to consider sequential assignment is Westkamp (2012), where the author studies the German college admissions system which operates through a combination of the Boston and the college proposing deferred acceptance mechanism. Similarly to Ergin and Sonmez (2006), Westkamp also characterizes the set of SPNE of this game as being the stable set.

Braun et al. (2011) compare the performance of the sequential German college admissions systems and a modified version of the DA mechanism through experiments. The results of the experiments show that the current practice in Germany harms the high-performing students and creates incentives for them to misreport their preferences. On the other hand, the modified DA mechanism improves the welfare of the high-performing applicants.

1.3 Outline of the paper

First we will provide an example to demonstrate that the mechanism used last year which is public school assignments followed with private school assignments results in wasteful outcomes assuming that reassignment rounds are not effective. Because of sequential nature of assignments, that is difference in the timing of admissions, the mechanism leads to serious coordination failures among students. Accordingly, we study Subgame Perfect Nash Equilibrium (SPNE) of the induced preference revelation game.
Our sequential assignment consists of two rounds of assignment – (centralized) public school assignment and (decentralized) private school assignment. First we model private school assignment as a decentralized admission game. In each round, each private school announces its cut-off score and after observing cut-off scores, each student applies to her acceptable schools. Next each school offers its remaining seats to the best applicants. Each student holds at most one offer among the ones offered in that round. We show that the Nash equilibrium outcome of this game is unique and it is equal to the Serial Dictatorship outcome under true preferences. After this observation we find the SPNE of the sequential assignment game. If students assigned in the first round are allowed to participate in the second round, the SPNE outcome of the game is fair, and individually rational under the true preferences regardless of the order of the assignments. Moreover the seats in the second round assignment are not wasted, yet the seats in the first round assignment can be wasted in the equilibrium outcome. This result shows that if the students assigned in the first round are not allowed to participate in the second round then the order between public and private schools would not prevent wastefulness. It would only change the set of wasteful outcomes, i.e. public school seats are wasted if the first round is centralized, and private school seats are wasted otherwise. In the next theorem, we didn’t allow the assignees of the first round to participate in the second round which turns out to be the critical change implemented this year. The SPNE outcome of the game is fair, and individually rational, and nonwasteful under the true preferences regardless of the order of the assignments. This result shows that the current system is an improvement over its predecessor given that it results in nonwasteful outcomes in the equilibrium. The reversal of the ordering of the rounds does not play a role here. It is mainly because the assignees of the first round is not allowed to participate in the second round,

Finally we investigate alternative mechanisms which do not require serious coordination which is assumed in the SPNE outcome. Our first alternative retains the previous order, i.e., Public – Private mechanism. The private schools announce their cut-off score. The students submit their preferences over all schools. We run student proposing deferred acceptance and assign public schools if the outcome of the matching gives a public school. For unassigned students we run Serial Dictatorship mechanism for private schools. We show that, under this system students do not benefit from misreporting their preferences in the first round and the final outcome is the student optimal stable matching.
2 Model

A school choice problem with private and state schools, or a problem for short, consists of

1. a set of schools $S = S_{pr} \cup S_{pu}$ where $S_{pr}$ is the subset of private schools and $S_{pu}$ is the subset of public schools and $S_{pr} \cap S_{pu} = \emptyset$,

2. a set of students $I = \{i_1, i_2, \ldots, i_n\}$,

3. a capacity vector of available seats in each school $q = (q_{s_1}, \ldots, q_{s_m})$,

4. a preference profile $P = (P_{i_1}, \ldots, P_{i_n})$ where $P_i$ is the preference relation of student $i$ over schools including being unassigned option denoted by $\emptyset$,

5. a list of priority order of students $\succ = (\succ_{s_1}, \succ_{s_2}, \ldots, \succ_{s_m})$ where $\succ_s$ is the priority order of agents in $I$ for school $s$.

In the rest of the paper we fix the set of students, schools, quota vector and priority order, and denote a problem with preference profile $P$. Let $R_i$ denote the associated at least as good as relation of student $i \in I$ where

$$s R_i s' \iff s P_i s' \text{ whenever } s \neq s'.$$

Each public school prioritizes students in the same order by using the standard test scores. Let $t^{pa}(i)$ be the standard test score of student $i$ for the public schools. For each $s \in S_{pu}$, $i \succ s j \iff t^{pa}(i) > t^{pa}(j)$. Hence, all public schools prioritize students in the same order.

Similarly, all private schools use the same weights to calculate their own test scores to prioritize students. However, each private school sets its own threshold scores for the acceptable students. Let $t^{pr}(i)$ be the test score of student $i$ for the private schools. For each $s \in S_{pr}$, $i \succ s j \iff t^{pr}(i) > t^{pr}(j)$. Note, it is possible that any two private school may have different priority order. In particular, it is possible that a student $i$ is acceptable for private school $s$ but unacceptable for private school $s'$.

A matching is a function $\mu : I \to S \cup \{\emptyset\}$ such that the number of students assigned to a school $s$ does not exceed the number of seats in school $s$ and each student can be assigned to at most one school, i.e., $|\mu^{-1}(s)| \leq q_s$ and $|\mu(i)| \leq 1$ for all $s \in S$ and $i \in I$. Let $\mathcal{M}$ be the set of all possible matchings.

Next we give the formal definitions of axioms which are used to compare matchings.
A matching $\mu$ is **non-wasteful** if there exists no student school pair $(i, s)$ such that $i \succ_s \emptyset$, $|\mu^{-1}(s)| < q_s$ and $sP_i \mu(i)$.

A matching $\mu$ is **individually rational** if no student is assigned to a school she finds worse than being unassigned. Formally, a matching $\mu$ is individually rational if whenever $\mu(i) \in S$, $\mu(i)R_i\emptyset$ and $i \succ_{\mu(i)} \emptyset$ for all $i \in I$.

A matching $\mu$ is **fair** if whenever a student prefers some other student’s assignment to his own, then the other student has a higher priority for that school than herself. Formally, for every $i, j \in I$, $\mu(j)P_i \mu(i)$ implies that, $\mu(j) \succ_{\mu(i)} i$.

A matching $\mu$ is **stable** if it is non-wasteful, individually rational and fair.

We say a student $i \in I$ prefers matching $\mu \in \mathcal{M}$ to $\nu \in \mathcal{M}$ if and only if she prefers $\mu(i)$ to $\nu(i)$. A matching $\mu$ is Pareto dominated by another matching $\nu \in \mathcal{M}$ if $\nu(i)R_i\mu(i)$ for all $i \in I$ and $\nu(j)P_j\mu(j)$ for at least one student $j \in I$. A matching $\mu$ is **Pareto efficient** if there does not exist another matching $\nu \in \mathcal{M}$ which Pareto dominates $\mu$.

A mechanism $\phi$ is a systematic procedure that selects a matching for each assignment problem. Denote the outcome selected by mechanism $\phi$ in problem $P$ by $\phi[P]$ and denote the match of student $i \in I$ by $\phi[P](i)$. A mechanism $\phi$ is non-wasteful <individually rational> [fair] {stable} (Pareto efficient) if it selects a non-wasteful <individually rational> [fair] {stable} (Pareto efficient) matching in any problem.

A mechanism $\phi$ is strategy-proof if $\phi[P](i)P_i \phi[P', P_{-i}](i)$ for any student $i$ and preference order $P'$.

A sequential assignment system $\Phi = (\phi^1, \phi^2)$ is a pair of assignment procedures such that

- $\phi^1$ operates on the restricted problem $(I, S^1, P^1, q|S^1, \succ |S^1)$ whose primitives are the set of all students, a subset of all schools available for assignment in the first round (defined by the application), and the preferences and priorities over available schools, and

- $\phi^2$ operates on the restricted problem $(I^2, S^2, P^2, q|S^2, \succ |S^2)$ whose primitives are the set of students who are allowed to participate in this round by the application, a subset of all schools available for assignment in the first round (defined by the application), and the preferences and priorities over available schools.

Under the school choice with private and state schools $S^1$ is either $S^{pr}$ or $S^{pu}$ and $S^2 = S \setminus S^1$. The set of students in $I^2$ may depend on the allocation selected in round 1. Hence, the sequential assignment system inherits dynamic properties.
Assignment of student $i$ for a problem under system $\Phi = (\phi^1, \phi^2)$ is formally defined as
\[
\Phi_i(I, S^1, S^2, P^1, P^2, q, \succ) = \begin{cases} 
\mu_2(i) & \text{if } i \in I^2 \text{ and } \mu_2(i)P_i\mu_1(i) \\
\mu_1(i) & \text{otherwise}
\end{cases}
\]
where $\mu_1 = \phi^1(I, S^1, P^1, q|S^1, \succ |S^1)$ and $\mu_2 = \phi^2(I^2, S^2, P^2, q|S^2, \succ |S^2)$.

3 Results

3.1 Decentralized Admission Game

The admission for the private schools are done through decentralized system which can be formalized as a sequential game. The players in this sequential game are the students and private schools. In each round $t$,

- Each private school $s$ announces a cut-off score $x^t_s \in [0, x_s^{t-1}]$ where $x_s^0$ is the maximum possible score that a student can take in the centralized exam.

- After observing the cut-off scores, each student $i$ chooses a subset of schools $A^t_i$ among the ones she has not rejected before, possibly emptyset, where $t^\text{pr}(i) \geq x^t_s$ for each $s \in A^t_i$.

- Given the applicants in this round each school $s$ offers to the top $\max\{a^t_s, b^t_s\}$ applicants where $b^t_s$ is the number of applicants in round $t$ and $a^t_s$ is the number of students rejected $s$’s offer in round $t - 1$ and $a^1_s = q_s$.

- Each student $s$ holds at most one offer among the ones she has received in that round and he has been holding before and rejects the rest.

Since the numbers schools and students are finite, this game terminates. The game ends when students do not reject any offer and each student is assigned to the school whose offer she has been holding when the game terminated. A school $s$ can be strategic only in the very first step of each round where it sets its cut-off point.

1 The decentralized admission game for the private schools has a unique SPNE outcome which is equivalent to the Serial Dictatorship outcome under true preferences.

**Proof.** Consider any preference and cut-off scores profiles $\tilde{P}$ and $\tilde{x} = (\tilde{x}_s)_{s \in S}$, respectively. Let $\tilde{\mu}$ be the outcome of serial dictatorship mechanism by using the test
scores in which each student picks the best remaining school considering her acceptable. Since each school has the same relative ranking over the acceptable students and under serial dictatorship mechanism students with higher test scores, $\bar{\mu}$ is the unique stable matching under $\bar{P}$ and $\bar{x}$. Let $P$ be the true preferences, $x = (x_s)_{s \in S}$ be the true cut-off scores and $\mu$ be a stable matching under $P$ and $x$. Then, $\mu$ is the outcome of the outcome of serial dictatorship mechanism by using the test scores and it is the unique stable matching under $P$ and $x$.

Fix a cut-off profile $\bar{x} = (\bar{x}_s)_{s \in S}$. Let $\bar{\mu}$ be the unique stable outcome under cut-off profile $\bar{x}$ and $P$. Let $Q = (Q_{i_1}, \ldots, Q_{i_n})$ be an arbitrary strategy profile and let $\nu$ be the resulting equilibrium outcome of the admission game. Here, $Q_i$ is the complete contingent plan of actions of student $i$. Suppose $\nu \neq \bar{\mu}$. Since $\bar{\mu}$ is the unique stable matching, $\nu$ is not stable under $P$ and $\bar{x}$. Suppose $\nu$ is individually irrational. Since, each school only offers to the acceptable students, there exists a student $i$ such that $\emptyset \overset{P}{\longrightarrow} \nu(i)$. This implies that $i$ does not reject all the offers she gets. Let $\bar{Q}_i$ be the strategy in which $i$ rejects all the offers she gets under preference profile $(\bar{Q}_i, Q_{-i})$ and cut-off profile $\bar{x}$. Under $(\bar{Q}_i, Q_{-i})$ student $i$ will be assigned to $\emptyset$. Therefore $Q$ cannot be an equilibrium. Suppose $\nu$ is wasteful or not fair. Then there exists a student-school pair $(i, s)$ such that $s \overset{P}{\longrightarrow} \nu(i)$ and either $|\nu^{-1}(s)| < q_s$ or there exists $j \in \nu^{-1}(s)$ and $i$ has higher test score than $j$. In both cases $i$ receives an offer from $s$ under $Q$ if she has applied to it. This implies that $i$ either does not apply to $s$ or rejects $s$ when he received an offer from $s$. Let $\bar{Q}_i$ be the strategy in which $i$ applies to $s$ and accepts the offer from $s$ whenever she gets and rejects all the offer she gets in the following rounds. Under $(\bar{Q}_i, Q_{-i})$ student $i$ will be assigned to $s$. Therefore $Q$ cannot be an equilibrium. That is, for any cut-off profile $\bar{x}$, we cannot have an equilibrium outcome which is different from the unique stable outcome under $P$ and $\bar{x}$.

Now suppose $\nu \neq \mu$. Then we can show that a school $s$ can be better off by deviating. In order to do this we go one by one from the student highest score. Let $i_k$ be the $k^{th}$ highest student. Suppose $\mu(i_1) \neq \nu(i_1)$. Then school $\mu(i_1)$ does not get any student. Hence, it can get $i_1$ by setting lower cut-off and be better off. Suppose $\mu(i_k) = \nu(i_k)$ for all $i_{k'}$ such that $k' \leq k$. Suppose $\mu(i_{k+1}) \neq \nu(i_{k+1})$. Then $\mu(i_{k+1})$ cannot get any student with lower score than $i_k$. Hence, it can get $i_{k+1}$ by setting lower cut-off and be better off. ■

### 3.2 Sequential Assignment

Since the unique equilibrium outcome of the decentralized admission game for the private schools is equivalent to the outcome of the serial dictatorship mechanism under true
preferences, in the rest of the paper we formalize the decentralized admission for the private schools as a serial dictatorship mechanism.

Students are assigned to the high schools in Turkey in two rounds in a sequential manner. In 2014, students first applied for the public schools and then for the private schools. When a student applied for private school, he had the right to keep his public school assignment. In 2015, the order of the assignment was changed. Moreover, only the unassigned students in the private school assignment were allowed to participate in the public school assignment. We formalize this two round assignment as a sequential game as follows:

**Round 1:**

- Let $S^1$ be the set of available schools in round 1. If $S^1 = S^{pr}$, then each $s \in S^{pr}$ announces its cut-off score $x_s$.
- Each student $i$ first decides to participate or not in this round. Denote the set of participants with $I^1$. If $i \in I^1$ then $i$ submits an ordered list over the available schools in $S^1 \cup \emptyset$ considering himself acceptable.
- Given the priority orders and submitted preferences, the participating students are assigned to schools in $S^1$ via serial dictatorship mechanism. Denote the assignment with $\mu^1$.
- If $i \notin I^1$ is not allowed to participate in the second round, then $\mu^1(i)$ is his final assignment.

**Round 2:**

- Let $S^2$ be the set of available schools in round 2. If $S^2 = S^{pr}$, then each $s \in S^{pr}$ announces its cut-off score $x_s$.
- Each student $i$ who is allowed to participate in Round 2 first decides to enter or not in this round. Denote the set of participants with $I^2$. If $i \in I^2$ then $i$ submits an ordered list over the available schools in $S^2 \cup \emptyset$ considering himself acceptable.
- Given the priority orders and submitted preferences, the participating students are assigned to schools in $S^2$ via serial dictatorship mechanism. Denote the assignment with $\mu^2$.
- If $i \notin I^1 \cap I^2$ then $i$ selects a school between $\mu^1(i)$ and $\mu^2(i)$. If $i \in I^2 \setminus I^1$ then $\mu^2(i)$ is his final assignment.
Consider the preference revelation game induced by the Public-Private (Private-Public) mechanism in which students assigned to a public (private) school can participate in the second round. Each Subgame Perfect Nash Equilibrium outcome of the game is fair and individually rational and the seats of private (public) schools is not wasted under the true preferences and the true cut-offs.

**Proof.** Let \( Q = (Q_1, \ldots, Q_n) \) be an arbitrary SPNE strategy profile and let \( \nu \) be the resulting equilibrium outcome of the Public-Private (Private-Public) mechanism. Let \( Q_j = (Q_j^1, Q_j^2) \) and \( Q_i^1 \) and \( Q_i^2 \) be the strategy played by student \( j \) in the public and private (private and public) school assignment procedure, respectively.

Suppose \( \nu \) is individually irrational. Then there exists \( i \in I \) such that \( \emptyset P_i \nu(i) \). Let \( \tilde{Q}_i = (\tilde{Q}_i^1, \tilde{Q}_i^2) \) be the strategy in which \( i \) ranks \( \emptyset \) at the top of \( \tilde{Q}_i^1 \) and \( \tilde{Q}_i^2 \) (alternatively we can say she rejects all offers she gets in the private school admissions). Under \((\tilde{Q}_i, Q_{-i})\) student \( i \) will be assigned to \( \emptyset \). Therefore \( Q \) cannot be an equilibrium.

Suppose \( \nu \) is not fair. Then there exists a student-school pair \((i, s)\) such that \( sP_i \nu(i) \) and either \( |\nu^{-1}(s)| \leq q_s \) or there exists \( j \in \nu^{-1}(s) \) and \( i \) has higher test score than \( j \). Suppose \( s \in S_{pu} \) (\( s \in S_{pr} \)). Let \( \tilde{Q}_i = (\tilde{Q}_i^1, \tilde{Q}_i^2) \) be the strategy in which \( i \) ranks \( s \) and \( \emptyset \) at the top of \( \tilde{Q}_i^1 \) and \( \tilde{Q}_i^2 \), respectively. Under \((\tilde{Q}_i, Q_{-i})\) student \( i \) will be assigned to \( s \). Suppose \( s \in S_{pr} \) (\( s \in S_{pu} \)). Let \( \tilde{Q}_i = (\tilde{Q}_i^1, \tilde{Q}_i^2) \) be the strategy in which \( i \) ranks \( s \) and \( \emptyset \) at the top of \( \tilde{Q}_i^1 \) and \( \tilde{Q}_i^2 \), respectively. Under \((\tilde{Q}_i, Q_{-i})\) student \( i \) will be assigned to \( s \). Hence, in both case \( Q \) cannot be an equilibrium.

In Theorem 2, we have shown that any SPNE of the preference revelation game induced by the Public-Private and Private-Public mechanism in which students assigned in the first round can enter the second round. However, we may have wasteful equilibrium outcome. We illustrate this situation in the following example.

There are one public and private school with capacity 1: \( S_{pu} = \{s\} \) and \( S_{pr} = \{s'\} \). There are two students: \( I = \{i, j\} \). The preferences and the priorities are: \( s'P_isP_i\emptyset \), \( sP_jsP_j\emptyset \), \( i \succ_s j \succ_s \emptyset \) and \( i \succ_{s'} j \succ_{s'} \emptyset \).

When students assigned in Round 1 are allowed to enter Round 2, under both Public-Private and Private-Public mechanisms, entering both assignment rounds and submitting true preferences and reporting true cut-offs is a SPNE. Under this SPNE strategy the seats at \( s \) and \( s' \) are wasted under Public-Private and Private-Public mechanisms, respectively.

When students assigned in Round 1 are not allowed to enter Round 1, under both Public-Private and Private-Public mechanisms, there exists a unique SPNE outcome in which student \( i \) is assigned to \( s \) and \( j \) is assigned to \( s' \). Hence, the unique SPNE of this game is stable.
3 Consider the preference revelation games induced by the Private-Public and Public-
Private mechanisms in which only the unassigned students can participate in the second
round. The set of Subgame Perfect Nash Equilibrium outcome of the game is equal to the
set of fair, individually rational, and non-wasteful matchings under the true preferences.


3.3 Alternatives

In this section, we propose two alternative assignment systems in order to overcome
some of the problems observed in the sequential systems.

3.3.1 Alternative System 1

Round 1: Assignment to the Public Schools

• Each private school \( s \in S^{pr} \) reports its base point \( x_s \).

• Each student \( i \) is required to submit her preferences over \( S^{pr} \cup S^{pa} \).

• Run DA mechanism in problem \((I, S^{pa} \cup S^{pr}, P, q, \succ)\). Denote the outcome of DA
mechanism with \( \nu_1 \). Let \( \nu_1(i) = \mu(i) \) if \( \mu(i) \in S^{pa} \) and \( \nu_1(i) = \emptyset \) otherwise.

Round 2: Assignment to the Private Schools

• Only unassigned students are allowed to apply to private schools in the second
round.

• Students assigned to the private schools through decentralized admission system
described in Section 3.1. Denote the outcome of the decentralized procedure with
\( \nu_2 \).

Since each school ranks any two students in the same order, running DA mechanism
in the first round is equivalent to running serial dictatorship mechanism according to
the test scores such that in each step the active student selects the best remaining school
considering herself acceptable.

We first show that, in the first round students cannot benefit from misreporting their
true preferences over the schools in the first round.

4 Under Alternative System 1, there exists a unique SPNE outcome which is equivalent
to the outcome of DA mechanism under problem \((I, S^{pa} \cup S^{pr}, P, q, \succ)\).
Proof. Since all schools rank any two students in the same order, the outcome of DA mechanism under problem $(I, S^p \cup S^p, P, q, \succ)$ is the unique stable outcome. Let $\mu$ denote the outcome of DA under problem $(I, S^p \cup S^p, P, q, \succ)$. Let $\nu$ be a SPNE outcome of Alternative System 1 and it is induced by complete contingent strategy profile $Q = (Q_x)_{x \in I \cup S}$. Let $i_k$ be the student with the $k^{th}$ highest test score. Since each school ranks students based on the test score, $i_k$ is the $k^{th}$ highest ranked student under all schools priorities. By induction we show that $\mu(i) = \nu(i)$ for all $i \in I$. First note that, under this system if $i$ is unacceptable for $s$, then $i$ cannot be assigned to $s$ in any equilibrium outcome. We start with $i_1$. Under preference profile $P_i$, $\mu(i_1)$ is the highest ranked school considering $i_1$ acceptable. Suppose $\mu(i_1) \neq \nu(i_1)$. There are two possible cases: either $i_1$ is not acceptable to $\mu(i_1)$ in both rounds of Alternative System 1 under strategy profile $Q$ or he is acceptable to $\mu(i_1)$ in both rounds of Alternative System 1 under $Q$.

Suppose all agents except $i_k$ act truthfully. For any $h < k$, since $i_k$ has lower priority than student $i_h$, $i_k$ cannot affect the first round assignment of $i_h$. Let $s_h$ denote the first round assignment of $i_h$ for all $h < k$. If $s_h \in S^p$, then $s_h$ is the final assignment of $i_h$. Otherwise, for all $h < k$, $i_h$ will be assigned to $s_h$ in any in round 2 when $h$. That is, submit where $h$. We start with $i_1$. Let $s_1$ denote the highest ranked school considering $i_1$ acceptable. When she reports $P_{i_1}$, her true preferences, she will be assigned to $s_1$ independent of the preferences submitted by other students in the first round. If $s_1 \in S^p$, then $s_1$ will be her final allocation. Otherwise, there will be at least one seat available in $s_1$ in the second round and she will get that seat when she applies to $s_1$. Hence, $i_1$ cannot be better off when she misreports in round 1. Let $\mu$ be the final outcome of the system when students play truthfully in Round 1 and play their equilibrium strategies in Round 2. Suppose student $i$ can benefit from misreporting. Since the assignees for the public schools are determined by the DA mechanism and DA mechanism is strategy proof, student $i$ cannot get a public school better than $\mu(i)$ when he misreports. Then, when student $i$ manipulates she gets a private school $s$ such that $sP_i \mu(i)$. Let $P'_i$ be the deviation strategy of student $i$.

Suppose $\mu(i) \in S^p$. Then, when students report true preferences $i$ is assigned to a private school in the first round. Since $s \in S^p$, First of all, $DA(P'_i, P_{-i})(i)$ cannot be a public school. Moreover $sP_i DA(P'_i, P_{-i})(i)$ due to the strategy-proofness of DA......

It is worth mentioning that Theorem 4 is not direct consequence of using a strategy-proof mechanism in both rounds. In particular, some sequential assignment systems, i.e. Appealing Process in NYC (see Dur and Kesten, 2015), make it profitable for students to misreport in Round 1 stay unassigned and participate in Round 2.
Next, we investigate whether it would be beneficial for a private school to allow the public school authority to run its assignment on behalf of itself. That is, whether a private school can do better than accepting the set of students assigned to it in the outcome of DA mechanism in Round 1.

5 If in the second round the participants play equilibrium strategy, then \( \mu^{-1}(s) = \nu_2^{-1}(s) \) for all \( s \in S^{pr} \). That is, any private school cannot make better than leaving the market in Round 1.

\textbf{Proof.} Let \( I_2 \) be the subset of students participating the second round. In Theorem 1, we show that there is a unique equilibrium of the decentralized admission game and it is equivalent to the unique stable outcome of the problem. In Theorem 4, we show that it is best response for students to report their true preferences in Round 1. Since DA is stable there does not exist any student \( i \) preferring being unassigned to \( \mu(i) \) and having justify envy over the set of private schools. That is, \( \mu(i) = \nu_2(i) \) for each \( i \in I_2 \). \( \blacksquare \)

6 There is a unique SPNE in undominated strategies of the Alternative System 1 which is the student optimal stable outcome of the problem \( (I, S, P, q, \succ) \).

\textbf{Proof.} Follows Theorem 4 and 1. \( \blacksquare \)

3.3.2 Alternative 2

Without loss of generality, we assume that private schools are inactive.

\textbf{Round 1: Decentralized Private School Assignment}

- All students are allowed to apply to private schools in this round.
- Students assigned to the private schools through decentralized admission system described in Section 3.1. Denote the outcome of the decentralized procedure with \( \nu_1 \).

\textbf{Round 2: Centralized Public School Assignment}

- All students are allowed to apply to private schools in this round. Let \( I^2 \) be the students who participate in this round and let \( P^2 = (P^2_i)_{i \in I^2} \) be the preference lists submitted by the participants.
- Run DA (or equivalently SD) mechanism in problem \( (I^2, S^{pu}, (q_s)_{s \in S^{pu}}, (\succ_s)_{s \in S^{pu}}) \) and denote the outcome with \( \nu_2 \).
Final Allocation:
Let $\nu$ denote the final outcome of this sequential procedure. If $i \in I^2$ and $\nu_2(i) \in S^u$, then $\nu(i) = \nu_2(i)$. Otherwise, $\nu(i) = \nu_1(i)$.

We denote this alternative system with $\Phi$. In the following theorem, we show that being straightforward is an equilibrium.

7 Let $\sigma$ be a strategy profile in which each student applies to her acceptable private schools, selects the best offer she gets in each round of decentralized private school assignment step and report the public schools that she prefers to her private school assignment truthfully. Then, $\sigma$ is a SPNE strategy profile, i.e., there does not exist any student $i$ and $\sigma'_i$ such that $\Phi(\sigma'_i, \sigma_{-i})(i) \neq \Phi(\sigma)(i)$.

Proof. On the contrary, suppose there exists a student $i$ and a strategy $\sigma'_i$ such that $\Phi(\sigma'_i, \sigma_{-i})(i) \neq \Phi(\sigma)(i)$. Let $s = \Phi(\sigma'_i, \sigma_{-i})(i)$. By the definition of $\Phi$, any student assigned to $s$ under $\Phi(\sigma)$ has higher test score than $i$. Since any other student plays the same strategy under $(\sigma'_i, \sigma_{-i})$ and $\sigma$ and $i$'s deviation does not affect the assignment of students with higher score than $i$, the set of students assigned to $s$ is the same under $\Phi(\sigma)$ and $\Phi(\sigma'_i, \sigma_{-i})$. This contradicts with $s = \Phi(\sigma'_i, \sigma_{-i})(i) \neq \Phi(\sigma)(i)$.

8 For a given problem $(I, S, q, P, \succ)$ when student play straightforwardly, let $\mu$ and $\nu$ be the outcome of $\Phi$ when private school $s$ participate the decentralized round and centralized round, respectively. School $s$ prefers $\nu$ to $\mu$.

Proof. Let $\mu^1(s)$ be the set of students assigned to $s$ in the first round of case 1. Under second case all students in $\mu^1(s)$ are assigned to a worse private school than $s$ in the first round (resource monotonicity). All the other students are assigned to weakly worse private school than $s$.

Let $\mu(s)$ and $\nu(s)$ be the final assignees of school $s$ under case 1 and case 2, respectively. Without loss of generality suppose $i$ is the highest score student with $i \in \mu(s)$ but $i \notin \nu(s)$ and $\nu(i)P_i s$. Let $s' = \nu(i)$. Note that $s'$ is a public school. Note that under case 1 student $i$ also applies to $s'$ in the second round. Since $i$ is not assigned to $s'$ in the first case all assignees of $s'$ has higher test score than $i$. Let $j \in \mu(s')$. Hence, $j$ has higher test score than $i$. Then $j$ has to be assigned to a better school than $s'$ under $\nu$ and it will be a public school. If we continue like this, we can show that there does not exist $i \in \mu(s)$ but $i \notin \nu(s)$ and $\nu(i)P_i s$. Hence if $i \in \mu(s)$ but $i \notin \nu(s)$ then $sP_i \nu(i)$. That is, all slots of $s$ is filled with students who have higher test score than $i$. Which means better students are assigned to $s$ under case 2.
References


[6] Sebastian Braun, Nadja Dwenger, Dorothea Kübler and Alexander Westkamp, "Implementing Quotas in University Admissions: An Experimental Analysis", working paper.


