Orness values for rank-dependent welfare functions and poverty measures

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Abstract

The rank-dependent welfare functions and the rank-dependent poverty measures are weighted sums of the income and the gap of an individual, respectively, where the weights only depend on the position of each individual. In this work we show that an OWA operator is underlying in the definition of every rank-dependent welfare function and every rank-dependent poverty measure. For each OWA operator assigned to the welfare functions and poverty measures we compute the corresponding orness value. Then, we establish a classification for the two classes of measures in terms of their orness value.

Welfare functions and poverty measures account for the distribution-sensitivity. That is, if a transfer of income takes place from an individual, poor in poverty, to another individual with less income, then, the magnitude of the increase on welfare or the decrease on poverty should be higher for lower incomes involved. Therefore, welfare functions and poverty measures can be classified in terms of their magnitude of change for different transfers. We prove that the orness classification of the welfare functions and the poverty measures can be interpreted as a classification in terms of their distribution-sensitivity.

Specifically, in this work, we prove that for a subset of these two classes, the measures for which the weights are linear, the orness classification and the distribution-sensitivity classification for some defined transfers are equivalent.

Keywords: Aggregation functions; OWA operators; Orness, Rank-dependent poverty measure.
JEL Classification: C02, C44, D63, I32.

1. Introduction

The family of ordered weighted averaging, hereafter OWA, operators was first introduced in decision theory by Yager [16] as a new aggregation technique that collects the multiple criteria to form an overall decision function. In the last years, OWA operators have received great attention and scholars have applied them in different contexts, such as decision making under uncertainty, fuzzy system, welfare and so on, see Yager and Kreinovich [17], Fodor and Roubens [9], Yager [24], Aristondo et al. [1] and Aristondo et al. [2].

In this paper we are interested in a numerical value that is joined to every OWA operator called orness. The orness of an OWA operator was introduced by Yager [16] in order to classify OWA operators in regard to their location between two extreme situations.

The aim of this paper is to propose a new classification for welfare functions and poverty measures in terms of the orness value of the OWA operators that are underlying on these measures.

The family of rank-dependent social welfare functions was originally defined by Yaari [23] as the weighted average of the variable of interest where the weight decreases with the position of the variable in the distribution. Some of them are the S-Gini class introduced by Weymark [22], the Bonferroni [3] welfare function, De Vergottini [21] welfare function and so on.
The rank-dependent poverty measures are those indices defined on gaps for which the weight attached to each individual depends on its position in the distribution. The most known of rank-dependent poverty measures are the poverty gap ratio, the two popular indices introduced by Sen [15], the index and the consequent class of indices proposed by Thon [19] and [20], the Kakwani index [10], the Shorrocks index [18] and the S-Gini class introduced by Weymark [22].

In this work, we show that all the rank-dependent welfare functions and rank-dependent poverty measures have underlying and OWA operator. Then we compute the orness values for all these OWA operators and we classify all the rank-dependent welfare functions and all the rank-dependent poverty measures in terms of their orness value.

Following Sen’s [15] pioneering work, researchers have proposed additional axiomatic requirements related with distribution-sensitivity and numerous classes of distribution-sensitive poverty indices have been developed, see Chakravarty [6] and Zheng [25]. In fact, it is widely accepted that welfare and poverty measurement should account for the distribution-sensitivity, Zheng [25]. Focusing on poverty, according to Kakwani [10], if a transfer of income takes place from a poor individual to a less poorer individual, then the magnitude of the increase in poverty must be larger for smaller incomes involved. Although it is well-establish that the degree of distribution-sensitivity is an important factor in welfare and poverty measurement, a formal definition of distribution-sensitivity comparisons of welfare or poverty measures has long been lacking. Zheng [26] provides a theoretical foundation to compare a class of additively separable poverty indices. However, as stressed by Bosmans [4] the criterion introduced by Zheng does not allow comparisons among rank-dependent poverty measures. For this reason, he introduces a condition based on the dominance of the vector of weights that allows comparisons between rank-dependent poverty measures.

In this work, we see that the orness classification has a direct link with the classification of the measures in terms of their distribution-sensitivity.

In the last part of this work, we follow Bosmans’ proposal and we propose in a similar way a distribution-sensitivity criteria for welfare functions. Then, we prove that the orness classification proposed in the previous sections and the distribution-sensitivity classification proposed in the last one are equivalent for a subset of the class of the rank-dependent welfare functions and another subset of the class of rank-dependent poverty measures. That is, for every rank-dependent welfare measures and every rank-dependent poverty measures for which the weights are linear, the two classifications coincide. We conclude telling that the orness value assigned to the welfare functions or the poverty measures could be interpreted as a distribution-sensitivity indicator.

The paper is organized as follows. Section 2 introduces notations and basic definition. In particular, we recall the definitions of OWA operators and their orness value and the class of rank-dependent welfare functions and the class of rank-dependent poverty measures. In section 3, we will compute the orness values for each class and we will classify them in term of this value. Finally, section 4 is devoted to show the existing link between the orness classification and the classification in term of their distribution-sensitivity of the welfare functions and poverty measures.

2. Notations and definitions

In what follows, we introduce some basic notations and definitions. We consider a population consisting of $n \geq 3$ individuals. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the income vector distribution, $x_i \in \mathbb{R}^+_{++}$ represent the income of the $i$-th individual and $D = \bigcup_{i \geq 3} \mathbb{R}^+_{++}$ be the set of all distributions. For a given $\mathbf{x} \in D$, let us denote by $x_1 \leq \cdots \leq x_n$ the non decreasing rearrangement of the coordinates of $\mathbf{x}$, respectively. Whereas, the arithmetic mean is denoted as $\mu(\mathbf{x}) = (x_1 + \cdots + x_n)/n$. Let us denote by $z \in \mathbb{R}^+_{++}$ the poverty line, such that, an individual $i \in \{1, \ldots, n\}$ is defined as poor if $x_i < z$ and as non poor if $x_i \geq z$. So doing, $q = q(\mathbf{x}, z)$ denotes the number of poor people and, for a given distribution $\mathbf{x}$, the poor distribution and its mean is defined as $\mathbf{x}_q = (x_1, \ldots, x_q)$ and $\mu(\mathbf{x}_q) = (x_1 + \cdots + x_q)/q$.

\footnote{Donaldson and Weymark [8] define two different ways to identify the poor: the weak and the strong definition. In particular, we use the weak form.}
respectively. Without loss of generality we establish that \( n > q \geq 2 \).

2.1. Ordered weighted averaging operators and their orness

In economic measurement using aggregation operators is quite common. Aggregation operators are mathematical objects that have the function of reducing a set of numbers into a unique representative (or meaningful) number. Formally,

**Definition 1.** A function \( A : [0,1]^n \rightarrow [0,1] \) is called as \( n \)-ary aggregation function if it is monotonic\(^2\) and \( A(0) = 0 \), \( A(1) = 1 \).

In what follows, the \( n \)-arity is omitted whenever it is clear from the context. A particular case of aggregation functions are the ordered weighted averaging operators, hereafter OWA operators, introduced by Yager [16].

**Definition 2.** Given a vector of weighs \( w = (w_1, \ldots, w_n) \in [0,1]^n \) satisfying \( \sum_{i=1}^{n} w_i = 1 \), the OWA operator associated with \( w \) is the aggregation function \( A_w : [0,1]^n \rightarrow [0,1] \) defined as follows,

\[
A_w(x) = \sum_{i=1}^{n} w_i x_{[i]} 
\]

The OWA operators provide a parameterized family of aggregation operators, which include many of the well-known operators such as the maximum, \( w_1 = 1 \) and 0 for the rest, the minimum, \( w_n = 1 \) and 0 for the rest, the k-order statistics, \( w_i = 1 \) and 0 for the rest, the arithmetic mean, \( w_i = \frac{1}{n} \) for every \( i \), and so on.

The Ordered Weighted Averaging operators are commutative\(^3\), monotonic, idempotent\(^4\), they are stable for positive linear transformations\(^5\) and they have a compensatory behavior\(^6\). The compensative property means that an OWA operator always is between the maximum and the minimum. Since this operator generalizes the minimum and the maximum, it can be seen as a parameterized way to go from the min to the max. Yager [16] formalize this concept introducing a degree of maxness, called orness, defined as follows:

**Definition 3.** Given an OWA operator \( A_w \) associated with a system of weights \( w = (w_1, \ldots, w_n) \in [0,1]^n \) satisfying \( \sum_{i=1}^{n} w_i = 1 \), the orness of an OWA operator is

\[
\text{orness}(A_w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i 
\]

It is easy to see that the orness of an OWA operator takes values in \([0,1]\). In addition, given an OWA operator associated to a distribution \( x \) and weights \( w \), the corresponding orness values will be higher for higher weights at the top of the distribution \( x \). The maximum, minimum and average value are obtained for \( w^* = (1,0,\ldots,0) \), \( w_\sigma = (0,\ldots,0,1) \) and \( w_A = (1/n,\ldots,1/n) \), respectively. Obviously, \( \text{orness}(w^*) = 1 \), \( \text{orness}(w_\sigma) = 0 \) and \( \text{orness}(w_A) = 1/2 \). In fact, the effect of the orness growth derived from higher weights at the top of the distribution can be interpreted as a distribution-sensitivity property.

\(^2\)A is monotonic if \( x \geq y \Rightarrow A(x) \geq A(y) \), for all \( x,y \in [0,1]^n \). Given \( x,y \in D \), by \( x \geq y \) we mean \( x_i \geq y_i \) \( \forall i \in \{1,\cdots,n\} \).

\(^3\)A is commutative if \( A(x_\sigma) = A(x) \), for any permutation \( \sigma \) on \( \{1,\ldots,n\} \) and all \( x \in [0,1]^n \).

\(^4\)A is idempotent if \( A(x \cdot I) = x \), for all \( x \in [0,1] \).

\(^5\)A is stable for translations if \( A(x + t \cdot I) = A(x) + t \), for all \( t \in R \) and \( x \in [0,1]^n \) such that \( x + t \cdot I \in [0,1]^n \). Whereas, \( A \) is stable for dilations if \( A(\lambda \cdot x) = \lambda A(x) \), for all \( \lambda > 0 \) and \( x \in [0,1]^n \) such that \( \lambda \cdot x \in [0,1]^n \).

\(^6\)A is compensative if \( x(1) \leq A(x) \leq x(n) \), for all \( x \in [0,1]^n \).
2.2. Rank-dependent welfare functions

A social welfare function is a function that resumes the economic welfare of the society. In this paper we assume the following definitions for welfare functions.

**Definition 4.** A welfare function is a function \( W : [0, \infty)^n \to [0, \infty) \) that is continuous, monotonic and strictly \( S \)-concave.

A particular class of welfare functions is shown in the following definition.

**Definition 5.** Given a weighting vector \( a = (a_1, \ldots, a_n) \in [0,1] \), with \( 0 < a_1 < a_2 < \cdots < a_n \) and \( \sum_{i=1}^n a_i = 1 \), the rank-dependent welfare function associated with \( a \) is the function \( W_a : \mathcal{D} \to [0, \infty) \) defined as

\[
W_a(x) = \sum_{i=1}^n a_i x_{[i]}
\]

where \( x_{[i]} \) is the income of individual \( i \) ranked in a non-increasing way, Chakravarty [7]. Positivity of \( a_i \) guarantees that \( W_a \) is monotonic, that is, it is increasing in \( x_i \). Unceasingness of the sequence of coefficients is necessary and sufficient for \( S \)-concavity of \( W_a \), see Bossert [5].

The following table collects some rank-dependent welfare functions proposed in the literature, the \( S \)-Gini family, hereafter \( W_{G\sigma} \) and the Bonferroni and the Vergottini welfare functions, \( W_B \) and \( W_V \) respectively.

<table>
<thead>
<tr>
<th>Measure</th>
<th>( a_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{G\sigma} )</td>
<td>( \frac{i-1}{n} ), ( \sigma \geq 2 )</td>
</tr>
<tr>
<td>( W_B )</td>
<td>( \frac{1}{n} \sum_{j=n+1-i}^{n} \frac{1}{j} )</td>
</tr>
<tr>
<td>( W_V )</td>
<td>( \frac{1+c}{n} + c \sum_{j=2}^{n} \frac{1}{j} ) where ( c = \frac{1}{n} \sum_{j=2}^{n} \frac{1}{j} )</td>
</tr>
</tbody>
</table>

Note that for \( \sigma = 2 \), \( W_{G2} \), we have the well known Gini welfare function, see Aristondo et al. [2].

It is easy to check that rank-dependent welfare functions are not directly OWA operators, since they are not restricted to the domain \([0,1]^n\). However, every welfare function could be restricted to \([0,1]^n\) domain in the following way. Consider \( \lambda \) an upper bound of the distribution \( x \in [0,\infty)^n \). Then, the welfare function can be defined as: \( W_a(x) = \lambda \cdot W_a(\frac{1}{\lambda} \cdot x) = \lambda W_a(y) \) where \( y = \frac{1}{\lambda} \cdot x \). Now, it is clear that if \( x \in [0,\infty)^n \) then \( y \in [0,1]^n \).

**Remark 1.** We assume that all the variables are restricted to the domain \([0,1]^n\). In this setting, every rank-dependent welfare function is an OWA operator. It is important to note that in this work the principal aim is to compute the orness values for those OWA operators, and in this case, the proposed normalization to \([0,1]^n\) interval does not change the orness value, since this value is defined only in terms of weights.

The following definition presents an special subgroup of rank-dependent welfare functions that will play an important role in this paper.

**Definition 6.** Linear rank-dependent welfare functions are the rank-dependent welfare functions for which the weights are linear with respect to the individual position, namely, the weights are \( a_i = b + (i-1)c \), where \( b \) and \( c \) do not depend on \( i \).

The unique linear welfare function shown before is the Gini welfare function.
2.3. Rank-dependent poverty measures

A poverty measure is a non-constant function \( P : D \times \mathbb{R}_+ \to \mathbb{R} \) whose value \( P(x, z) \) denotes the poverty level associated with an income distribution \( x \) and the poverty line \( z \). The normalized gaps for incomes below the poverty line are defined as the relative distance between the income value and the poverty line and for incomes above, it is defined as zero. Formally: \( g_i = \max \{ \frac{x_i - z}{x_i}, 0 \} \). Finally, \( g = (g_1, \ldots, g_n) \) denotes censored normalized income gap vector and \( g_q = (g_1, \ldots, g_q) \) the normalized gap vector of the poor.

In the poverty measurement, a rank-dependent poverty measures is a poverty measure whose individual’s weight depends only on its place in the distribution with respect to the others. Formally:

**Definition 7.** A poverty measure \( P(x, z) : D \times \mathbb{R}_+ \to \mathbb{R} \) is rank-dependent if for each income distribution \( x \in D \) and any fixed poverty line \( z \in \mathbb{R}_+ \), it takes the following expression

\[
P(x, z) = \sum_{i=1}^{q} w_i \frac{z - x(i)}{z} = \sum_{i=1}^{q} w_i g(i)
\]

where \( w_1 \geq w_2 \geq \cdots \geq w_q \). In addition, if the weights decrease strictly then the transfer axiom\(^7\) is satisfied.

The following table collects the rank-dependent poverty measures proposed in the literature, namely the poverty gap ratio, hereafter \( PGR \), the Sen and modified Sen indices, \( S \) and \( S' \) respectively, the Thon index, \( T \), the Shorrocks index, \( SST \), the Kakwani’s family of indices, \( K_k \), the Thon class, \( T_r \), and the \( S \)-Gini class of poverty measures, \( G_\sigma \).\(^8\)

<table>
<thead>
<tr>
<th>Measure</th>
<th>( w_i )</th>
<th>Measure</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( PGR )</td>
<td>( \frac{1}{n} )</td>
<td>( S )</td>
<td>( \frac{2(q+1)(q+1)k}{(q+1)n} )</td>
</tr>
<tr>
<td>( S )</td>
<td>( \frac{2(q+1)(q+1)}{(q+1)n} )</td>
<td>( K_k )</td>
<td>( \frac{2(q+1)(q+1)k}{n} )</td>
</tr>
<tr>
<td>( S' )</td>
<td>( \frac{2(q+1)(q+1)}{(n+1)n} )</td>
<td>( T_r )</td>
<td>( \frac{2(q+1)(q+1)}{(n+1)n} )</td>
</tr>
<tr>
<td>( T )</td>
<td>( \frac{2(q+1)(q+1)}{(n+1)n} )</td>
<td>( G_\sigma )</td>
<td>( \frac{(q+1)^2 - (q+1)\sigma}{(q+1)^2} )</td>
</tr>
</tbody>
</table>

A particular subset of the rank-dependent poverty measures is when weights are linear with respect to the place they take up in the distribution.

**Definition 8.** Linear rank-dependent poverty measures are the rank-dependent poverty measures for which the weights are linear with respect to their position in the distribution, namely, the weights are \( w_i = e + (i - 1)d \), where \( e \) and \( d \) do not depend on \( i \).

Some examples of linear rank-dependent poverty measures are \( PGR \), \( S \), \( S' \), \( T \), \( SST \) and \( G_2 \).

However, following definition (7), it is easy to check that rank-dependent poverty measures are not directly \( OWA \) operators, since they do not automatically fulfill the \( \sum_{i=1}^{q} w_i = 1 \) requirement. However, every rank-dependent poverty measure \( P(x, z) \) can be written as the product of a normalization factor and the normalized poverty index, as follows:

\[
P = C_P \cdot A_P(g_q) = C_P \cdot \sum_{i=1}^{q} w_i g_i
\]

where \( C_P \) is the normalization factor, \( \frac{w_i}{C_P} \) is the normalized weight and \( A_P(g_q) \) is the corresponding normalized poverty index. Now, it is easy to see that the normalized poverty index will be an \( OWA \) operator based on the poverty gap vector.

For each poverty index listed in Table 2, Table 3 shows the corresponding normalization factors and the \( OWA \) operators.

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\(^7\)The transfer axiom states that, given other things, a pure transfer of income from a poor individual to any other individual that is richer must increase the poverty measure.

\(^8\)See Bosmans [4].
Table 3: Normalization factor and normalized poverty measures

<table>
<thead>
<tr>
<th>Measure</th>
<th>$c_P$</th>
<th>$\bar{w}_i$</th>
<th>Measure</th>
<th>$c_P$</th>
<th>$\bar{w}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PGR</td>
<td>$H$</td>
<td>$\frac{1}{2}$</td>
<td>SST</td>
<td>$H(2 - H)$</td>
<td>$\frac{2(a+1)-H}{2}$</td>
</tr>
<tr>
<td>S</td>
<td>$H$</td>
<td>$\frac{2(a+1)q}{2(a+1)}$</td>
<td>$K_k$</td>
<td>$H$</td>
<td>$\frac{g(a+2)-H}{2(a+1)}$</td>
</tr>
<tr>
<td>$S'$</td>
<td>$H$</td>
<td>$\frac{2(a+0.5)q}{2(a+1)}$</td>
<td>$T_r$</td>
<td>$\frac{H(r+1)}{r+1}$</td>
<td>$\frac{g(r+2)-1}{2}$</td>
</tr>
<tr>
<td>T</td>
<td>$\frac{H(2-H)+1}{2}$</td>
<td>$\frac{2(a+1)-1}{2(a+1)+1}$</td>
<td>$G_o$</td>
<td>$1 - (1 - H)^{\sigma}$</td>
<td>$\frac{g(r+2)-1}{2}$</td>
</tr>
</tbody>
</table>

3. Orness classification

As we have mentioned before, the concept of orness can be interpreted as a degree that is related to each weight vector or to each measure. In this section we will compute the orness values for the rank-dependent welfare functions and the rank-dependent poverty measured introduced in the previous sections. Then, we will rank all these indices in terms of their orness value.

3.1. Rank-dependent welfare functions

As mentioned, every rank-dependent welfare function is an OWA operator. Consequently, it would be possible to compute the orness values for all of them. Following the definition of the rank-dependent welfare measures we know that all the weights are ordered in a non increasing way. Consequently, the corresponding orness values will be between 0 and $\frac{1}{2}$, see Liu and Lou [14].

We will start computing the orness values of the rank-dependent welfare functions shown in this paper. Then, we will classify them in terms of their orness value.

The following proposition shows the orness values for the S-Gini welfare functions.

Proposition 1. The orness values for the S-Gini family of welfare functions are:

$$orness(W_{G\sigma}) = \frac{1}{(n-1)n^\sigma} \sum_{i=1}^{n-1} i^\sigma$$

Proof of Proposition 1: See Appendix.

Now, we will focus on the welfare functions of the Bonferroni and the Vergottini. The next propositions shows the orness values of these two.

Proposition 2. The orness values for the Bonferroni and the Vergottini welfare functions are the following:

$$orness(W_B) = \frac{1}{4}$$

and

$$orness(W_V) = \frac{1}{2} - \frac{1}{4c}$$

Proof of Proposition 2: See Aristondo et al. [2].

Now we will try to classify the below functions in terms of their orness value. We will start by classifying the S-Gini family of welfare functions. The following proposition shows the orness classification for the S-Gini family.

Proposition 3. The orness of the S-Gini welfare functions decreases for higher values of the parameter $\sigma$. That is:

$$orness(W_{G\sigma}) > orness(W_{G\beta})$$

for $\forall \sigma, \beta \in \mathbb{R}$ and $\beta > \sigma \geq 2$.  

6
Proof of Proposition 3: See Appendix.

In the following proposition we will order the Bonferroni and the Vergottini welfare functions and two members of the S-Gini family with respect to their orness value.

Proposition 4. The S-Gini welfare functions for $\sigma = 2$ and $\sigma = 3$ and the Bonferroni and the Vergottini welfare functions can be ordered with respect to their orness values as follows:

$$\text{orness}(W_{G2}) < \text{orness}(W_B) < \text{orness}(W_{G3}) < \text{orness}(W_V)$$

Proof of Proposition 4: See Appendix.

3.2. Rank-dependent poverty measures

As we have mentioned before, every normalized rank-dependent poverty measure will be an OWA operator and it would be possible to compute the orness values for all of them. Following the definition of the rank-dependent poverty measures, all the weights are ordered in a non decreasing way, consequently they will take an orness value higher or equal to $1/2$, see Liu and Lou [14].

By definition, the orness values will be higher for higher weights at the top of the distribution $g$. That is, we obtain higher orness values when higher weights are applied to lower income values. This sensibility on the bottom of the distribution could be interpreted as a distribution sensitivity of the poverty measures. We will focus on this in the following section.

In what follows, we add the prefix $N$ to the name of each rank-dependent poverty index in order to refer to the normalized rank-dependent poverty measures obtained applying equation 4.

The following proposition shows the orness values for the normalized Poverty Gap Ratio, the normalized Sen and the normalized Second Sen poverty measures, the normalized Thon poverty measure, the normalized Shorrocks poverty measure and the normalized Thon’s family of poverty indices.

Proposition 5. The orness values for $NPGR$, $NS$, $NS'$, $NT$, $NSST$ and $NT_\tau$ are:

$$\text{orness}(NPGR) = \frac{1}{2} \quad \text{orness}(NS) = \frac{2}{3} \quad \text{orness}(NS') = \frac{2}{3} + \frac{1}{6q}$$

$$\text{orness}(NT) = \frac{3n - q + 2}{3(2n - q + 1)} \quad \text{orness}(NSST) = \frac{6n - 2q + 1}{6(2n - q)} \quad \text{orness}(NT_\tau) = \frac{3\tau n - 2q + 1}{6(\tau n - q)}$$

Proof of Proposition 5: See Appendix.

Next proposition will order a first group of linear rank-dependent poverty measures in terms of their orness value.

Proposition 6. The orness values for the normalized rank-dependent poverty measures $NPGR$, $NT$, $NSST$, $NS$ and $NS'$ can be ordered with respect to their orness value as follows:

$$\text{orness}(NPGR) \leq \text{orness}(NT) \leq \text{orness}(NSST) \leq \text{orness}(NS) \leq \text{orness}(NS')$$

(8)

Proof of Proposition 6: It is straightforward from Proposition 5 and by operating.

Next proposition is devoted to rank the normalized class of Thon indices.

Proposition 7. The following orness ordenation holds for the Thon class of poverty indices:

$$\text{orness}(NT_\beta) < \text{orness}(NT_\theta) \text{ for } \beta > \theta$$

$$\text{orness}(NT_\infty = NPGR) < \text{orness}(NT_3) < \text{orness}(NT_{2+\frac{1}{n}}) < \text{orness}(NT_\mu) < \text{orness}(NT_2 = SST)$$

(10)

for every $\beta, \theta, \delta, \mu \geq 2$ and $\delta > 2 + \frac{1}{n}$, $\mu < 2 + \frac{1}{n}$. 7
Proof of Proposition 7: See Appendix.

Summarizing, if we want to rank the Thon class for natural values of $\tau$, we have:

$$orness(NT_{\infty} = NPGR) < \cdots < orness(NT_{4}) < orness(NT_{3}) < orness(NT) < orness(NT_{2} = NSST)$$

In a similar way, we can rank the members of the Kakwani and the S-Gini families of poverty indices. The next proposition shows the orness values for each family of indices.

Proposition 8. The orness values for the normalized poverty measures of Kakwani family ($NK_k$) and the orness values for the normalized poverty measures of S-Gini ($NG_{\sigma}$) are, respectively:

$$orness(NK_k) = \frac{1}{(q-1)} \sum_{i=1}^{q} (i^{k+1} - i^k)$$

$$orness(NG_{\sigma}) = \frac{1}{(n^\sigma - (n-q)^{\sigma})} \frac{1}{(q-1)} \sum_{i=1}^{q} [(n+1-i)^{\sigma} - (n-i)^{\sigma}] (q-i)$$

Proof of Proposition 8: It is straightforward by operating.

The following proposition establishes the rank in terms of orness values for the Kakwani family.

Proposition 9. The orness values for the normalized Kakwani family increases when the parameter $k$ increases, as follows:

$$orness(NK_k) < orness(NK_{k+1})$$

for every $k, q, n \in N$ that $k \geq 1$ and $n > q \geq 2$.

Proof of Proposition 9: See Appendix.

Combining results of Proposition 6, Proposition 7 and Proposition 9, the following result holds:

Proposition 10. The following inequalities hold:

$$orness(NK_0 = NPGR) < orness(NK_1 = NS) < orness(NK_2) < orness(NS') < orness(NK_3)$$

for every $k, n \in N$ and $n > q \geq 2$.

Proof of Proposition 10: See Appendix.

Now we will classify the S-Gini family of indices in terms of their $\sigma$ parameter.

Proposition 11. The orness values for the normalized S-Gini family increases when the $\sigma$ parameter increases as follows:

$$orness(NG_{\sigma}) < orness(NG_{\sigma+1})$$

for $\forall \sigma, q, n \in N$ and $n > q \geq 2, \sigma \geq 2$.

Proof of Proposition 11: See Appendix.

In this section we have ranked almost all the rank-dependent poverty measures in terms of the orness values associated to the normalized rank-dependent poverty measures. In the following section we will show the existing link between the orness values of welfare and poverty measures and the effect of the measures with respect to some income specific transfers.
4. Distribution-sensitivity

If we consider an income transfer from an individual to another poorer individual, a progressive transfer, the S-concavity of every rank-dependent welfare functions $W_a$ entails an improvement on the welfare level. On the other hand, the S-convexity of every rank-dependent poverty measures will oblige poverty measures to decrease with progressive income transfers between poor individuals. Nevertheless, if the amount given by the donor is higher than the amount the recipient gets, then only some welfare measures will improve and some poverty measures will decrease after the transfer. Atkinson (1973) and Okun (1975) pioneered the use of this kind of transfers called lossy transfers.

We define also another kind of transformations called lossy equalization. This transformation equalizes all the income values, but diminishes the total income for every individual or every poor individual for welfare functions or for poverty measures, respectively.

In this work, we will follow the Bosmans proposal to compare welfare measures and poverty measures on the basis of distribution-sensitivity by comparing their reactions to lossy transfers and lossy equalizations.

4.1. Rank-dependent welfare functions

In what follows any income distribution $x \in D$ will be defined in a non increasing way, that is, $x_1 \geq x_2 \geq \cdots \geq x_n$.

We define the lossy transfers and the lossy equalizations among individuals as follows:

**Definition 9.** Let $x$ and $y$ be two income distributions in $D$. Then $x$ is obtained from $y$ by a lossy transfer among the individuals if $n_x = n_y = n$ and $x = (y_1, y_2, \ldots, y_i - \beta, \ldots, y_j + \alpha, \ldots, y_n)$ where $0 < \alpha < \beta$ and $y_i < y_i - \beta \leq y_j + \alpha < y_j$.

**Definition 10.** Let $x$ and $y$ be two income distributions in $D$. Then $x$ is obtained from $y$ by a lossy equalization among the individuals if $n_x = n_y = n$ and $x = (\theta, \ldots, \theta)$ where $\theta n < \sum_{i=1}^{n} y_i$.

Then, we will say that a welfare function $W$ is at least as distribution-sensitive for lossy transfers or equalizations as a welfare function $V$ if $W$ registers a welfare increment for each lossy transfer or equalization transfer for which $V$ does. The following definitions establish the distribution-sensitivities for these operations.

**Definition 11.** Let two welfare functions $W$ and $V$ and suppose that $x$ is obtained from $y$ by a lossy transfer among the individuals. Then, $W$ is at least as distribution-sensitive for lossy transfers as $V$ if $V(x) > V(y)$ implies $W(x) > W(y)$.

A similar definitions holds for lossy equalization transfers.

**Definition 12.** Let two welfare functions $W$ and $V$ and suppose that $x$ is obtained from $y$ by a lossy equalization among the individuals. Then, $W$ is at least as distribution-sensitive for lossy equalizations as $V$ if $V(x) > V(y)$ implies $W(x) > W(y)$.

Following the two distributions-sensitivities proposed, welfare functions can be ordered in terms of their distribution-sensitivity for lossy transfers or lossy equalizations. We have just found an ordination for the welfare functions in terms of their orness value. The following proposition holds that for linear rank-dependent welfare function the orness classification and the classifications for lossy transfers and lossy equalizations are the same.

---

9See Liu [12] and Liu and Chen [13]
Proposition 12. Let $W = \sum_{i=1}^{n} a_i x[i]$ and $V = \sum_{i=1}^{n} e_i x[i]$ be two linear rank-dependent welfare functions. Then three conditions are equivalent:

(i) $W$ is at least as distribution-sensitive for lossy transfers as $V$.

(ii) $W$ is at least as distribution-sensitive for lossy equalizations as $V$.

(iii) $\forall k = 1, \ldots, n$ we have $k \sum_{i=1}^{k} a_i \leq k \sum_{i=1}^{k} e_i$.

(iv) $a_i - a_{i+1} \leq e_i - e_{i+1} \forall i = 1, \ldots, n$.

(v) $\frac{a_i}{a_j} \leq \frac{e_i}{e_j} \forall i < j < n$.

(vi) orness$(W) \leq$ orness$(V)$

Proof of Proposition 12: See Appendix.

Consequently, we can conclude that the orness value corresponding to each rank-dependent welfare function can be interpreted as a distribution-sensitivity measure. Following this criteria, for higher orness values we will have measures that are more sensitive for transfers with lower income values. Specifically, we have proved that for welfare functions with linear weights, the orness value is directly linked with two types of income transfers that are introduced to define distribution-sensitivity of welfare functions. That is, Proposition 12 conclude that if orness$(W)$ is lower than orness$(V)$, following statement (iii), the generalized Lorenz curve of the vector of weights $a$ is dominated for the generalized Lorenz curve of the vector of weights $e$. Consequently, we can conclude that the vector of weights $e$ can be obtained from the vector of weights $a$ by a finite sequence of replications and/or permutations and/or progressive transfers, see Marshall and Olkin [11].

4.2. Rank-dependent poverty measures

Similarly, we define the lossy transfers and the lossy equalizations among poor as follows:

Definition 13. Let $x$ and $y$ be two income distributions in $D$. Then $x$ is obtained from $y$ by a lossy transfer among the poor if $n_x = n_y = n$, $q_x = q_y = q$, and $x = (y_1, y_2, \ldots, y_i, y_i - \beta, \ldots, y_j + \alpha, y_{j+1}, \ldots, y_q, y_{q+1}, \ldots, y_n)$ where $0 < \alpha < \beta$ and $y_i < y_i - \beta \leq y_j + \alpha < y_j < z$.

Definition 14. Let $x$ and $y$ be two income distributions in $D$. Then $x$ is obtained from $y$ by a lossy equalization among the poor if $n_x = n_y = n$, $q_x = q_y = q$, and $x = (\theta, \ldots, \theta, y_{q+1}, y_{q+2}, \ldots, y_n)$ where $\theta q < \frac{1}{q} \sum_{i=1}^{q} y_i$.

Definition 15. Let two poverty measures $P$ and $Q$ and suppose that $x$ is obtained from $y$ by a lossy transfer among the poor. $P$ is at least as distribution-sensitive* for lossy transfers among the poor as $R$ if $R(x) < R(y)$ implies $P(x) < P(y)$.

Definition 16. Let two poverty measures $P$ and $Q$ and suppose that $x$ is obtained from $y$ by a lossy equalization among the poor. $P$ is at least as distribution-sensitive* for lossy equalizations among the poor as $R$ if $R(x) < R(y)$ implies $P(x) < P(y)$.

Following Bosmans [4], we know poverty measures can be in terms of their distribution-sensitivity for lossy transfers or lossy equalizations. In this work, we have proposed a new classification in terms of the orness values of their normalized measures. The next proposition holds that for linear rank-dependent poverty indices the three classifications proposed the measures equally.
Proposition 13. Let \( P_w = \sum_{i=1}^{q} w_i g[i] \) and \( P_v = \sum_{i=1}^{q} v_i g[i] \) be two linear rank-dependent poverty measures. Then three conditions are equivalent:

(i) \( P_W \) is at least as distribution-sensitive* for lossy transfers as \( P_V \).

(ii) \( P_W \) is at least as distribution-sensitive* for lossy equalizations as \( P_V \).

(iii) \( \forall q, n, 1 \leq q \leq n \) and \( \forall k = 1, \ldots, q \) we have \( \sum_{i=1}^{k} w_i \geq \sum_{i=1}^{k} v_i \).

(iv) \( \bar{w}_i - \bar{w}_{i+1} \geq \bar{v}_i - \bar{v}_{i+1} \) \( \forall i = 1, \ldots, q \)

(v) \( \frac{w_i}{w_j} \geq \frac{v_i}{v_j} \) \( \forall i = 1, \ldots, q \)

(vi) \( \text{orness}(NP_w) \geq \text{orness}(NP_v) \)

Proof of Proposition 13: See Appendix.

In this work, the orness value corresponding to each rank-dependent poverty measure will be interpreted as a distribution-sensitivity measure. Following this criteria, for higher orness values we will have measures that are more sensitive for transfers with lower income values. Specifically, we have proved that for poverty measures with linear weights, the orness value is directly linked with two types of income transfers that are introduced to define distribution-sensitivity of poverty measures. That is, Proposition 12 conclude that if orness\((W)\) is higher than orness\((V)\), following statement (iii), the generalized Lorenz curve of the vector of weights \( w \) dominates the vector of weights \( v \). Consequently, we can conclude that the vector of weights \( w \) can be obtained from the vector of weights \( v \) by a finite sequence of replications and/or permutations and/or progressive transfers, see Marshall and Olkin [11].

5. Concluding remarks

In this work we prove that every rank-dependent welfare functions and every rank-dependent poverty measures have underlying an OWA operator. Then, we compute the orness value for the OWA operators and we classify the rank-dependent welfare functions and the rank-dependent poverty measures in terms of this numerical value. We prove that the orness value assigned to these measures could be interpreted as a numerical value that measures the degree of sensitivity of the measures for transfers at the bottom of the distribution. Finally, we show that for linear measures, the orness classification is exactly the same as an existing classification of the measures in terms of their sensibility for some known transfers of incomes.
PROOF OF PROPOSITION 4:

\[ \text{orness}(W_{Gx}) = \frac{1}{n-1} \sum_{i=1}^{n} \frac{i^2 - (i-1)^2}{n^2} (n-i) = \frac{1}{n-1} \sum_{i=1}^{n} \frac{i^2 - (i-1)^2}{n^2} n - \frac{1}{n-1} \sum_{i=1}^{n} \frac{i^2 - (i-1)^2}{n^2} i = \]

\[ = \frac{n}{n-1} - \frac{1}{n-1} \sum_{i=1}^{n} \frac{i^2 + (i-1)^2}{n^2} = \frac{n}{n-1} - \frac{1}{(n-1)n^2} (n^2 - \sum_{i=1}^{n-1} i^2) = \frac{1}{n-1} \sum_{i=1}^{n-1} i^2 . \]

**Proof of Proposition 3:** For \( \beta > \sigma \) we want to prove that:

\[ \text{orness}(W_{Gx}) = \frac{1}{(n-1)n^2} \sum_{i=1}^{n} i^2 \sigma^2 > \text{orness}(W_{Gx}) = \frac{1}{(n-1)n^2} \sum_{i=1}^{n-1} i^2 \beta \]

Operating, \( \sum_{i=1}^{n-1} (\frac{\sigma}{\beta})^2 > \sum_{i=1}^{n} (\frac{i}{n})^2 \) that it is true for \( i = 1, \cdots, n-1 \) since \( \frac{i}{n} \in (0,1) \).

**Proof of Proposition 4:** By mathematical manipulation and remembering that \( \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 \) and \( \sum_{i=1}^{n} i^3 = (n(n+1)/2)^2 \) we have that \( \text{orness}(W_{Gx}) = \frac{1}{4} - \frac{1}{6n} \) and \( \text{orness}(W_{Gx}) = \frac{1}{4} - \frac{1}{6n} \). It is clear that \( \text{orness}(W_{Gx}) = \frac{1}{4} - \frac{1}{6n} < \text{orness}(W_{Gx}) = \frac{1}{4} \). \( \text{orness}(W_{Gx}) = \frac{1}{4} - \frac{1}{6n} \) implies that \( \frac{2}{3n} < 0 \) that is true for every \( n > 2 \). The last statement we need to prove is that \( \text{orness}(W_{Gx}) = \frac{1}{4} - \frac{1}{6n} < \text{orness}(W_{Gx}) = \frac{1}{2} - \frac{1}{4n} \). Operating we need to prove that \( \frac{3n}{2} \frac{q}{q} < \frac{3n}{2} \frac{q}{q} \). As \( c = \sum_{j=2}^{\infty} \frac{n}{n-1} j \), then \( \sum_{j=2}^{\infty} \frac{n}{n-1} \frac{3}{n+1} \) must be satisfied. For \( \forall n \geq 7 \) we have that \( \sum_{j=2}^{n} \frac{n}{n-1} \frac{3}{n+1} \). We have proved that it is true for every \( n \geq 7 \). The prove for every \( 3 \leq n \leq 6 \) is straightforward simply operating.

**Proof of Proposition 5:** By mathematical manipulation and remembering that \( \sum_{i=1}^{q} i = q(q+1)/2 \) and \( \sum_{i=1}^{q} i^2 = q(q+1)(2q+1)/6 \) we have:

\[ \text{orness}(NPGR) = \frac{1}{q-1} \sum_{i=1}^{q} \frac{i^2}{q} \frac{1}{q(q-1)} \left( q^2 - \frac{q(q+1)}{2} \right) = \frac{1}{q(q-1)} \left( q^2 - \frac{q(q+1)}{2} \right) = \frac{1}{2} . \]

\[ \text{orness}(NS) = \frac{1}{q-1} \sum_{i=1}^{q} \frac{2(q+1-i)(q-i)}{q(q+1)} = \frac{2}{q(q-1)(q+1)} \sum_{i=1}^{q} \frac{(q+1-i)(q-i)}{q} = \frac{2}{q(q-1)(q+1)} \sum_{i=1}^{q} \frac{q^2 + q - (2q+1)i + i^2}{q} = \frac{2}{q(q-1)(q+1)} \frac{q(q+1)(q-1)}{3} = \frac{2}{3} . \]

\[ \text{orness}(NS) = \frac{1}{q-1} \sum_{i=1}^{q} \frac{2(q+0.5-i)(q-i)}{q^2} = \frac{2}{q(q-1)(q+1)} \sum_{i=1}^{q} \frac{q^2 + q - (2q+0.5)i + i^2}{q} = \frac{2}{q(q-1)} \frac{q}{q-1} \frac{q}{q-1} \frac{4q^2 - 3q^2 - q}{12} = \frac{1}{2} . \]

\[ \text{orness}(NT) = \frac{1}{q-1} \sum_{i=1}^{q} \frac{2(n+1-i)(q-i)}{q(n-2q+1)} = \frac{2}{q(q-1)(2n-q+1)} \sum_{i=1}^{q} \frac{(n+1-i)(q-i)}{q} = \frac{2}{q(q-1)(2n-q+1)} \sum_{i=1}^{q} \frac{q(n+1) - (n+q+1)i + i^2}{q} = \frac{2}{q(q-1)(2n-q+1)} \frac{q(q+1)(q+2)}{6q(q-1)(2n-q+1)} = \frac{1}{2} . \]

\[ \text{orness}(NSST) = \frac{1}{q-1} \sum_{i=1}^{q} \frac{2(n+0.5-i)(q-i)}{q(2n-q)} = \frac{2}{q(q-1)(2n-q)} \sum_{i=1}^{q} \frac{(n+0.5-i)(q-i)}{q} = \frac{2}{q(q-1)(2n-q)} \sum_{i=1}^{q} \frac{q(n+0.5) - (n+q+0.5)i + i^2}{q} = \frac{2}{q(q-1)(2n-q)} \frac{q(q+1)(q+2)}{6q(q-1)(2n-q)} = \frac{1}{2} . \]
orness \( (NT_2) = \frac{1}{q-1} \sum_{i=1}^{q} \frac{(\tau n+1-2i)(q-1)}{(\tau n-1)} = \frac{1}{q-1} \left( \frac{q}{(\tau n-1)} \sum_{i=1}^{q} (\tau n+1-2i) - \frac{q}{(\tau n-1)} \sum_{i=1}^{q} (\tau n + 1 - (\tau n + 2q + 1)i) \right) \)

= \frac{1}{q(q-1)(\tau n-1)} \left( q^{2}(\tau n + 1) - (\tau n + 2q + 1)(q(q+1))(2q+1) + \tau n \right) - \frac{q}{q(q-1)(\tau n-1)} \left( \frac{3n-2q+1}{6q-1} \right) = \frac{q(q-1)(3\tau n - 2q + 1)}{6q(q-1)(\tau n-1)} = \frac{1}{2} + \frac{q+1}{6 \tau n-q}.

**Proof of Proposition 7:** For \( \beta > \theta \) we want to prove:

orness \( (NT_\beta) = \frac{3(n-2q+1)}{4(n-2q+1)} = \text{orness}(NT_0) \)

Operating, we need to prove, \(-n(q+1)(\beta - \theta) < 0\) that is straightforward.

For equation 10, we want to find the \( q \) value that satisfies \( \text{orness}(T_q) = \frac{3(n-2q+1)}{6q+1} = \frac{3n-q+2}{3(2n-q+1)} = \text{orness}(T) \)

Operating, we have, \(-3(q+1)((\tau-2)n-1) = 0\), then \( \tau = 2 + \frac{1}{n} \).

The remaining rankings are straightforward. Finally, applying equation 9 we conclude the proof.

**Proof of Proposition 9:** We will prove this proposition by mathematical induction on \( q \).

For \( q=2 \) (basic step):

We have: \( \text{orness}(NK_k) = \frac{1}{2} \sum_{i=1}^{2} i^k = \frac{2^k}{1 + 2^k} \) and \( \text{orness}(NK_{k+1}) = \frac{2^{k+1}}{1 + 2^{k+1}} \). We need to prove that \( \frac{2^k}{1 + 2^k} \leq \frac{2^{k+1}}{1 + 2^{k+1}} \), or equivalently, \( 2^k \left( 1 + 2^{k+1} \right) \leq 2^{k+1} \left( 1 + 2^k \right) \). By mathematical manipulation we obtain: \( 2^k \leq 2^{k+1} \) that is true for every \( k \geq 1 \).

Let us assume that it is true for \( q \) (inductive step):

\[
\frac{1}{(q-1)} \sum_{i=1}^{q} i^k \leq \frac{1}{(q-1)} \sum_{i=1}^{q} (i^{k+1} - i^k)
\]

Analogously

\[
\sum_{i=1}^{q} i^{k+1} \leq \sum_{i=1}^{q} \left( i^{k+2} - i^{k+1} \right)
\]

We need to show that it is true for \( q+1 \). That is:

\[
\sum_{i=1}^{q+1} i^{k+1} \leq \sum_{i=1}^{q} \left( i^{k+2} - i^{k+1} \right)
\]

Again, operating, we have:

\[
\left( \sum_{i=1}^{q} i^{k+1} + (q+1)^{k+1} \right) \leq \left( \sum_{i=1}^{q} i^{k+1} \right) + q(q+1)^k + (q+1)^k \sum_{i=1}^{q} i^{k+1} \leq \left( \sum_{i=1}^{q} i^{k+1} \right) + q(q+1)^{k+1}
\]

Analogously,

\[
\sum_{i=1}^{q} i^{k+1} \leq q(q+1)^k \sum_{i=1}^{q} i^{k+1} + (q+1)^k \sum_{i=1}^{q} i^{k+1} \sum_{i=1}^{q} (i^{k+1} - i^k) + q(q+1)^k \sum_{i=1}^{q} i^{k+1} - (q+1)^k \sum_{i=1}^{q} (i^{k+2} - i^{k+1}) \leq 0
\]
Finally, by induction

\[ \sum_{i=1}^{q} i^{k+1} \cdot \sum_{i=1}^{q} (i^{k+1} - i^k) + q(q+1)^k \sum_{i=1}^{q} i^{k+1} + (q+1)^{k+1} \sum_{i=1}^{q} (i^{k+1} - i^k) - \sum_{i=1}^{q} i^k \cdot \sum_{i=1}^{q} (i^{k+2} - i^{k+1}) - q(q+1)^k \sum_{i=1}^{q} i^k - (q+1)^{k+1} \sum_{i=1}^{q} (i^{k+2} - i^{k+1}) \leq \sum_{i=1}^{q} i^k \cdot \sum_{i=1}^{q} (i^{k+2} - i^{k+1}) + q(q+1)^k \sum_{i=1}^{q} i^{k+1} + (q+1)^{k+1} \sum_{i=1}^{q} (i^{k+1} - i^k) + - \sum_{i=1}^{q} i^k \cdot \sum_{i=1}^{q} (i^{k+2} - i^{k+1}) - q(q+1)^k \sum_{i=1}^{q} i^k - (q+1)^{k+1} \sum_{i=1}^{q} (i^{k+2} - i^{k+1}) = = q(q+1)^k \sum_{i=1}^{q} i^{k+1} + (q+1)^{k+1} \sum_{i=1}^{q} (i^{k+1} - i^k) - q(q+1)^k \sum_{i=1}^{q} i^k + -(q+1)^k \sum_{i=1}^{q} (i^{k+2} - i^{k+1}) = 2(q+1)^{k+1} \sum_{i=1}^{q} i^{k+1} - (q+1)^{k+2} \sum_{i=1}^{q} i^k + - (q+1)^k \sum_{i=1}^{q} i^{k+2} = (q+1)^k \sum_{i=1}^{q} i^k (-i^2 + 2(q+1)i - (q+1)^2) = = -(q+1)^k \sum_{i=1}^{q} i^k (i - (q+1))^2 \leq 0 \]

and it is trivial to prove that the last line is true.

**Proof of Proposition 10:**

\[ \text{orness}(NK_3) = \frac{1}{(q-1)} \sum_{i=1}^{q} (i^3 - i^2) = \frac{12q^2+15q+2}{15q(q+1)} \text{ and } \text{orness}(NS') = \frac{1+4q}{6q}. \]

We need to prove that \( \frac{12q^2+15q+2}{15q(q+1)} < \frac{1+4q}{6q} \). Operating, we have \( 4q^2 + 5q - 1 > 0 \) that is true for every \( q \geq 2 \).

**Proof of Proposition 11:** Before starting with the proof, we need to prove the following two Lemmas.

**Lemma 1.** The following inequality holds:

\[ 2n^\sigma (n-2)^\sigma \leq n^\sigma (n-1)^\sigma + (n-1)^\sigma (n-2)^\sigma \quad (13) \]

for any \( \sigma, n \in N \) and \( n \geq 3 \).

**Proof of Lemma 1:** We will prove this proposition by mathematical induction on \( \sigma \).

For \( \sigma = 1 \) (basic step) we have:

\[ 2n(n-2) \leq n(n-1) + (n-1)(n-2), \]

operating we have \( 0 \leq 2 \) that is true.

Let assume that it is true for \( \sigma \) (inductive step):

\[ 2n^\sigma (n-2)^\sigma \leq n^\sigma (n-1)^\sigma + (n-1)^\sigma (n-2)^\sigma. \]

We will show that it is true for \( \sigma + 1 \). That is, \( 2n^{\sigma+1}(n-2)^{\sigma+1} \leq n^{\sigma+1}(n-1)^{\sigma+1} + (n-1)^{\sigma+1}(n-2)^{\sigma+1} \leq 0 \).

By induction, we have:

\[ 2n^{\sigma+1}(n-2)^{\sigma+1} - n^{\sigma+1}(n-1)^{\sigma+1} + (n-1)^{\sigma+1}(n-2)^{\sigma+1} \leq n(n-2)(n^\sigma (n-1)^\sigma + (n-1)^\sigma (n-2)^\sigma) - n^{\sigma+1}(n-1)^{\sigma+1} + (n-1)^{\sigma+1}(n-2)^{\sigma+1} + (n-1)^\sigma [(n-2)^{\sigma+1} - n^{\sigma+1}] \leq 0 \] that is true for \( n \geq 3 \).

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\[ ^{10} \text{We have used that } \sum_{i=1}^{q} i = q(q+1)/2, \sum_{i=1}^{q} i^2 = q(q+1)(2q+1)/6, \sum_{i=1}^{q} i^3 = (q(q+1)/2)^2 \text{ and } \sum_{i=1}^{q} i^4 = q(6q^4+15q^2+10q^2-1)/30. \]
Lemma 2. The following inequality holds:

\[(q + 1)n^{\sigma}(n - q - 1)^{\sigma} \leq (n - q)^{\sigma}(n - q - 1)^{\sigma} + qn^{\sigma}(n - q)^{\sigma}\]  \hspace{1cm} (14)

for \(\forall \sigma, n, q \in \mathbb{N}\) and \(n > q \geq 2\).

Proof of Lemma 1: We will prove this proposition by mathematical induction on \(\sigma\).

For \(\sigma = 1\) (basic case):
\[(q + 1)n(n - q - 1) \leq (n - q)(n - q - 1) + qn(n - q).\] Operating we have, \(0 \leq q^2 + q\) that is true for \(q \geq 1\).

Let us assume it is true for \(\sigma\) (inductive step):
\[(q + 1)n^{\sigma}(n - q - 1)^{\sigma} \leq (n - q)^{\sigma}(n - q - 1)^{\sigma} + qn^{\sigma}(n - q)^{\sigma}.\]

We need to show that it is true for \(\sigma + 1\). That is,
\[(q + 1)n^{\sigma + 1}(n - q - 1)^{\sigma + 1} \leq (n - q)^{\sigma + 1}(n - q - 1)^{\sigma + 1} + qn^{\sigma + 1}(n - q)^{\sigma + 1}.\]

Operating
\[(q + 1)n^{\sigma}(n - q - 1)^{\sigma}n(n - q - 1) - (n - q)^{\sigma + 1}(n - q - 1)^{\sigma + 1} - qn^{\sigma + 1}(n - q)^{\sigma + 1} \leq 0.

Finally, by induction
\[(q + 1)n^{\sigma}(n - q - 1)^{\sigma}n(n - q - 1) - (n - q)^{\sigma + 1}(n - q - 1)^{\sigma + 1} - qn^{\sigma + 1}(n - q)^{\sigma + 1} \leq (n - q)^{\sigma}(n - q - 1)^{\sigma} + qn^{\sigma}(n - q)^{\sigma}n(n - q - 1) - (n - q)^{\sigma + 1}(n - q - 1)^{\sigma + 1} - qn^{\sigma + 1}(n - q)^{\sigma + 1} = q(n - q)^{\sigma}[(n - q - 1)^{\sigma + 1} - n^{\sigma + 1}] \leq 0\] that is true.

Lemma 1 and 2 allow us to compute the orness value for the normalized poverty measures.

Following with proposition 6 proof:

We want to prove the following:
\[
\sum_{i=1}^{q} \frac{[(n + 1 - i)^{\sigma} - (n - i)^{\sigma}] \cdot (q - i)}{(n^{\sigma} - (n - q)^{\sigma})(q - 1)} \leq \sum_{i=1}^{q} \frac{[(n + 1 - i)^{\sigma + 1} - (n - i)^{\sigma + 1}] \cdot (q - i)}{(n^{\sigma + 1} - (n - q)^{\sigma + 1})(q - 1)}
\]

Analogously
\[
(n^{\sigma + 1} - (n - q)^{\sigma + 1}) \sum_{i=1}^{q} [(n + 1 - i)^{\sigma} - (n - i)^{\sigma}] \cdot (q - i) \leq \]
\[
\leq (n^{\sigma} - (n - q)^{\sigma}) \sum_{i=1}^{q} [(n + 1 - i)^{\sigma + 1} - (n - i)^{\sigma + 1}] \cdot (q - i)
\]

We prove this proposition by mathematical induction on \(q\).

For \(q = 2\) (basic step):
\[(n^{\sigma + 1} - (n - 2)^{\sigma + 1}) (n^{\sigma} - (n - 1)^{\sigma}) \leq (n^{\sigma} - (n - 2)^{\sigma}) (n^{\sigma + 1} - (n - 1)^{\sigma + 1})\]

Operating,
\[2n^{\sigma}(n - 2)^{\sigma} \leq n^{\sigma}(n - 1)^{\sigma} + (n - 1)^{\sigma}(n - 2)^{\sigma}\]
that is true from Lemma 1.
Let us assume that it is true for \( q \) (inductive step):

\[
\frac{\sum_{i=1}^{q} [(n + 1 - i)\sigma - (n - i)\sigma] (q - i)}{(n\sigma - (n - q)\sigma) (q - 1)} \leq \frac{\sum_{i=1}^{q} [(n + 1 - i)\sigma^{+1} - (n - i)\sigma^{+1}] (q - i)}{(n\sigma^{+1} - (n - q)\sigma^{+1}) (q - 1)}
\]

We need to show that it is true for \( q + 1 \). That is:

\[
(n^{\sigma^{+1}} - (n - q - 1)^{\sigma^{+1}}) \sum_{i=1}^{q+1} [(n + 1 - i)^{\sigma} - (n - i)^{\sigma}] (q + 1 - i) \leq (n^{\sigma} - (n - q - 1)^{\sigma}) \sum_{i=1}^{q} [(n + 1 - i)^{\sigma^{+1}} - (n - i)^{\sigma^{+1}}] (q + 1 - i)
\]

Operating,

\[
(n^{\sigma^{+1}} - (n - q - 1)^{\sigma^{+1}}) \sum_{i=1}^{q} [(n + 1 - i)^{\sigma} - (n - i)^{\sigma}] (q - i) + (n^{\sigma^{+1}} - (n - q - 1)^{\sigma^{+1}}) (n^{\sigma} - (n - q)^{\sigma}) \leq (n^{\sigma} - (n - q - 1)^{\sigma}) \sum_{i=1}^{q} [(n + 1 - i)^{\sigma^{+1}} - (n - i)^{\sigma^{+1}}] (q - i) + (n^{\sigma} - (n - q - 1)^{\sigma}) (n^{\sigma^{+1}} - (n - q)^{\sigma^{+1}})
\]

Equivalently,

\[
(n^{\sigma^{+1}} - (n - q - 1)^{\sigma^{+1}}) \sum_{i=1}^{q} [(n + 1 - i)^{\sigma} - (n - i)^{\sigma}] (q - i) - (n^{\sigma} - (n - q - 1)^{\sigma}) \sum_{i=1}^{q} [(n + 1 - i)^{\sigma^{+1}} - (n - i)^{\sigma^{+1}}] (q - i) + (q + 1)n^{\sigma}(n - q - 1)^{\sigma} - (n - q)^{\sigma}(n - q - 1)^{\sigma} - qn^{\sigma}(n - q)^{\sigma} \leq 0
\]

By induction,

\[
[q + 1]n^{\sigma}(n - q - 1)^{\sigma} - (n - q)^{\sigma}(n - q - 1)^{\sigma} - qn^{\sigma}(n - q)^{\sigma} + \left[ (n^{\sigma^{+1}} - (n - q - 1)^{\sigma^{+1}}) \frac{(n^{\sigma} - (n - q)^{\sigma})}{(n^{\sigma^{+1}} - (n - q)^{\sigma^{+1}})} - (n^{\sigma} - (n - q - 1)^{\sigma}) \right] \cdot \sum_{i=1}^{q} [(n + 1 - i)^{\sigma^{+1}} - (n - i)^{\sigma^{+1}}] (q - i) \leq 0
\]

Operating,

\[
[q + 1]n^{\sigma}(n - q - 1)^{\sigma} - (n - q)^{\sigma}(n - q - 1)^{\sigma} - qn^{\sigma}(n - q)^{\sigma} \leq 0
\]

By Lemma 2 we have:

\[
[q + 1]n^{\sigma}(n - q - 1)^{\sigma} - (n - q)^{\sigma}(n - q - 1)^{\sigma} - qn^{\sigma}(n - q)^{\sigma} \leq 0
\]
And by definition we know that

\[
\left[ 1 + \frac{\sum_{i=1}^{q} [(n + 1 - i)^{\sigma + 1} - (n - i)^{\sigma + 1}] (q - i)}{(n^{\sigma + 1} - (n - q)^{\sigma + 1})} \right] = \text{orness}(1 + (q - 1)G_{\sigma + 1}) \geq 0
\]

that concludes the proof.

**Proof of Proposition 12:** First of all we need to prove another Lemma.

**Lemma 3.** For every \( x_1 \geq x_2 \geq \cdots \geq x_n \). If \( w_1 \geq w_2 \geq \cdots \geq w_n \) then we have that, \( \sum_{i=1}^{n} w_i x_i \geq \frac{\sum_{i=1}^{n} w_i}{n} \cdot \sum_{i=1}^{n} x_i \)

and if \( w_1 \leq w_2 \leq \cdots \leq w_n \) then \( \sum_{i=1}^{n} w_i x_i \leq \frac{\sum_{i=1}^{n} w_i}{n} \cdot \sum_{i=1}^{n} x_i \) is satisfied.

**Proof of Lemma 3:**

We need to prove that,

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} w_i (x_i - x_j) + \sum_{i=1}^{n} \sum_{j>i}^{n} w_i (x_i - x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i (x_i - x_j) + \sum_{i=1}^{n} \sum_{j>i}^{n} w_i (x_i - x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_j (x_i - x_j)
\]

Operating,

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} w_i (x_i - x_j) + \sum_{i=1}^{n} \sum_{j>i}^{n} w_i (x_i - x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i (x_i - x_j) + \sum_{i=1}^{n} \sum_{j>i}^{n} (w_i - w_j) (x_i - x_j)
\]

We have that for \( \forall i, j = 1, \ldots, n \) and \( j > i \) then \( x_i \geq x_j \) and for \( w_i \geq w_j \) we obtain the first condition and for \( w_i \leq w_j \) the second one. Following with Proposition 12:

(ii) \(\Rightarrow\) (iii) Assume that (ii) holds. Seeking a contradiction, assume that (iii) does not hold. Let \( k \) and \( n \) be such that \( 1 \leq k \leq n \) and \( \sum_{i=1}^{k} a_i > \sum_{i=1}^{k} e_i \). Let \( x \) and \( y \) two income distributions such that \( n_x = n_y = n \).

Let \( y_i = \alpha \) for \( i \leq k \) and \( y_i = \beta \) for \( i > k \) where \( \alpha > \beta \). Let \( x = (\theta, \ldots, \theta) \) where \( \theta = \alpha \sum_{i=1}^{k} a_i + \beta \sum_{i=k+1}^{n} a_i - \gamma \) with \( \gamma > 0 \). We need to prove that \( x \) is obtained from \( y \) by a lossy equalization. That is, \( \theta n < \sum_{i=1}^{n} y_i \).

From Lemma 3 we have that \( \theta = \sum_{i=1}^{n} a_i y_i - \gamma < \sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} \frac{a_i}{n} \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \frac{y_i}{n} \).

Now we have that \( W(y) = \theta + \gamma \) and \( W(x) = \theta \), consequently \( W(x) < W(y) \). Computing the same for \( V \),

\[
V(x) = \theta \quad \text{and} \quad V(y) = \alpha \sum_{i=1}^{k} e_i + \beta \sum_{i=k+1}^{n} e_i - \theta = \alpha \sum_{i=1}^{k} e_i + \beta \sum_{i=k+1}^{n} e_i - \alpha \sum_{i=1}^{k} a_i - \beta \sum_{i=k+1}^{n} a_i + \theta + \gamma = \alpha \sum_{i=1}^{k} e_i + \beta (1 - k) e_i - \alpha \sum_{i=1}^{k} a_i - \beta (1 - k) a_i + \theta + \gamma = (\alpha - \beta) \sum_{i=1}^{k} (e_i - a_i) + \theta + \gamma < \theta + \gamma.
\]

Hence, for a sufficient small \( \gamma > 0 \), we have \( V(x) > V(y) \). This contradicts (ii).

(iii) \(\Rightarrow\) (ii) Assume that (iii) holds. We define as \( s_k = \sum_{i=1}^{k} a_i \) and \( t_k = \sum_{i=1}^{k} e_i \) and we have for \( k = 1, \ldots, n \)
that $s_k \leq t_k$. Then, $W(x) - V(x) = \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} e_i x_i = \sum_{i=1}^{n-1} (s_i - l_i)(x_i - x_{i+1})$

As we have $s_k \leq t_k$ for $\forall k = 1, \ldots, n$ then $W(x) - V(x) \leq 0$ for $\forall x \in D$. Then, if $V(x) - V(y) = \theta - V(y) > 0$ then $W(x) - W(y) = \theta - W(y) \geq \theta - V(y) > 0$. Consequently, (ii) holds.

(iii) $\Rightarrow$ (vi) We have for $k = 1, \ldots, n$ that $s_k \leq t_k$. Then, $W(x) - V(x) = \sum_{i=1}^{n-1} (s_i - l_i)(x_i - x_{i+1})$

We have $s_k \leq t_k$ for $\forall k = 1, \ldots, n$, then $W(x) - V(x) \leq 0$ for $\forall x \in D$. Choosing $x = \left(\frac{n-1}{n-1}, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0\right)$ we have that $orness(W) \geq orness(V)$.

(iv) $\Rightarrow$ (iii) Starting from $a_i - a_{i+1} \leq e_i - e_{i+1}$ and operating we have that, $a_i - e_i \leq a_{i+1} - e_{i+1}$. Then, $a_i - e_i \leq a_{i+1} - e_{i+1}$ for $\forall a_i \leq \ldots, a_0$ we have $c, c' \geq 0$.

The orness values are, $orness(W) = \frac{6 - cn(n+1)}{12}$ and $orness(V) = \frac{6 - c'n(n+1)}{12}$. If $orness(W) \leq orness(V)$ then $\frac{6 - cn(n+1)}{12} \leq \frac{6 - c'n(n+1)}{12}$ that is $c \geq c'$. Operating, $a_i - a_{i+1} = -c$ and $e_i - e_{i+1} = -c'$, consequently we have that $a_i - a_{i+1} \leq e_i - e_{i+1}$.

(iv) $\Rightarrow$ (v) We know that $a_i - a_{i+1} \leq e_i - e_{i+1}$ holds. Since the weights are linear, $a_i = b + (i - 1)c$ and $e_i = b' + (i - 1)c'$, rewriting we have that $b + (i - 1)c - (b + ci) \leq b' + (i - 1)c' - (b' + ci')$ and manipulating $c \geq c'$. In addition, since (iv) $\Leftrightarrow$ (iii) we have that $a_1 \leq e_1$ that implies $b \leq b'$.

Now, we need to prove that $\frac{a_i}{e_i} \leq \frac{e_i}{a_i}$ for $\forall i \leq j \leq n$.

We prove that inequality for $j = i + 1$ since, by manipulation we always can write $\frac{a_i}{e_i}$ as follows:

$$\frac{a_i}{e_i} = \frac{a_i}{a_{i+1}} \cdot \frac{a_{i+1}}{a_{i+2}} \cdot \ldots \cdot \frac{a_{j-1}}{a_{j}}$$

So we will prove that $\frac{a_i}{e_{i+1}} \leq \frac{e_i}{a_i}$ for $i = 1, \ldots, n - 1$, that is, $\frac{b + (i - 1)c}{b' + ci} \leq \frac{b' + ci}{b + (i - 1)c'}$

Operating

$$(b + (i - 1)c)(c' + d'i) \leq (b + ci)(b' + (i - 1)c') \Rightarrow be' \leq b'c$$

that is true since $c \geq c'$ and $b \leq b'$.

(v) $\Rightarrow$ (vi) Assume that (v) holds. This is equivalent to satisfies $a_i e_j \leq e_i e_j$ for all $1 < j < n$. Then,

$$W(x) - V(x) = \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} e_i x_i = \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} e_i x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i x_i e_j - \sum_{i=1}^{n} \sum_{j=1}^{n} e_i x_i a_j =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_i + \sum_{i=1}^{n} \sum_{j<i}^{n} (a_i e_j - e_i a_j) x_i =$$
\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i e_j - e_i a_j) x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (e_j a_i - a_j e_i) x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (e_j a_i - a_j e_i) (x_{ij} - x_{ij})
\]

Since, for \( j > i \) we know that \( a_i e_j - e_i a_j \leq 0 \) and \( x_i > x_j \), this implies that \( W(x) \leq V(x) \).

Finally, choosing \( x = (\frac{n-1}{n-1}, \frac{n-1}{n-1}, \cdots, 0 \frac{n-1}{n-1}) \) we have that \( oren(x) \leq oren(y) \).

(i) \(\Rightarrow\) (v) Assume that (i) holds. Seeking a contradiction assume that (v) does not hold. Let \( i, j \) and \( n \) be such that \( 1 \leq i < j \leq n \) and \( \frac{a_i}{a_j} > \frac{e_i}{e_j} \). Let \( y \) be an income distribution in \( D \) such that \( x = (y_1, y_2, \cdots, y_j - \beta, \cdots, y_j + \alpha, \cdots, y_n) \) where \( y_i - \beta > y_{i+1} \geq y_j > y_j + \alpha \) and \( \frac{y_j}{\beta} = \frac{a_i}{a_j} \). Note that \( x \) is obtained from \( y \) by a lossy transfer among individuals. Then, \( W(x) = W(y) \), since \( a_i e_j - \beta y_i = 0 \). Now, if we compute \( V(x) - V(y) = \alpha e_j - \beta e_i \). Since, \( \beta = \frac{y_j}{\beta} > \frac{a_i}{a_j} \) then \( \alpha e_j - \beta e_i > 0 \). Consequently, \( V(x) - V(y) > 0 \) that contradicts (i).

(v) \(\Rightarrow\) (i) Assume that (v) holds. Let \( x \) and \( y \) two income distributions in \( D \) such that \( x \) is obtained from \( y \) by a lossy transfer among individuals, that is, \( x = (y_1, \cdots, y_j - \beta, \cdots, y_j + \alpha, \cdots, y_n) \) where \( 0 < \alpha < \beta \) and \( y_i > y_j - \beta > y_j + \alpha > y_j \). Assume that the transfer changes the position of individual \( i \) to \( i + l \) and the position of \( j \) to \( j - m \), therefore, \( y_i - \beta = x_{i+l} \) and \( y_j + \alpha = y_{j-m} \). We decompose the amounts \( \beta \) and \( \alpha \) in \( l \) and \( m \) amounts on terms for each position change, respectively. That is, \( \beta = \beta_0 + \beta_1 + \cdots + \beta_l \) and \( \alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_m \) with \( \beta_l = y_{i+l} - y_{i+l+1} \) for each \( t = 0, 1, \cdots, 1 \) and \( \alpha_t = y_{j-1} - y_{j-r} \) for each \( r = 0, 1, \cdots, m-1 \) and \( \beta_l = y_{i+l} - x_{i+l+1 \cdots} \) and \( \alpha_m = x_{j-m} - y_{j-m} \).

Now, if we have that \( V(x) - V(y) > 0 \)

\[
V(x) - V(y) = (v_{j-i} + v_{i-j+1} + \cdots + v_j) - (v_{i+d} + v_i + v_{i+1} + \cdots + v_i + v_{i+m} \beta_m) > 0
\]

Since (ii) holds, we have that \( \frac{w_i}{v_j} \leq \frac{w_j}{v_j} \) then

\[
(w_i, w_{i+1}, \cdots, w_j) = (v_i, v_{i+1}, \cdots) \]

We have defined \( \beta_l, \alpha_l \) for \( t = 0, 1, \cdots, 1 \) and \( r = 0, 1, \cdots, m \), then

\[
W(x) - W(y) = (w_{j-i} + w_{i-j+1} + \cdots + w_j) - (w_{i+d} + w_i + \beta_1 + \cdots + w_i + \beta_m) = \sum_{i=1}^{q} (s_i - t_i)(g_{ij}) \]

As we have that \( s_k \geq t_k \) for \( \forall k = 1, \cdots, q \), then, \( P_{\pi}(x) - P_{\pi}(x) \geq 0 \) for \( \forall x \in D \). Choosing \( x = (\frac{\beta_1}{\beta_2}, \frac{\beta_2}{\beta_3}, \cdots, \frac{\beta_l}{\beta_m}, 0) \) we have that \( oren(x) \geq oren(x) \).

(iii) \(\Rightarrow\) (iv) We define as \( s_k = \sum_{i=1}^{k} \pi_i \) and \( t_k = \sum_{i=1}^{k} \pi_i \) and we have for \( k = 1, \cdots, q \) that \( s_k \geq t_k \). Then,

\[
P_{\pi}(x) - P_{\pi}(x) = \sum_{i=1}^{q} \pi_i g_{ij} - \sum_{i=1}^{q} \pi_i g_{ij} = \sum_{i=1}^{q} (s_i - t_i)(g_{ij}) \]

As we have that \( s_k \geq t_k \) for \( \forall k = 1, \cdots, q \), then, \( P_{\pi}(x) - P_{\pi}(x) \geq 0 \) for \( \forall x \in D \). Choosing \( x = (\frac{\beta_1}{\beta_2}, \frac{\beta_2}{\beta_3}, \cdots, \frac{\beta_l}{\beta_m}, 0) \) we have that \( oren(x) \geq oren(x) \).

(iv) \(\Rightarrow\) (iii) If weights are linear, the normalized weights will be also linear. We define as \( \pi_i = a + (i-1)d \) and \( \pi_i = a' + (i-1)d' \) for \( 1 \leq i \leq q \) where \( a = \frac{2+q-d-a}{2q}, a' = \frac{2+d-d'a}{2q}, a + (q-1)d \geq 0, a' + (q-1)d' \geq 0 \)
and $d, d' \leq 0$.

The orness values are $\text{orness}(NP_w) = \frac{6-dq(q+1)}{12}$ and $\text{orness}(NP_v) = \frac{6-d'q(q+1)}{12}$.

If $\text{orness}(NP_w) \geq \text{orness}(NP_v)$ then $\frac{6-dq(q+1)}{12} \geq \frac{6-d'q(q+1)}{12}$ that is $d \leq d'$.

Operating, $\bar{w}_i - \bar{w}_{i+1} = -d$ and $\bar{v}_i - \bar{v}_{i+1} = -d'$, consequently we have that $\bar{w}_i - \bar{w}_{i+1} \geq \bar{v}_i - \bar{v}_{i+1}$.

(iii) $\Rightarrow$ (ii) Starting from $\bar{w}_i - \bar{w}_{i+1} \geq \bar{v}_i - \bar{v}_{i+1}$ and operating we have that, $\bar{w}_i - \bar{v}_i \geq \bar{w}_{i+1} - \bar{v}_{i+1}$.

Then, $\bar{w}_i - \bar{v}_i \geq \bar{w}_2 - \bar{v}_2 \geq \cdots \geq \bar{w}_n - \bar{v}_n$.

As $x_1 \geq \cdots \geq x_n$ and from Lemma 1 we have that

$$P_{\bar{w}}(x) - P_{\bar{v}}(x) = \sum_{i=1}^{q} (\bar{w}_i - \bar{v}_i) g_i \geq \sum_{i=1}^{n} (\bar{w}_i - \bar{v}_i) = 0$$

Then, as $P_{\bar{w}}(x) \geq P_{\bar{v}}(x)$ and choosing $g = (1, \ldots, 1, 0, \ldots, 0)$ as $g_i = 1$ for $i \leq 1$ and $g_i = 0$ for $i > k$ for every $k = 1, \ldots, q$ we have that $s_k \geq t_k \forall k = 1, \ldots, q$.

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