HEDGE FUNDS MANAGEMENT WITH LIQUIDITY CONSTRAINT

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Abstract. We propose a model for a manager of a hedge fund with a liquidity constraint, where the manager is seeking to optimise his utility of wealth, with one and multiple period horizons. By using stochastic control techniques we state the corresponding multi-dimensional Hamilton-Jacobi-Bellman partial differential equation and we use a robust numerical approximation to obtain its unique viscosity solution. We examine the effects of the liquidity constraint on managerial trading decisions and optimal allocation, finding that the manager behaves in a less risky manner. We also calculate the cost of being at sub-optimal positions as the difference in the certainty equivalent payoff for the manager. Moreover we compare the values for the hedge fund with another one having a risky asset with a higher rate of return but less liquidity.

1. Introduction

Academic research into hedge funds has increased as assets under such management have risen to $2.712 trillion in the third quarter of the 2015 according to BarclayHedge database\(^1\). Hedge funds are virtually unregulated investments with intricate payment fees, which commonly include a manager partnership and incentive fees. According to Fung and Hsieh [1997], hedge funds are private investment partnerships, where the managing partner is given a broad investment mandate; in addition these investments are restricted to high-worth investors. The manager of the fund is granted the authority to make trading decisions on behalf of investors or owners in the so-called managed account. Hedge funds have the ability to trade on an increasing number of instruments (such as world interest rates, currencies, real estate, etc), which could be illiquid, and in an increased number of ways such as leverage, short selling and other hedging strategies, as is also stated in Jaeger [2003]. Therefore, the returns of a hedge fund will not depend solely on the initial allocation or quantity of assets but also on the strategy and is where the liquidity or lack of it plays a fundamental role. As a consequence there is a need to study the effect of liquidity restrictions on hedge funds in some detail.

Hedge funds have been an interesting topic for research because of their distinctive form of manager compensation, their investment objectives, and their strategies, which are very different from more traditional funds, and because of the huge amounts of money these days managed in this way. One of the key features of the manager’s compensation in hedge funds is the performance-based fee or incentive fee, which is often related to a high-water mark. Typically this high-water mark is the starting fund value, and the performance fee is linked to the excess of the fund value over the high-water mark, creating an option-like payment. Many authors have studied the consequences of this type of compensation such as Goetzmann et al. [2003], Koijen [2012] and Buraschi et al. [2014]. In this paper we

\(^{1}\)http://www.barclayhedge.com
build on these papers although we include into the model a key feature which has not previously been added, namely the capability of selling or/and buying a desired amount of shares within a specific time period, i.e. the liquidity.

A major concern about optimal portfolio allocation problems is that they assume that assets are fully liquid and this is not always the case when investing. Therefore, we study the effects that liquidity restrictions or constraints may have on the portfolio allocation decisions of the fund’s manager, and how restrictions affect his decisions. There are different types of liquidity, but we are interested in stock restrictions, or illiquidity in allocation, which can be modelled as in Longstaff [2001], who models illiquidity as the inability to trade the desired amount of shares (between a risky and a riskless investment) within any time interval. Longstaff [2001] gives us an appropriate definition of illiquidity and a problem for the optimal portfolio management and valuation of illiquid securities. Longstaff’s model weakness is that it has correlated terms and hence its numerical solution becomes very challenging. In this paper, we propose a novel approach to model illiquidity not involving correlation variables and hence leads to a numerically tractable problem.

Our base model is structured in accordance with the paper on hedge fund management by Hodder and Jackwerth [2007], where the fund value consists of a risky and a risk-less investment, where the manager controls, or dynamically alters, the allocation between these two investments and the fund has normally distributed log returns. The manager’s compensation depends on three factors: he owns a fraction of the fund (typically about 10%), on the remaining fraction of the fund he charges an annual fee as a percentage (usually less than 5%) and earns an incentive fee over the excess of a prescribed goal or high-water mark (typically about 20% of the excess of the fund value over the high-water mark), this incentive fee mirrors the behaviour of an European call option. In a multiple period setting, the high-water mark is adjusted depending on the manager’s performance in the sense that, if at the payment time (i.e. at the end of the contract or payment period) the fund’s value surpasses the high-water mark then it must be updated to a new goal (or high-water mark) for the next period, behaviour that resembles a discrete look-back option.

Furthermore, the fund may be closed by the manager (termed endogenous closure) or by the investor (termed exogenous closure). In the former, the manager chooses to shut down the fund, when he identifies an outside opportunity that gives him more earnings. This generates a free-boundary problem which is akin to an American option. With the exogenous closure the investor closes the fund because of poor performance. That is, if the fund falls below some established liquidation boundary (corresponding in this paper to half of the high-water mark), the investor chooses to close the fund and this type of closure induces a discontinuity in the value function.

The problem of hedge-fund management under liquidity constraints has hardly been considered in the literature previously. In this paper we extend the hedge-fund model for the manager considered in Hodder and Jackwerth [2007] (described earlier) to include liquidity constraints. We add this extra feature to the model in order to capture the fact that hedge fund managers invest in assets with greater trading restrictions in order to gain higher returns. Aragon [2007] provides supporting results stating that greater trading restrictions are consistent with higher expected returns, which suggests that managers of hedge funds tend to invest in illiquid assets, because managers searching for higher returns would be likely to invest in some instruments that are traded less frequently (that is, more illiquid assets). To model liquidity constraints we proceed in similar way as Longstaff [2001] who models illiquidity as the inability to trade the desired amount of shares (between a
risky and a riskless investment) in a specific time interval. We are able to ensure that we have a unique viscosity solution and we solve the problem we use a robust numerical scheme, i.e. The Semi-Lagrangian Crank-Nicolson. Apart from performance and returns, we explore the manager’s optimal allocation, because investing in illiquid assets involves additional risk exposures (such as the inability to unwind a position before a crash). We are able to check the main differences between the value function having a fully liquid fund and having a liquidity restriction, finding that the cost of having the liquidity constraint grows with the distance from the optimal portfolio position. We also consider the effects of the liquidity restriction on the manager’s optimal allocation where we find that the manager behaves in a less risky manner. We also find that investing in assets with higher expected returns but less liquidity is not always better for the manager and depends on the allocation and on the fund’s value relative to the high-water mark.

This paper is organised as follows: in section 2 we describe the model and state the problem in form of a Hamilton Jacobi Bellman (HJB) Partial Differential Equation (PDE). Section 3 presents a numerical approximation to solve the multidimensional PDE. Section 4 gives some results showing the effect of the liquidity constraint and some comparative results concerning with the effects of liquidity restrictions on the optimal portfolio and the manager’s wealth. In section 5 we describe an extension of the model to allow multiple periods of time, with high-water mark resetting, which we implement as a discrete look back option. Finally, in section 6 we give some conclusions and ideas to continue this research.

2. The Model

Let us consider a fund comprising one risky investment and one riskless investment; we treat the wealth of the hedge fund manager as a function of the fund’s wealth and we crucially add a liquidity constraint, in the sense that we fix the maximum amount of money that can be interchanged between these two investments during a time period. Specifically, we let the manager invest in two (independent) instruments, the first is a money account which earns at a continuously compounded interest rate $r$ (the riskless investment), and the second represents the total amount of money invested in the risky asset. This risky investment relies on shares of an asset, which are usually modelled as a geometric Brownian motion with average rate of return $\mu$ and volatility $\sigma$. That is, our model distinguishes between the amount of wealth deposited on the risky investment (company shares, equity or even a portfolio of shares) and on the risk-free investment (bank account). Let us call the risky instrument $R_t$ and the riskless $M_t$.

The liquidity variable or trading strategy $\Gamma$ represents the amount of money going in or out the risky investment in a time instant, so that we are restricted by the total cost of the purchase not the volume as was assumed in Longstaff [2001]. Thus at any time instant, the manager chooses the value of $\Gamma$ (i.e. the amount to buy or sell) as his strategy. And the liquidity restriction is then the inability to sell more than $|\Gamma_{\text{min}}|$ or buy more than $\Gamma_{\text{max}}$, i.e. $\Gamma \in [\Gamma_{\text{min}}, \Gamma_{\text{max}}]$. In this way the dynamics of the risky technology are given by

\begin{equation}
\begin{aligned}
dR_t &= \mu R_t dt + \sigma R_t dB_t + \Gamma dt ,
\end{aligned}
\end{equation}

where $B_t$ is a standard Brownian motion, and since the money is moving between the two investments, this implies the following dynamics on the riskless instrument

\begin{equation}
\begin{aligned}
dM_t &= r M_t dt - \Gamma dt .
\end{aligned}
\end{equation}
The fund value is given by $X_t = R_t + M_t$, and the manager’s wealth will be given by the sum of the following:

1. The manager owns a portion $a$ of the fund, that is $aX_t = a(R_t + M_t)$, which is a common practice in hedge funds. In our case $a = 0.1$ for an owned portion of 10%.

2. The management fee is a percentage $b$ of the non-owned portion of the fund and is paid over a period of management (i.e. from 0 to $T$), $(1 - a)(R_T + M_T)bT$ where $b = 0.02$ corresponds to the 2% per period. This fee will be only paid at time $T$. In the case when the fund is closed before $T$, the management fee will be prorated according to the actually managed time.

3. The incentive fee (or performance based fee) $c = 0.2$ is paid on the excess of the fund value over some pre-established mark, the so called high-water mark $H$, so the manager earns the fee only when the fund value $X_T$ is greater than $H$, that is $(1-a)c(R_T + M_T - H)^+$. This fee will only be paid at the end of the period, or not paid at all if the fund is closed.

The manager’s incentive corresponds to a call option based on the high-water mark $H$, and then the manager’s terminal wealth is given by

$$W_T = aX_T + (1-a)bTX_T + (1-a)c(X_T - H)^+.$$  

Although, if the fund performs poorly investors may require closure of the fund. The simplest approach is to have a lower boundary for liquidation $\Phi$, set to half of the high-water mark, i.e. $\Phi = 0.5H$. In the case of an early closure of the fund because of poor performance, the manager ends up with his personal investment plus the proportional management fee, that is

$$W_\tau = aX_\tau + \tau(1-a)b\Phi$$  

for $0 \leq \tau < T$ here, $\tau$ represents the time at which the fund hits the lower boundary.

Furthermore, we also assume the manager has some endogenous reason to close the fund, such as: going to work for another organisation or having a new investment. Endogenous shutdown represents an American-style option where the manager chooses when to liquidate the fund. We follow Hodder and Jackwerth [2007] and model the manager’s outside opportunities using $L$ to represent an annual external compensation rate, paid only on liquidation and not depending on the fund value. If the manager chooses to liquidate the fund at time $\tau$, he receives

$$W_\tau = aX_\tau + (1-a)b\tau X_\tau + L(T-\tau)$$  

for $0 \leq \tau < T$.

Consequently under the incentive of winning $L(T-\tau)$, the manager is willing to liquidate if he believes that the fund is going to give an incentive less than $L(T-\tau)$.

The manager seeks to maximise the expected utility of terminal wealth $W_T$ and has a CRRA (Constant Relative Risk Aversion) utility function with risk aversion parameter $\gamma$, namely

$$U(W_t) = \frac{W_t^{1-\gamma} - 1}{1 - \gamma}.$$  

In this problem, for each terminal fund value above $\Phi$ we calculate the utility of the obtained wealth $W_T$ as in (2.3), but if at any time the fund is closed, then we calculate the utility for $W_\tau$ as in (2.4) or (2.5).

Hence we seek to solve the value function

$$J(t, R, M) = \max_{\Gamma} \left\{ \mathbb{E}^{t, R, M} \left[ U(W_T) \right] \right\},$$

where $\Gamma$ denotes the set of all stopping times, $t$ is the current time, $R$ is the realized return, and $M$ is the realized management fee.
where $W_T$ is as in (2.3) and $X_T = R_T + M_T$, subject to final condition
\begin{equation}
J(T, R, M) = U(W_T).
\end{equation}

We use stochastic control techniques to deduce the corresponding multi-dimensional HJB equation, which reflects the dynamics of the fund’s value subject to a final condition at $T$ (this final condition is the utility of the manager’s wealth), that is
\begin{equation}
\frac{\partial J}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 J}{\partial R^2} + \max_{\Gamma} \left\{ \left( \mu R + \Gamma(t) \right) \frac{\partial J}{\partial R} + \left( r M - \Gamma(t) \right) \frac{\partial J}{\partial M} \right\} = 0,
\end{equation}
with
\begin{equation}
J(T, R, M) = U(W_T).
\end{equation}

Remark 2.1. Under this setting and because of the discontinuity imposed at $X = \Phi$ we must assure there is a unique solution to the problem, which is shown in appendix A.

An analytical solution seems not possible because of the free boundary due to the endogenous closure condition, henceforth we use a numerical approach, which is explained next.

### 3. Numerical approximation

Notice that we treat the risk-less ($M$) and risky ($R$) investments as independent entities, and under this model the HJB PDE does not involve correlated terms that would appear if we consider the wealth $X$ and constrain the number of shares to trade as in Longstaff [2001] ($\Gamma$ constraint the total cost of the shares). Therefore, our model is numerically tractable with finite differences.

We solve the stochastic control problem (2.9) subject to (2.10) using a numerical approximation based on finite differences. Specifically we use the Semi-Lagrangian scheme, as used in Stainforth and Côté [1991] and D’Halluin et al. [2005], for the advection terms, and the Crank-Nicolson scheme for the diffusion terms. Then, we base our numerical scheme on Spiegelman and Katz [2006] who implement the Semi-Lagrangian Crank-Nicolson (SLCN) scheme, which achieves the second order accuracy in time.

First to apply the Semi-Lagrangian scheme, we use the total derivative of $J$, so that
\begin{equation}
\frac{DJ}{Dt} = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R} \frac{dR}{dt} + \frac{\partial J}{\partial M} \frac{dM}{dt}
\end{equation}
and to match this to the advection terms of the HJB equation (2.9), we take the gradient vector coordinates to be
\[ \frac{dR}{dt} = \mu R + \Gamma(t), \quad \frac{dM}{dt} = r M - \Gamma(t), \]
and we calculate the value of $J$ at take-off points
\[ (R^*, M^*) = (R + \alpha_1(R, \Gamma(t)), M + \alpha_2(M, \Gamma(t))) \]
for “each” $\Gamma(t)$ value, where $\alpha_1$ and $\alpha_2$ represent the instantaneous change $\frac{dR}{dt}(R, \Gamma)$ and $\frac{dM}{dt}(M, \Gamma)$ respectively. Then, we determine the maximum value of $J$ over all the take-off points and call it $J^*$, that is
\begin{equation}
J^*(t + \Delta t, R, M) = \max_{\Gamma} \{ J(t + \Delta t, R^*, M^*) \}.
\end{equation}
Once we have calculated the value of the function $J^*$, we proceed to solve the differential equation

$$
\frac{DJ}{Dt} + \frac{1}{2}\sigma^2 R^2 J_{RR} = 0.
$$

(3.3)

Next we consider the boundary conditions, and the imposed liquidation boundaries for the exogenous and endogenous shutdown.

### 3.1. Boundary conditions.

We truncate the domain by considering the region $R \in [0, R_{max}]^2$ and $M \in [M_{min}, M_{max}]$, for suitable values of $R_{max}$, $M_{min}$ and $M_{max}$. For a more detailed explanation about how to obtain this boundary limits refer to Ramirez [2016].

If $R = 0$, there is no stochasticity on the fund meaning that all the money is in the bank account, i.e. $X_T = M_T$, then the manager’s wealth depends only on the money account and we may use a discounted rate (of the bank account) as the final condition. To obtain a Neumann boundary condition, we consider that for a small change in time $\Delta t$

$$
(e^{\Delta t} M_t - H) \approx e^{\Delta t}(M_t - H)
$$

(3.4)

and therefore for the CRRA utility function (2.6) we can write the boundary condition as

$$
J(0, M, t - \Delta t) = e^{r(1-\gamma)\Delta t} J(0, M, t),
$$

(3.5)

or as a Neumann condition

$$
\frac{\partial J}{\partial t} \bigg|_{R=0} = -r(1-\gamma)J.
$$

(3.6)

If $R >> 1$, we assume that $(R_t + M_t) - H \approx R_t + M_t$, since $R_t \to \infty$, hence the fund behaves as in Merton [1969], thus we reduce our model to that of Merton, i.e. the manager’s wealth can be written as

$$
W_t = A X_t - D \approx A X_t = A(R_t + M_t)
$$

(3.7)

where $A = a + (1-a)bT + (1-a)c$, and $D = (1-a)cH$. Using the dynamics of the fund, and setting $p = \frac{AR_t}{AX_t} \approx \frac{AR_t}{W_t}$ and $1-p \approx \frac{AM_t}{W_t}$ we have

$$
dW_t^p = AdX_t \approx (r + p(\mu - r))W_t dt + \sigma p W_t dB_t,
$$

(3.8)

where $B_t$ represents the standard Brownian motion and $W_t^p$ is the manager’s wealth subject to $p$ (percentage) invested on $R_t$. Then the value function, for the Merton case, is

$$
J(W, t) = \max_p \left\{ \mathbb{E}^t [U(W_T^p)] \right\}.
$$

(3.9)

Note that, for simplicity, we use the same notation for the function $J$ in one variable $W_t$ and in several variables $R_t$ and $M_t$, i.e. $J(W, t) = J(R, M, t)$, which are directly related by (3.7) and thus, in this case, we obtain the following HJB equation

$$
\max_p \left\{ J_t + \frac{1}{2}\sigma^2 p^2 W^2 J_{WW} + (r + p(\mu - r))W J_W = 0 \right\},
$$

(3.10)

which is the optimal portfolio problem described in Merton [1969], and thus has a known solution for the case of CRRA utility functions. In Merton we obtain closed form solutions for the optimal allocation $p^*$ and the value function $J$. In our setting we use this as a
boundary condition but instead of maximising the portfolio \( p \), we use the close form solution of \( J \) for different constant values of \( p \), that is

\[
J(W,t) = \exp \left\{ \left( r + p(\mu - r) - \frac{1}{2}\gamma \sigma^2 p^2 \right)(1 - \gamma)(T - t) \right\} \frac{W^{1 - \gamma}}{1 - \gamma}.
\]

The exponential in (3.11) can be regarded as the Merton discount rate, so we convert this to a Neumann-type condition as

\[
\frac{\partial J}{\partial t} = - \left( (r + p(\mu - r) - \frac{1}{2}\gamma \sigma^2 p^2)(1 - \gamma)\Delta t \right) J.
\]

Neumann conditions (3.6) and (3.12) are the actual conditions to be used in the respective boundaries.

### 3.2. Liquidation Boundaries.

We must also consider the fund’s closure boundaries or liquidation boundaries of the rectangular domain \( [M_{\text{min}}, M_{\text{max}}] \times [0, R_{\text{max}}] \) for suitable values of \( M_{\text{min}}, M_{\text{max}}, \) and \( R_{\text{max}} \), which allow sufficient values above the high-water mark \( H \) and to ensure we include sufficient leveraged positions, we have a maximum leverage of \( p_{\text{max}} = 4.5 \) times the money in the risky asset (which is undertaken by borrowing money from the bank, i.e. \( M < 0 \)). We emphasize that we only consider positive values of wealth since the exogenous boundary condition constraints the wealth to be greater than \( \Phi = \frac{1}{2} H \).

**Exogenous liquidation** is modelled as a boundary condition, in the sense that when the fund value is less than half the high-water mark, we close the fund with utility of manager’s wealth given by the utility of the proportional manager’s wealth, at the closure time, \( W_t \) as in (2.5).

**Endogenous liquidation** is a more difficult case, since at each time period we must determine the value for \( J(R,M,t) \) for each value of \( (R,M) \) to decide whether to close or keep the fund active, hence this problem is a free boundary between the regions to continue the fund and to close it.

If shut down is optimal, we close with fund value \( X_t = R_t + M_t \) and the manager’s terminal wealth

\[
\tilde{W}_T = aX_t + (1 - a)btX_t + L(T - t)
\]

thus \( J(R,M,t) = U(\tilde{W}_T) \) and if to continue the fund is optimal, we will close the fund at the final time \( T \), and then

\[
J(R,M,t) = \max_\Gamma E^\Gamma [U(W_T)]
\]

where \( W_T \) is given by (2.3), which means that \( J(R,M,t) \) is described by the dynamics in (2.9) and (2.10). To solve this problem we treat it as an American option, as in Wilmott [1995], and solve the HJB equation with a Semi-Lagrangian PSOR scheme.

### 4. Results

In this section we describe and explain results for the problem of the manager’s optimal expected utility of wealth, using a CRRA utility as the function showing the preferences of the manager. To be able to compare our results with those in Hodder and Jackwerth [2007], we use their same set of parameters shown in table 1. In this table, we add the the constant \( p \) to resemble the optimal allocation for the simplified problem (i.e. Merton’s problem), and we perform a segment search for the maximum argument of \( \Gamma \) in the appropriate range. For the purposes of this paper we have a liquidity constraint of the form \( -\Gamma_{\text{max}} = \Gamma_{\text{min}} \), although our model supports liquidity restrictions where \( -\Gamma_{\text{max}} \neq \Gamma_{\text{min}} \).
Table 1. Model parameters, as in Hodder and Jackwerth [2007]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>$T_{max}$</td>
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<tr>
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<tr>
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Before considering our results we note that Hodder and Jackwerth [2007] solve the manager’s optimal portfolio allocation $p^*(t, X)$ (i.e. the percentage invested in the risky investment that gives the manager the maximum utility of wealth for a given fund value at a given time) having endogenous and exogenous shut down constraints. In this way the natural connection between our model’s optimal trading decision $\Gamma$ with the optimal portfolio of Hodder and Jackwerth is performed by searching the maximum utility of wealth over the diagonal $R = X - M$ for each fixed fund value $X$, that is

$$p^*(t, X) = \left\{\frac{R}{R + M} : (t, R, M) = \arg\max_{R + M = X} \{J(t, R, M)\}\right\}.$$  

Figure 1 shows the optimal trading strategy $\Gamma^*$ for a manager with liquidity constraint $\Gamma_{max} = 0.3$, at two different times. First notice that the plotted line corresponds to the optimal position for the manager, which corresponds to $\Gamma^* = 0$, because his best trading decision consists on staying at his position (not selling or buying) and the plotted dashed line encloses the endogenous closure region. The curve corresponding to the manager’s optimal portfolio position is also $\Gamma^* = 0$ (i.e. the optimal portfolio $p^*$), because no change in position means that there are no possible improvements and hence the position is optimal. This curve may be referenced by the high-water mark $R + M = H = 1$ line and by Merton’s optimal strategy $M = -2R$, because strategies with fund values above the high-water mark behave similarly to the optimal portfolio allocation described by Merton but with a lower leverage (i.e. $p^* \approx 1.55 < p$). The figure may be interpreted in the following way, if the values correspond to a position with portfolio greater than $p^*$ then the decision is to sell risky assets $\Gamma^* = -\Gamma_{max}$ (to the top left of the graphs), and if the position has a portfolio lower than $p^*$ then buy $\Gamma^* = \Gamma_{max}$ (bottom right of each graph). For values below the high-water mark diagonal $R + M = H$, the strategy varies in time as we can see in the graphs at times $t = 0.8$ and $t = 0$. The difference between these graphs is that in the optimal curve (corresponding to $\Gamma^* = 0$ or $p^*$) the spike at the high-water mark diagonal becomes less pronounced and thicker, which means that at initial time the manager, who is close to the high-water mark, behaves in a way that is less risky.

For calculations of equation (4.1) and the comparison to the optimal trading decisions, we can see that the optimal portfolio $p^*$ allocations coincide with the trading decision of $\Gamma = 0$. Moreover, in this setting our results agree with those found in Hodder and Jackwerth [2007]. Figure 1 also shows that trading decisions are dynamic and depend in time and allocation of the wealth, contrary to Merton’s optimal portfolio problem where
the optimal allocation does not change through time and hence the decisions, at any time, are to get closer to Merton’s optimal allocation $p$. This dynamic strategy differentiates the kinds of problems faced in hedge funds and mutual funds, which agrees with the empirical data analysis shown in Fung and Hsieh [1997].

4.1. **Liquidity effects on the optimal portfolio.** Next we compare our results with those in the fully liquid case. To do so we consider the model following Hodder and Jackwerth [2007] and solve it via stochastic control to obtain the following HJB PDE

$$\frac{\partial J_L}{\partial t} - \max_p \left\{ \frac{1}{2} p^2 \sigma^2 X^2 \frac{\partial^2 J_L}{\partial X^2} + (r + p(\mu - r)) X \frac{\partial J_L}{\partial X} \right\} = 0,$$

where we use the notation $J_L$ for the value function, which is now fully liquid and depends on the total fund $X$ and the portfolio $p$. To solve this equation subject to the same closure conditions and payment fees as before, incurs some numerical difficulties such as using the upwind differencing, which are discussed in Ramirez [2016] but go beyond the scope of this paper. Specifically, the main technique used to solve this PDE is the PSOR with finite differences.

In figure 2 we show the graph for the optimal allocation of the manager at time $t = 0$, for a hedge fund with and without the liquidity constraint. We can observe that adding the liquidity constraint affects the manager’s optimal allocation in the sense that near the endogenous closure a manager having a liquidity restriction takes less risky optimal positions. This is due to the fact that in the case of adverse movements of the assets, the liquidity does not let him react fast enough to change his position. We also compare the optimal $p^*$ at different times ($t > 0$) and check that the differences between the optimal portfolios with and without liquidity constraint decrease with time. This result is expected because the liquidity restriction affects more the rate at which the manager’s position approaches the optimal than the optimal position itself.

4.2. **Liquidity effects on the utility of wealth.** To show the effect the liquidity constraint has on the manager’s value function (i.e. expected utility of wealth), we fixed the fund value at $X = 0.75$ and plot the value function at time $t = 0$ changing the liquidity constraint parameter $\Gamma_{max}$. Figure 3 illustrates the value function $J$ for a small fraction of the range around the optimal $p^*(t, X) \approx 2.1$ (respectively $R \approx 1.6$) so that we can see better the shape of the curve. As the restriction becomes more severe (i.e. smaller values of $\Gamma_{max}$), the values of $J$ decrease proportionally with the distance to the maximum point, implying that the restriction affects more the values farther from the optimal allocation.

Motivated by this, we define the cost of being away from the optimal position by $C$ as the difference between the certainty equivalent values having different $\Gamma_{max}$ restrictions (the certainty equivalents values are the inverse values of the value function, and are feasible to calculate since we use a CRRA utility function). So $C$ represents how much the manager is losing when investing in more illiquid assets, when all other parameters remain the same, that is

$$C(\Gamma_1, \Gamma_2, R, M) = U^{-1} \left( J(0, R, M) \bigg|_{\Gamma_{max} = \Gamma_1} \right) - U^{-1} \left( J(0, R, M) \bigg|_{\Gamma_{max} = \Gamma_2} \right).$$

In figure 4 we plot the cost function $C$ for fixed fund values $X = 0.75$ (below $H01$) and $X = 1.2$ (above $H$), and through different portfolio choices $p$, where we set $\Gamma_1 = 0.16$ as the benchmark. As a result we find that the highest cost corresponds to funds being leveraged more than is optimal. This is because the fund can have a negative variation (i.e. a drop in the value of the risky asset in an instant of time), and this variation takes
Figure 1. Map plots of the $\Gamma^*$ decisions with restriction $\Gamma_{max} = 0.3$, for a CRRA utility function, at times $t = 0.8$ (top), and $t = 0$ (bottom), where $\Gamma^* = -\Gamma_{max}$ implies selling shares of risky investment and $\Gamma^* = \Gamma_{max}$ implying buying shares. The region enclosed by dashed lines represents endogenous closure.
a manager in a high leverage position to a position leveraging more than the maximum permitted (and hence the fund must be closed). In our case since leveraging more than 4.5 means closure of the fund and therefore all values close to \( p = 4.5 \) have a higher cost. Furthermore, figure 4 shows that, as the fund value decreases, the cost in non-leveraged positions (i.e. \( p < 1 \)) increases (as we can see in the top graph), which agrees with figure 2, since funds with values less than unity (i.e. \( X < 1 = H \)) have higher optimal positions than wealthier funds (i.e. \( X > 1 \)). Also in figure 4, for the plot corresponding functions having \( X = 1.2 \) (bottom graph), we can see that the cost for values of \( p < 3 \) does not change dramatically, and is less significant at \( p \approx 1.1 \), which is the optimal position for \( X = 1.2 \). So the cost of being below the optimal position affects more managers who have not achieved the high-water mark and hence do not receive the incentive fee.

We also see in figure 4 that at the level of wealth \( X = 0.75 \), the maximum cost of being away of the optimal (due to the liquidity) is about \( C = 0.0002 \), which means that for an investment of $750,000 the cost (or deduction from the manager’s payment) would be $200. In the same manner, from the bottom graph we deduce that an investment of $1,200,000 the cost would be $3,500, note that in this case the restriction varies from being able to trade a maximum of $160,000 per year to a maximum of $2,000 per year.

4.3. More severe liquidity restriction with higher expected returns. Next we explore the scenario when managers invest in more illiquid assets in order to obtain more
return (or *alpha*) from these assets. To do so we compare a hedge fund having restriction \( \Gamma_{\text{max}} = 0.3 \) with the same fund but having half of the restriction \( \frac{1}{2} \Gamma_{\text{max}} = 0.15 \) (implying a more severe restriction). As expected, if the rate of return \( \mu \) remains the same the value function with the more severe restriction \( \left( \frac{1}{2} \Gamma_{\text{max}} \right) \) have a lesser value. Therefore if we increase the value of the expected rate of return \( \hat{\mu} \) (and correspondingly \( \sigma \) to keep Merton’s optimal portfolio \( p \) constant), we find that the value function increases as well, although contrary to intuition the value function (and hence the certainty equivalent) does not increase proportionally for all the values of the fund. In figure 5 we show the certainty equivalent values for the fixed wealth value \( X = 1.1 \) with the original restriction of \( \Gamma_{\text{max}} = 0.3 \) and with half of this restriction where we choose to increase the expected rate of return to \( \hat{\mu} = 0.1 \) to resemble the higher return (i.e. \( \hat{\mu} > \mu \)). We observe in this figure that the certainty equivalent value for the fund with more severe restriction (and hence with higher expected rate of return) has less payoff in the more leveraged positions, although it has higher payoff in the positions with less risk, i.e. smaller \( p \). This is a direct consequence of the liquidity restriction and may be observed from this results that with more liquidity restriction the optimal should be moved away from leverage.

To complement these results, in figure 6 we show the difference between the certainty-equivalent payoffs for the manager, and we see that the negative values correspond to the region where investing in a more restricted asset is more beneficial because of the higher returns, and the positive values correspond to the region where investing in less-restricted assets is better because of the leverage (riskier) position taken by the manager. Figure 6
Figure 4. Cost due to liquidity constrained functions for fixed fund values $X = 0.75$ on the top and $X = 1.2$ on the bottom. The cost is calculated by subtracting certainty equivalent values with different liquidity constraints $\Gamma_2 = 0.08$, 0.04, 0.02 and 0.002 from the value function with liquidity constraint $\Gamma_1 = 0.16$. 
includes the plots for different values of wealth \( X = 0.8, 0.9, 1, 1.1 \) and \( 1.2 \), and we observe that the positive value region grows larger as the fund value \( X \) increases. This is because as the fund’s wealth increases (specifically above the high-water mark) the manager prefers safer investments and since the positive values occurs in the more leveraged positions \( (p > 2) \) the manager will prefer an investment with a lower liquidity restriction (higher value of \( \Gamma_{\text{max}} \)) to be able change his position fast enough to a safer position (moving to a lower value of \( p \)).

5. Multiperiod Case

In this section we extend the model discussed previously, to allow it to incorporate multiple periods. In particular we reset the high-water mark yearly (for example). Since the same model holds, we show how to extend our numerics and describe numerical results. Furthermore, comparisons with the multi-period model shown in Hodder and Jackwerth [2007] are given.

5.1. Model extension. Let us first explain what happens at the end of a year (period) for a multiple-year hedge fund. Suppose ending-year times are \( T_1, T_2, \ldots, T_n \) so that year \( i \) ends at time \( T_i \), these times are also known as sampling times. Under our assumptions at the end of the year the manager receives payment for managerial and incentive fees. After the manager is paid, the high-water mark is reset (if the fund’s value is higher than \( H \)),
and then the next year is treated as a single year until the next year end, that is to say between sampling times the fund’s behaviour is the same as the single period case.

Since the high-water mark is, generally, reset at each year period, then our model changes in the sense that $H$ is no longer a constant but a variable depending on the fund’s value and time. For instance, at the end of the time period $[T_{i-1}, T_i)$ we have two cases for the high-water mark:

- If the performance of the fund is positive, the fund value is then greater than the previous high-water mark (i.e. $H_{i-1} \leq X_{T_i}$); thus the high-water mark is reset, that is $H_i := X_{T_i}$.
- If the fund’s value is below the previous high-water mark (i.e. $H_{i-1} > X_{T_i}$) then the high-water mark is not reset $H_i := H_{i-1}$.

Specifically, given the dates of evaluation $\{T_1, T_2, \ldots, T_n\}$, the high-water mark at time $t$ prior expiry (i.e. $T_k \leq t < T_{k+1}$, for some $0 \leq k < n$) is

$$H_k = \max\{X_{T_i} : i = 0, \ldots, k\}$$

where we assume $X_{T_0}$ to be the initial high-water mark $H$ (the high-water mark for the first time period, i.e. $T_0 = 0$). This behaviour of the high-water mark over the years can be treated as a discrete time look-back option, hence the value function $J$ now depends on the extra variable $H$ (i.e. $J(R, M, H, t)$).
Moving backwards in time, at sample time \( T_i \), the values of \( X_{T_i} \) greater than \( H_i \) should no exist. Thus for any point satisfying \( X_{T_i} > H_i \), we change the value function with the value function having the correct high-water mark, i.e. the value function \( J(R_{T_i}, M_{T_i}, H_{i-1}, T_i) \) must be replaced by \( J(R_{T_i}, M_{T_i}, X_{T_{i+}}^{+}, T_i) \), that is \( H_i = X_{T_i}^{+} = R_{T_i} + M_{T_i}^{+} \). The next step in this setting is to add the payment of the managerial fees.

Let us assume that the value function is given in terms of the manager’s wealth and the high-water mark (we suppose that \( \hat{J}(W_i, H) = J(R_i, M_i, H, t) \)), and we add the fees’ payment at time \( T_i \), then we must have the following

\[
\hat{J}(W_{T_i}, H_i) := \hat{J}(W_{T_i} + FP(X_{T_{i-1}}, H_{i-1}), X_{T_{i+}}^{+}) ,
\]

where

\[
FP(X_{T_i}, H_{i-1}) = (1 - a) b X_{T_i} + (1 - a) c (X_{T_i} - H_{i-1})^+
\]

represents the fee payment, which is added to the manager’s wealth. Note that the new notation \( \hat{J} \) represents the value function in terms of \( W \), the manager’s wealth.

Then, in the case that \( X_{T_i}^{+} < H_{i-1} \) there is nothing to do but pay fees and continue, that is

\[
\hat{J}(W_{T_i}, H_i) := \hat{J}(W_{T_i} + F_p, H_{i-1}) .
\]

Since the payment of the manager’s fees is taken from the the fund, we deal next with the small adjustments needed. We have that at sample time \( T_i \), the fund value has a jump condition because of the payment of the fees to the manager, so we have that

\[
X_{T_i}^{+} = X_{T_i}^{−} + FP(X_{T_{i-1}}, H_{i-1})
\]

and immediately after this payment is made we may reset the high-water mark to \( H_i = \max \{H_{i-1}, X_{T_i}^{+}\} \). Notice that equation (5.2) still holds but, in this case, with \( X_{T_i}^{+} \) as in (5.5).

5.2. Numerical approach. In this section we describe the numerical techniques needed to extend the approach given in section 5.1, this means we are only concerned with the sampling times.

Suppose that the final time is \( T = T_n \), so the computation starts with final condition

\[
J(R, M, H; T) = U(\alpha(R + M) + (1 - \alpha) b(T - T_{n-1})(R + M) - (1 - \alpha) c(R + M - H)^+) \]

and solve this problem in the same way as for a single period up to \( T_{n-1} \), i.e. this means to consider \( H \) as constant through the interval \( [T_{n-1}, T_n] \). At time \( T_{n-1} \), we continue to the payment of the fees as follows

\[
J_{to}(R, M, H_{n-1}, T_{n-1}) = J(R_{FP}, M_{FP}, H_n, T_{n-1}) ,
\]

where \( H_n = \max \{H_{n-1}, X - FP(X_{T_{n-1}}, H_{n-1})\} \), \( X = R_{T_{n-1}} + M_{T_{n-1}} \), the fee’s pay \( FP(X_{T_{n-1}}, H_{n-1}) \) as in (5.3), and \( R_{FP}, M_{FP} \) represent the amount of money in the risky and risk-less technologies after paying fees respectively. If the manager is paid with assets and money, then we regard \( p \) as the proportion taken from the risky asset to pay the fees, thus \( R_{FP} = R - p FP(X, H_{n-1}) \) and \( M_{FP} = M - (1 - p) FP(X, H_{n-1}) \). In the case when \( X_{T_i} > H_i \), since the \( H_i \) has just been reset the value of \( X \) cannot be greater than the new high-water mark (i.e. the maximum), we must obtain the value from the corresponding value according to the high-water mark reset, that is \( H_n = X - FP(X_{T_{n-1}}, H_{n-1}) \).

Note that equation (5.7) does not involve the fee payment to the manager, although it is useful as a take-off point (denoted by \( J_{to} \) in (5.7)). Also notice that even though we know
the value of the expected utility of wealth for the manager, we do not know the actual manager’s wealth at that time. Therefore following Hodder and Jackwerth [2007] we use the certainty equivalent value. We add the fees payment to this certainty equivalent value and calculate the corresponding utility of this value, that is

\( J(R, M, H, T_{n-1}) = U\left(FP(X, H_{n-1}) + U^{-1}(J_{to}(R, M, H, T_{n-1}))\right). \)

5.3. Multi-period Results. In this section we show results for the multi-period case and as a benchmark we use the results presented in Hodder and Jackwerth [2007], and we focus on the differences between their model and our model.

5.3.1. Hodder and Jackwerth scheme. In Hodder and Jackwerth [2007] we find a model without liquidity constraints, thus they model the fund wealth \( X \) as a single variable. They distinguish between the risky and risk-free investments by letting \( pX \) be the risky investment and \((1 - p)X\) be the risk-less investment and they control the value of \( p \) (that is the portfolio), compared to our model, meaning that \( R = pX \) and \( M = (1 - p)X \). In their approach, the value \( X \) does not represent the fund value but the fund value relative to the high-water mark, and all other parameters are also relative to \( H \). By this we mean that they work under the change of variables \( \tilde{X} = \frac{X}{H} \) and all the values are regarded being scaled to the high-water mark, this change is made for simplicity of the model specially when evaluating multiple periods of time. Our approach is more realistic in the sense that it does not work with relative values but instead with absolute values, and we overcome the difficulties in Hodder and Jackwerth [2007] by adding an extra dimension to our continuous PDE and solve it using Semi-Lagrangians and finite differences. On the other hand, Hodder and Jackwerth use a scheme that is a discrete dynamic programming problem, which is similar to the quadrature method (for more information about the quadrature method of Andricopoulos et al. [2003]).

5.4. Multi-period with liquidity constraint. Since we do not work with values relative to the high-water mark, we need to add the extra dimension on \( H \), which allows us to work with the actual fund value \( X \) (represented by \( R \) and \( M \)), so the path dependence is modelled as an extra dimension.

Since the fees are paid from within the fund, we have to calculate the take-off point as the point corresponding to the hedge fund’s wealth after the fees are paid. This value is not unique since it lies on the diagonal \( R + M = X - FP(R, M, H) \). Therefore we calculate for the pair \((R, M)\) the corresponding proportional take-off value \((R^*, M^*)\) on the line \( R^* + M^* = X - FP(R, M, H) \). By proportional we mean that we do not affect the fund’s distribution and after the fees are paid, the proportion invested in risky and risk-free assets remains the same. Hence we must

1. Calculate the proportional take off points for the balancing

\[ R^* = R - pFP(R, M, H), \]
\[ M^* = M - (1 - p)FS(R, M, H) \]

and then interpolate to obtain the value function \( J(R^*, M^*, t_i) \).

2. Calculate the new final value having different certainty equivalent, that is

\[ J(R, M, H, t_i) = U\left(FP(R, M, H) + U^{-1}(J(R^*, M^*, H^*, t_i))\right) \]

where \( H^* = \max\{H, R^* + M^*\} \).
Figure 7. Surface plots ($t \times X$ vs $p^*$) for the optimal allocation position taken by the manager $p^*_L(t, X)$ for the fully liquid case (left graph) and $p^*(t, X)$ for the case with liquidity restriction (right graph), in the time period $t \in [27, 28]$ for $T = 30$.

The previous numeric approach is then applicable.

In figure 7, we show a comparison between the fully liquid hedge fund, similar to Hodder and Jackwerth [2007], (on the left graph) and the version with liquidity constraint (on the right graph), and we show the dynamic strategies from three years to two years prior to the terminal time, i.e. from $t = 27$ to $t = 28$ (for $T = 30$). We can see in this figure that the maximum leverage (near the endogenous liquidation at about $X = 0.55$) without the liquidity constraint has $p^* = 4.5$ and with liquidity constraint has about $p^* = 3.5$. Therefore, in the presence of a liquidity constraint the manager takes less risk, specially near the endogenous closure, due to the additional inherent risks of illiquidity. For times farther away from closure, the peak of higher risks near the endogenous closure is reduced in both cases to the point that its effect is insignificant (about 10 years prior to closure, i.e. $T - 10$).

5.4.1. Alternatives for the payment of fees. It is not always the case where the manager of the fund is paid proportional in shares and in money (i.e. cash), which means that we do not alter the fund’s portfolio, as is presented earlier. It could be the case that the manager is paid only in money, only in shares or any combination of those.

Next, we look at the difference in portfolio allocations by changing the proportion in which the manager is paid, between Hodder and Jackwerth’s multiperiod replication with liquidity constraint, paying the manager fees from the fund proportionally (as before), and paying the manager fees in money (from the fund’s bank account), which corresponds to

$$J(t_i, R, M, H) = U(FP(R, M, H) + U^{-1}(J(t_i,\text{cherani}, R, M^*, H^*))),$$

where $H^* = \max\{H, R + M^*\}$ and $M^* = M - FP(R, M, H)$, i.e. $p = 0$ in (5.9) and (5.10).

To do so we set $T = 3$ and sampling times $T_1 = 1$, $T_2 = 2$. In figure 8 we plot the manager’s optimal allocation for a fixed time between $T_1$ and $T_2$, i.e. $t = 1.7$ or $T - 1.3$. We note that the difference between paying the fees proportionally and from the bank account ($M_t$), is that in the latter the manager is less aggressive for fund values above the high-water mark and as the fund value increases it stabilises at approximately 1.65, which is less than Merton’s optimal value ($p = 2$). Consequently, paying the fees proportionally induces
the manager to replicate Merton’s optimal position. Furthermore, we find a different behaviour in our version of Hodder and Jackwerth with liquidity constraint because for values of the fund $X$ above $H$ the manager takes less risk, due to the balancing condition (5.11).

For fund values $X < H$, we see a similar behaviour as that described for $X > H$. Such that when the fees are paid proportionally, the manager takes more risk, since after the payment of the fees the fund has less value ($R + M - FP(R, M)$ or $X - FP(X)$) which could lead to closure. This can be clearly seen in figure 9, where we plot the optimal allocation position for the manager for the case where the fees are paid from the fund at time $t = 0.7$ (i.e. $t = T - 2.3$), and again results when the fees are paid proportionally show that the manager behaves more aggressively in his strategy.

6. Conclusions

In this paper we present an original model for hedge-fund management with a liquidity constraint. The modelling involves combining the work of Hodder and Jackwerth [2007] and that of Longstaff [2001]. We use stochastic control techniques, which results in a value function with three state variables ($R$ representing the amount of money invested in a risky asset, $M$ representing the amount of money in a bank account and $H$ the high-water mark) plus time, where the liquidity constraint is given by restricting the amount of money $\Gamma$ that can be traded between $R$ and $M$ in a small period of time $\Delta t$ (namely $\Gamma \in [-\Gamma_{\text{max}}, \Gamma_{\text{max}}]$).

Under this setting we are able to show uniqueness of the solution and we use the modelling tools for American and Look-Back options to represent the endogenous closure and the high-water mark respectively. This model has a novel approach to numerically solving this...
class of problem in a way that it is numerically tractable with finite differences, using a combined Semi-Lagrangian Crank-Nicolson PSOR scheme.

The first results in this paper tells us how to trade, depending on the time state position, which is comparable with Hodder and Jackwerth [2007], although we are able to establish a less risky trading behaviour of the manager in the presence of liquidity constraints (Which has its own risks), specifically that decreases the manager’s risk-exposure to endogenous closure. Thereafter, we measure the cost of being in a sub-optimal allocation position as the difference between the certainty equivalent values have different liquidity restrictions and we observe that this difference is more profound in the leveraged zones and in funds that have not achieved the incentive fee ($X < H$).

We reaffirm the statement that managers opt to invest in assets with higher expected rates of return but are less liquid because they produce more returns although in our results the manager does not always benefit from this investment, because the liquidity incurs in additional risks and therefore risk positions should be taken more carefully.

Finally we study the manager’s optimal allocation in multiple years, where we reinforce our theory that the manager acts less aggressively and takes less risks with liquidity constraints, compared with the fully liquid case. We also check the differences between paying the manager in different proportions (i.e. paying the manager in cash) and with paying the manager proportionally between cash and assets, and we find that the manager takes more risks in the latter scenario.

Our research has led us to see that managers of hedge funds do not report all of their movements and the investor (most likely) does not know about managers’ strategy. Hence there is an unexplored area of research on the consequences of having partial information.
or partial hedging by investors (and managers as well) in hedge funds. Since having only partial information could lead to incomplete markets within a non-Markovian setting, this area of research would necessitate different mathematical techniques such as Malliavin calculus tools, and hence would lead to a whole new model/project. In the literature, there has already been some work capturing similar characteristics to this; Menoukeu Pamen et al. [2013] derive a general stochastic maximum principle under partial information and Di Nunno and Øksendal [2009] consider the optimal portfolio problem for a manager who does not have full access to market information; this theory could be extended to hedge funds.

Obviously there is still much work to be done on modelling hedge funds and this will become more important in the future as investors and fund managers search for the highest returns and lowest risk in an increasingly uncertain world.

**Appendix A. Unique Solution**

Following the lines of Touzi [2013] (chapter 7), we know that the value function $J$ is a viscosity solution and for a uniqueness result we have to check certain assumptions. Therefore to prove uniqueness for viscosity solutions to the final value problem in equations (2.9) and (2.10), we only need to prove the following assumptions:

1. $\exists \vartheta > 0$ such that
   \[ F(t, X, r, v, \vec{q}, A) - F(t, X, r', v, \vec{q}, A) \geq \vartheta (r - r') \]

   For all $r \geq r'$, and $(t, X, v, \vec{q}, A) \in Q \times \mathbb{R} \times \mathbb{R}^n \times S_n$.

2. There exists $\overline{w} : \mathbb{R}^+ \to \mathbb{R}^+$ with $\overline{w}(0^+) = 0$, such that
   \[ F(t, y, r, v, \alpha(x - y), B) - F(t, x, r, v, \alpha(x - y), A) \leq \overline{w}(\alpha|x - y|^2 + |x - y|) \]

   For all $(t, x), (t, y) \in Q, r \in \mathbb{R}$ and $A, B$ such that

   \[ -\left(\frac{1}{\epsilon} + |\alpha|\right)I_{2n} \preceq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \preceq (\alpha + \epsilon|\alpha|^2) \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} \]

   Particularly letting $\alpha = \frac{1}{\epsilon}$

   \[ -3\alpha I_{2n} \preceq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \preceq 3\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} \]

   Note that (A.4) implies that $A \preceq B$ since the right side has norm 0 (the matrix norm in the usual sense $|A| = \sup\{A\eta \cdot \eta : |\eta| = 1\}$).

   Thus in our case, if we let $x = (x_R, x_M)$ and $y = (y_R, y_M)$, then we may define the HJB functional as

   \[ F(t, x, J, \frac{\partial J}{\partial t}, \frac{\partial J}{\partial x}, \frac{\partial^2 J}{\partial x^2}) \]
   \[ = -\frac{\partial J}{\partial t} - \frac{1}{2} x\sigma \frac{\partial^2 J}{\partial x^2} (x\sigma)^\prime r - \max \left\{ (\mu x_R + \Gamma(t), rx_M - \Gamma(t)) \cdot \frac{\partial J}{\partial x} \right\} \]
where
\[
\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
\frac{\partial J}{\partial x} = \begin{pmatrix} \frac{\partial J}{\partial R} \frac{\partial J}{\partial M} \end{pmatrix},
\]
\[
\frac{\partial^2 J}{\partial x^2} = \begin{pmatrix} \frac{\partial^2 J}{\partial R^2} \frac{\partial^2 J}{\partial R \partial M} \frac{\partial^2 J}{\partial M^2} \end{pmatrix}.
\]

We notice that the function \( F \) does not depend on \( J(t, X) \) explicitly; a minor modification of the assumption is therefore needed. According to Crandall et al. [1992], notice that for every \( \epsilon > 0 \) and viscosity subsolution \( \tilde{J} \) of (2.9), \( \tilde{J}_\epsilon := \tilde{J} + \epsilon t \) is also a sub-solution of (2.9) since
\[
F\left(t, X, \tilde{J}_\epsilon, \frac{\partial \tilde{J}_\epsilon}{\partial t}, \frac{\partial \tilde{J}_\epsilon}{\partial X}, \frac{\partial^2 \tilde{J}_\epsilon}{\partial X^2}\right) \leq -\epsilon < 0.
\]

Therefore, we obtain the result in (A.1) by choosing \( \theta \leq \frac{1}{T} \) and using \( \tilde{J} \) and \( \tilde{J}_\epsilon \) since
\[
\frac{\partial \tilde{J}_\epsilon}{\partial t} = \frac{\partial \tilde{J}}{\partial t} + \epsilon.
\]

If we let \( Q = [0, T] \times O \), where \( O \) is the state space, this modification is enough for the comparison theorem since by letting \( J \in LSC(Q) \) and \( \tilde{J} \in USC(Q) \), we have that
\[
\tilde{J}_\epsilon - J \leq \sup_{\partial Q} \{ \tilde{J}_\epsilon - J \} \quad \text{and} \quad J - J \leq \sup_{\partial Q} \{ J - \tilde{J} \} + \epsilon T,
\]
which gives the comparison theorem as \( \epsilon \to 0 \).

The proof of the second assumption: For all \((t, x), (t, y) \in Q, r \in \mathbb{R} \) and \( A, B \) such that
\[
-\left( \frac{1}{\epsilon} + |\alpha| \right) I_4 \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq (\alpha + \epsilon |\alpha|^2) \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}.
\]

Note that we have adapted the matrix to our case where the space dimension is 2, and by letting \( \alpha = \frac{1}{\epsilon} \) we obtain
\[
(A.6) \quad \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq 2\alpha \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}.
\]

We can check this by changing the corresponding values of \( F \), that is
\[
F(t, y, r, v, \alpha(x - y), B) - F(t, x, r, v, \alpha(x - y), A) =
\]
\[
-v - \frac{1}{2} y \sigma B(y \sigma)^t \max \{ (\mu y_R + \Gamma, r y_M - \Gamma) \cdot \alpha(x - y) \}
\]
\[
+ v + \frac{1}{2} x \sigma A(x \sigma)^t \max \{ (\mu x_R + \Gamma, r x_M - \Gamma) \cdot \alpha(x - y) \}.
\]

Using that
\[
\frac{1}{2} ((x \sigma) A(x \sigma)^t - (y \sigma) B(y \sigma)^t) = \frac{1}{2} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
\[
\leq \alpha \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
\[
= \alpha \sigma^2 (x_R - y_R)^2 \leq \alpha \sigma^2 \|x - y\|^2,
\]
we obtain
\[ F(t, y, r, v, \alpha(x - y), \mathbf{B}) - F(t, x, r, v, \alpha(x - y), \mathbf{A}) \leq \alpha \sigma^2 \|x - y\|^2 - \max \left\{ \left( \mu y R + \Gamma, ry M - \Gamma \right) \cdot (x - y) \right\} + \max \left\{ \left( \mu x R + \Gamma, rx M - \Gamma \right) \cdot (x - y) \right\} + \max \left\{ \left( \mu y R + \Gamma^* x, ry M - \Gamma^* x \right) \cdot (x - y) \right\} + \max \left\{ \left( \mu x R + \Gamma^* x, rx M - \Gamma^* x \right) \cdot (x - y) \right\}. \]

Now using that the maximum of \( \Gamma \) is obtained over a compact set (In our case a bounded and closed segment), we denote the maximum for \( x \) as \( \Gamma^* x \) and then
\[ F(t, y, r, v, \alpha(x - y), \mathbf{B}) - F(t, x, r, v, \alpha(x - y), \mathbf{A}) \leq \alpha \sigma^2 \|x - y\|^2 - \alpha \left( \mu y R + \Gamma^* x, ry M - \Gamma^* x \right) \cdot (x - y) + \alpha \left( \mu x R + \Gamma^* x, rx M - \Gamma^* x \right) \cdot (x - y), \]

and thus
\[ F(t, y, r, v, \alpha(x - y), \mathbf{B}) - F(t, x, r, v, \alpha(x - y), \mathbf{A}) \leq \alpha (\sigma^2 + 2\mu) \|x - y\|^2. \]

Then by letting \( \varpi(z) = Cz \), where \( C > 0 \) is a constant greater than \( \sigma^2 - 2\mu \), we obtain the desired result in equation (A.2).

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