VALUATION OF MULTI-ASSET BARRIER OPTIONS

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Abstract. We obtain closed form expressions for the exact no-arbitrage prices, as well as estimates, of some types of multivariate options with barriers that are generated by hyperplanes placed on a collection (or vector) of stock prices (they are not placed individually on each stock of the collection). A novelty for the estimates is that we combine ideas of convex analysis with tools of stochastic theory.

1. INTRODUCTION

European barrier options in the multivariate case are multivariate European options which involve barriers constraints tailored for a certain behavior of the underlying assets along the option’s lifetime. If this behavior is what the investor thinks will happen, then he may pay less buying the barrier option instead of its standard counterpart, obtaining the same result whenever his beliefs meet reality. Otherwise the option’s payoff cancels.

Since the seminal works [1] and [7], option pricing theory experienced a significant thrust in obtaining closed-form expressions for the exact prices and hedges, as more complicated models were considered. However, such achievements did not replicate when entering with more sophisticated derivatives as path dependent options, particularly barrier options (see, e.g., [6]).

In the multi-asset scenario little work has been done to derive prices of path-dependent options. Quoting [8], an important reason for this is because the matter poses significant technical difficulties that do not appear in the single-asset case.

The work in [5] and [10] presents a relatively simple method for obtaining closed form valuation formulae for one-touch, multi-asset options where a single barrier is a function of two stocks simultaneously, via method of images. We can also find analytical price for two-asset barrier options where the barriers on each asset are monitored at distinct time periods in [9].

Still in the Black-Scholes framework and addressing the case of one barrier for each stock, analytical prices of multi-asset options, can be found in [8]. These prices are obtained by combining ideas from the generalized reflection principle of Brownian Motion with ideas from algebraic geometry.

Addressing multivariate American option pricing in the Black-Scholes world, [11] provides numerical solutions via a finite difference method. In [12], a numerical method is provided to address multivariate European option pricing problems with jumps entering in the dynamics.

A study on multi-asset Black-Scholes model with correlated underlyings is found in [13]. Pricing is shown to depend on a certain partial differential equation. Correlation is the main subject of the study. Namely, the study shows that limits for the existence of a valid solution for the correlated multi-asset Black-Scholes model occurs. Particularly, usual formulas that appears in the literature are no longer valid.

Via the risk neutral pricing technique and combining ideas of convex analysis, we present elegant pricing formulas, as well as estimates, for multi-asset up-and-out call options.

The barriers can be viewed as exponential modifications of hyperplanes that test the price vector of the underlyings, or else, hyperplanes that test the logarithmic vector of the prices of the underlyings.

Key words and phrases. no arbitrage pricing, multi-asset barrier options, Martingale measure.

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We are allowed to test if a certain collection or portfolio of assets breaches the barrier. In turn, the payoffs per se, i.e., without the barrier as specified above, are as follows:

(i): on a stock that is not verified by the barrier specified above, however it is tested by an individual barrier \( B_0 \) which is per se interesting from a financial perspective: it is constant if denominated on the numeraire taken to be the price of the risk-free asset. namely,

\[
B_0(u) = c_0 \beta(u) \quad \text{and} \quad \beta(u) = e^{\int_0^u r(s)ds},
\]

where \( c_0 \) is the value of the barrier \( B_0 \) at time zero and \( r \) is the interest rate - Section 2;

(ii): on a geometric basket of stocks - Section 3;

(iii): on an arbitrary combination of stocks (portfolio) - Section 4.

We provide, for arbitrary \( t \), exact prices for the first two items above, while an estimate is given for the third item in terms of a lower bound.

In our thrust to finding closed form solutions for exact prices we addressed the uncorrelated scenario. It is worth noticing that a seminal tool in stochastic theory, namely, the time change for martingales theorem, is only presented in the literature in the uncorrelated form. Moreover, dramatic problems seem to appear when correlation is allowed as pointed out in [13], as mentioned above.

Let us consider a multidimensional market model on \((\Omega, \mathcal{F}, \mathbb{P})\) where the risky assets prices evolve according to the stochastic differential equations

\[
dS_0(u) = \mu_0(u)S_0(u)du + \sigma_0(u)S_0(u)dW_0(u),
\]

\[
dS_i(u) = \mu_i(u)S_i(u)du + \sigma(u)S_i(u)dW_i(u),
\]

\( i = 1, 2, \ldots, m \). On the probability space \((\Omega, \mathcal{F}, \tilde{\mathbb{P}})\), where \( \tilde{\mathbb{P}} \) is the risk-neutral probability for the market described above, the dynamics reads

\[
d\bar{S}_0(u) = r(u)S_0(u)du + \sigma_0(u)S_0(u)d\tilde{W}_0(u),
\]

\[
d\bar{S}_i(u) = r(u)S_i(u)du + \sigma(u)S_i(u)d\tilde{W}_i(u).
\]

The risk-free interest rate \( r \), the instantaneous rates of return \( \mu_i \), and the volatilities \( \sigma_0 \), \( \sigma \) are deterministic functions of time subject to the mild constraint of square integrability (jumps are allowed, for instance). \( W = (W_0, W_1, \ldots, W_m) \) and \( \tilde{W} = (\tilde{W}_0, \tilde{W}_1, \ldots, \tilde{W}_m) \) are \( m + 1 \)-dimensional Brownian Motions on \((\Omega, \mathcal{F}, \mathbb{P})\) and \((\Omega, \mathcal{F}, \tilde{\mathbb{P}})\) respectively. \( \bar{S}(u) = (\bar{S}_1(u), \ldots, \bar{S}_m(u)) \) denotes the discounted asset price vector, where

\[
\bar{S}_i(u) = \frac{S_i(u)}{\beta(u)} \quad \text{and} \quad \beta(u) = e^{\int_0^u r(s)ds}, \quad i = 1, \ldots, m.
\]

\( \beta \) is the price of the risk-free asset.

Compared with the constant parameters scenario, significant difficulties appear when entering with deterministic time dependent parameters. However, the latter case improves calibration and pricing. Moreover, it is worth noticing that markets forecasts stem primarily from practitioners believes and intuition, which lead to deterministic scenarios. Also, while stressing the notion of stochastic volatility, [2] asserts that "various quantities of interest (such as option prices) may sometimes be computed as though future volatility were deterministic rather than stochastic". These aspects motivate the use of deterministic time dependent models.

2. Pricing up-and-out call options with payoff on the stock \( S_0 \)

In this section, we compute the price of an up-out-call option whose payoff reads

\[
H(T) = (S_0(T) - K)^+ \mathbb{1}_{\{S_0(u) \leq B_0(u), \bar{T} \in S(u) \leq h \quad \forall \; u \in [0,T]\}},
\]

i.e., the payoff is on a stock \( S_0 \) which is not tested by our barrier. \( K > 0 \) is the strike price, \( T > 0 \) the expiration time and \( \bar{T} = (1, \ldots, 1) \in \mathbb{R}^n \).
At time \( t \in [0, T] \) arbitrarily fixed, the no-arbitrage price for this option is (see, e.g., in [3])
\[
H(t) = \hat{E}[H(T) \beta(t)/\beta(T) \mid \mathcal{F}(t)],
\]
with \( \mathcal{F}(t) \subset \mathcal{F}, 0 \leq t \leq T, \) denoting the filtration generated by \( W. \)

In order to deal with a zero-origin starting point, define \( \eta'(u) = \eta(t + u), 0 \leq u \leq T - t, \) where \( \eta \) stands for \( \sigma, S \) and \( \hat{W}. \) It follows that
\[
dS_0(u) = r^t(u)S_0(u)du + \sigma^t(u)S_0(u)d\hat{W}^0(u),
\]
\[
dS_i(u) = r^t(u)S_i(u)du + \sigma^t(u)S_i(u)d\hat{W}^i(u),
\]
\( S_0(0) = S_0(t) \) and \( S_i(0) = S_i(t), i = 1, \ldots, m. \)

We define the \( t \)-discounted risky assets prices processes \( \bar{S}_0(u) = S_0(u)(\beta(t)/\beta(t + u)) = S_0(t)\exp\{\bar{I}_0(u)\} \)
and \( \bar{S}_i(u) = S_i(t)\exp\{\bar{I}_i(u)\}, \) martingales under \( \bar{P}, \) where
\[
\bar{I}_0(u) = \int_0^u \sigma^t_0(s)d\hat{W}^0(s) - \frac{1}{2} \int_0^u (\sigma^t_0(s))^2 ds,
\]
\[
\bar{I}_i(u) = \int_0^u \sigma^t_i(s)d\hat{W}^i(s) - \frac{1}{2} \int_0^u (\sigma^t_i(s))^2 ds,
\]
\( 0 \leq u \leq T - t, i = 1, \ldots, m. \) We also define
\[
\bar{M}_0 = \max_{0 \leq u \leq T - t} \bar{I}_0(u), \text{ so that }
\]
\[
\max_{0 \leq u \leq T - t} \bar{S}_0(u) = S_0(t)\exp\{\bar{M}_0\},
\]
and
\[
\bar{M}'_i = \max_{0 \leq u \leq T - t} \bar{I}_i(u)
\]
with \( \bar{I}_i(u) = (\bar{I}_1(u), \ldots, \bar{I}_m(u)). \)

Bearing in mind (2.7), (2.8) and the barriers in (2.1), we have that
\[
\{S_0(u) \leq B_0(u) \ \forall u \in [t, T]\} = \{\bar{S}_0(u) \leq B_0(t) \ \forall u \in [0, T - t]\} = \{S_0(t)\exp\{\bar{M}_0\} \leq B_0(t)\},
\]
and
\[
\{\bar{I} \cdot \ln \bar{S}(u) \leq h \ \forall u \in [t, T]\} = \{\bar{I} \cdot \ln \bar{S}(u) \leq h \ \forall u \in [0, T - t]\} = \{\bar{M}' \leq \bar{h}\},
\]
where \( \bar{h} = h - \bar{I} \cdot \ln \bar{S}(t). \) So, the \( t \)-discounted payoff \( \bar{H}'(T) \) reads
\[
\bar{H}'(T) = H(T)\frac{\beta(t)}{\beta(T)} = \left(S_0(t)\exp\{\bar{I}_0(t - T)\} - K(t)\right)\mathbb{1}\{\bar{I}_0(t - T) \geq b_0(t), \bar{M}_0 \leq b_0(t)\}\mathbb{1}\{\bar{M}' \leq \bar{h}\}
\]
with \( K(t) = K\frac{\beta(t)}{\beta(T)}, b_0(t) = \ln \left(\frac{K(t)}{S_0(t)}\right) \) and \( b_0(t) = \ln \left(\frac{B_0(t)}{S_0(t)}\right). \)

**Lemma 2.1.** Under \( \bar{P}, \) the joint density of \( \left( \bar{M}'_i, \bar{I}_i(t - T) \right) \) is given by
\[
\bar{f}_{\bar{M}'_i, \bar{I}_i(t - T)}(x_0, y_0) = \begin{cases} 
\frac{(2x_0^2 - y_0)}{Q_0(t)\sqrt{2\pi Q_0(t)}} \exp \left(-\frac{1}{2}y_0 - \frac{1}{2y_0}Q_0(t) - \frac{(2x_0^2 - y_0)^2}{2y_0}\right), & y_0 \leq x_0, \ x_0 > 0 \\
0, & \text{otherwise},
\end{cases}
\]
where \( Q_0(t) = \int_t^T \sigma^2_0(u)du. \)
Proof. Relying on the square integrability of \( \sigma_0(u) \), define the Radon-Nikodym derivative process

\[
Z_0^*(u) = \exp \left\{ \int_0^u \frac{\sigma_0^2(s)}{2} \, dW_0^*(s) - \frac{1}{2} \int_0^u \frac{(\sigma_0(s))^2}{4} \, ds \right\},
\]

0 \leq u \leq T - t, as well as the new measure \( \tilde{P}(A) = \int_A Z_0^*(T - t) \, d\tilde{P} \); \( A \in \mathcal{F} \), under which (we invoke Girsanov’s Theorem) \( \tilde{W}_0(u) = \tilde{W}_0^*(u) - \frac{1}{2} \int_0^u \sigma_0(s) \, ds \) is a Brownian Motion and

\[
\tilde{h}_0^*(u) = \int_0^u \sigma_0^2(s) \, d\tilde{W}_0(s), \quad 0 \leq u \leq T - t,
\]
is a continuous martingale. We may further express \( \tilde{h}_0^* \) as

\[
\tilde{h}_0^*(u) = \mathcal{W}_0(\tilde{h}_0^*(u)), \quad 0 \leq u \leq T - t,
\]
where \( \mathcal{W}_0 \) is a Brownian Motion under \( \tilde{P} \), and

\[
\tilde{h}_0^*(u) = \left[ \tilde{I}_0^t, \tilde{I}_0^t \right](u) = \int_0^u \left( \sigma_0^2(s) \right) ds
\]
is the quadratic variation of \( \tilde{h}_0^* \) (for this, see the time-change for Martingales theorems, e.g. in [4]). Moreover, since \( \tilde{h}_0^* \) is continuous and increasing, (2.7) gives us that \( \tilde{M}_0^* = \max_{0 \leq s \leq h_0^*(T - t)} \mathcal{W}_0(s) \). This and (2.23) allow us to state the following version of the Reflection Principle extended for continuous-time Martingales with deterministic quadratic variation (in this case given by (2.24)):

\[
\tilde{P} \left\{ \tilde{M}_0^* \geq x_0, \tilde{I}_0^*(T - t) \leq y_0 \right\} = \tilde{P} \left\{ \tilde{I}_0^*(T - t) \geq 2x_0 - y_0 \right\},
\]
yo \leq x_0, x_0 > 0. Differentiating (2.25) and bearing in mind that \( \tilde{I}_0^*(T - t) \) is normally distributed with zero mean and variance \( \tilde{h}_0^*(T - t) \), we obtain

\[
\tilde{f}_{\tilde{h}_0^*(T - t)}(x_0, y_0) = \frac{2(2x_0 - y_0)}{\tilde{h}_0^*(T - t) \sqrt{2\pi \tilde{h}_0^*(T - t)}} \exp \left\{ -\frac{(2x_0 - y_0)^2}{2 \tilde{h}_0^*(T - t)} \right\},
\]
yo \leq x_0, x_0 > 0. On the other hand,

\[
\tilde{I}_0^*(T - t) = \exp \left\{ \frac{1}{2} \tilde{I}_0^*(T - t) + \frac{1}{8} \tilde{h}_0^*(T - t) \right\},
\]
so we may write

\[
\tilde{P} \left\{ \tilde{M}_0^* \leq x_0, \tilde{I}_0^*(T - t) \leq y_0 \right\} = \tilde{E} \left[ \exp \left\{ -\frac{1}{2} v - \frac{1}{8} \tilde{h}_0^*(T - t) \right\} \tilde{f}_{\tilde{h}_0^*(T - t)}(w, v) \right] \in \tilde{f}_{\tilde{M}_0^*, \tilde{I}_0^*(T - t)}(w, v) \, dw \, dv.
\]

Differentiating the above expression gives us the desired result. }

**Lemma 2.2.** Under \( \tilde{P} \), the joint density of \( \left( \tilde{M}^*, \tilde{I} \cdot \tilde{I}^*(T - t) \right) \) is given by

\[
\tilde{f}_{\tilde{M}^*, \tilde{I}^*(T - t)}(x, y) = \left\{ \begin{array}{ll}
\frac{2(2x - y)}{m \tilde{Q}(t) \sqrt{2\pi m \tilde{Q}(t)}} \exp \left\{ -\frac{1}{2} y - \frac{m \tilde{Q}(t)}{2} \left( \frac{(2x - y)^2}{2m \tilde{Q}(t)} \right) \right\}, & y \leq x, \ x > 0 \\
0, & \text{otherwise},
\end{array} \right.
\]

where \( Q(t) = \int_0^T \sigma^2(u) \, du \).
Proof. Relying on the square integrability of $\sigma(u)$, define the Radon-Nikodým derivative process

$$Z^u(t) = \exp \left\{ \sum_{i=1}^m \int_0^u \frac{\sigma^i(s)}{2} d\hat{W}^i_s - \frac{m}{8} \int_0^u (\sigma^i(s))^2 ds \right\},$$

$0 \leq u \leq T - t$, as well as the new measure $\hat{P}(A) = \int_A Z^u(T - t) d\bar{\mathbb{P}}$, $A \in \mathcal{F}$, under which (we invoke Girsanov’s Theorem) $\hat{W}(u) = (\hat{W}_1(u), \ldots, \hat{W}_m(u))$ with $\hat{W}_i(u) = \hat{W}^i_t(u) - \frac{1}{2} \int_0^u \sigma^i(s) ds$, $i = 1, \ldots, m$, is a $m$-dimensional Brownian Motion and

$$\hat{I}^i(u) = (\hat{I}^i_1(u), \ldots, \hat{I}^i_m(u)), \quad 0 \leq u \leq T - t,$$

is a $m$-dimensional continuous martingale where

$$\hat{I}^i_t(u) = \int_0^u \sigma^i(s) d\hat{W}^i_s,$$

$i = 1, \ldots, m$.

By virtue of the time-change for Martingales theorem, we may further express $\hat{I}$ as

$$\hat{I}(u) = \mathcal{W}(h'(u)), \quad 0 \leq u \leq T - t,$$

where $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m)$ is a $m$-dimensional Brownian Motion under $\hat{P}$, and

$$h'(u) = \left[ \hat{I}, \hat{I} \right](u) = \int_0^u (\sigma^i(s))^2 ds$$

is the quadratic variation of $\hat{I}^i(u)$ for all $i = 1, \ldots, m$. Moreover, since $h'(u)$ is continuous and increasing, (2.8) gives us that $M^t = \max_{0 \leq s \leq h'(T - t)} \hat{I}^i \mathcal{W}(s)$. This, (2.23) and Proposition 3.25 in [8] allow us to state the following version of the generalised Reflection Principle extended for continuous-time Martingales whose components are uncorrelated and have the same deterministic quadratic variation function (in this case given by (2.24)):

$$\hat{P}\left\{ M^t \geq x, \hat{I} \mathcal{W}(T - t) \leq y \right\} = \hat{P}\left\{ \hat{I} \mathcal{W}(T - t) \geq 2x - y \right\},$$

$y \leq x, x > 0$. Differentiating (2.25) and bearing in mind that $\hat{I} \mathcal{W}(T - t)$ is normally distributed with zero mean and variance $m h'(T - t)$, we obtain

$$\hat{f}_{\hat{M}^t, \hat{I} \mathcal{W}(T - t)}(x, y) = \frac{2(2x - y)}{m h'(T - t) \sqrt{2\pi m h'(T - t)}} \exp \left\{ -\frac{(2x - y)^2}{4mh'(T - t)} \right\},$$

$y \leq x, x > 0$. On the other hand,

$$Z^u(T - t) = \exp \left\{ \frac{1}{2} \left( \hat{I} \mathcal{W}(T - t) \right) + \frac{m}{8} h'(T - t) \right\},$$

so we may write

$$\hat{P}\left\{ M^t \leq x, \hat{I} \mathcal{W}(T - t) \leq y \right\} = \hat{E}\left[ \frac{1}{Z^u(T - t)} \right| M^t \leq x, \hat{I} \mathcal{W}(T - t) \leq y \right],$$

$$= \int_{-\infty}^y \left( I^x \mathcal{W}(T - t) \right) \hat{f}_{\hat{M}^t, \hat{I} \mathcal{W}(T - t)}(w, v) dv.$$

Differentiating the above expression gives us the desired result. \hfill \Box

The pricing result is the content of the following theorem.

**Theorem 2.1.** If the up-and-out call option whose payoff is given by (2.1) has not knocked out prior to time $t \in [0, T)$, then its no-arbitrage price $H(t)$ is given by

$$H(t) = \left\{ S_0(t)[N(z_{t,a}^{0,+}) - N(z_{t,b}^{0,0})] - K(t)[N(z_{t,a}^{0,-}) - N(z_{t,b}^{0,-})] - B_0(t)[N(z_{t,c}^{0,+}) - N(z_{t,c}^{0,-})] \right\}$$

$$+ K(t)\{N(z_{t,c}^{0,-}) - N(z_{t,c}^{0,0})\} \{N(z_{t,h}^{+}) - e^{-h}N(z_{t,h}^{-})\},$$
where \( S_0(t) \) is the price of the risky asset \( S_0 \) observed at time \( t \), \( K(t) = \frac{\beta(t)}{\beta(T)}K \), \( Q_0(t) = \int_t^T \sigma^2(u)du \),
\[
\begin{align*}
 z^0_{t,s} &= \frac{1}{\sqrt{Q_0(t)}} \left[ \ln s \pm \frac{1}{2}Q_0(t) \right], \\
 Q(t) &= \int_t^T \sigma^2(u)du, \\
 z^\pm_{t,s} &= \frac{1}{\sqrt{mQ(t)}} \left[ mQ(t) \pm s \right], \\
 N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{z^2}{2}} \, dz,
\end{align*}
\]
\( a_0 = S_0(t)/K(t) \), \( b = S_0(t)/B_0(t) \), \( c = B_0^2(t)/(K(t)S_0(t)) \), \( d = B_0(t)/S_0(t) \) and \( h = h - \bar{1} \cdot \ln S(t) = h - \sum_{i=1}^m \ln S_i(t) \).

**Proof.** Bearing in mind that \( S_0(t) \) is \( \mathcal{F}(t) \)-measurable and \( \bar{H}_t' \) and \( \bar{M}_t \) are independent of \( \mathcal{F}(t) \), it follows, from (2.2) and (2.10), that
\[
H(t) = \bar{E}[(S_0(t) \exp\{\bar{I}_H'(T-t)\} - K(t))I_{\bar{I}_H'(T-t)\geq h_0(t), \bar{M}_t\leq h_0}] | \mathcal{F}(t), \bar{P} \left\{ \bar{M}_t \leq \bar{h} \right\} = \text{A.B},
\]
where
\[
A = \int_{h_0}^{b_0} \int_{b_0}^{h_0} (S_0(t)e_{h_0}^y - K(t)) \bar{I}_M'(0)(x_0, y_0) \, dx_0 \, dy_0,
\]
and
\[
B = \int_{-\infty}^{\bar{h}} \int_{x^+}^{\bar{h}} \bar{I}_M'(0)(x, y) \, dx \, dy
\]
where \( \bar{I}_M'(0)(T-t) \) and \( \bar{I}_M'(0)(T-t) \) are given by (2.11) and (2.19) respectively, and \( v^+ = \max\{v, 0\} \).

The pricing formula (2.28) follows after some algebraic manipulation. \( \square \)

### 3. Pricing up-and-out call options with payoff on a basket of stocks

In this section we consider an up-and-out call option with payoff on a geometric basket of stocks given by
\[
\mathcal{H}(T) = \left( \prod_{i=1}^m (S_i(T))^{1/m} - K \right)^+ \mathbb{I}_{\{ \bar{1} \cdot \ln \bar{S}(u) \leq \bar{h} \forall u \in [t, T) \}},
\]
where \( K > 0 \) is the strike price. The \( t \)-discounted payoff of this derivative reads
\[
\bar{H}'(T) = \left( \prod_{i=1}^m (\bar{S}_i(T-t))^{1/m} - K(t) \right)^+ \mathbb{I}_{\{ \bar{1} \cdot \ln \bar{S}(u) \leq \bar{h} \forall u \in [t, T) \}},
\]
where \( \bar{S}_i(T-t) = S_i(T)/(\beta(t)/\beta(T)) \) and \( K(t) = K(\beta(t)/\beta(T)) \). Its no-arbitrage price at \( t \in [0, T] \) is
\[
\mathcal{H}(t) = \bar{E} \left[ \bar{H}'(T) | \mathcal{F}(t) \right].
\]

Since \( \bar{S}_i^t(u) = S_i(t) \exp\{\bar{I}_H'(u)\} \) and from (2.9) we may rewrite (3.2) as follows
\[
\begin{align*}
\bar{H}'(T) &= \left( \prod_{i=1}^m (S_i(t)e^{\bar{I}_H'(T-t)})^{1/m} - K(t) \right)^+ \mathbb{I}_{\{ \bar{M}_t \leq \bar{h} \}} \\
&= \left( \prod_{i=1}^m (S_i(t))^{1/m} e^{\sum_{i=1}^m \bar{I}_H'(T-t)/m} - K(t) \right)^+ \mathbb{I}_{\{ \bar{M}_t \leq \bar{h} \}} \\
&= \left( S \cdot e^{\left( \sum_{i=1}^m \bar{I}_H'(T-t)/m \right)} - K(t) \right)^+ \mathbb{I}_{\{ \bar{M}_t \leq \bar{h} \}} \\
&= \left( S \cdot e^{\left( \sum_{i=1}^m \bar{I}_H'(T-t)/m \right)} - K(t) \right) \mathbb{I}_{\{ \bar{M}_t \leq \bar{h} \}}. 
\end{align*}
\]
Proof. Bearing in mind that

\[
S(t) = (S_1, \ldots, S_m)
\]

is the vector price of the risky assets observed at time \( t \), \( S = (\prod_{i=1}^{m} S_i(t))^{1/m} \) and \( k = m \ln (K(t)/S) \).

**Theorem 3.1.** If the derivative whose payoff is given by (3.1) has not knocked out prior to time \( t \in [0, T] \), then its no-arbitrage price \( \mathcal{H}(t) \) is given by

\[
\mathcal{H}(t) = Se^{\left(\frac{1}{2m} \right) \sigma^2(t) Q(t)} \left\{ N(z_{t,h+Q(t)}) - N(z_{t,h+Q(t)}) \right\}
- K(t) \left\{ N(\delta_{t,h}) - N(\delta_{t,h}) \right\}
+ K(t)e^{-h} \left\{ N(\delta_{t,h}^-) - N(\delta_{t,h^-}^-) \right\}
- Se^{(\frac{2m}{m})h+(\frac{1}{2m})Q(t)h}
\cdot \left\{ N(z_{t,h+Q(t)}) - N(z_{t,h+Q(t)}) \right\},
\]

where \( (3.5) \) follows after some algebraic manipulation.

The pricing formula (3.5) follows after some algebraic manipulation.

4. Obtaining Lower Bounds for the Price of Up-And-Out Call Options with Payoff on a Combination of Stocks

First, let us consider a \( t \)-dependent function given by

\[
\tilde{v}^T(t) = (\tilde{I} \cdot \ln \tilde{S}^t(\tilde{T} - t) - \tilde{K})^+ I_{\{\tilde{I} \cdot \ln \tilde{S}(u) \leq h \ \forall \ u \in [t, T]\}},
\]

where \( \tilde{S}^t(T) = S(T) (\beta(t)/\beta(T)) \) and \( \tilde{K} \in \mathbb{R} \), as well as

\[
v(t) = \tilde{E} [\tilde{v}^T(T) \mid \mathcal{F}(t)].
\]

From (2.8) and (2.9) we may rewrite (4.1) as follows

\[
\tilde{v}^T(T) = \left( \tilde{I} \cdot \ln (\tilde{S}(t) \exp(\tilde{I}(T - t))) - \tilde{K} \right)^+ I_{\{\tilde{M} \leq h\}}
= \left( \tilde{I} \cdot \tilde{I}(T - t) - (\tilde{K} - \tilde{I} \cdot \ln \tilde{S}(t)) \right)^+ I_{\{\tilde{M} \leq h\}}
= \left( \tilde{I} \cdot \tilde{I}(T - t) - \tilde{K} \right) I_{\{\tilde{I} \cdot \tilde{I}(T - t) \geq \tilde{K}\}} I_{\{\tilde{M} \leq h\}},
\]

(4.3)
where $\hat{K} = \bar{K} - \bar{I} \cdot \ln S(t)$ and $h = h - \bar{I} \cdot \ln S(t)$.

**Theorem 4.1.** If $\{\bar{I} \cdot \ln S(u) \leq h \; \forall \; u \in [t, T]\}$ holds, then

\[
v(t) = \sqrt{\frac{m Q(t)}{2\pi}} \left\{ e^{\frac{1}{2} \left( z_{t,K}^+ \right)^2} - e^{\frac{1}{2} \left( z_{t,K}^- \right)^2} \right\}
- \sqrt{\frac{m Q(t)}{2\pi}} \left\{ e^{-h - \frac{1}{2} \left( z_{t,K-2h}^+ \right)^2} - e^{-h - \frac{1}{2} \left( z_{t,K}^- \right)^2} \right\}
- \left( \frac{m Q(t)}{2} + \hat{K} \right) \left\{ N(z_{t,h}^+ - N(z_{t,K}^-) \right\}
+ \left( \frac{m Q(t)}{2} + \hat{K} - 2\hat{h} \right) e^{-h}
\]

(4.4)

where $S(t) = (S_1, \ldots, S_m)$ is the vector price of the risky assets observed at time $t$, $\hat{K} = \bar{K} - \bar{I} \cdot \ln S(t)$, $h = h - \bar{I} \cdot \ln S(t)$, $Q(t) = \int_t^T \sigma^2(u) du$, $z_{t,s}^\pm = \frac{1}{\sqrt{m Q(t)}} \left[ \frac{m Q(t)}{2} \pm s \right]$, $N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-y^2} dy$.

**Proof.** Bearing in mind that $\bar{I} \cdot \hat{I}(T - t)$ and $\hat{M}^t$ are independent of $\mathcal{F}(t)$, it follows, from (4.2) and (4.3), that

\[
v(t) = \tilde{E} \left[ \left( \bar{I} \cdot \hat{I}(T - t) - \hat{K} \right) I_{[\bar{I} \cdot \hat{I}(T - t) \geq \hat{K}] \cap (\hat{M}^t \leq \hat{h})} \mid \mathcal{F}(t) \right]
\]

= $\int_{\hat{K}}^h \int_{y^+} \left( y - \hat{K} \right) \tilde{f}_{\hat{M}^t, \bar{I} \cdot \hat{I}(T - t)}(x, y) dx dy$

(4.5)

where $\tilde{f}_{\hat{M}^t, \bar{I} \cdot \hat{I}(T - t)}$ is given by (2.19) and $y^+ = \max\{y, 0\}$. The formula (4.4) follows after some algebraic manipulation. \qed

Now, let us consider the up-and-out call option whose payoff is on a combination of stocks. It reads

\[
V(T) = (\bar{u} \cdot S(T) - K)^+ I_{[\bar{I} \cdot \ln S(t) \leq h \; \forall \; t \in [0, T]}$

(4.6)

where $u \in \mathbb{R}_+^m$, $K > 0$ is the strike price and $T > 0$ is the expiration time. Its $t$-discounted payoff is given by

\[
V^t(T) = (\bar{u} \cdot S^t(T - t) - K(t))^+ I_{[\bar{I} \cdot \ln S(t) \leq h \; \forall \; t \in [0, T]}$

(4.7)

where $K(t) = K(\beta(t)/\beta(T))$.

Combining Theorem 4.1 with ideas of convex analysis, we have the following theorem.

**Theorem 4.2.** A maximum lower bound for the price of the up-and-out call option with payoff (4.6) is given by

\[
\mathcal{V}(t) = \sqrt{\frac{m Q(t)}{2\pi}} \left\{ e^{\frac{1}{2} \left( z_{t,K_2}^+ \right)^2} - e^{\frac{1}{2} \left( z_{t,K_2}^- \right)^2} \right\}
- \sqrt{\frac{m Q(t)}{2\pi}} \left\{ e^{-h - \frac{1}{2} \left( z_{t,K-2h}^+ \right)^2} - e^{-h - \frac{1}{2} \left( z_{t,K}^- \right)^2} \right\}
- \left( \frac{m Q(t)}{2} + K_2 \right) \left\{ N(z_{t,h}^+ - N(z_{t,K_2}^-) \right\}
\]

where $\hat{K} = \bar{K} - \bar{I} \cdot \ln S(t)$ and $h = h - \bar{I} \cdot \ln S(t)$.
\[ S(t) = (S_1, \ldots, S_m) \] is the vector price of the risky assets observed at time \( t \), \( K(t) = \frac{\partial h}{\partial t} K, \bar{h} = h - \bar{1} \cdot \ln S(t), \]

\[ Q(t) = \int_t^T \sigma^2(u) du, z_{1,t}^\pm = \frac{1}{\sqrt{mQ(t)}} \left[ \frac{mQ(t)}{2} \pm s \right], N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{s}{\sqrt{2}}} e^{-\frac{y^2}{2}} dy. \]

**Proof.** By virtue of Theorem 4.3 and Corollary 4.1 with \( K_3 = K(t) \), we have that (4.1), with \( \bar{K} = K_1^* \), is the maximum lower bound for (4.7) in the class of the \( K_1 \)-parameterized functions

\[ \bar{1} \cdot \ln \bar{S}(T-t) - K_1^+, K_1 \in \mathbb{R}. \]

Hence, (4.4) with

\[ \bar{K} = \left( \sum_{i=1}^n v_i \left( \frac{1}{u_i} \ln \eta - \eta \right) \right) + K(t), \]

is a lower bound for the no-arbitrage price of (4.6). This concludes the proof. \( \square \)

**Theorem 4.3.** [Maximum lower bound for \( \max \{0, l\} \), \( l \) an affine, via a class of concave functions] Let \( h : \mathbb{R}^n \to \mathbb{R} \) be differentiable and concave, \( l : \mathbb{R}^n \to \mathbb{R} \) be affine, and admit that \( \frac{\partial h(x)}{\partial x} \), as well as \( \lim_{x_i \to \infty} \frac{\partial h(x)}{\partial x_i} \) with fixed \( x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \), are nonzero, \( i = 1, \ldots, n \). Let \( x^* \in \mathbb{R}^n \). It follows that

1. \( l(x^*) = 0 \) and \( \nabla h(x^*) = a \nabla l \) for some \( a \in (0,1] \) (co-linear gradients), or
2. \( l(x^*) > 0 \) and \( \nabla h(x^*) = \nabla l \).

if and only if

\[ (h(x) - K_1^*)^+, K_1 = h(x^*) - l(x^*) \]

is the maximum lower bound (MLB) for \( l(x^+) \) in the class of the \( K_1 \)-parameterized functions

\[ (h(x) - K_1)^+, K_1 \in \mathbb{R}. \]

If \( x^* \) exists, \( x^* \) defines a touching point between the graphs of \( l^+ \) and \( (h(\cdot) - K_1)^+ \) (i.e., \( (h(x^*) - K_1^*)^+ = l(x^*)^+ \)).

**Corollary 4.1.** [Maximum lower bound for the multi-asset call option \( (u \cdot x - K_3)^+ \)] For given \( u, v \in \mathbb{R}_{\geq 0}^n, K_3 \geq 0 \) and denoting (with slight abuse of notation) \( h \equiv \ln : \mathbb{R}_{\geq 0}^n \to \mathbb{R} \) such that

\[ \ln(x) = (\ln(x_1), \ldots, \ln(x_n)), \]

define

\[ b = \frac{1}{K_3} \sum_{i=1}^n v_i, \eta = \begin{cases} 1/b & \text{if } 0 \leq b < 1 \\ 1 & \text{if } b \geq 1, \end{cases} \]

\[ K_1^* = \left( \sum_{i=1}^n v_i \left( \frac{1}{u_i} \ln \eta - \eta \right) \right) + K_3 \text{ and} \]

\[ x^* = \left( \frac{v_1}{u_1} \eta, \ldots, \frac{v_n}{u_n} \right)^T. \]
Then, \((v \cdot \ln(x) - K_1)^+\) is the maximum lower bound (MLB) for \((u \cdot x - K_3)^+\) in the class of the \(K_1\)-parameterized functions

\[
(v \cdot \ln(x) - K_1)^+, \quad K_1 \in \mathbb{R}, \quad x \in \mathbb{R}_{>0}. 
\]

**Remark 4.1.** We may view

\[
v(T) = \left( \mathbf{1} \cdot \beta(T) \ln \left( \frac{S(T)^{\beta(t)}}{\beta(t)} \right) \right)^+ \mathbf{1}_{\{1 \ln S(u) \leq h \ \forall \ u \in [t,T]\}},
\]

where \(h > 0\), as the payoff of a possible financial derivative. In the light of (4.1) and (4.2), its exact price is given by (4.4) with \(\tilde{K} = m \ln(K)\), if the option has not knocked out prior to time \(t\).

5. Conclusion

We obtain closed form expressions for the exact prices of a variety of multi-asset up-and-out call barrier options. The barriers can be viewed as exponential modifications of hyperplanes that test the price vector of underlyings, or else, hyperplanes that test the logarithmic vector of the prices of underlyings. Therefore, the barriers are placed on a collection of stock prices simultaneously. It is not the case of testing each asset against a corresponding barrier independently. The payoffs per se (i.e., without the barrier), are (i) on a stock that is verified by a distinct barrier, namely, one that is per se interesting from a financial perspective: it is constant if denominated on the numeraire taken to be the price of the risk-free asset, and (ii) on a geometric basket of stocks. In the case where the payoff per se is on an arbitrary combination of stocks (portfolio), we obtain closed form expressions for estimates of the exact prices. It seems to be the first time that ideas of convex analysis and risk-neutral pricing technique are combined, which we did in obtaining the estimates.

It is worth noticing that deriving prices in the multi-asset case poses significative difficulties that do not appear in the single-asset case, as mentioned in [8].

We addressed the uncorrelated scenario. Dramatic problems seem to appear when correlation among the assets are allowed, as mentioned in [13]. Actually, the work condemned the validity of most results dealing with the correlated scenario, even in the Black-Scholes framework.

**References**


