PRICING PATH-DEPENDENT DERIVATIVES IN FIXED INCOME MARKETS:  
A NEW APPROACH

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Abstract. The goal of this paper is to present a new form to pricing derivatives in fixed income markets. Our idea is to produce a time and value discretization of the stochastic process that represents the instantaneous interest rate (which solely subscribes all fixed income derivatives), in order to obtain an approximate solution for the derivative price. Our method admits all sort of diffusions consistent to the Feynman-Kac representation formula. The payoff format can also be quite general, for continuous time or discrete monitoring schemes. We illustrate the method pricing an Asian interest rate option under the (more realistic) discrete interest rate compounding scheme.

1. Introduction

The pricing and hedging of interest rate derivatives is a challenging task both from a theoretical and empirical point of view. Although in last decades we observe a tremendous development of techniques devoted to this end, there are many issues that remain open, or at least poorly resolved. In this paper, we address this question by proposing a new method to pricing fixed income derivatives that relies on the discretization of the stochastic process followed by the instantaneous interest rate.

From the seminal work of Harrison and Kreps ([9], 1979), we know that the price of an asset is equal to the risk-neutral expectation of its discounted future payoff conditional to present information. In other words, the discounted payoff is a martingale under the risk-neutral measure. While the physical distribution represents the likelihood of the future states of the economy, the risk-neutral measure is a distribution law equivalent to the physical measure under which the investors are insensitive to risk. This finding has represented a revolution in the finance theory since the effort to pricing assets is reduced to two operations: an average (expectation) and a discount.

Although the martingale property of asset prices is a well-established theory, its practical application is not trivial, especially in the fixed income derivatives market. In this case, the risk-neutral pricing technique boils down to modeling the short-term rate since both the payoff and discounted factor can be written as a function of it. However, investment decisions involve gains and losses, thus it is necessary to capture as accurately as possible the dynamics of the short-term interest rate. Therefore, one should consider processes that take into account stylized facts such as stochastic volatility and jumps. Moreover, products available in the market can be very complex with a diversity of exotic payoffs. These two features complicate enormously the empirical application of the risk-neutral pricing method.

A large amount of academic literature has been dedicated to propose models to circumvent the practical issues of the pricing and hedging fixed income derivatives. The traditional idea followed by most papers consists in specifying a dynamic to the instantaneous interest rate and solve the risk-neutral pricing equation under this framework. The affine class of term structure model studied by Duffie and Kan ([6], 1996) is an example of this approach. In this model the instantaneous interest rate is a linear combination of the state process which follows an affine diffusion (which means that...
the drift and the diffusion terms of the state process are affine functions of the interest rate). In the recent years the affine class has become the main tool to analyze the yield curve. The affine class encompasses a number of famous models such as the Vasicek model ([5], 1977), the Cox-Ingersoll-Ross ([7], 1985) model and the Hull and White model ([8], 1990). If the affine process is Gaussian and the payoff of the derivative is vanilla, then it is often possible to find a closed formula to its price. However, in more general setup this is not the case and one must rely in numerical methods to find the derivative price. There are at least three alternatives to numerically solve the risk-neutral equation. Firstly, by writing the risk-neutral equation in terms of Fourier transforms, and secondly, via Monte Carlo simulation. Finally, using the Feymann-Kac formula which is equivalent to solve a deterministic differential equation. Unfortunately, none of these methods is perfectly reliable. For a description of these methods and some caveats of them see, for instance, [13], [15] and [12].

In this paper we tackle this issue through a new method. The idea is to implement a time and value discretization of the stochastic process that represents the instantaneous interest rate. The method is quite general, since it deals with continuous time as well as discrete monitoring path dependent payoffs. It requires only two weak assumptions. First, the market is arbitrage-free which allow us to use the risk-neutral pricing technique. Second, the diffusion must be consistent to the Feynman-Kac representation formula, which admits a broad range of dynamics. In each step of the discretization process we have to find the prices of Arrow-Debreu securities, i.e., assets with a fixed payout of one unit in a specific state and paying zero in all other states. The price of this asset is the risk-neutral probability of the particular state of the world in which it pays and can be easily obtained by a novel way to exploit the Feynman-Kac formula proposed by us. To complete the procedure, we show that the price of a derivative with an arbitrary payoff is the limit of the sum of the values of a synthetic derivative whose payoff is the same as the original derivative in the discretization slot and zero otherwise.

We demonstrate the power of the method proposed by a simple example in which we have a closed formula to the value of the derivative. Thus we can compare our approach to what is considered the best solution by the literature. We price an Asian interest rate option when the instantaneous rate follows a Vasicek model ([5]), and compare the prices to those analytically solved according to [3]. We find interesting results. In a nutshell, we show that the method presents good performance in a full range of the initial condition for the interest rate. Moreover, the discretization does not exhibit problems related to low volatility regimes. Unlike the stock market, the volatility of the interest-rate is very small (around 1% per year or even lower). Although progress has been obtained in [4] (focusing the Vasicek and CIR model), it is well known that when the volatility is very low, numerical methods to pricing Asian options present instability (see, for instance, [10], [14], and [11]). Now, in the simulations provided herein, we find that the difference between the (analytical) price of Vieira and Neto (1999) and the price provided by our method is a decreasing function of the interest rate volatility. This may indicate that, unlike the usual numerical methods, ours performs well in the low range of volatility. Finally, it is worth mentioning that our method admits parallel computing, which is not the case of most standard methods, including that of [4].

2. Pricing Generic Interest-Rate Derivatives

Let us consider a complete fixed income market evolving in a probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})\) equipped with a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}\). We set \(\mathcal{F} = \mathcal{F}_T\) and write \((\Omega, \mathcal{F}, \mathbb{Q})\), rather than \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})\). We admit that under \(\mathbb{Q}\) all prices of derivatives denominated on the numeraire taken to be the share of the risk free asset are arbitrage free, and therefore Martingales. We write \(r = (r_u)_{u \in [0,T]}\) and \(B = (B_u)_{u \in [0,T]}\), to denote the adapted stochastic process that represents the instantaneous interest rate - which subscribes all derivatives, and the value of the non tradable risk free asset, respectively (often, \(B_t = \exp \left( \int_0^t r(s)ds \right) \)).

Consider a fixed income derivative whose payoff at expiration date \(T\) may depend on the whole trajectory of the interest rate \(r\), namely the random variable \(v(r) \equiv v(r(\omega)) \equiv v((r_u(\omega))_{u \in [0,T]}), \ \omega \in \Omega\) with a filtration \(F = (\mathcal{F}_t)_{t \in [0,T]}\) and write \(\mathcal{F} = \mathcal{F}_T\) and write \((\Omega, \mathcal{F}, \mathbb{Q})\), rather than \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})\). We admit that under \(\mathbb{Q}\) all prices of derivatives denominated on the numeraire taken to be the share of the risk free asset are arbitrage free, and therefore Martingales. We write \(r = (r_u)_{u \in [0,T]}\) and \(B = (B_u)_{u \in [0,T]}\), to denote the adapted stochastic process that represents the instantaneous interest rate - which subscribes all derivatives, and the value of the non tradable risk free asset, respectively (often, \(B_t = \exp \left( \int_0^t r(s)ds \right) \)).

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\[ \delta r_t = \beta(t, r_t)dt + \gamma(t, r_t)dW_t, \quad t \in [0, T], \]

where \( W \) is a Brownian Motion in \((\Omega, \mathcal{F}, \mathbb{P})\), \( \beta \) and \( \gamma \) are Borel-measurable functions, and the initial condition is \( r_0 \) at \( t = 0 \). Hence, the above equation is a diffusion consistent to the Feynman-Kac formula, and it covers most one-factor short-rate models, e.g.,

- models with parameters that do not depend on \( t \): the Ornstein-Uhlenbeck mean reverting process, i.e., the Vasicek model (1977), the Rendleman-Barter model (1980) and the Cox-Ingersoll-Ross mean reverting model (1985),
- models with parameters that do not depend on \( r_t \): the Ho-Lee model (1986) and the Kalotay-Williams-Fabozzi model (1993),
- models with parameters that depend on \( t \) and \( r_t \): the Hull-White model (1990).

We address the problem of pricing such interest rate derivatives at time zero, via a time and value discretization of the stochastic process \( r \). This is done through the well known risk neutral formula

\[ V(0, \mathcal{F}_0) = \mathbb{E}^Q \left[ \frac{B_0}{B_T(r)} v(r) \mid \mathcal{F}_0 \right]. \]

2.1. The Discretization Method. Let \( \delta \) and \( \varepsilon \) be positive constants, and define \( N = T/\delta \) and the \( \varepsilon \)-indexed discrete state space \( S^\varepsilon = \{j\varepsilon : j \in \mathbb{Z}\} \). Also consider the discrete-valued process \( \delta r_t = (\delta r_t)_{t \in [0, T]} \) where

\[ \delta r_t = \sum_{i=0}^{N-1} r_i \mathbb{1}_{[i\delta, (i+1)\delta)}(t), \]

the discrete-valued grid process \( \delta r^\varepsilon_t = (\delta r_t^\varepsilon)_{t \in [0, T]} \) where

\[ \delta r_t^\varepsilon = \sum_{i=0}^{N-1} \sum_{s \in S^\varepsilon} s \mathbb{1}_{[s, s+\varepsilon)}(r_i \mathbb{1}_{[i\delta, (i+1)\delta)}(t)), \]

as well as the corresponding discrete-time processes

\[ r^\delta_t \triangleq (r^\delta_i)_{i=0, \ldots, N-1} \quad \text{where} \quad r^\delta_i = \delta r_i = r_i \delta, \]

and

\[ x^\delta,\varepsilon \triangleq (x^\delta,\varepsilon_i)_{i=0, \ldots, N-1} \quad \text{where} \quad x^\delta,\varepsilon_i = \delta x^\varepsilon_i. \]

We have that \( \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \delta r_t^\varepsilon(\omega) = r_t(\omega) \) for all \((\omega, t) \in \Omega \times [0, T]\). Hence, an approximation for \( r_t(\omega) \) is the piecewise constant trajectories \( \delta r_t^\varepsilon(\omega) \) with values in \( S^\varepsilon \). Expression (2.4) can also be characterized by

\[ \delta r_t^\varepsilon = s \iff \omega \in \{r_{i\delta} \in [s, s+\varepsilon)\} \quad \text{and} \quad t \in [i\delta, (i+1)\delta), \]

for some \( i = 0, \ldots, N - 1 \).

For \( x_0 \in \mathbb{R} \), also define

\[ x^\varepsilon_0 \triangleq \sum_{s \in S^\varepsilon} s \mathbb{1}_{[s, s+\varepsilon)}(x_0). \]

Clearly, \( x^\varepsilon_0 \to x_0 \) as \( \varepsilon \downarrow 0 \). Expression (2.7) can also be characterized by

\[ x^\varepsilon_0 = s \iff x_0 \in [s, s+\varepsilon). \]

We assume that \( v \) is pointwise continuous in that \( \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} v(\delta r^\varepsilon) = v(r) \ \forall \omega \in \Omega \). Analogously, we assume that \( \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} B_T(\delta r^\varepsilon) = B_T(r) \) for \( B_T : \mathbb{R}^{[0, T]} \to \mathbb{R} \).
There is a clear 1 – 1 correspondence, say \( z \), between \( \delta r^\varepsilon \) and \( r^\delta, \varepsilon \). So, we have that \( \delta r^\varepsilon = Z(r^\delta, \varepsilon) \) and we define \( v_d : \mathbb{R}^N \rightarrow \mathbb{R} \) and \( B_T^T : \mathbb{R}^N \rightarrow \mathbb{R} \) as

\[
v_d = v \circ Z \quad \text{and} \quad B_T^T = B_T \circ Z,
\]

or else,

\[
v_d((r^\delta, \varepsilon)_i) = v((\delta r^\varepsilon)_u)_{0,T}
\]

\[
B_T^T((r^\delta, \varepsilon)_i) = B_T((\delta r^\varepsilon)_u)_{0,T}.
\]

Hence,

\[
limit \lim \frac{v_d((r^\delta, \varepsilon)_i)_{0,T}}{d \delta} = v((r_u)_{0,T})
\]

\[
\lim \lim \frac{B_T^T((r^\delta, \varepsilon)_i)_{0,T}}{d \delta} = B_T((r_u)_{0,T}).
\]

For instance, in the case of the well known format

\[
B_T(r) = B_0 e^{\int_0^T r_u \, du},
\]

we have that

\[
B_T(\delta r^\varepsilon) = B_0 e^{-\int_0^T \delta r^\varepsilon \, du} = B_0 e^{\sum_{i=0}^{N-1} \delta r^\varepsilon, \delta} = B_T^T(r^\delta, \varepsilon).
\]

2.2. Approximating Prices in Discrete time and Discrete State Space. Let \( r \) be given by the SDE (2.1) and initial condition \( r_0 = x_0 \in \mathbb{R} \) and, for positive constants \( \delta \) and \( \varepsilon \), let \( x_{0}^\delta \) be as in (2.7) and denote \( r_0^\delta = r_0^\delta, \varepsilon \). Now define

\[
V_1^{\delta, \varepsilon}(0, x_0^\delta) = \mathbb{E}^Q \left[ \frac{B_0}{B_T^T(r^\delta, \varepsilon)} v_d(r^\delta, \varepsilon) \, | \, \{ r_0^\delta = x_0^\delta \} \right],
\]

and

\[
V_2^{\delta, \varepsilon}(0, x_0^\delta) = \mathbb{E}^Q \left[ \frac{B_0}{B_T^T(r^\delta, \varepsilon)} v_d(r^\delta, \varepsilon) \, | \, \{ r_0^\delta = x_0^\delta \} \right] = \mathbb{E}^Q \left[ h_\varepsilon^\delta(r_i) \, | \, \mathcal{F}(i-1) \right],
\]

where \( h_\varepsilon^\delta : \mathbb{R} \rightarrow \mathbb{R} \) is the unique Borel-measurable function that performs

\[
q_i^{\delta, \varepsilon}(r_{(i-1)\delta}) = \mathbb{E}^Q \left[ h_\varepsilon^\delta(r_i) \, | \, \mathcal{F}(i-1) \right],
\]

We specify \( h_\varepsilon^\delta \) by

\[
h_\varepsilon^\delta(y) = 1_{[s,s+\varepsilon]}(y).
\]

**Theorem 2.1.** For every \( \delta > 0 \),

\[
\lim \frac{V_1^{\delta, \varepsilon}(0, x_0^\delta)}{\varepsilon^\delta} = \lim \frac{V_2^{\delta, \varepsilon}(0, x_0^\delta)}{\varepsilon^\delta},
\]

where limits are in \( \mathbb{R} \) and a truncation of the process \( r \) in \([-L, L] \) is admitted, \( L \) arbitrarily large.

**Proof.** We have that

\[
V_1^{\delta, \varepsilon}(0, x_0^\delta) = \sum_{s_i \in S^\varepsilon} \frac{B_0}{B_T^T(r^\delta, \varepsilon)} v_d(x_0^\delta, s_1, \ldots, s_{N-1})
\]

\[
\mathbb{Q} \{ \sum_{i=1}^{N-1} \{ r_i^\delta = s_i \} | \{ r_0^\delta = x_0^\delta \} \}
\]

\[
\lim \frac{V_1^{\delta, \varepsilon}(0, x_0^\delta)}{\varepsilon^\delta} = \lim \frac{V_2^{\delta, \varepsilon}(0, x_0^\delta)}{\varepsilon^\delta},
\]

where limits are in \( \mathbb{R} \) and a truncation of the process \( r \) in \([-L, L] \) is admitted, \( L \) arbitrarily large.
But
\[
Q(\cap_{t=1}^{N-1}\{r_{t}^{\delta,\varepsilon} = s_{i}\}|\{r_{0}^{\varepsilon} = x_{0}^{\varepsilon}\})
= Q(\{r_{1}^{\delta,\varepsilon} = s_{1}\}|\{r_{0}^{\varepsilon} = x_{0}^{\varepsilon}\})
\quad \times \prod_{t=2}^{N-1}Q(\{r_{t}^{\delta,\varepsilon} = s_{i}\}|\{r_{t-1}^{\delta,\varepsilon} = s_{i-1}\})
= Q(r_{18} \in [s_{1}, s_{1} + \varepsilon]|\{r_{0}^{\varepsilon} = x_{0}^{\varepsilon}\})
\quad \times \prod_{t=2}^{N-1}Q(r_{t\delta} \in [s_{1}, s_{1} + \varepsilon]|\{r_{t-1}^{\delta,\varepsilon} = s_{i-1}\})
= \mathbb{E}^{Q}[\mathbb{I}_{[s_{1}, s_{1} + \varepsilon]}(r_{1\delta})|\{r_{0}^{\varepsilon} = x_{0}^{\varepsilon}\}]
\quad \times \prod_{t=2}^{N-1}\mathbb{E}^{Q}[\mathbb{I}_{[s_{1}, s_{1} + \varepsilon]}(r_{t\delta})|\{r_{t-1}^{\delta,\varepsilon} = s_{i-1}\}],
\]
where the first equality relies on the fact that the discrete process \(r_{t}^{\delta,\varepsilon}\) remains Markovian, since this is the case of \(r\), and the second equality is due to having \(\{r_{t}^{\delta,\varepsilon} = s_{i}\} = \{r_{t\delta} \in [s_{1}, s_{1} + \varepsilon]\}\). So,
\[
V_{1}^{\delta,\varepsilon}(0, x_{0}) = \sum_{s_{1} \in S} \prod_{t=2}^{N} \mathbb{E}^{Q}[\mathbb{I}_{[s_{1}, s_{1} + \varepsilon]}(r_{t\delta})|\{r_{0}^{\varepsilon} = x_{0}^{\varepsilon}\}]\cdot \prod_{t=2}^{N-1}\mathbb{E}^{Q}[\mathbb{I}_{[s_{1}, s_{1} + \varepsilon]}(r_{t\delta})|\{r_{t-1}^{\delta,\varepsilon} = s_{i-1}\}],
\]
and we must prove that
\[
\lim_{\varepsilon \downarrow 0} \sum_{s_{1} \in S} \prod_{t=2}^{N} \mathbb{E}^{Q}[\mathbb{I}_{[s_{1}, s_{1} + \varepsilon]}(r_{t\delta})|\{r_{0}^{\varepsilon} = x_{0}^{\varepsilon}\}]\cdot \prod_{t=2}^{N-1}\mathbb{E}^{Q}[\mathbb{I}_{[s_{1}, s_{1} + \varepsilon]}(r_{t\delta})|\{r_{t-1}^{\delta,\varepsilon} = s_{i-1}\}]
= q_{s_{1}}^{\delta,\varepsilon}(x_{0}) \prod_{t=2}^{N-1} q_{s_{1}}^{\delta,\varepsilon}(s_{i-1}) = 0.
\]
Let us now admit that the interest rate \(r\) is confined in \([-L, L], L > 0\) arbitrarily fixed. We may then rewrite (2.21) as
\[
\lim_{\varepsilon \downarrow 0} \sum_{j_{1} \in \mathbb{Z}_{L} \cup \{0\}} \prod_{j_{n} \in \mathbb{Z}_{L} \cup \{0\}} \mathbb{E}^{Q}[\mathbb{I}_{[j_{1}, j_{1} + \varepsilon]}(r_{t\delta})|\{r_{0}^{\varepsilon} = x_{0}^{\varepsilon}\}]\cdot \prod_{t=2}^{N-1}\mathbb{E}^{Q}[\mathbb{I}_{[j_{1}, j_{1} + \varepsilon]}(r_{t\delta})|\{r_{t-1}^{\delta,\varepsilon} = j_{i-1}\}]
= q_{j_{1}}^{\delta,\varepsilon}(x_{0}) \prod_{t=2}^{N-1} q_{j_{1}}^{\delta,\varepsilon}(j_{i-1}) = 0,
\]
where \(\mathbb{Z}_{L} \cup \{0\} = \{0, \pm 1, \ldots \pm N_{L,\varepsilon}\}\), with \(N_{L,\varepsilon} = L/\varepsilon\), is a finite set. The proof now relies on Lemmas A.1 and A.2, and the fact that \(r_{t}^{\delta,\varepsilon}\) (respectively \(h^{\varepsilon}\) given by (2.16)) is an increasing (respectively decreasing) sequence of r.v.s. as \(\varepsilon\) decreases, bounded from above by \(r_{t}^{\delta,\varepsilon}\) (respectively the zero function).

**Corollary 2.1.** Define \(z : \mathbb{R} \rightarrow \mathbb{R}\) as
\[
z(x_{0}) \triangleq \lim_{\delta \downarrow 0, j_{1} \downarrow 0} V_{1}^{\delta,\varepsilon}(0, x_{0}^{\varepsilon}).
\]
Then \(z\) is the unique Borel measurable function such that
\[
V(0, \mathcal{F}_{0}) \triangleq \mathbb{E}^{Q}\left[\frac{B_{0}}{BT(r)}v(r)|\mathcal{F}_{0}\right] = z(r_{0}),
\]
the price at time zero of the interest rate derivative paying \( v(r) \) at expiration time \( T \). Since \( z \) is \( r_0 \)-measurable, the above price reads

\[
V(0, x_0) = z(x_0) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} V_{\epsilon, \delta}^{i} (0, x_0)
\]

(2.25)

\[
= \lim_{\epsilon \to 0} \lim_{\delta \to 0} \sum_{s_1 \in \mathbb{S}^r, \ldots, s_{N-1} \in \mathbb{S}^r} \frac{B_s}{B_T(x_0, s_1, \ldots, s_{N-1})} v_d(x_0^s, s_1, \ldots, s_{N-1}) q_{i, \epsilon, \delta}^s (x_0^s) \prod_{l=2}^{N-1} q_{s_l, \epsilon, \delta}^l (s_{l-1}).
\]

for observed \( r_0 = x_0 \).

The calculations are given in the theorem below.

**Theorem 2.2 (Feynman-Kac Formula for calculating \( q_{i, \epsilon, \delta} \)).** Let \( q_{i, \epsilon, \delta}^t (t, x) \) be the unique solution of the PDE

\[
g_{i, \epsilon, \delta}^t (t, x) + \beta(t, x) q_{i, \epsilon, \delta}^t (t, x) + \frac{1}{2} \gamma^2(t, x) q_{i, \epsilon, \delta}^{t, xx} (t, x) = 0
\]

(2.26)

with terminal condition at time \( i \delta \) given by

\[
g_{i, \epsilon, \delta}^t (i \delta, x) = h_i^t (x) = \mathbb{I}_{[t,t+\epsilon]}(x), \quad i = 1, \ldots, N - 1 \text{ and } l \in \mathcal{S}^r.
\]

Then

\[
g_{i, \epsilon, \delta}^t (i \delta, s_{l+1}) = E^Q \left[ h_i^t (r_{l+1}) \mathbb{I}_{[t,t+\epsilon]}(r_{l+1}) \right] = q_{i, \epsilon}^t (r_{l+1}),
\]

so clearly

\[
q_{i, \epsilon}^t (x_0^s, s_1, \ldots, s_{N-1}) = g_{i, \epsilon, \delta}^t (0, x_0^s) \quad \text{and} \quad q_{i, \epsilon}^t (s_{l-1}) = g_{i, \epsilon, \delta}^t (i \delta, s_{l-1}).
\]

**Proof.** For each \( i = 1, \ldots, N - 1 \) and \( s \in \mathcal{S}^r \), consider the SDE (2.1) evolving in \([0, i \delta] \) with same initial condition \( r_0 \) at \( t = 0 \). \( \square \)

### 3. The Discrete Monitoring Case

In this case the payoff \( v \) in (2.2) is discretely monitored while \( B_T \) still depends continuously on \( r \). As in (2.5), for given \( \Delta > 0 \) and the integer \( M := T/\Delta \), define \( r_\Delta := (r_\Delta)_i = 0, \ldots, M \), with \( r_\Delta = r_{i \Delta} \). The payoff is now given by

\[
v_d (r_\Delta),
\]

where \( v_d : \mathbb{R}^{M+1} \to \mathbb{R} \) is continuous.

The price now reads

\[
V(0, \mathcal{F}_0) = \mathbb{E}^Q \left[ \frac{B_0}{B_T(r_\Delta)} v(r_\Delta) \mid \mathcal{F}_0 \right].
\]

The developments here follows the same lines as those of the previous section. The change affect the abscessa \( \delta \) while the ordinate \( \epsilon \) remains as it was.

3.1. **Discrete Setup.** We replace \( \delta \) by \( \Delta/N \) for some \( N \in \mathbb{N} \), so now we have \( \Delta/N \to 0 \) as \( N \to \infty \) (instead of \( \delta \to 0 \)) and as in the last section consider now the next processes \( N r = (N r_i)_{i \in [0,T]} \) where

\[
N r_i = \sum_{i=0}^{M-1} r_{i \Delta} \mathbb{I}_{[i \Delta, (i+1) \Delta]}(t),
\]

(3.2)

and \( N r^e = (N r^e_i)_{i \in [0,T]} \) where

\[
N r^e_i = \sum_{i=0}^{M-1} \sum_{s \in \mathcal{S}^r} s \mathbb{I}_{[i \Delta, (i+1) \Delta]}(t) \mathbb{I}_{[s, s+\epsilon]}(r_{i \Delta}),
\]

(3.3)

so, we define the discrete processes

\[
r^N_i \triangleq (r^N_i)_{i=0, \ldots, MN},
\]

(3.4)
where \( r_i^N = N r_i \), for \( i = 0, ..., MN - 1 \) and \( r_{MN}^N = r_{M+1} = r_T \) and 
\[
(3.5) \quad r_i^{N,ε} = (r_i^N)_{i=0, ..., MN},
\]
where \( r_i^{N,ε} = N r_i \), for \( i = 0, ..., MN - 1 \) and \( r_{MN}^{N,ε} = r_T \), with
\[
r_T^ε = \sum_{s ∈ S^ε} s [s, s+ε](r_T).
\]
Since \( r_i^N = r_i^ε \), \( r_i^{N,ε} = r_i^N \), \( ∀ i = 0, ..., M \), we have that \( r^N \) absorbs \( r^ε \) and therefore we ensure the inclusion of \( r^ε \) in the indexes of the sum of (2.14). Also define
\[
(3.6) \quad r_i^{Δ,ε} = (r_i^{Δ,ε})_{i=0, ..., MN},
\]
where \( r_i^{Δ,ε} = r_i^{N,ε} \). Also denote \( r_0^ε \).

3.2. Approximating payoffs. We have that \( \lim_{T→0} v_d(s, ε) = v_d(r^ε) \). Analogously, we require having \( \lim_{N→∞} \lim_{T→0} B_T(Nr^ε) = B_T(r) \) for \( B_T : \mathbb{R}^{[0, T]} → \mathbb{R} \).

Again, there is a 1-1 correspondence between \( N r^ε \) and \( r_i^{N,ε} \), so there is \( B_T^ε : \mathbb{R} → \mathbb{R} \) such that
\[
(3.7) \quad B_T^ε((r_i^{N,ε})_{i=1, ..., MN-1}) = B_T((N r^ε)_{i∈[0, T]}).
\]
Hence,
\[
(3.8) \quad \lim_{N→∞, ε→0} B_T^ε((r_i^{N,ε})_{i=1, ..., MN-1}) = B_T((r^ε)_{i∈[0, T]}).
\]

Analogously, (3.7) and (3.8) hold suppressing \( ε \).

3.3. Approximating prices. Assume, as before, that \( r \) is a Markov process on \( (Ω, F, Q) \) and for arbitrary \( x_0 ∈ \mathbb{R}, N ∈ \mathbb{N} \) and \( ε > 0 \), let \( x_0^ε \) be as in (2.7). Define
\[
(3.9) \quad V_1^{N,ε}(0, x_0^ε) = E^Q \left[ \frac{B_0}{B_T^ε((r_i^{N,ε})_{i=0, ..., MN-1})} v_d(r^Δ, ε) \right| \{ r^ε = x_0^ε \},
\]
and
\[
V_2^{N,ε}(0, x_0^ε) = \sum_{s_1 ∈ S^ε, ..., s_{MN} ∈ S^ε} \frac{B_0}{B_T^ε(x_0^ε, s_1, ..., s_{MN-1})} v_d(x_0^ε, s_1, ..., s_{MN}) q_{s_1^ε}^ε(x_0^ε) \prod_{i=2}^{MN} q_{s_i^ε}^ε(s_{i-1}),
\]
where \( q_{s_i^ε}^ε : \mathbb{R} → \mathbb{R} \) is the unique Borel-measurable function such that
\[
(3.11) \quad q_{s_i^ε}^ε(r_{(i-1)^ε}) = E^Q[h_i^ε(r_{(i-1)^ε}) | F_{(i-1)^ε}],
\]
where \( h_i^ε(r_{(i-1)^ε}) = E^Q[h_i^ε(r_{(i-1)^ε}) | F_{(i-1)^ε}], i = 1, ..., MN \) and \( s ∈ S^ε \).

We specify \( h_i^ε \) by
\[
(3.12) \quad h_i^ε(r_{(i-1)^ε}) = I[\{ r_{(i-1)^ε} ∈ [s, s+ε) \}].
\]
The theorem and corollary that follow are the analogous of the Theorem 2.1 and Corollary 2.1.
Theorem 3.1. For every $N \in \mathbb{N}$,
\begin{equation}
\lim_{\varepsilon \downarrow 0} V_{N,\varepsilon}^1(0, x_0^\varepsilon) = \lim_{\varepsilon \downarrow 0} V_{N,\varepsilon}^2(0, x_0^\varepsilon),
\end{equation}
where limits are in $\mathbb{R}$ and a truncation of the process $r$ in $[-L, L]$ is admitted, $L$ arbitrarily large.

Corollary 3.1. Define $z : \mathbb{R} \to \mathbb{R}$ as
\begin{equation}
z(x_0) \triangleq \lim_{N \to \infty} \lim_{\varepsilon \downarrow 0} V_{N,\varepsilon}^2(0, x_0^\varepsilon).
\end{equation}
Then $z$ is the unique Borel measurable function such that
\begin{equation}
V(0, F_0) \triangleq \mathbb{E}_Q \left[ \frac{B_0}{B_T(r)} v(r^\Delta) | F_0 \right] = z(r_0),
\end{equation}
the price at time zero of the interest rate derivative paying $v^\Delta(r)$ at expiration time $T$. Since $z$ is $r_0$-measurable, the above price reads
\begin{equation}
V(0, x_0) = z(x_0) = \lim_{N \to \infty} \lim_{\varepsilon \downarrow 0} V_{N,\varepsilon}^2(0, x_0^\varepsilon)
= \lim_{N \to \infty} \lim_{\varepsilon \downarrow 0} \sum_{s_1 \in S^\varepsilon, ..., s_M \in S^\varepsilon} \frac{B_0}{B_T(r)} B_{s_1N} \cdot v_d(x_0^\varepsilon, s_1N, s_2N, ..., s_MN) q_1(\varepsilon) \prod_{i=2}^{MN} q_i(\varepsilon)(s_i - 1),
\end{equation}
for observed $r_0 = x_0$.

4. The Case Of An Asian Option

We address the particular case of a financial option of Asian type named IDI, shorthand for Interbank Deposit Index Rate. The index accumulates discretely according to
\begin{equation}
y(n) = B(0) \prod_{i=1}^{n} (1 + r_i^\Delta)^{252},
\end{equation}
where $n = 1, 2, ..., M$, corresponds to the $n$-th day, and $r_i^\Delta$ denotes, at the end of the day $i$, the average of the interbank rate of a one day period ($\Delta = 1$ day) calculated daily and expressed as the effective rate per annum. Correspondingly, the discretely monitoring payoff for the call option with maturity $T = \Delta M$ and strike $K$ is given by
\begin{equation}
v_d(r^\Delta) = \max \left\{ B_0 \prod_{i=1}^{M} (1 + r_i^\Delta)^{252} - K, 0 \right\}.
\end{equation}

We take as a particular case of (2.1), the Vasicek Model
\begin{equation}
dr_t = \alpha(\theta - r_t) dt + \sigma dW_t, \quad t \in [0, T].
\end{equation}

In turn, a discretization that is commonly exploited in finance for the discounting factor is
\begin{equation}
B_T^d(r^{N,\varepsilon}) = B_0 \prod_{i=1}^{MN} (1 + \frac{\Delta}{N} r_i^{N,\varepsilon}).
\end{equation}
Hence, Corollary 3.1 holds with
\begin{equation}
\psi_d(x_0, s_{1N}, s_{2N}, \ldots s_{MN}) = \max \left\{ B_0 \prod_{i=1}^{M} (1 + s_{iN})^{\frac{x_i}{\Delta N}} - K, 0 \right\}
\end{equation}

and
\begin{equation}
B_d^T(x_0, s_1, \ldots s_{MN}) = B_0 \prod_{i=1}^{MN} \left( 1 + \frac{\Delta}{N} s_i \right).
\end{equation}

In this simple exercise the functions \( q \) found in (3.11) can be calculated either via the Feynman-Kac Theorem (2.2) or directly as follows
\begin{equation}
q^{i,j N, \varepsilon}(r_{(i-1)N}, \varepsilon) = \mathbb{E}_Q \left[ h^{(i,j N, \varepsilon)}(r_{(i-1)N}) | F_{(i-1)N} \right] = \int h^{(i,j N, \varepsilon)}(x) dx
\end{equation}

where according to [2]
\begin{align*}
\mu &= \mathbb{E}_Q \left[ r_{(i-1)N} | F_{(i-1)N} \right] = r_{(i-1)N} e^{-\alpha \Delta N} + \theta (1 - e^{-\alpha \Delta N}), \\
\tilde{\sigma}^2 &= \text{Var}_Q \left[ r_{(i-1)N} | F_{(i-1)N} \right] = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha \Delta N}).
\end{align*}

5. Numerical Results

We numerically solve the price of an IDI option under the Vasicek model. We set the long term mean level \( \theta = 0.1 \), the instantaneous volatility \( \sigma = 0.05 \) and the speed of reversion \( \alpha = 0.2 \). The IDI starts at \( t = 0 \) with 100000 points and the strike is given by \( K = 100100 \). For the sake of comparison, we also provide prices according to [3], obtained via closed form expressions. In [3], an idealization for mathematical tractability, much less realistic than (4.1) is assumed, and consists on the fact that the IDI accumulates continuously, i.e,
\begin{equation}
y(t) = B(0) \exp(\int_0^T r_u du).
\end{equation}

Concerning our method, the discretization slot in time is one day (\( \Delta = 1 \)). Figs. 1 to 4 show the prices calculated according to [3] (dashed line) and according to our method (continuous line). The prices are given as a function of the observed value of the interest rate \( r(0) \) at time zero.

The curves representing the prices according to [3], in Figs. 1 and 2, are the same. The ones representing our method correspond to a discretization step of the interest rate given by \( \varepsilon = 0.01 \), i.e., 1% per year (Fig. 1) and a more refined one given by \( \varepsilon = 0.0075 \), i.e., 0.75% per year (Fig. 2). They reveal the good performance of our scheme because of the following.

- Our prices track those of [3] - assumed as benchmark, in the full range of the interest rate initial condition \( r(0) \).
Figure 1. IDI Call Option Prices with grid spacing $\epsilon = 0.01$ for the discretization method

Figure 2. IDI Call Option Prices with grid spacing $\epsilon = 0.0075$ for the discretization method

- The updating scheme in our case is discrete and that of [3] is continuous. Hence, our prices have to be cheaper than those of [3], which actually occurs $^1$.

$^1$When using our method, the discretely compounding updating scheme of the IDI was adopted, since the method allow us for doing so. Such updating scheme is more realistic than the continuous one - which has to be adopted in [3] for solving prices.
Figs. 3 and 4, in turn, differ one from the other by the volatility parameter, given by $\sigma = 0.01$ i.e., 1%, and $\sigma = 0.001$ i.e., 0.1%, respectively. This corresponds to low volatilities, typical in the fixed income markets, where most of the numerical methods are unstable for price calculations. Other parameters are as those of Fig. 1. Again, our prices track those of [3] in the full range of the interest
rate initial condition \( r(0) \), and no erratic behaviour can be observed on both curves corresponding to our numerical scheme.

With a view to speeding the computational processing, a lifetime of five days was assumed for the derivative. We recall that even with short periods, existing numerical methods to pricing Asian options still present instability (see, e.g., [10], [14], and [11], so the setup of our exercise does not detract the results of the simulations.

6. Conclusions

We provide a new method for pricing derivatives of fixed income markets. It is based on a discretization procedure on the time line and on the path of the interest rate process, in conjunction with a peculiar use of the Feynman-Kac representation formula. Due to its nature, the method adapts indifferently to a broad range of derivatives and short-rate dynamics. Namely, payoffs of various kinds that may depend on the path of the short-rate alone - including discrete monitoring. It also adapts to any diffusion consistent to the Feynman-Kac representation formula.

The numerical results focus on a simple case where a closed form solution for the price of the derivative exists already ([3]). They unveil prices close to but cheaper than those of [3] in spite of changes of the interest rate initial data, as it should be. From the pricing results it appears that the method may not present instability in the range of low volatilities - a disadvantage often encountered in existing numerical methods.

6.1. Ongoing works. With a view to improving the computational tractability, we are developing specific jumping counting methods in order to boost the time performance of the algorithm.

Appendix A. Support lemmas

Lemma A.1. Let \( X \) and \( Y \) be random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\) taking values on \( \mathbb{R} \), and let \( g \) be the unique Borel measurable function that performs \( g(X) \triangleq \mathbb{E}^\mathbb{Q}[Y | X] \). Then, assuming that \( \mathbb{Q}\{X \in [s, s + \epsilon)\} > 0 \), \( s \in \mathbb{R} \), and \( g \) is continuous, it follows that

\[
\mathbb{E}^\mathbb{Q}[Y | \{X \in [s, s + \epsilon)\}] \to g(s) \quad \text{as} \quad \epsilon \downarrow 0.
\]

In turn, define

\[
X^\epsilon \triangleq \sum_{s \in S^\epsilon} s \mathbb{1}_{[s, s + \epsilon)}(X), \quad S^\epsilon = \{j\epsilon : j \in \mathbb{Z}\},
\]

or equivalently, let \( X^\epsilon \) be such that

\[
X^\epsilon = s \Leftrightarrow X \in [s, s + \epsilon).
\]

Then

\[
\mathbb{E}^\mathbb{Q}[Y | \{X^\epsilon = s\}] \to g(s) \quad \text{as} \quad \epsilon \downarrow 0.
\]

Proof. Define

\[
I_{g,s}(\epsilon) \triangleq \int_{\{X \in [s, s + \epsilon)\}} g(X) d\mathbb{Q} \quad \text{and} \quad I_{\mathbb{Q},s}(\epsilon) \triangleq \mathbb{Q}\{X \in [s, s + \epsilon)\}
\]
First note that for any continuous (Borel measurable) function $g$, we have that $\forall \epsilon > 0 \exists \gamma > 0$ such that

\[
\int_{\{X \in [s, s+\epsilon]\}} g(s) dQ - \epsilon \leq I_g(s) \leq \int_{\{X \in [s, s+\epsilon]\}} g(s) dQ + \epsilon, \quad \text{whenever } \epsilon \leq \gamma.
\]

(A.6)

Now,

\[
\lim_{\epsilon \to 0} \frac{I_g(s)}{\epsilon} = \lim_{\epsilon \to 0} \frac{dI_g(s)}{d\epsilon} = \lim_{\epsilon \to 0} g(s + \epsilon) = g(s).
\]

(A.7)

In particular, this holds for $g(X) \triangleq \mathbb{E}^Q[Y | X]$, still assuming that $g$ is continuous. Moreover, since $\{X \in [s, s + \epsilon]\}$ clearly belongs to the $\sigma$-algebra generated by $X$, it follows that

\[
\mathbb{E}^Q [Y | \{X \in [s, s + \epsilon]\}] = \frac{\int_{\{X \in [s, s + \epsilon]\}} g(X) dQ}{\mathbb{Q}[\{X \in [s, s + \epsilon]\}]},
\]

(A.8)

Substitution on (A.7) leads to (A.1).

□

For the following lemma see, e.g., [1].

**Lemma A.2.** Let $Y$ and $Y^n$, $n \in \mathbb{N}$, be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ taking values on $\mathbb{R}$, and suppose that let $Y^n \downarrow Y$ a.s. as $n \to \infty$. Then $E[Y^n | \mathcal{G}] \downarrow E[Y | \mathcal{G}]$, $\mathcal{G} \subset \mathcal{F}$ a.s..

**References**


