FOREIGN EXCHANGE EXPECTATION ERRORS AND FILTRATION ENLARGEMENTS

Abstract. Extrapolations of future markets forward rates are a better predictor of the 30-days ahead BRL-USD exchange rate than forecasts from the Focus survey of Brazilian market participants. This is puzzling because market participants observe forward rates as they submit predictions. Dandapani and Protter (2016) describe a mechanism through which new information enlarges the information set (a filtration), changing the underlying risk neutral measure and inducing a drift into the martingale process, turning the process into a strict local martingale. We argue this mechanism can explain our rational conundrum. To empirically test the plausibility of such connection we first employ a nonparametric identification test based on Jarrow et al. (2011a), then estimate the stochastic volatility model of Andersen and Piterbarg (2007). Results suggest that Focus survey forecasts indeed display characteristics of a strict local martingale, while spot exchange rates and forward rates are consistent with a martingale process.

Keywords: Strict Local Martingales; Filtration Enlargement; Foreign Exchange Markets; Expectation Errors.

1. Introduction

Imagine a rational agent that observes some information set (a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$) and is interested in forecasting values of some random variable. As new information becomes available (in the form of a random variable $L$), such that we have a new, enlarged filtration $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ where $\mathcal{G}_t = \bigcap_{\epsilon > 0} (\mathcal{F}_{t+\epsilon} \vee \sigma(L))$ one would expect correction of systematic errors and/or some reduction of prediction variance. Through the information accumulation process useless information is discarded, and only information that yields true prediction gains is incorporated — one thus expects forecasts based on the enlarged information set $\mathcal{G}$ to be at least as good as forecasts based on the smaller information set $\mathcal{F}$.

We compare survey-based forecasts from the Market Expectations System of the Central Bank of Brazil (henceforth Focus) with BMF&BOVESPA 30-days forward rates; and find that forward rates are unbiased predictor of future spot rates, but Focus survey forecasts are biased predictors through a substantial portion of the observed sample. This is counter intuitive because forward rates are obviously within the information set of keen market participants. No arbitrage theory asserts that exchange rates should be well characterized by a martingale process under the equivalent risk neutral measure $P$ — i.e., the best possible predictor of future exchange rates is the present value of forward rates. Results in Table 3 suggest it is indeed the case in Brazilian markets: it is not possible to reject the martingale hypothesis for either the spot exchange rate, or future market forward rates. Yet Focus survey based forecasts seem inconsistent with a martingale process. One could say that markets participants are employing a prediction scheme systematically incompatible with the observed stochastic process of interest. How to explain such phenomenon?
We will argue this puzzle phenomenon can be interpreted through strict local martingale theory — specifically through the mechanism detailed by Dandapani and Protter (2016). Dandapani and Protter (2016) explain how enlarging filtration $F$ to $G$ changes the underlying risk neutral measure (say from $P$ to $Q$), which can induce a stochastic drift in the volatility of $S$, turning $S$ from an $F$ martingale to a $G$ strict local martingale.

The idea of smaller and larger information sets is natural to problems in finance. Corcuera and Vadivia (2012) study the cases in which an $F$ semimartingale remains a semimartingale under filtration $G$, and show how filtration enlargements can be applied to model default risk and insider trading problems. Strict local martingales are at the core of a blooming real-time bubble detection literature. For example, Jarrow et al. (2011b) applies a nonparametric methodology (which we borrow in Section 4) to show that Linked-In stock exhibited price bubble behavior immediately after the firms IPO.

Using two different identification techniques we explore the strict local martingale hypothesis. In Section 4 we employ a nonparametric kernel approach based on Jarrow et al. (2011a) to study how the volatility of our series of interest behave as price varies, and access the convergence of condition (2.5). We then turn to an alternative, parametric strict local martingale test in Section 5.

To our knowledge this is the first work to relate systematic prediction errors and the filtration enlargement mechanism. While the nonparametric identification procedure in Section 4 draws heavily from Jarrow et al. (2011a), formal univariate estimation of the stochastic volatility model in Section 5 is itself novel. Standard practice in financial industry is to calibrate parameter values from option price information (see for example Mikhailov and Nögel (2004)), and classical estimates (like by Aï et al. (2007)) employ volatility proxies to circumvent the latent variable problem — we tackle the latent variable issue by employing bayesian simulation techniques to estimate model (5.1), so that it is possible to make inference regarding model parameters from a single time series.

The remainder of this paper is structured as follows. Section 2 presents some key definitions and a discussion about the filtration enlargement mechanism. Empirical evidence of our main points are found in Section 3. Nonparametric strict local martingale identification is presented in Section 4. Section 5 details our parametric estimation. We discuss results in Section 6, and Section 7 concludes.

2. Filtration Enlargement and Strict Local Martingales

2.1. Strict Local Martingales. Consider an economy in which are traded a risky asset and a money market account. We begin with a complete probability space $(\Omega, \mathcal{F}, P)$, and a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. Let $r = (r_t)_{t \geq 0}$ denote the instantaneous default-free spot interest rate, then the money market account value at time $t$ is

\begin{equation}
B_t = \exp \left( \int_0^t r_u du \right).
\end{equation}

No arbitrage in the sense of “no free lunch with vanishing risk” (NFLVR) implies that there exists a probability measure, $Q$, equivalent\(^1\) to $P$ such that

\begin{equation}
\frac{S_t}{B_t}
\end{equation}

\(^1\)Such that $Q$ and $P$ “agree” on zero-probability events.
is a local martingale. This is the first theorem of asset pricing.  

A local martingale is a stochastic process, $M = (M_t)_{t \geq 0}$, for which there exists an increasing sequence of stopping times $\tau_n \to \infty$ with probability one such that the stopped process $M_{\min\{\tau, \tau_n\}}$ is a martingale. The key insight is that if $M$ is a local martingale, then it is either a true martingale or a strict local martingale.

The risky asset (normalized) price, $S = (S_t)_{t \geq 0}$, is assumed nonnegative and driven by a standard stochastic differential equation

\begin{equation}
    dS_t = \sigma(S_t)dW_t + \mu(S_t)dt.
\end{equation}

No arbitrage in the sense of “No Free Lunch with Vanishing Risk” implies that there exists a risk neutral measure under which the SDE in (2.3) simplifies to

\begin{equation}
    S_t = S_0 + \int_0^t \sigma(S_s)dW_s.
\end{equation}

A very important property (to be further exploited) is that $S$ is a strict local martingale if and only if

\begin{equation}
    \int_{\alpha}^{\infty} \frac{x}{\sigma(x)^2}dx < \infty
\end{equation}

for all $\alpha > 0$. Jarrow (2012) notes that condition (2.5) states that an asset is a strict local if and only if its volatility, as function of its price, increases faster than linearly.

Jarrow et al. (2011a) discuss the intuition behind the distinction between a nonnegative martingale and a strict local martingale, $S \geq 0$, follows from the fact that $S$ is always a supermartingale and is a true martingale if and only if it has constant expectation. Then if $S$ is a strict local martingale its expectation decreases over time. The typical behavior of a SLM process is to shoot up to high values, to then drop down and stay low.

2.2. Filtration Enlargement. Dandapani and Protter (2016) detail a mechanism through which new information introduced into the observed information set (the initial filtration $\mathcal{F}$) induce a change in the risk neutral measure, from $P$ under $\mathcal{F}$ to $Q$ under the enlarged filtration $\mathcal{G}$, such that a $\mathcal{F}$ martingale process $S$ can become a $\mathcal{G}$ strict local martingale for a random time interval.

To understand how the filtration enlargement mechanism connects strict local martingale bubbles and systematic foreign exchange forecast errors, imagine some agent One wants to predict future values of a risky asset $S = (S_t)_{t \geq 0}$ (for example the 30-days spot BRL-USD exchange rate). Agent One observes an information set (a filtration) $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, such that we have a complete filtered probability space $(\Omega, \mathcal{F}, P)$. Suppose $S$ is a $\mathcal{F}$ true martingale under an equivalent risk neutral measure $Q^1$.

Now consider some other agent Two that also aims to forecast $S$. At time $t = 0$ only agent Two observes a random variable $L$, that enlarges his or her information set, from $\mathcal{F}$ to $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ — i.e., $L \in \mathcal{G}$ but $L \notin \mathcal{F}$. This changes the underlying equivalent risk neutral measure from $P$ to $Q$. Dandapani and Protter (2016) show that it is possible for, due to such change, process $S$ to be a $\mathcal{G}$ strict local martingale.

We reproduce the main points of their discussion below.

\footnote{Note that from now on when we refer to $S_t$ we actually mean $S_t/B_t$ — this is loose notation, but implies no confusion and contributes to a more straightforward exposition.}
Consider the Heston type model studied by Lions and Musiela (2007)

\[ dS_t = S_t v_t dB_t \]  
\[ dv_t = \sigma(v_t) dW_t + b(v_t) dt. \]  

Denote the new, enlarged filtration \( G \) as

\[ G_t = \bigcap_{\epsilon > 0} (F_{t+\epsilon} \vee \sigma(L)). \]

Let \( \eta \) be the distribution of \( L \), and \( Q_t(w, dw) \) be the regular conditional distribution of \( L \) given \( F_t \). There exists an \( F \) martingale \( q \), such that

\[ \mathbb{E}_t \left[ \int_0^t q^x S_s dS_s \right] = (k^x q^x) \cdot \mathbb{E}_t [S, S] \]  

Jacod (1985) shows that the following process \( \hat{S}_t \) is a \( G \) local martingale

\[ \hat{S}_t = S_t - \int_0^t k^L_s d[S, S]_s. \]

Enlarging filtration \( F \) to \( G \) changes the semimartingale decomposition of both \( S \) and \( v \). Grisanov’s theorem allows us to restore the local martingale properties of \( S \) by changing the equivalent probability measure to \( Q \).

Under \( (Q, G) \), \( S \) can be written as

\[ S_t = \int_0^t (S_s v_s) dB_s - \int_0^t k^L_s (S_s v_s)^2 ds - \int_0^t ((S_s v_s) H_s + \rho J_s) ds + \int_0^t k^L_s (S_s v_s)^2 ds + \int_0^t ((S_s v_s) H_s + \rho J_s) ds \]

to ensure that \( S \) is a \( Q \) local martingale we choose

\[ k^L_t (S_t v_t)^2 = -(S_t v_t) H_t - \rho J_t. \]

The drift component \( b(v_t) \) in volatility equation (2.7) also changes

\[ \hat{b}(v_t) = b(v_t) + k^L_t \sigma^2(v_t) + (\rho H_t + J_t) \sigma(v_t). \]

This is the change in the drift of the volatility equation that enables the change from a martingale to a strict local martingale.

We can now state Theorem 3 of Dandapani and Protter (2016) p.p. 10, which relates \( S \) local martingale under \( (F, P) \) to \( S \) strict local martingale under \( (G, Q) \).

**Theorem.** Dandapani and Protter (2016). Assume that \( k, H \) and \( J \) have right continuous paths almost surely, and \( Q(w : k^L_0 > 0) > 0 \). Suppose functions \( \sigma(\cdot) \) and \( b(\cdot) \) are \( C^1 \), and the following conditions hold:

(i)  \[ \limsup_{x \to +\infty} \frac{\rho x \sigma(x) + b(x)}{x} < \infty; \]

(ii)  \[ \liminf_{x \to +\infty} \left( \rho x \sigma(x) + b(x) \right) + \min(\varepsilon^{(1)}, \varepsilon^{(2)}) \sigma^2(x) - \max(\varepsilon^{(1)}, \varepsilon^{(2)}) \sigma(x) \phi^{-1}(x) \geq 0; \]
where \( \phi(x) \) is an increasing, positive, smooth function satisfying
\[
\int_{\alpha}^{\infty} \frac{1}{\phi(x)} \, dx < \infty.
\]

Let \( S \) be the unique strong solution of the SDE
\[
dS_t = S_t v_t dB_t \\
dv_t = \sigma(v_t) dW_t + b(v_t) dt
\]
on \((\mathcal{P}, \mathcal{F})\), where \( B_t \) and \( W_t \) are Brownian motions with correlation \( \rho > 0 \). Then process \( S \) is also the solution of
\[
dS_t = S_t v_t dB_t \\
dv_t = \sigma(v_t) dW_t + b(v_t) dt + k_t \sigma^2(v_t) dt + \left( \rho H_t + J_t \right) \sigma(v_t) dt.
\]

Proof: See Dandapani and Protter (2016).

Therefore if functions \( \sigma(\cdot) \) and \( b(\cdot) \) respect conditions (i) and (ii), then under the enlarged filtration \( \mathcal{G} \) the drift component \( b(\cdot) \) changes to \( \hat{b}(\cdot) \) (as in equation (2.12)), which turns \( S \) from an \( \mathcal{F} \) (true) martingale into a \( \mathcal{G} \) strict local martingale.

3. EMPIRICAL EVIDENCE

Our sample consists of (30-days ahead) exchange rate predictions of Brazilian market participants (the Focus survey); future markets closing forward rates traded at Bovespa BM&F; and the official Brazilian exchange rate (PTAX). Sample range and frequency are limited by Focus data such that 179 monthly observations, from 2001/11 to 2016/09, are available. Focus forecasts regard the last week day of the corresponding month, while future markets forward rates are for the first week day of the same month. We thus obtain the “predicted forward rate at the last day of the month” by using the CDI (Brazilian Interbank Deposit Certificate) rate. We also consider two alternative formulations — first using the forward rate the day before Focus release, and then using the discounted second forward maturity. Results are similar to the baseline formulation and therefore not reported. Trajectories are plotted in Figure 3, and descriptive statistics are displayed in Table 3.

The Focus Bulletin is a survey released every Monday by the Central Bank of Brazil. The report, a daily survey of forecasts of about 120 banks, funds and other institutions summarizes the market expectations for the Brazilian economy. The Market Expectations System, which collects information reported in the Focus Bulletin, consists of a web interface where agents report their expectations for the macroeconomic variables of interest. The system considers the last information reported in the past 30 days by market participants. If an agent does not revise his or her expectations in the last thirty days, this information is not considered in statistic calculations. An important incentive of this system is the monthly release of best performing institutions, indicated in the top five predictors for each macroeconomic variable monitored in the system (see Mariani and Laurini (2016) for a thorough description of the Focus survey).

We now verify whether processes describing forward rates, Focus forecasts, and the spot exchange rate are consistent with a martingale — as expected by the first theorem of asset pricing. For such we employ test based on Cramer-von Mises (CvM) and Kolmogorov-Smirnov (KS) statistics proposed by Domínguez and López (2003), augmented with bootstrap finite sample correction of Charles et al.
Figure 3.1. Spot rate, Focus and future market forecasts

Note: This figure plots observed values of BRL-USD (PTAX) spot exchange rates, BMF&BOVESPA CDI-discounted forward rates, and Focus forecasts.

Table 3.1. Descriptive statistics

<table>
<thead>
<tr>
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<th>sd</th>
<th>max</th>
<th>min</th>
<th>mfe</th>
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<tr>
<td>Spot</td>
<td>2.397</td>
<td>0.630</td>
<td>4.117</td>
<td>1.564</td>
<td>0.000</td>
</tr>
<tr>
<td>Focus</td>
<td>2.393</td>
<td>0.625</td>
<td>4.080</td>
<td>1.570</td>
<td>-0.004</td>
</tr>
<tr>
<td>Future</td>
<td>2.414</td>
<td>0.636</td>
<td>4.079</td>
<td>1.572</td>
<td>0.017</td>
</tr>
</tbody>
</table>

Note: This table reports descriptive statistics for BRL-USD (PTAX) spot exchange rates, BMF&BOVESPA CDI-discounted forward rates, and Focus forecasts. Mean forecast error is with respect to spot exchange rates.

(2011). Test statistics and equivalent p-values are reported in Table 3. Both tests fail to reject the martingale null hypothesis for the spot exchange rate and future markets forward rates, but strongly reject it when considering Focus forecasts. The same tests are applied to prediction errors (Future-spot and Focus-spot). We find that the difference martingale hypothesis is rejected for Focus prediction errors, and not rejected for future market prediction errors.

Figure 3 plots p-values for a forecast bias test. We regress 30-days ahead exchange rates against Focus forecasts and a constant, i.e. $y_t = \alpha + \beta x_t + \epsilon_t$, and jointly test if $\alpha = 0$ and $\beta = 1$. From 2001/11 to 2008/07 it is possible to reject the null $\alpha = 0$ and $\beta = 1$ hypothesis at a 10% significance level.

Table 3 displays an illustrative prediction performance comparison between Focus survey forecasts and future market rates. We repeat the procedure for the "biased" and "unbiased" Focus subsamples (2001/11/30-2008/07/31 and 2008/09/29-2016/09/30). Forward rates are generally a better predictor than the 30-days ahead spot interest rate. When considering the whole sample or any subsample forward
Table 3.2. Martingale tests

<table>
<thead>
<tr>
<th></th>
<th>CvM</th>
<th>CvM p-val</th>
<th>KS</th>
<th>KS p-val</th>
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<tbody>
<tr>
<td>Focus</td>
<td>2.569</td>
<td>0.000</td>
<td>2.282</td>
<td>0.000</td>
</tr>
<tr>
<td>Future</td>
<td>0.042</td>
<td>0.948</td>
<td>0.553</td>
<td>0.897</td>
</tr>
<tr>
<td>Spot</td>
<td>0.070</td>
<td>0.810</td>
<td>0.731</td>
<td>0.616</td>
</tr>
<tr>
<td>Δ Focus</td>
<td>2.730</td>
<td>0.000</td>
<td>2.150</td>
<td>0.000</td>
</tr>
<tr>
<td>Δ Future</td>
<td>0.026</td>
<td>0.990</td>
<td>0.549</td>
<td>0.910</td>
</tr>
</tbody>
</table>

Note: This table reports martingale tests of BRL-USD (PTAX) spot exchange rates, BMF&BOVESPA CDI-discounted forward rates, and Focus survey forecasts. Cramer von Mises (CvM) and Komolgorov-Smirnov statistics calculation were based in 2000 bootstrap replications.

Figure 3.2. Rolling sample bias test

Note: This figure plots rolling sample bias test for Focus forecasts. P-values based on a rolling sample estimation.

rates fare better when considering RMSE or MAE, while Focus mean prediction errors are smaller in absolute terms.

4. Non Parametric Estimation

Recall from our discussion in Section 2 that a stochastic process is a strict local martingale if its volatility increases fast enough with higher level, such that

\[
\int_{\alpha}^{\infty} \frac{x}{\sigma^2(x)} dx < \infty
\]

for all \( \alpha > 0 \). We try to empirically characterize the volatility term \( \sigma(x)^2 \) in order to study the convergence properties of the integral above. In order to estimate \( \sigma^2(S_t) \) and relation (2.5) we employ the kernel based estimator of Florens-Zmirou
Table 3.3. Forecast comparison

<table>
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<th></th>
<th>ME</th>
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<th>MAE</th>
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<tr>
<td>Focus</td>
<td>-0.004</td>
<td>0.159</td>
<td>0.102</td>
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<tr>
<td>Future</td>
<td>0.017</td>
<td>0.146</td>
<td>0.096</td>
</tr>
<tr>
<td>Focus first subsample</td>
<td>0.011</td>
<td>0.182</td>
<td>0.110</td>
</tr>
<tr>
<td>Future first subsample</td>
<td>0.035</td>
<td>0.167</td>
<td>0.105</td>
</tr>
<tr>
<td>Focus second subsample</td>
<td>-0.016</td>
<td>0.136</td>
<td>0.095</td>
</tr>
<tr>
<td>Future second subsample</td>
<td>0.002</td>
<td>0.126</td>
<td>0.088</td>
</tr>
</tbody>
</table>

Note: This table reports BRL-USD (PTAX) spot exchange rates prediction performance comparison statistics of BMF&BOVESPA CDI-discounted forward rates, and Focus forecasts. We split the observed sample in two subsamples, 2001/08 to 2008/07, and from 2008/08 onwards. Column one displays mean error, second column root mean squared error, third column mean absolute error.

Note: This figure plots volatility function $\sigma(S_t)$ (top panel) and relation (2.5) (bottom panel) of spot exchange rates, discounted forward rates, and Focus forecasts.

\[
\hat{\sigma}^2(x) = \frac{\sum_{i=0}^{n-1} K\left(\frac{x-X_i}{h_n}\right) (X_{i+1} - x)^2}{\Delta_n \sum_{i=0}^{n-1} K\left(\frac{x-X_i}{h_n}\right)}
\]

where $K(\cdot)$ is a kernel function, here Gaussian.

Figure 4 plots estimated volatility functions (top panel) and relation $x/\sigma^2(x)$ (bottom panel), for our three processes of interest. One readily notices that Focus forecasts are less volatile than spot exchange rates or future market predictions for every observed price realization; and that volatility functions of both spot rates and

(1993) and Jiang and Knight (1997)
Figure 4.2. Vertically rescaled variance functions, whole sample

Note: This figure plots vertically rescaled volatility function $\sigma(S_t)$ (top panel) and relation (2.5) (bottom panel) of spot exchange rates, discounted forward rates, and Focus forecasts.

future markets stop increasing somewhere past $x = 3.75$. The relation $x/var(x)$ is at stable (one might even say increasing) at higher price ranges for spot and futures processes, which allows to a violation of condition (2.5) (i.e., the integral is infinite) — indicating that Spot rates and future market predictions are true martingales.

The curvature of the volatility function of Focus forecasts decreases slower, such that the function is always increasing for observed prices. This reduces the term $x/var(x)$ and supports the convergence of the integral in (2.5). Visual comparison might be easier if one disregards vertical scale, as in Figure 4. We argue systematic Focus prediction errors are associated with incompatibility between the stochastic process that drives forecasts and exchange rates. Figure 4.3 plots $var(x)$ and $x/var(x)$ for the first subsample period. Note the absence of inflection at the end of Focus’ variance function, and how sharply $x/var(x)$ decreases. This can be taken as evidence that Focus forecasts behave like strict local martingales from 2001/11 to 2008/08. Conversely, when considered the second subsample, as in Figure 4.4, it becomes hard to distinguish between Focus and spot processes — both are consistent with what is to expect from true martingales.

5. Parametric Estimation

Consider the stochastic volatility model studied by Andersen and Piterbarg (2007)

\begin{align*}
    dS_t &= \lambda X_t \sqrt{V_t} dW^1_t \\
    dV_t &= \kappa (\theta - V_t) dt + \epsilon V^p_t dW^2_t
\end{align*}

where $\lambda, \kappa, \theta > 0$ and $(W^1_t, W^2_t)$ is a standard two dimensional Brownian motion with correlation coefficient $\rho$. Note that if $p = 1/2$ the model in (5.1) is known as
Figure 4.3. Vertically rescaled variance functions, first subsample

Note: This figure plots vertically rescaled volatility function $\sigma(S_t)$ (top panel) and relation (2.5) (bottom panel) of spot exchange rates and Focus forecasts for the first subsample period.

Figure 4.4. Vertically rescaled variance functions, second subsample

Note: This figure plots vertically rescaled volatility function $\sigma(S_t)$ (top panel) and relation (2.5) (bottom panel) of spot exchange rates and Focus forecasts for the second subsample period.

Heston (1993) model. This model is equivalent to the one discussed in Section 2.2, but allows for testable implication of the Martingale hypothesis. Results of Andersen and Piterbarg (2007) and Protter (2013) describe how the parameter space of model (5.1) can be divided in two disjoint sets, such that a specific parameter range
characterizes stochastic process $S$ as a strict local martingale (see Protter (2013) for a proof):

**Theorem 1.** For the model in (5.1): if $\rho \leq 0$, $S$ is a true martingale; if $\rho > 0$ and $p \leq 1/2$ or $p > 3/2$, $S$ is a true martingale; if $\rho > 0$ and $1/2 < p < 3/2$, then $S$ is a strict local martingale. For the case $p = 3/2$ $S$ is a true martingale if $\rho \leq 1/2\epsilon\lambda^{-1}$, and $S$ is a strict local martingale if $\rho > 1/2\epsilon\lambda^{-1}$.

Equipped with Theorem 1 we employ Bayesian techniques to estimate the stochastic volatility model of Andersen and Piterbarg (2007) to investigate whether Focus forecasts are compatible with a true martingale or a strict local martingale process.

Estimation was carried out in Stan (called through R-package “rstan”). Stan employs a Hamiltonian Monte Carlo (HMC) algorithm rather than the traditional Metropolis Hastings (MH). HMC augments the usual random walk posterior sampling scheme with Hamiltonian Dynamics, such that new proposed values can be further apart than they would be under a random walk exploration scheme, but have a high acceptance probability nevertheless. HMC is then more efficient than MH in the sense that it takes fewer algorithm iterations in order to achieve chain convergence. This is specially relevant in our stochastic volatility context because Bayesian statistics treats latent variables as it treats unknown parameters, what can be computationally expensive.

We elected prior distributions given restrictions on parameter values in model (5.1). A Uniform distribution between 0 and 1.5 was specified for $\lambda$ and $\epsilon$; $\kappa$ was assumed Normal, with mean 0.9 and standard deviation 0.5; $\theta$ was assumed Normal, with mean 0.01 and standard deviation 0.05; $p$ was also assumed Normal, with mean 0.5 and standard deviation 0.05; correlation $\rho$ was specified as $2\rho^* - 1$, $\rho^* \sim Beta(5, 2.5)$.

Table 5.1 reports prior specification and estimation results, posterior densities are displayed in Figure 5.1. Notice correlation coefficient $\rho$ is estimated positive with at least 97.5% probability — this is desirable since if $\rho \leq 0$, $S_t$ is a true martingale for all values of $p$, and further analysis would be unnecessary.

**Table 5.1. Estimation results, unrestricted model**

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<th>.5q</th>
<th>.975q</th>
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<td>0.137</td>
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<td>$\kappa$</td>
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<td>0.194</td>
<td>0.280</td>
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<tr>
<td>$\theta$</td>
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<td>0.119</td>
<td>0.048</td>
<td>0.123</td>
<td>0.468</td>
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<tr>
<td>$\epsilon$</td>
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<td>0.273</td>
<td>0.799</td>
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<td>$\rho$</td>
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<td>0.180</td>
<td>0.026</td>
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<tr>
<td>$p$</td>
<td>0.365</td>
<td>0.204</td>
<td>0.028</td>
<td>0.364</td>
<td>0.743</td>
</tr>
</tbody>
</table>

Note: This table reports results for the nonrestricted estimation of model (5.1). First column displays posterior means, second column standard errors, and columns three through five display selected quantiles. Estimation was carried out by Hamiltonian Monte Carlo with 100000 replications.

The posterior mean of parameter $p$ is estimated around 0.365, which according to Theorem 1 characterizes Focus forecasts as a true martingale process. However, the posterior distribution of $p$ is quite disperse and imply substantial uncertainty.
Figure 5.1. Posterior Densities, unrestricted model

Note: This figure plots posterior distributions of the unrestricted model (5.1), estimated via Stan with 100000 replications.

regarding parameter value. Also, prior and posterior distributions of parameters $\lambda$ and $\epsilon$ are very similar, thus it is questionable whether those parameters are identifiable from data. We recognize that available Focus data is of monthly frequency and sample size is limited, which makes estimating model (5.1) an ambitious enterprise. In face of this identification difficulty we decide to fix $\lambda = 1$ and focus attention on estimates of $p$ and $\rho$, which are at the core of our parametric approach.

One can see from Table 5.2 that the restricted model indeed yields more precise estimates. Comparing the second column of Table 5.1 and Table 5.2 one can see that posterior standard errors of $\theta$, $\rho$, and $p$ are substantially smaller; while $\lambda$ estimates are essentially unchanged. Figure 5.2 displays bell-shaped, less disperse, posterior distributions of $\kappa$, $\rho$, and $p$.

Table 5.2. Estimation results, restricted model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Std</th>
<th>.025q</th>
<th>.5q</th>
<th>.975q</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>0.743</td>
<td>0.199</td>
<td>0.262</td>
<td>0.788</td>
<td>0.991</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.095</td>
<td>0.037</td>
<td>0.048</td>
<td>0.088</td>
<td>0.186</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.789</td>
<td>0.291</td>
<td>0.340</td>
<td>0.747</td>
<td>1.408</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.194</td>
<td>0.093</td>
<td>0.062</td>
<td>0.178</td>
<td>0.415</td>
</tr>
<tr>
<td>$p$</td>
<td>0.507</td>
<td>0.046</td>
<td>0.413</td>
<td>0.507</td>
<td>0.597</td>
</tr>
</tbody>
</table>

Note: This table report results for the restricted ($\lambda = 1$) estimation of model (5.1). First column displays posterior means, second column standard errors, and columns three through five display selected quantiles. Estimation was carried out by Hamiltonian Monte Carlo with 100000 replications.

Parameter $p$ is now estimated between 0.4 and 0.6 with 97.5% probability, a much narrower credibility interval. Still, because substantial probability is attributed to
Figure 5.2. Posterior Densities, restricted model

Note: This figure plots posterior distributions of the restricted $\lambda = 1$ model (5.1), estimated via Stan with 100000 replications.

the $p \leq 0.5$ range, it is not possible to convincingy assert that $S_t$ is a strict local martingale process from Table 5.2 alone. A more formal test is appropriate. We ask: does imposing that $S_t$ is a strict local martingale worsen out-of-sample forecast performance? for such we further restrict model (5.1) such that $p \in [0.5, 1.5]$. Then compare this formulation to the one in Table 5.2 and Figure 5, for which $p \in [0.5, \infty)$. We evaluate model performance through the Leave-one-out cross-validation (LOO) and the widely applicable information criterion (WAIC) of Vehtari et al. (2016) — those are pointwise out-of-sample prediction accuracy tests of a fitted Bayesian model, and are based on the log-likelihood evaluated at the posterior simulation of the parameter values.

Table 5.3 reports WAIC and LOO information criteria for the unrestricted, $p \in [0, \infty)$, and restricted, $p \in [0.5, 1.5]$ models. Imposing SLM does not worsen out of sample forecast behavior, as both WAIC and LOO attribute a smaller information criterion to the restricted model. Figure 5 plots fitted versus realized Focus volatility for both models in Table 5.3.

Table 5.3. Model evaluation

<table>
<thead>
<tr>
<th></th>
<th>$p \in [0, \infty)$</th>
<th>$p \in (0.5, 1.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>WAIC</td>
<td>-659.8</td>
<td>-663.8</td>
</tr>
<tr>
<td></td>
<td>[28.3]</td>
<td>[28.9]</td>
</tr>
<tr>
<td>LOO</td>
<td>-613.3</td>
<td>-620.1</td>
</tr>
<tr>
<td></td>
<td>[36.3]</td>
<td>[36.4]</td>
</tr>
</tbody>
</table>

Note: This table reports WAIC and LOO information criteria estimates (standard errors in square brackets). First column display results for the $p$-unrestricted model, and second column displays results for the SLM imposed model.
6. Discussion

We view the results of Sections 4 and 5 as evidence that Focus survey forecasts are better characterized by a strict local martingale than a driftless true martingale process, specially through the first subsample from 2000/08 to 2008/07 — although one must recognize methodological choices and statistical uncertainty.

The nonparametric approach of empirically evaluating condition (2.5) is fundamentally limited by the range of observed values $S_t$ in a given time interval. We are trying to extrapolate from a compact set of $R_+$ properties of $\sigma(S_t)$ for all of $R_+$, and as discussed in Jarrow et al. (2011b) results will invariably depend on the extension method employed.

Theorem 1 gives us empirically testable objective criteria that characterize a price process $S_t$ as either a true martingale or a strict local martingale. But formally estimating heston-type models is not standard practice for understandable reasons. Classical likelihood inference has major difficulties when dealing with latent variables, what makes Bayesian simulation methods appealing.

When we compromise and fix $\lambda = 1$, our estimates are still inconclusive with respect to Theorem 1 — because although one can say with reasonable certainty that $\rho$ is positive, the posterior of $p$ attributes significant probability to the true
martingale range, what makes a more formal test of the strict local martingale hypothesis warranted.

We draw from developments in bayesian model selection (Vehtari et al. (2016)) and devise an experiment in which one imposes that $S_t$ is a strict local martingale (i.e. $\rho > 0$ and $0.5 < p < 1.5$), and evaluate whether this restriction hinders out-of-sample forecast performance. Imposing SLM behavior does not worsen out-of-sample forecast, but it is hard to distinguish between models. In hindsight this result is not unsurprising because even in the $p$-unrestricted model more, probability is attributed to the SLM range of $p \in (0.5, 1.5)$ than otherwise.

7. Concluding Remarks

We investigate the puzzling poor forecast performance of Brazilian market participants. Empirical illustration in Section 3 show that extrapolated forward rates are a better predictor of the 30-days ahead spot exchange rate than Focus survey forecasts — which are biased through a substantial sample portion.

The filtration enlargement mechanism of Dandapani and Protter (2016) is presented as an interpretation of such phenomenon. Nonparametric evidence presented in Section 4 revolves around estimating the volatility term $\sigma^2(S_t)$ and relation (2.5) — and results suggest spot exchange and forward rates are true martingales, while Focus forecasts are strict local martingales. In Section 5 we estimate the stochastic volatility model of Andersen and Piterbarg (2007), whose parameter range either determines whether the process is a true martingale or a strict local martingale. Results indicate it is not possible to discard the strict local martingale hypothesis for Focus forecasts.

References


