Robust Selling Mechanisms*

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Abstract

We consider the problem of a seller who faces a privately informed buyer and only knows one (in the more general case, arbitrary) moment of the distribution from which values are drawn. In face of this uncertainty, the seller maximizes his worst-case expected profits. We show that a robustness property of the optimal mechanism imposes restrictions on the seller's ex-post profit function. These restrictions are used to derive the optimal mechanism. The optimal mechanism entails distortions at the intensive margin, e.g., except for the highest value buyer, sales will take place with probability strictly smaller than one. The seller can implement such allocation by committing to post prices drawn from a non-degenerate distribution, so that randomizing over prices is an optimal robust selling mechanism. We extend the model to deal with the case in which: (i) M goods are sold and the buyer's private information is multidimensional and (ii) the seller and the buyer interact for several periods. In the case of multiple goods, selling the goods in a fully separable way is always optimal. When the consumer's expected value for each of the M goods is the same, bundling all goods in fixed proportions is also optimal. In the dynamic model, we show that repetition, period by period, of the static-optimal mechanism is optimal.

Keywords: Robust Mechanism Design, Monopolistic Pricing under Uncertainty, Price Discrimination

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Contents

1 Introduction

This paper considers the problem faced by a monopolist (he), who – except for knowing one moment of the distribution – is unaware of the distribution from which the consumer's (she) value is drawn. In face of such ambiguity, he seeks to design a mechanism that is robust, in the sense of maximizing his worst-case expected profits over all distributions satisfying the moment condition. The analysis encompasses the cases in which $M > 1$ goods are sold (therefore, valuation is multidimensional) and the monopolist and the consumer interact for multiple periods.

For most of the exposition we consider a seller that is informed about the first moment of the type distribution. The assumption that the seller knows at least the first moment of the distribution of values makes the problem interesting. Under full ignorance of the distribution, the optimal mechanism would be trivial: set a price equal to the lowest value the consumer might attribute to the good. Moreover, it might well be the case that the seller, proceeding as an econometrician as proposed by Segal (2003), has access to enough data to estimate (say, through an hedonic regression) the mean of the distribution of values, but his sample falls short of data for a non-parametric (consistent) estimation of the whole distribution. Alternatively, it might be common knowledge that consumer expects her value to be $k > 0$ but, before purchase and after the seller offers her a mechanism, might acquire relevant information about her value. In such interpretation, due to Carroll (2013), if the seller is uncertain about the agent's information acquisition technology, all he will know is that values are drawn from a distribution with mean k.

We start, to fix ideas, by considering a simple one-period case with single-dimensional private information. Since the seminal paper Myerson (1981), much is known about such case when the consumer has a unit demand, and the monopolist knows the distribution F from which this value is drawn. Bulow and Roberts (1989) demonstrate that the optimal selling mechanism resembles the solution of a standard monopolist's problem. In fact, for a given price p, if one defines "quantities" sold as the probability of sales under F – namely, $1 - F(p)$ – total revenues can be written as $p(1 - F(p))$. A monopolist would then compute marginal revenues, $p - \frac{1-F(p)}{f(p)}$, and sell if and only if they are larger than costs. The rule coincides with the one derived by Myerson (1981). Moreover, when marginal revenues are monotone, the seller can implement such optimal mechanism by posting a single price.

Although quite simple, the solution requires full knowledge of "demand" – the distribution F – and cannot be pursued if the monopolist only knows the expected value (denote it by $k > 0$) of the consumer's valuation. Solving for the mechanism that maximizes expected profits under the worst-case distribution involves a couple of steps. First of all, it is convenient to think of the robust mechanism design problem as being the seller's best response to an adversarial nature, who seeks to minimize his expected profits by choice of distributions. This minimization problem is constrained by the fact that the distribution must integrate to one, and, by assumption, its expected value is k. Once one incorporates those constraints in a Lagrangian functional, nature minimizes the expected value of the seller's profits subtracted by the inner product of the constraints and their shadow costs (the Lagrangian multipliers).

In our first result, we show that nature will only place positive likelihood on values $\theta \in [0, 1]$ for which the seller's profits equal $\xi\theta - \lambda$, where ξ is the shadow cost of the constraint that imposes that the average of the distribution must be k, and λ is the shadow cost of the constraint that the distribution must integrate to one. As a corollary of such result, we show that seller's profits are piecewise linear: they are equal to zero over the region in which there are no sales, and is linear, with slope ξ, when the mechanism entails sales with positive probability. It follows that the robust mechanism imposes restrictions on the seller's payoff levels. One interpretation is that, knowing only the first moment of the distribution of values, the seller can only explore linearly higher consumer's values; else nature could move likelihood weights in a way that preserves the shadow costs imposed by its restrictions and reduces the seller's expected profits. Alternatively, under the information acquisition interpretation of the model, the stage at which the consumer obtains new information adds volatility to the seller's (degenerate) prior. A profit function with a call option format is then optimal.

Having established that profits are linear in values conditional on sales, finding the robust allocation is simple. Indeed, by imposing that the derivative of the profit function with respect to values is ξ conditional on sales, we obtain an ordinary differential equation whose solution yields the robust sales' decision. As opposed to what prevails in the standard Bayesian problem, in the robust mechanism sales take place with probability smaller than one for consumers with valuation below the highest one. Put differently, there are distortions at the intensive margin. The interpretation is simple. To insure against uncertainty, the seller finds desirable to sell to a marginal consumer with a low value $\theta' < 1$. Nevertheless, if he were to sell with probability one to this marginal consumer, the maximum he could charge from infra marginal consumers (those with values $\theta > \theta'$) would be θ'. By distorting the consumption of lower types, the seller can charge more from higher types: a standard discriminatory practice, although with a very different rationale from standard price discrimination theories.[1] Price discrimination is the way by which the seller simultaneously insures against uncertainty and charges high prices from infra marginal consumers.

There are two different ways to indirectly implement such robust allocation. Since the robust allocation displays distortions at the intensive margin, we can use standard non-linear prices

[1] In particular, in a setting like ours, if the seller is a standard expected utility maximizer, he will only find it optimal to discriminate if either the consumer's payoff or his cost function have some curvature.

to implement it. The second, and more interesting, way to implement the robust allocation explicitly uses the lack of curvature in the consumer's payoff. In fact, the seller can implement the robust allocation by committing to pick a price from a well designed, non-degenerate, probability distribution of prices. Therefore, randomizing over posted prices is an optimal robust mechanism.

We extend the simple model in three directions. First, we allow for the seller to have knowledge of an arbitrary moment condition of the distribution of values. The condition we consider is general enough to encompass, as particular cases, knowledge by the principal of the n-th moment, the median or an arbitrary quantile of the distribution of values. As we shall see, the main features just described remain qualitatively the same; in particular, the robust allocation entails distortions at the intensive margin. Second, we consider the case in which the monopolists sells M goods and all he knows is that the consumer has unit demand for each of the goods, and that her (multidimensional) value lies in $[0, 1]^M$ and has expected value of $k \in [0, 1]^M$. We assume that the monopolist faces no technological constraints and can produce up to one unit of each good. In Bayesian settings, not much is known about the solution of such a problem and even standard features of single-good sales mechanisms, such as monotonicity of profits in consumer's values and the optimality of deterministic mechanisms, do not extend in general to multidimensional profit maximizing mechanisms (see, for example, Manelli and Vincent (2007) and Hart and Reny (forthcoming)).

In contrast, a general characterization of the robust multidimensional mechanism is not only feasible, but follows from similar arguments used in the single-dimensional problem. In fact, we consider the problem faced by nature in minimizing the seller's expected profits and construct a distribution for which the the seller's best response attains at most the value he would obtain by selling the goods in an entirely separable way. As a mechanism that entails selling each good separately is always feasible to the seller, this shows that separate sales is an optimal mechanism. For the case in which the consumer's expected valuations for the goods are the same, selling them in fixed proportions in a bundle is also optimal. This is established by a simply verifying that such mechanism yields the same (worst-case) expected profits for the seller as separate sales.

Third, we consider the case in which the consumer and the monopolist interact for multiple periods and the consumer's values might evolve over time. There are two main reasons to consider the role of dynamics in our model. Even if the seller has little information about the buyer's value, extra information might become available in future periods, leading to better revenue extraction. However, ambiguity on the side of the seller regarding this additional information makes the result non-trivial. In our dynamic model this is incorporated by looking at pricing rules that potentially depend on previous consumption behavior by the buyer. Also, knowledge of average valuations is naturally connected to learning and information acquisition, as conditional expectations with increasing information sets follow a martingale. However, information acquisition is a dynamic phenomenon that should be discussed in a fully dynamic model.

A central part of our analysis deals with the key distinction between the case of multiple goods and multiple periods: the sequential revelation of information. A consequence of the sequential revelation of information is a failure of the revelation principle in the presence of ambiguity. One methodological contribution of this paper is to present a way of dealing with dynamics in a world with ambiguity. The seller chooses from the larger set of indirect mechanisms and, in the face of large ambiguity, considers all possible type distribution as well the whole set of optimal reporting strategies induced by each distribution. Our main result is the irrelevance of dynamics in the presence of large ambiguity, i.e., repeated static optimal pricing is optimal.

Related literature

Our paper is part of a growing literature on mechanism design with principals with maximin preferences.[2] Frankel (2014) and Carrasco and Moreira (2013) consider decision-making problems with non-transferable utility in which a maximin principal is unaware of the agent's bias (in Frankel (2014)) or the distribution of states (Carrasco and Moreira (2013)). In an otherwise standard procurement setting à la Laffont and Tirole (1986), Garrett (2014) considers the case of a principal who does not know the producer's disutility of effort, and show that a simple fixed-price-cost-reimbursement (FPCR) menu minimizes the principal's maximum expected payment to the agent. In Carroll (2015a), the principal only partially knows the set of actions available to the agent; he shows that if the principal maximizes expected profits under worst-case set of actions, the optimal contract is linear in output. Our work differs from those listed above by considering a seller's pricing decision.

There are, nevertheless, a set of papers that focus on pricing with unknown distribution of values and posit that the seller has preferences for robustness.[3] Bergemann and Schlag (2008, 2011) are perhaps the first to do so. In their first paper, they consider the case in which the seller designs a mechanism to minimize the maximum regret, whereas in the second they also consider a maximin procedure. In both cases, they work within a static and single-dimensional case. Similar to what we find, they show that randomizing over prices is a way to insure against uncertainty in the minimax problem. Pointing out that, without further restrictions,

[2]There is also a growing literature with maximin agents. Bose et al. (2006) and Wolitsky (2014) are examples of analysis of, respectively, optimal auctions and bilateral trade when agents have maximin preferences.

[3]Segal (2003) also considered the case in which a monopolist did not know the distribution from which values were drawn. Rather than positing that the seller has preferences for robustness, he considered a seller who proceeds as an econometrician and estimates, from the mechanisms offered to subset of consumers, the distribution of values.

the maximin problem entails the trivial solution of charging the lowest possible valuation with probability one, they consider "local robustness", that is, maximin pricing over neighborhoods around a given distribution. They show that the optimal is to post a single price and establish that, starting at the certainty case, the charged price decreases as uncertainty (measured by the size of the neighborhood) increases. Instead of considering a minimax criterion or working with neighborhoods around a given distribution, to avoid a trivial solution for the maximin problem, we assume that a given moment of the distribution of types is known. This complements their minimax analysis, on the one hand, and allows for an analysis of the multidimensional and repeated cases, on the other.

In independent work, Carroll (2013) considers a setting in which a buyer knows the expected value of its willingness to pay, but can acquire information about it before purchasing the good. The seller, who only knows the prior from which the expected valuation is drawn, designs a mechanism to maximize the worst-case (over information acquisition technologies, which amounts to choosing among mean preserving spreads of the prior) expected profits.[4] Our single-good case corresponds to a special case of his when the seller's prior is degenerate. For the case of a single good, however, we also consider the case of knowledge of an arbitrary moment by the seller. We also extend the analysis to $M > 1$ goods (and multidimensional values) and repeated sales.

Handel and Misra (forthcoming) consider the problem of a monopolist who launches a new product and, without knowledge of (the time invariant) demand, decides – restricting attention to price posting mechanisms – on intertemporal prices to minimize maximum intertemporal regret. They show that prices decrease over time if consumers are homogenous, and increase if consumers are heterogeneous. Caldentey et al. (2015) also consider minimax intertemporal pricing for the case in which seller restrict attention to posting price mechanisms. However, on top of not knowing demand, the seller does not know the arrival process of consumers in their paper. They also establish that optimal price paths are decreasing when buyers are rational. In contrast to those papers, we consider maximin design (with the restriction that expected values follows martingale[5]) and allow for general mechanisms. We show that the optimal dynamic mechanism is time-invariant. Such time-invariance result can be related to the recent literature on (Bayesian) dynamic mechanism design. In particular, Pavan et al. (2015) show that the amount of informational rents that must be left to an agent whose private information follows an AR(1) process depends on its the degree of persistence. It should then be no surprise that the adversarial nature the seller faces chooses consumer's types to be fully persistent in our model.

Not much is known in general for Bayesian multidimensional design (see, for instance,

[4]In the introduction, we have borrowed Carroll's story to justify why the seller might know the mean of the distribution from which values are drawn.

[5]Again, this can be justified by an information acquisition story, since, if information is acquired over time, at any given time the martingale property must hold.

Hart and Reny (forthcoming) and references therein). By looking at worst-case selling procedures, we are able to fully derive optimal mechanisms and show that they involve full separation (and full bundling for the symmetric case), in contrast, for example, to the mixed-bundling solution of McAfee et al. (1989) and the literature that followed. In Carroll (2015b)[6] an additive separable multidimensional mechanism design problem with general (quasi-linear) preferences is considered for the case in which the designer only knows the marginal distributions and maxminimizes expected payoffs. Our setting is much more restrictive in terms of the payoff structure, but allows for a larger degree of ignorance by the designer (since he only knows a single moment of the joint distribution of the agent's private information, rather than the marginals). Both papers derive a full separability result. For what we call the symmetric case – in which the known moment is the same in each dimension –, our proof relies on an explicit computation of a Nash Equilibrium of the zero sum game played by nature and the seller. For such particular case, we also establish that full bundling is optimal. Our proof of separability for the asymmetric case is a variant, for the case in which only the expected value of the distribution of types is known, of what Carroll (2015b) describes as the "maximal positive correlation" case, in his Section 3.2. In fact, we use the worst case distributions derived for the single-good case to construct what Carroll (2015b) calls a comonotonic joint distribution that induces expected profits by the seller which are bounded above by what he obtains under full separability. Our result that the dynamic mechanism is the period by period repetition of the static mechanism can be interpreted – under the assumptions of a martingale condition and linear payoffs – as a stronger form of separability in the multidimensional problem than the one we derive in Section 4 and the one in Carroll (2015b). Indeed, one could allow for a seller to decide on the sequence in which it sells different goods to a consumer who might obtain information about her valuation of a given good if she consumes other goods. Consider the case in which a consumer wants to buy different quantities of goods x, y, z and t. The monopolist might sell first good x, and then bundle the sale of goods y, z and t. Alternatively, he might sell x and y separately (and in sequence) and then bundle z and t, or adopt any possible combination of those selling strategies. Our result establishes that (again, under a martingale condition and with linear payoffs), full separation will be an optimal mechanism.

Organization

Section 2 lays down the general model. In Section 3, we derive the optimal robust mechanism for the static setting in which the monopolist sells one good. We tackle the robust design problem

[6]After the first version of this paper was written, Carroll (2015b) was brought to our attention. We thank Lucas Maestri for calling our attention to this recent paper.

using two different approaches. In the first one, we use standard Lagrangian technique, whereas, in the second, we recast the robust design problem in terms of a zero-sum game played by the monopolist and an adversary nature who chooses distribution to minimize his expected profits. This latter approach proves useful to derive the optimal robust mechanism for an arbitrary moment condition that might be known by the monopolist. In Section 4 we consider the robust design for the case in which the monopolist sells M goods. We move to the case of a $T-$period robust design in Section 5. We draw our concluding remarks in Section 6.

2 Model

A monopolist (or seller) can produce $M \geq 1$ indivisible and non-storable goods at zero cost in each period $t \in \{1,..,T\}$, $T \geq 1$. The seller faces a consumer who has valuation for the good in period t denoted as $\theta_t = \left(\theta^1, \ldots, \theta^M\right) \in [0,1]^M$. A sequence of valuations is denoted by $\theta^t = (\theta_1, ..., \theta_t) \in [0,1]^{Mt}$, for any $t \leq T$. If quantity $\mathbf{q}_t \in [0,1]^M$ (we use bold to represent the vector of quantities) and transfers p_t are made in each period $t \in T$, the utility obtained by the consumer is given by

$$\sum_{t=1}^{T} \delta^{t-1} \left(\mathbf{q}_t \cdot \theta_t - p_t\right),$$

and the seller's profits are

$$\sum_{t=1}^{T} \delta^{t-1} p_t.$$

The set of direct mechanisms is defined as

$$\mathcal{M} \equiv \left\{ m = (\mathbf{q}_t, p_t); (\mathbf{q}_t, p_t) : [0,1]^{MT} \to [0,1] \times \mathbb{R} \text{ is } \theta^t\text{-measurable} \right\}.$$

The set of mechanisms with arbitrary message spaces is given by $\overline{\mathcal{M}} \supsetneq \mathcal{M}$. For any mechanism m with message space A in each period, a reporting strategy is $\sigma = (\sigma_t)_{t=1}^T$ with function $\sigma_t : [0,1]^{MT} \to A$ which is measurable with respect to $\theta^t \in [0,1]^{Mt}$. The set of all reporting strategies is Σ_m and, for any direct mechanism, the truth-telling strategy is denoted as σ^{TT}.[7] The realized payoff of an agent following strategy $\sigma \in \Sigma_m$ in mechanism $m \in \overline{\mathcal{M}}$ is

$$\mathcal{U}_m \left(\sigma \mid \theta^T\right) \equiv \sum_{t=1}^{T} \delta^{t-1} \left[\mathbf{q}_t \left(\sigma \left(\theta^T\right)\right) \cdot \theta_t - p_t \left(\sigma \left(\theta^T\right)\right)\right],$$

[7] We assume that the set of allocations that are feasible for a buyer is compact, i.e., $\{(q^\infty, p^\infty); \exists a^\infty \in A^\infty \text{ such that } (q_t(a^\infty), p_t(a^\infty)) = (q_t^\infty, p_t^\infty), \forall t \in T\}$ is compact in the product topology. This is required to guarantee the buyer always has an optimal reporting strategy.

and the realized firm profits are given by

$$\Pi_m\left(\sigma \mid \theta^T\right) \equiv \sum_{t=1}^{T} \delta^{t-1} p_t\left(\sigma\left(\theta^T\right)\right).$$

When dealing with a direct mechanism we will also use $\mathcal{U}_m\left(\theta^T\right)$ and $\Pi_m\left(\theta^T\right)$ to denote realized payoffs for the buyer (the rent function) and the seller (the profit function) under truth-telling.

We assume that the type distribution $F \in \Delta\left([0,1]^{MT}\right)$ is known by the buyers but unknown by the seller. The seller only knows that the set of possible distributions is $\mathcal{F} \subseteq \Delta\left([0,1]^{MT}\right)$ but does not have a probability distribution over this set. Instead, the seller is ambiguity averse. The seller's problem amounts to designing a mechanism to maximize his worst-case expected profits over all distributions in $\mathcal{F}$. In this paper the defining property of this set it assumed to be a set of moment conditions, which is the only market information available to the seller. The definition of this set is presented for each case of interest studied. Over the next sections, to build the ideas that will allow us to find the robust mechanism for the general model we just laid out, we will consider the seller's problem for some special cases of interest.

3 One period and one good

We first consider the static case where the seller holds one indivisible good and only has information about the average of the type distribution. Formally, the set of possible distributions is

$$\mathcal{F} \equiv \left\{F \in \Delta([0,1]);\int \theta dF(\theta) = k\right\},$$

for some $k \in (0,1)$.

The seller wants to maximize the revenue guarantee given by this moment condition alone. This is done by considering the worst-case expected profits from the set of possible distributions. Since the buyer has complete information at announcement stage, the set of optimal reporting strategies is independent of the actual type distribution $F \in \mathcal{F}$. As a consequence, any selection from this set that is independent of the distribution choice can be implemented through a direct mechanism with truthful strategies. Hence, we restrict attention to incentive compatible direct mechanisms. The seller's problem becomes:

$$\max_{m\in\mathcal{M}} \min_{F\in\mathcal{F}} \int_0^1 p(\theta)\, dF(\theta), \tag{1}$$

subject to participation constraint

$$\mathcal{U}_m(\theta) \geq 0,$$

and incentive compatibility constraints

$$\mathcal{U}_m(\theta) \geq \theta q(\widehat{\theta}) - p(\widehat{\theta}),$$

for all $\theta, \hat{\theta} \in [0,1]$.

As usual under the single-crossing condition, incentive compatibility is equivalent to the envelope condition:

$$\mathcal{U}_m(\theta) = \mathcal{U}_m(0) + \int_0^\theta q(\tau)\, d\tau, \tag{2}$$

and the monotonicity condition: $q(\cdot)$ is non-decreasing. Substituting equation (2) in the objective and noticing that, regardless of worst-case distribution, the seller will always pick a mechanism with $\mathcal{U}_m(0) = 0$, his problem can be equivalently rewritten as:

$$\max_{m \in \mathcal{M}} \min_{F \in \mathcal{F}} \int_0^1 \Pi_m(\theta) dF(\theta)$$

subject to $q(\cdot)$ non-decreasing and

$$\Pi_m(\theta) := p(\theta) = \theta q(\theta) - \int_0^\theta q(\tau)\, d\tau$$

is the profit function associated to mechanism $m = (q, p)$.

We follow the classical approach in looking at the relaxed problem without monotonicity constraints and showing that the solution to this problem indeed satisfies the ignored constraints.

3.1 Worst-case payoff

We start by fixing an arbitrary incentive compatible mechanism and the associated profit function $\Pi(\theta)$, i.e., any function $\Pi(\theta)$ that satisfies $\Pi(0) = 0$ and $\Pi(\theta)$ is non-decreasing[8]. Because of this latter property, it is without loss of generality to assume that $\Pi(\theta)$ is left-continuous.

[8] It follows that, whenever it exists, the derivative of the profit function $\Pi(\theta) = \theta q(\theta) - \int_0^\theta q(\tau)\, d\tau$ associated to the incentive compatibility allocation $q(\cdot)$ is a.e. $\Pi'(\theta) = q(\theta) + \theta q'(\theta) - q(\theta) = \theta q'(\theta) \geq 0$. Therefore, $\Pi'(\theta) \geq 0$ a.e. if and only if $q'(\theta) \geq 0$ a.e.

Let us relax, for a moment, the constraints which require that the cumulative distributions are probability distributions that have mean k and consider the following problem of finding the worst-case expected profits:

$$\min_{F\in\mathcal{D}} \int \Pi(\theta)\, dF(\theta) \tag{3}$$

subject to aggregate mass being smaller than one

$$\int dF(\theta) \leq 1,$$

and the average condition

$$k - \int \theta dF(\theta) \leq 0,$$

where $\mathcal{D} = \{F : [0,1] \to [0,1]$ is non-decreasing and right continuous$\}$. This, in turn, allows us to show:

Lemma 3.1. *(Existence) There is a solution to the problem (3). At the robust mechanism, the solution belongs to $\mathcal{F}$.*

Proof of Lemma 3.1. The set $\mathcal{D}$ is compact and the constraints of problem (3) are closed with respect to the weak topology. It is straightforward to see that the objective function is also lower semi-continuous with respect to the weak topology ($\Pi(\theta)$ is a left-continuous and non-decreasing function). The second constraint must bind. Otherwise, using the Lagrangian approach presented below, we know that if $\xi^* = 0$, then the distribution that attains the minimum would be concentrated at $\theta = 0$ since $\Pi(\theta) + \lambda^*$ is a positive function for $\theta \in (0, 1]$. However, this violates the second constraint (unless $k = q = \lambda^* = 0$, in which case the Dirac measure concentrated at $\theta = 0$ is the optimal distribution). Hence, $\xi^* > 0$ and the second constraint being binding. In the proof of Proposition 3.1 we argue that if $\xi^* > 0$, then $\lambda^* > 0$ and, consequently, the first constraint should bind at the robust mechanism. □

Let $F^* \in \mathcal{F}$ be a solution of the problem in (3). Standard arguments (see, for instance, Luenberger (1969)) imply that there exists a Lagrangian functional $L : \mathcal{D} \times \mathbb{R}^2_+ \to \mathbb{R}$ of problem (3) defined by

$$L(F;\lambda,\xi) = \int \Pi(\theta)dF(\theta) + \lambda\left(\int dF(\theta) - 1\right) + \xi\left(k - \int \theta dF(\theta)\right)$$

so that the worst-case distribution F^* and multipliers $\lambda^* \geq 0$ and $\xi^* > 0$ satisfy the saddle point condition:

$$L(F^*;\lambda,\xi) \leq L(F^*;\lambda^*,\xi^*) \leq L(F;\lambda^*,\xi^*),$$

for all $(F, \lambda, \xi) \in \mathcal{D} \times \mathbb{R}_+^2$. Hence, ignoring constant terms,

$$\int \Pi(\theta)\, dF^*(\theta) = \min_{F \in \mathcal{D}} \int [\Pi(\theta) + \lambda^* - \xi^* \theta]\, dF(\theta).$$

Clearly, problem (3) is well defined only if $\Pi(\theta) + \lambda^* - \xi^*\theta \geq 0$, for all $\theta \in [0,1]$. In fact, if for some $\theta \in [0,1)$ such that $\Pi(\theta) + \lambda^* - \xi^*\theta < 0$, one would have

$$L(N.H_\theta; \lambda^*, \xi^*) \to -\infty \text{ when } N \to \infty,$$

where H_θ is the Heaviside function at θ, i.e., $H_\theta(x) = 0$ for $x \in [0, \theta]$ and $H_\theta(x) = 1$ for $x \in (\theta, 1]$. Since $\Pi(\theta)$ is left-continuous and non-decreasing, then $\Pi(1) + \lambda^* - \xi \geq 0$. We can then define the sets of types where the profit function is the affine envelope function $\xi^*\theta - \lambda^*$, $I = \{\theta \in [0,1]; \Pi(\theta) = \xi^*\theta - \lambda^*\}$, and its complement $I^c = \{\theta \in [0,1]; \Pi(\theta) > \xi^*\theta - \lambda^*\}$. At the optimal distribution F^*, the mass of types where the profit is above the affine envelope must be zero, i.e., $dF^*(I^c) = 0$. Otherwise, by moving weight from I^c to I, the objective of the minimization problem could be improved by reducing expected profit.

More interestingly, at the robust mechanism, the worst-case distribution must assign positive likelihood for all types above a certain cutoff type. The intuition is that "nature",[9] who is minimizing the seller's shadow profits, assigns positive likelihood for types in I – for which shadow profits, $\Pi(\theta) + \lambda^* - \xi^*\theta$, are zero –, and assigns zero likelihood for types in I^c, for which shadow profits are positive. Hence, if an interval, say, in the form $[\bar{\theta}, 1]$ were to have positive shadow profits, the seller could reduce the amount sold to those types (and therefore, profits) in a way that, simultaneously, brings shadow profits slightly below zero (and, therefore, induces nature to assign positive likelihood to $[\bar{\theta}, 1]$) and still yields profits that are larger than the ones obtained from types smaller than $\bar{\theta}$ (since $\Pi(\theta)$ is non-decreasing). This would ensure larger expected profits for the seller.

In fact, the next proposition formally establishes that, at the robust mechanism, $I = [\theta^*, 1]$, where $\xi^*\theta^* = \lambda^*$. It actually does more: it proves that it is without loss of optimality to restrict attention to profit functions that are piecewise linear. Indeed, since $\Pi(\theta) + \lambda^* - \xi^*\theta \geq 0$ and nature only places positive likelihood on $I = \{\theta \in [0,1]; \Pi(\theta) = \xi^*\theta - \lambda^*\}$, for any mechanism and resulting profit function $\Pi(\theta)$, there exists a piecewise linear lower envelope of $\Pi(\theta)$. The next results show that such lower envelope is itself a profit function associated with an alternative allocation which also satisfies the condition that sales are smaller or equal to 1.

Lemma 3.2. *(Auxiliary) Let $q(\cdot)$ be any allocation such that $0 \leq q(\theta) \leq 1$ and for some*

[9] In Subsection 3.3 we will explore this interpretation to derive that the robust mechanism as a part of a Nash equilibrium between the seller and the adversary nature in zero-sum game.

$\xi^* > 0$ *and* $\lambda^* \geq 0$

$$\Pi(\theta) := \theta q(\theta) - \int_0^\theta q(\tau)d\tau \geq \xi^*\theta - \lambda^*,$$

for all $\theta \in [0,1]$. *Then,*

$$q(\theta) \geq \frac{1}{\theta^*}\int_0^{\theta^*} q(\tau)d\tau + \xi^* \; ln \; \left(\frac{\theta}{\theta^*}\right),$$

for all $\theta \in [0,1]$*, where* $\xi^*\theta^* = \lambda^*$.

Proof of Lemma 3.2. Defining $\psi(\theta) = \int_0^\theta q(\tau)d\tau$, the hypothesis of the lemma is equivalent to $\theta\psi'(\theta) - \psi(\theta) \geq \xi^*(\theta - \theta^*)$, for all $\theta \in [0,1]$. Now, notice that

$$\begin{aligned} \frac{\psi(\theta)}{\theta} - \frac{\psi(\theta^*)}{\theta^*} - \lambda^*\left(\frac{1}{\theta} - \frac{1}{\theta^*}\right) &= \int_{\theta^*}^\theta \left(\frac{\psi(\tau)}{\tau}\right)' d\tau + \lambda^* \int_{\theta^*}^\theta \frac{1}{\tau^2}d\tau \\ &= \int_{\theta^*}^\theta \frac{\tau\psi'(\tau)-\psi(\tau)}{\tau^2}d\tau + \lambda^* \int_{\theta^*}^\theta \frac{1}{\tau^2}d\tau \\ &\geq \int_{\theta^*}^\theta \frac{\xi^*(\tau-\theta^*)+\lambda^*}{\tau^2}d\tau = \xi^* \ln \left(\frac{\theta}{\theta^*}\right). \end{aligned}$$

Hence,

$$\begin{aligned} q(\theta) &\geq \frac{\psi(\theta)}{\theta} + \xi^*\left(1 - \frac{\theta^*}{\theta}\right) \\ &\geq \frac{\psi(\theta^*)}{\theta^*} + \xi^* \ln \left(\frac{\theta}{\theta^*}\right) + \xi^*\left(1 - \frac{\theta^*}{\theta}\right) + \lambda^*\left(\frac{1}{\theta} - \frac{1}{\theta^*}\right) \\ &= \frac{1}{\theta^*}\int_0^{\theta^*} q(\tau)d\tau + \xi^* \ln \left(\frac{\theta}{\theta^*}\right). \end{aligned}$$

□

Proposition 3.1. *(Piecewise linear envelope) Suppose that* $\Pi^*(\theta)$ *is the profit function of a robust mechanism. Then,* $\Pi^*(\theta) = \max\{\xi^*\theta - \lambda^*, 0\}$*, for some* $\xi^* > 0$ *and* $\lambda^* \geq 0$ *(i.e., it is piecewise linear).*

Proof of Proposition 3.1. Let $m = (q,p)$ be any feasible mechanism and $\Pi(\theta)$ the associated profit function. By the Lagrangian approach we described above, we know that there exist $\xi^* > 0$ and $\lambda^* \geq 0$ such that $\Pi(\theta) \geq \xi^*\theta - \lambda^*$, for all θ. By Lemma 3.2, $q(\theta) \geq \frac{1}{\theta^*}\int_0^{\theta^*} q(\tau)d\tau + \xi^* \ln \left(\frac{\theta}{\theta^*}\right)$, for all θ, where $\theta^* = \lambda^*/\xi^*$. In particular, since $q(1) \leq 1$, we have that $\xi^* \ln \left(\frac{1}{\theta^*}\right) \leq 1$. Therefore, if $q^*(\theta) = \max \left\{\xi^* \ln \left(\frac{\theta}{\theta^*}\right), 0\right\}$, then $q^*(1) \leq 1$ and

$$\Pi(\theta) \geq \Pi^*(\theta) := \theta q^*(\theta) - \int_0^\theta q^*(\tau)d\tau = \max \left\{\xi^*(\theta - \theta^*), 0\right\},$$

i.e., the allocation $q^*(\cdot)$ is feasible allocation and attains the lower envelope profit, which proves the result. Finally, notice that the robust mechanism, the constraint $\xi^* \ln \left(\frac{1}{\theta^*}\right) \leq 1$ should bind, which implies that $\theta^* > 0$, once $\xi^* > 0$. Hence, $\lambda^* > 0$. □

Proposition 3.1 shows that a robust mechanism imposes restrictions on payoff levels. Only knowing the first moment of the distribution from which the consumer's valuation is drawn, the best the seller can do to insure against ambiguity is to design a mechanism that induces profits which are, conditional on sales, linear in valuations. Under the interpretation that, ex-ante, the seller knows the consumer's expected willingness to pay, but the latter acquires additional information after the mechanism is offered, the convexity (or call-option format) of the profit function is the seller's optimal response to the added volatility in the consumer's valuation stemming from the information acquisition stage.

3.2 Optimal mechanism

Letting $F^* \in \mathcal{F}$ be a worst-case distribution, from Proposition 3.1, the seller's expected profits $\Pi^*(\theta)$ at a robust mechanism is

$$\int \Pi^*(\theta)\, dF^*(\theta) = \int \max\{\xi^*\theta - \lambda^*, 0\}\, dF^*(\theta) = \xi^*[k - \theta^*],$$

where $\theta^* = \lambda^*/\xi^*$ is the marginal type – i.e., the type such that the mechanism prescribes sales for all $\theta > \theta^*$. Moreover, at a robust mechanism $q^*(\cdot)$,

$$\Pi^*(\theta) \equiv \theta q^*(\theta) - \int_0^\theta q^*(\tau)\, d\tau = \xi^*\theta - \lambda^*, \text{ for all } \theta \in [\theta^*, 1].$$

Differentiating the above condition, we get

$$\theta \frac{dq^*}{d\theta}(\theta) = \xi^*, \text{for all } \theta \in [\theta^*, 1]$$

which implies that

$$q^*(\theta) = \xi^* \ln\left(\frac{\theta}{\theta^*}\right), \text{ for all } \theta \in [\theta^*, 1].$$

The monopolist's problem can then be simplified to

$$\max_{\theta^* \in [0,1], \xi^* \geq 0} \xi^*[k - \theta^*] \tag{4}$$

subject to

$$\xi^* \ln\left(\frac{\theta}{\theta^*}\right) \leq 1, \text{ for all } \theta \in (0,1]. \tag{5}$$

For the next proposition let us consider the (implicit) solution $\tilde{k} \in (0, k)$ of the following equation:[10]

$$\tilde{k}\left(1-\ \ln \tilde{k}\right)=k. \tag{6}$$

Then, the solution to problem (4) yields:

Proposition 3.2. *(Robust mechanism) The optimal robust allocation is given by*

$$q^*(\theta)=\begin{cases}0, & \text{if } \theta\le\widetilde{k}\\ 1-\frac{\ln\theta}{\ln\widetilde{k}}, & \text{if } \theta\ge\widetilde{k}\end{cases}$$

and the robust profit is given by $\tilde{k}$.

Proof of Proposition 3.2. Notice that, at optimal solution, the constraint (5) of problem (4) must be binding exactly at $\theta=1$, i.e., $\xi^*=-1/\ \ln\theta^*$. The problem then simplifies to

$$\max_{\theta^*\in[0,1]}\varphi(\theta^*),$$

where $\varphi(\theta^*)=\frac{\theta^*-k}{\ln\theta^*}$. The first-order condition amounts to $\varphi'(\theta^*)=\frac{1}{\ln\theta^*}+\frac{k-\theta^*}{\theta^*(\ \ln\theta^*)^2}=0$ which is equivalent

$$\theta^*(1-\ \ln\theta^*)=k.$$

Since φ is strictly concave (φ' is strictly decreasing), this last equation has a unique solution, which we call $\tilde{k}$, and is the solution of our maximization problem. Now the expected profit is

$$\xi^*[k-\theta^*]=-\frac{1}{\ln\tilde{k}}[k-\tilde{k}]=\tilde{k},$$

which concludes the proof. □

While, as in standard Bayesian selling mechanisms, there are no distortions at the top ($q^*(1)=1$), the robust mechanism entails sales with probability smaller than one for all valuations $\theta<1$. Hence, the mechanism distorts the allocation at the intensive margin. Although coming from a different source (uncertainty, rather than the curvature of payoffs as in Bayesian settings), the reason for this distortion is to price discriminate different consumer types. Considering the worst-case scenario, the seller will find it desirable to sell to consumers with low valuations. If he was, however, to sell with probability one to them, the amount he would be able to charge from infra marginal consumers would be small. By selling with probability smaller than one to low valuation consumers, the seller can charge more from infra marginal buyers. Price discrimination is the way by which the seller simultaneously insures against uncertainty and charges high prices from infra marginal consumers.

[10]From the proof of Proposition 3.2, it is clear that $\tilde{k}$ exists and is unique.

Implementation

There are many ways to implement the allocation in Proposition 3.2. Perhaps the most immediate one is through a non-linear tariff. In fact, for

$$q^*(\theta) = \begin{cases} 0, & \text{if } \theta \leq \widetilde{k} \\ 1 - \frac{\ln(\theta)}{\ln(\widetilde{k})}, & \text{if } \theta \geq \widetilde{k} \end{cases}. \tag{7}$$

Let

$$p^*(\theta) = \theta q^*(\theta) - \int_0^\theta q^*(\tau)\, d\tau, \tag{8}$$

we can then make use of the taxation principle to implement the robust direct mechanism through an indirect mechanism $(q, P^*(q))$ with $P^*(q) = p^*(\theta)$ for $q = q^*(\theta)$.

A more interesting way – and that explicitly uses the fact that the consumer's payoff is linear – to implement the robust mechanism is, however, through a *distribution* of posted prices. Indeed, notice that the direct mechanism calls for a consumer of type θ to be assigned assigned the good with probability $q^*(\theta)$. At any given price p, the consumer will buy if and only if $p \leq \theta$. Now, assume that the seller commits to posting a price $p \in \left[\widetilde{k}, 1\right]$ drawn from the cumulative distribution

$$q^*(p), \text{ for all } p \in \left[\widetilde{k}, 1\right],$$

with $q^*(\cdot)$ from equation (7). It is easy to see that, if prices are drawn from $q^*(p)$, the probability that a consumer of valuation θ buys is exactly $q^*(\theta)$. Hence, we have:

Proposition 3.3. *(Implementation) Committing to posting a price drawn from the distribution* $q^*(p)$*, for all* $p \in \left[\widetilde{k}, 1\right]$*, is a robust selling mechanism.*

3.3 Robust mechanism as a Nash equilibrium

An alternative to the approach that makes use of Lagrangian techniques is to recast, as Bergemann and Schlag (2008) do in their minimax pricing problem, the robust design problem in terms of a zero-sum game between the monopolist and an adversarial nature.

In such game, the monopolist's von-Neumann-Morgenstern utility is $\Pi(\theta)$, whereas nature's is $-\Pi(\theta)$; the monopolist chooses incentive compatible mechanisms in $\mathcal{M}$ and nature selects distributions in $\mathcal{F}$.

As argued in Bergemann and Schlag (2008), if (m^*, F^*) is a Nash equilibrium of such game, then m^* is a robust mechanism and F^* is a worst-case distribution. Consider the density function on $[0,1]$:

$$f^*(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \widetilde{k}) \\ \frac{\widetilde{k}}{\theta^2}, & \text{if } \theta \in [\widetilde{k}, 1] \end{cases}. \tag{9}$$

The following proposition shows that this density characterizes the absolutely continuous part of the distribution F^*:

Proposition 3.4. *(Nash equilibrium) Let $m^* = (q^*, p^*)$ be the mechanism characterized by (7) and (8) and the distribution F^* with absolutely continuous part described by (9) and singular part characterized by the Dirac measure at $\theta = 1$ with mass of $\widetilde{k}$. Then, (m^*, F^*) is the unique Nash equilibrium of the zero-sum game played by the nature and the monopolist.*

Proof of Proposition 3.4. 1) Characterization. We start by guessing that the cumulative distribution F^* of the worst-case measure has a density f^*, except possibly at $\theta = 1$ where it may have a mass point. Notice that, for any implementable allocation $q(\cdot)$, integration by parts yields

$$\begin{aligned} &\int_0^1 \left[\theta q(\theta) - \int_0^\theta q(\tau)\, d\tau\right] f^*(\theta)\, d\theta \\ = &\int_0^1 \left[\theta - \frac{F^*_-(1) - F^*(\theta)}{f^*(\theta)}\right] q(\theta) f^*(\theta)\, d\theta + q(1) - \int_0^1 q(\tau)\, d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^1 \left[\theta q(\theta) - \int_0^\theta q(\tau)\, d\tau\right] dF^*(\theta) \\ = &\int_0^1 \left[\theta - \frac{F^*_-(1) - F^*(\theta)}{f^*(\theta)}\right] q(\theta) f^*(\theta)\, d\theta + \left(1 - F^*_-(1)\right) \left[q(1) - \int_0^1 q(\tau)\, d\tau\right] \\ = &\int_0^1 \left[\theta - \frac{1 - F^*(\theta)}{f^*(\theta)}\right] q(\theta) f^*(\theta)\, d\theta + \left(1 - F^*_-(1)\right) q(1). \end{aligned}$$

By making

$$\theta - \frac{1 - F^*(\theta)}{f^*(\theta)} = 0, \tag{10}$$

nature guarantees that the seller will be indifferent among any incentive compatible mechanism with $q(1) = 1$. In particular, the one in equation (7) is a best reply by the monopolist if equation

(10) is satisfied. Solving (10) amounts to solving

$$\frac{d}{d\theta}\left[\theta F^*(\theta)\right] = 1,$$

one then has

$$F^*(\theta) = 1 - \frac{a}{\theta},$$

and $f^*(\theta) = \frac{a}{\theta^2}$, for all $\theta \in [a, 1]$, where $a = 1 - F^*_-(1)$. Now, we know that $F^* \in \mathcal{F}$ and hence

$$k = \int_0^1 \theta dF^*(\theta) = \int_a^1 \theta f^*(\theta)\, d\theta + 1 - F^*_-(1),$$

which implies that $k = a \int_a^1 \frac{d\theta}{\theta} + a$, or $k = a(1 - \ln\, a)$, which implies that $a = \widetilde{k}$.

The discussion that precedes Proposition 3.1 and the derivation of the mechanism in (7) Proposition 3.2 establishes that nature is indifferent among any distribution in $\mathcal{F}$; if the monopolist chooses the mechanism in equations (7) and (8). Hence, F^* is a nature's best response.

2) Uniqueness. Given the Nash equilibrium $((q^*, p^*), F^*)$ characterized above, for all $F \in \mathcal{F}$

$$\tilde{k} = \int \Pi^*(\theta)dF^*(\theta) \leq \int \Pi^*(\theta)dF(\theta),$$

and for all non-decreasing positive allocation $q(\cdot)$ such that $q(1) \leq 1$

$$\int \Pi(\theta)dF^*(\theta) = \tilde{k}\left(q(1) - \int_0^{\tilde{k}} q(\tau)\, d\tau\right) \leq \tilde{k}, \tag{11}$$

where $\Pi(\theta) = \theta q(\theta) - \int_0^\theta q(\tau)d\tau$.

Suppose that there exists another Nash equilibrium $((\bar{q}, \bar{p}), \bar{F})$. The mechanism q^* assures at least an expected profit of $\tilde{k}$ and therefore the mechanism $\bar{q}$ gives

$$\int \bar{\Pi}(\theta)d\bar{F}(\theta) \geq \tilde{k},$$

where $\bar{\Pi}(\theta) = \theta\bar{q}(\theta) - \int_0^\theta q(\tau)d\tau$. On the other hand, since the nature is minimizing the expected profit, it cannot attain a payoff lower than $\tilde{k}$ when deviating to distribution F^*. Then, using (11) we must have

$$\int \bar{\Pi}(\theta)dF^*(\theta) = \tilde{k}\left(\bar{q}(1) - \int_0^{\tilde{k}} \bar{q}(\tau)\, d\tau\right) = \tilde{k}.$$

We then necessary have $\bar{q}(1) = 1$ and $\bar{q}\left(\tilde{k}_-\right) = 0$. Hence, by continuity $\bar{\Pi}(\tilde{k}) = 0$ and, by Proposition 3.1, $\tilde{k} \leq \bar{k}$, where $\bar{I} = [\bar{k}, 1]$ contains the support of $\bar{F}$. This implies that $\bar{F}_-\left(\tilde{k}\right) = 0$. If $x\left(1 - \bar{F}(x)\right) > \tilde{k}$, then the mechanism

$$q(\theta) = \begin{cases} 0 & \text{se } \theta \leq x \\ 1 & x < \theta \leq 1 \end{cases},$$

gives profit $\int \left(\theta q(\theta) - \int_0^\theta q(\tau)\, d\tau\right) d\bar{F}(\theta) = \int_x^1 x d\bar{F}(x) = x\left(1 - \bar{F}(x)\right) > \tilde{k}$. Hence, $x\left(1 - \bar{F}(x)\right) \leq \tilde{k}$, for all x. Now

$$k = \int_0^1 \left(1 - \bar{F}(x)\right) dx = \tilde{k} + \int_{\tilde{k}}^1 \left(1 - \bar{F}(x)\right) dx \leq \tilde{k} + \int_{\tilde{k}}^1 \frac{\tilde{k}}{x} dx = \tilde{k} + \tilde{k} \ln\left(\frac{1}{\tilde{k}}\right) = k.$$

Therefore, $x\left(1 - \bar{F}(x)\right) = \tilde{k}$, for all $x > \tilde{k}$, and hence $\bar{F} = F^*$.

From Proposition 3.1, there exist $\bar{\xi} > 0$ and $\bar{\lambda} \geq 0$ such that $\bar{\Pi}(\theta) \geq \bar{\xi}\left(\theta - \bar{k}\right)$, for all θ, where $\bar{k} = \frac{\bar{\lambda}}{\bar{\xi}}$. By Lemma 3.2, we have

$$\bar{q}(\theta) \geq \frac{\int_0^{\bar{k}} \bar{q}(\tau)\, d\tau}{\bar{k}} + \bar{\xi} \ln\left(\frac{\theta}{\bar{k}}\right), \text{ for all } \theta.$$

In particular, $1 \geq \bar{\xi} \ln\left(\frac{1}{\bar{k}}\right)$. Since,

$$\bar{\xi}\left(k - \bar{k}\right) \leq \frac{k - \bar{k}}{\ln\left(\frac{1}{\bar{k}}\right)} < \frac{k - \tilde{k}}{\ln\left(\frac{1}{\tilde{k}}\right)} = \tilde{k}$$

if $\bar{k} \neq \tilde{k}$ we conclude that $\bar{k} = \tilde{k}$. Hence, $\bar{q}(\theta) \geq q^*(\theta)$, for all θ. Let $dG = a\delta_{\tilde{k}} + (1-a)\,\delta_1$ where $a = \frac{1-k}{1-\tilde{k}}$ for $\int \theta dG(\theta) = k$ and δ_θ is the Dirac measure at θ. Then,

$$\begin{aligned} \int \bar{\Pi}(\theta)\, dG(\theta) &= (1-a)\left(\bar{q}(1) - \int_0^1 \bar{q}(x)\, dx\right) + a\tilde{k} = (1-a)\left(1 - \int_{\tilde{k}}^1 \bar{q}(x)\, dx\right) + a\tilde{k} \\ &\leq (1-a)\left(1 - \int_{\tilde{k}}^1 q^*(x)\, dx\right) + a\tilde{k} = \int \tilde{\Pi}(\theta)\, dH(\theta) = \tilde{k}. \end{aligned}$$

Therefore, $\bar{q}(x) = q^*(x)$ almost surely and then $\bar{q} = q^*$. □

3.4 Arbitrary moment condition

In this section, we show that the above analysis extends to the case in which all the seller knows is an arbitrary moment condition. Consider a continuously differentiable function $\kappa : \mathbb{R} \to \mathbb{R}$ satisfying $\kappa' (\cdot) > 0$,[11] and assume that all the seller knows is that $\int \kappa (\theta) \, dF (\theta) = k$. In this case, the set of feasible distributions is

$$\mathcal{F} = \left\{ F \in \Delta [0,1] ; \int \kappa (\theta) \, dF (\theta) = k \right\}, \tag{12}$$

for some fixed $k \in (\kappa (0) , \kappa (1))$.

We derive the robust mechanism by constructing a strategy profile of the zero sum game between nature and the monopolist and then showing that it is a Nash equilibrium.

Distribution

Define the distribution $F^{\underline{k}}$ as

$$F^{\underline{k}} (\theta) \equiv \begin{cases} 0, & \text{if } \theta < \underline{k}, \\ 1 - \frac{\underline{k}}{\theta}, & \text{if } \theta \in [\underline{k}, 1) , \\ 1, & \text{if } \theta = 1. \end{cases}$$

A higher $\underline{k}$ leads to a first-order stochastic increase in the distribution $F^{\underline{k}}$. Hence, the function that takes $\underline{k}$ to $\int \kappa (\tau) \, dF^{\underline{k}} (\tau)$ is continuous and strictly increasing. Also, it converges to $f(i)$ when $\underline{k} \to i$, for $i \in \{0, 1\}$. By the intermediate value theorem, there is a unique point $\tilde{k} \in (0, 1)$ that satisfies

$$\int \kappa (s) \, dF^{\tilde{k}} (s) = k.$$

Let us denote $F^* = F^{\tilde{k}}$.

Mechanism

Define $\chi^a : \left[\tilde{k}, 1\right] \to \mathbb{R}$ as follows[12]

$$\chi^a (\theta) \equiv \begin{cases} 0, & \text{if } \theta < \tilde{k}, \\ 1 - a \int_\theta^1 \frac{\kappa'(\tau)}{\tau} d\tau, & \text{if } \theta \geq \tilde{k}. \end{cases}$$

[11] The argument can be extended to the more general case of bounded non-decreasing functions. Therefore, our analysis encompasses, as special cases, not only knowledge of, for example, an n-th moment – $\kappa (\theta) = \theta^n$, for $n \geq 1$ – but also of arbitrary percentiles – $\kappa (\theta) = \mathbf{1}_{(\theta_0, 1]}$.

[12] The function χ^a solves the differential equation $\theta q' (\theta) = a\kappa' (\theta)$, with final condition $q (1) = 1$.

Simple derivations imply that the profit of the seller implied by the allocation χ^a is given by

$$\Pi^a(\theta) = \begin{cases} 0, & \text{if } \theta < \tilde{k} \\ \chi^a(\tilde{k}) + a\left[\kappa(\theta) - \kappa(\tilde{k})\right], & \text{if } \theta \geq \tilde{k}. \end{cases}$$

Define $\tilde{a} \equiv \left[\int_{\tilde{k}}^{1} \frac{\kappa'(\tau)}{\tau} d\tau\right]^{-1}$ and let $\Pi^* \equiv \Pi^{\tilde{a}}$, $q^* = \chi^{\tilde{a}}$, transfer p^* as

$$p^*(\theta) \equiv \theta q^*(\theta) - \int_0^\theta q^*(\tau)\,d\tau,$$

and mechanism $m^* = (q^*, p^*)$. This mechanism leads to a profit function that is continuous and satisfies

$$\Pi^*(\theta) = \tilde{a}\left[\kappa(\theta) - \kappa(\tilde{k})\right].$$

The next two lemmata provide a characterization of a Nash equilibrium of this game.

Lemma 3.3. *(Nature's problem) The robust distribution F^* solves the problem*

$$\min_{F\in\mathcal{F}} \int \Pi^*(\theta) dF(\theta).$$

Proof of Lemma 3.3. Notice that $\Pi^*(\theta) = \tilde{a}\left[\kappa(\theta) - \kappa(\tilde{k})\right]$ if $\theta \geq \tilde{k}$ and $\Pi^*(\theta) > \tilde{a}\left[\kappa(\theta) - \kappa(\tilde{k})\right]$ if $\theta < \tilde{k}$. Hence, it follows that

$$\int \Pi^*(\theta)\,dF(\theta) \geq \int \tilde{a}\left[f(\theta) - f(\tilde{k})\right] dF(\theta) = \tilde{a}\left[k - \kappa(\tilde{k})\right],$$

for any $F \in \mathcal{F}$. The first inequality holds as an equality if and only if $supp\,(F) \subseteq \left[\tilde{k}, 1\right]$.

The following lemma shows that m^* solves the seller's problem. □

Lemma 3.4. *(Seller's problem) The robust mechanism m^* solves the problem*

$$\max_{m=(q,p)\in\mathcal{M}} \int \Pi_m(\theta)\,dF^*(\theta),$$

subject to $p(\theta) = \Pi_m(\theta) \equiv \theta q(\theta) - \int_0^\theta q(\tau)\,d\tau$ and $q(\cdot)$ is non-decreasing.

Proof of Lemma 3.4. Notice that for any incentive compatible mechanism $m = (q, p) \in \mathcal{M}$

$$p(\theta) \equiv \theta q(\theta) - \int_0^\theta q(\tau)\,d\tau,$$

which, using the distribution F^*, leads to

$$\int p(\theta)\, dF^*(\theta) = (1 - F^*(1))\, q(1).$$

Any feasible mechanism with $q(1) = 1$ maximizes revenue. $\square$

These two results lead immediately to the following result.

Proposition 3.5. *(Robust mechanism - arbitrary moment) The mechanism* $m^* = (q^*, p^*)$ *solves the robust revenue maximization problem with moment condition* $\mathbb{E}\left[\kappa\left(\tilde{\theta}\right)\right] = k$.

4 One period and M goods

We now move to the case in which the monopolist can sell $M > 1$ goods. For the case in which $M > T = 1$, the seller has to design a mechanism $m = (\boldsymbol{q}, p) \in \mathcal{M}$, where $\boldsymbol{q} = (q^1, ..., q^M)$, to maximize his worst-case expected profits under all cumulative probability distributions on $[0,1]^M$ with the vector of mean $\boldsymbol{k} = (k^1, ..., k^M) \in [0,1]^M$, i.e., the set of possible distributions is given by

$$\mathcal{F} \equiv \left\{ F \in \Delta\left([0,1]^M\right); \int \theta^i dF(\theta) = k^i, \text{ for } i = 1\ldots, M \right\}.$$

The seller's problem reads

$$\max_{m\in\mathcal{M}} \min_{F\in\mathcal{F}} \int p(\theta)\, dF(\theta)$$

subject to participation constraint

$$\mathcal{U}_m(\theta) := \theta \cdot \boldsymbol{q}(\theta) - p(\theta) \geq 0,$$

and incentive compatibility constraint

$$\mathcal{U}_m(\theta) \geq \theta \cdot \boldsymbol{q}(\widehat{\theta}) - p(\widehat{\theta}),$$

for all $\theta, \widehat{\theta} \in [0,1]^M$. We refer to a mechanism satisfying these as a feasible mechanism.

It is standard to show that a mechanism $m = (\boldsymbol{q}, p)$ is incentive compatible if and only if

$$\nabla\mathcal{U}_m(\theta) = \boldsymbol{q}(\theta) \text{ for a.e. } \theta \in [0,1]^M \tag{13}$$

and

$$\mathcal{U}_m(\theta) \text{ is convex,}$$

where ∇ represents the gradient symbol.

Using equation (13), one can write incentive compatible payments made to the seller as the difference between total surplus and the buyer's utility:

$$p(\theta) = \theta \cdot \boldsymbol{q}(\theta) - \mathcal{U}_m(\theta) = \theta \cdot \nabla \mathcal{U}_m(\theta) - \mathcal{U}_m(\theta). \tag{14}$$

Therefore, each feasible mechanism $m = (\boldsymbol{q}, p)$ can be associated to a non-negative convex function rent $\mathcal{U}_m$ that satisfies (13). Reciprocally, given a non-negative convex function rent $\mathcal{U}_m$ that satisfies (13), then the mechanism $m = (\boldsymbol{q}, p)$ satisfying (13) and (14) is feasible.

Plugging this is in the seller's objective we get

$$\max_{\{\mathcal{U}_m(\theta)\geq 0 \text{ and convex}\}} \min_{F\in\mathcal{F}} \int \underbrace{[\theta \cdot \nabla \mathcal{U}_m(\theta) - \mathcal{U}_m(\theta)]}_{\Pi_m(\theta)} dF(\theta).$$

It is convenient to derive a lower bound for the expected profits the seller can get when facing an adversarial nature. Toward that, let$\tilde{k}^i$ be the unique solution of the equation

$$\tilde{k}^i\left(1 - \ln \tilde{k}^i\right) = k^i, \tag{15}$$

for all $i = 1, ..., M$ and define $\tilde{k}_A = \sum_{i=1}^{M} \tilde{k}^i$. We then have:

Proposition 4.1. *The seller can guarantee at least expected profits $\tilde{k}_A$ in a robust mechanism (i.e. $\tilde{k}_A$ is a lower bound for the maxmin value of the zero sum game between the seller and the nature).*

Proof of Proposition 4.1. Let $m^* = (\boldsymbol{q}^*, p)$ be a mechanism such that the robust mechanism where q^{i*} and p are:

$$q^{i*}(\theta) = \begin{cases} 0, & \text{if } \theta^i < \widetilde{k}^i \\ 1 - \frac{\ln \theta^i}{\ln \widetilde{k}^i}, & \text{if } \theta^i \geq \widetilde{k}^i, \end{cases}$$

for $i = 1, ..., M$, and the price is defined by (13) and (14). Because of (13), we have that

$$\mathcal{U}_{m^*}(\theta) = \sum_{i=1}^{M} U_i^*(\theta^i),$$

where $U_i^*(\theta^i) = \int_0^{\theta_i} q^{*i}(\tau)d\tau$. Therefore,

$$\Pi_{m^*}(\theta) = \sum_{i=1}^{M} \Pi_i^*(\theta^i).$$

Let $F \in \mathcal{F}$ any distribution. We have that

$$\int \Pi_{m^*}(\theta) dF(\theta) = \sum_{i=1}^{M} \int \Pi_i^*(\theta^i) dF_i(\theta^i),$$

where $\Pi_i^*(\theta^i) = \theta^i q^{*i}(\theta^i) - U_i^*(\theta^i)$ and $dF_i(\theta^i) = \int_{\theta^{-i}} dF(\theta^i, \theta^{-i})$ is the marginal distribution. Let F_i^* be the robust distribution of the one good case with mean k^i characterized by proposition the characterizes the Nash equilibrium in the one good case when $k = k^i$. Since the marginal distribution $F_i(\theta^i)$ has mean k^i and by robustness of the mechanism in the one good case we have

$$\int \Pi_i^*(\theta^i) dF_i(\theta^i) \geq \int \Pi_i^*(\theta^i) dF_i^*(\theta^i) = \tilde{k}^i.$$

Therefore,

$$\int \Pi_{m^*}(\theta) dF(\theta) \geq \sum_{i=1}^{M} \tilde{k}^i = \tilde{k}_A.$$

This implies that by selling the goods separately, the seller can guarantee payoff at least $\tilde{k}_A$. That is, the maxmin value of the the game is at least $\tilde{k}_A$. □

Proposition 4.1is a simple implication of the fact that the seller always has the option to sell each of the goods separately. In such case, he is bounded to obtain as expected profits at least the sum of the single-good worst-case expected profits derived in Proposition 3.4. Put differently, nature's equilibrium payoff is bounded below by $\tilde{k}_A$. What we now show is that nature can construct distributions that attain such bound, so that $\tilde{k}_A$ is the value of the zero sum game it plays with the seller. We split the analysis in two separate cases. First, we consider the case where $(k^1, ..., k^M) = k\,(1, ...1)$. Then, using the insight of such symmetric case, we consider the case of general $(k^1, ..., k^M)$.

4.1 Nash equilibrium in the symmetric case

Consider the following distribution

$$F^*(\theta) \equiv G(\min_{1 \leq i \leq M} \theta^i),$$

where

$$G(\theta) \equiv \begin{cases} 0, & \text{if } \theta \in \left[0, \tilde{k}\right), \\ 1 - \frac{\tilde{k}}{\theta}, & \text{if } \theta \in [\tilde{k}, 1) \\ 1 & \text{if } \theta = 1 \end{cases}$$

and $\tilde{k}$ is characterized by (6).

This distribution leads to perfectly correlated types. Notice, moreover, that the marginal distribution of each coordinate is equal to the distribution derived in Proposition 3.4, for the one dimensional case. Also define $m^* = (\mathbf{q}^*, p^*)$ as the following

$$p^*(\theta) = \sum_{i=1}^{M} \left[\theta_i q^{i*}(\theta) - \int_0^{\theta^i} q^{i*}\left([\theta, s]^i\right) ds \right], \tag{16}$$

where

$$q^{i*}(\theta) \equiv \begin{cases} 1 - \frac{\ln \theta^i}{\ln \tilde{k}}, & \text{if } \theta^i \geq \tilde{k} \\ 0, & \text{if } \theta^i < \tilde{k}, \end{cases} \tag{17}$$

$$[\theta]^i \equiv \left(\theta^1, \ldots, \theta^i, 0, \ldots, 0\right),$$

and

$$[\theta, s]^i \equiv \left(\theta^1, \ldots, \theta^{i-1}, s, 0, \ldots, 0\right)$$

are specific truncations of the vector θ

In the following we will show that (m^*, F^*) is a Nash equilibrium of the zero sum game between nature and the seller.

Lemma 4.1. *(Distribution optimality) The optimal distribution F^* solves*

$$\min_{F \in \mathcal{F}} \int p^*(\theta)\, dF(\theta).$$

Proof of Proposition 4.1. Using (16) and (17) we have that

$$p^*(\theta) = -\sum_{i=1}^{M} \frac{\left(\theta^i - \tilde{k}\right)^+}{\ln \tilde{k}}.$$

Notice that, for any $\theta \in [0,1]^M$

$$p^*(\theta) \geq -\sum_{i=1}^{M} \frac{\theta^i - \tilde{k}}{\ln \tilde{k}},$$

and the above holds as an inequality if $\theta \notin \times_{i=1}^{M} \left[\tilde{k}, 1\right]$.

Also, for any $F \in \mathcal{F}$ we have that

$$\int p^*(\theta) dF(\theta) \geq -\int \sum_{i=1}^{M} \frac{\theta^i - \tilde{k}}{\ln \tilde{k}} dF(\theta) = -\sum_{i=1}^{M} \frac{k - \tilde{k}}{\ln \tilde{k}} = M\tilde{k}.$$

The first inequality holds as an equality if $supp\,(F) \subseteq \times_{i=1}^{M} \left[\tilde{k}, 1\right]$ and as a strict inequality otherwise. Finally, notice that $F^* \in \mathcal{F}$ and $supp\,(F^*) = \times_{i=1}^{M} \left[\tilde{k}, 1\right]$. □

We now show that, given distribution $F^* \in \mathcal{F}$, the seller finds it optimal to choose mechanism m^*.

Lemma 4.2. *(Mechanism optimality) The mechanism m^* solves the revenue maximization problem defined by F^**

$$m^* \in \arg\max_{m \in \mathcal{M}} \int p(\theta)\, dF^*(\theta).$$

Proof of Proposition 4.2. Notice that using incentive constraints relative to types the diagonal vectors $\theta^M := \theta\mathbf{1}$, $\theta'^M := \theta'\mathbf{1} \in [0,1]^M$, where θ and θ' are scalar now with some abuse of notation, we have that

$$\sum_i \theta q_i\left(\theta^M\right) - p\left(\theta^M\right) = -p\left(0^M\right) + \int_0^{\theta} q^A(s)\, ds,$$

where $q^A(s) \equiv \sum_i q_i\left(s^M\right)$. Hence, through standard arguments we can rewrite profits as

$$\int \left[\theta q^A(\theta) + p\left(0^M\right) - \int_0^{\theta} q^A(s)\, ds\right] dG(\theta) = \int q^A(\theta)\left[\theta - (1 - G(\theta))\right] dG(\theta) + p\left(0^M\right)$$

which is equal to

$$(1 - G(1))\, q^A(1) + p\left(0^M\right).$$

Taking into account the constraint $q^A(1) \leq M$ and $p\left(0^M\right) \leq 0$, we have that the mechanism is indeed optimal. □

It follows from the above results that selling each of the goods separately is an optimal robust mechanism. Rather than selling each good separately, the seller could bundle the goods. Somewhat surprisingly, the seller can fare equally well by selling each of the goods in fixed proportions in a bundle. In fact, let $\widetilde{\theta}, \xi \in [0,1]^M$and consider the allocation

$$\boldsymbol{q}(\theta) = \begin{cases} 0, & \text{if } \xi \cdot \left(\theta - \widetilde{\theta}\right) < 0 \\ \ln\left(\frac{\xi \cdot \theta}{\xi \cdot \widetilde{\theta}}\right)\xi, & \text{if } \xi \cdot \left(\theta - \widetilde{\theta}\right) \geq 0 \end{cases} \tag{18}$$

coupled with prices that satisfy equation 14. By construction, such mechanism is incentive compatible. It can be readily seen that induces profits equal to $\Pi(\theta) = \max\left\{\xi . \left(\theta - \widetilde{\theta}\right), 0\right\}$. Moreover, by similar arguments to the ones used in Section 3.1, if facing shadow costs ξ on the constraints that impose $\int \theta^i dF(\theta) = k^i$, for $i = 1, ..., M$, and shadow cost $\lambda = \xi \cdot \widetilde{\theta}$ on the constraint that $\int dF(\theta) = 1$, Nature is willing to choose any distribution F on θ satisfying $\xi \cdot \left(\theta - \widetilde{\theta}\right) \geq 0$. If F^* is one of such distributions, the seller's worst-case profits under the described mechanism is

$$\int \Pi(\theta)\, dF^*(\theta) = \int \xi \cdot \left(\theta - \widetilde{\theta}\right) dF^*(\theta) = \xi \cdot \left[\mathbf{k} - \widetilde{\theta}\right].$$

Therefore, the best among all such mechanism solves

$$\max_{\xi, \widetilde{\theta} \geq 0} \xi \cdot \left[\mathbf{k} - \widetilde{\theta}\right] \tag{19}$$

subject to

$$\ln\left(\frac{\xi \cdot \theta}{\xi \cdot \widetilde{\theta}}\right) \xi \leq \mathbf{1}, \text{ for all } \theta,$$

where $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^M$ is the vector formed by "1" in all entries.

The next proposition characterizes the optimal mechanism in such class.

Proposition 4.2. *(Full bundling) The solution to problem (19) entails full bundling in fixed proportions:*

$$q^i(\theta) = \begin{cases} 0, & \textit{if } \mathbf{1} \cdot \theta \leq M\theta^* \\ \frac{\ln\left(M\tilde{k}_M\right) - \ln \bar{\theta}}{\ln \tilde{k}_M} \mathbf{1}, & \textit{if } \mathbf{1} \cdot \theta > M\theta^* \end{cases}$$

where $\theta^* = -\frac{M\tilde{k}_M}{\ln \tilde{k}_M}$ and $\tilde{k}_M$ is the unique solution of (6) for $k = \frac{\mathbf{1} \cdot \boldsymbol{k}}{M}$. Moreover, the robust profit is given by $M.\tilde{k}_M$.

Proof. Defining $\eta = \xi \cdot \mathbf{1}$ and $\theta^* = \xi \cdot \tilde{\theta}$, the problem (19) is then equivalent to

$$\begin{aligned} &\max_{\eta, \xi, \theta^* \geq 0} \xi \cdot \boldsymbol{k} - \theta^* \\ &s.t.\ \mathbf{1} - (\ln \eta - \ln \theta^*)\, \xi \geq 0 \\ &\qquad \eta - \xi \cdot \mathbf{1} \geq 0. \end{aligned}$$

Let $a \in \mathbb{R}^M_+$ and $b \geq 0$ be the Lagrangian multipliers of the constraints. The following first-order conditions are necessary for optimality:[13]

$$\begin{cases} -\frac{\xi \cdot a}{\eta} + b \leq 0 \\ -1 + \frac{\xi \cdot a}{\theta^*} \leq 0 \\ \boldsymbol{k} - (\ln \eta - \ln \theta^*)\, a - b\mathbf{1} \leq 0 \end{cases}$$

and the usual slackness conditions.

Guessing (and then verifying) that the first-order conditions and the first slackness conditions are binding, we have that $\theta^* = \xi \cdot a = b\eta$,

$$a = \frac{1}{\ln \eta - \ln \theta^*}(\boldsymbol{k} - b\mathbf{1}) \text{ and } \xi = \frac{1}{\ln \eta - \ln \theta^*}\mathbf{1}.$$

Hence,

$$\eta = \xi \cdot \mathbf{1} = \frac{M}{\ln \eta - \ln \theta^*} \text{ and } \theta^* = \xi \cdot a = \frac{\sum_{i=1}^{M} k^i - bM}{(\ln \eta - \ln \theta^*)^2} = b\eta = \frac{bM}{\ln \eta - \ln \theta^*}.$$

This implies that

$$\frac{\bar{\boldsymbol{k}}}{b} - 1 = \ln \eta - \ln \theta^* = \frac{M}{\eta}.$$

Hence, $\theta^* = \eta \exp\,[-M/\eta]$. However, $\theta^* = b\eta$ implies that $b = \exp\,[-M/\eta]$. Plugging this back into the previous expression, we get

$$b(1 - \ln b) = \frac{\mathbf{1} \cdot \boldsymbol{k}}{M},$$

which gives a unique solution for b, which pins down the values of θ^* and ξ:

$$\theta^* = -\frac{Mb}{\ln b} \text{ and } \xi = -\frac{1}{\ln b}\mathbf{1}.$$

Notice that substituting these values in (18) we get the expression of the optimal allocation and the optimal profit is given by

$$\xi \cdot \boldsymbol{k} - \theta^* = -\frac{\boldsymbol{k} \cdot \mathbf{1}}{\ln b} + \frac{Mb}{\ln b} = \frac{Mb - \mathbf{1} \cdot \boldsymbol{k}}{\ln b} = Mb.$$

□

[13]Notice that, for each $\eta \geq 0$, the Lagrangian is a concave functional of (ξ, θ^*). Therefore, at the optimum choice of η, the first-order and slackness conditions for (ξ, θ^*) are sufficient for optimality. We are deriving the necessary conditions of Lagrangian when the optimum $\eta$is interior. For sufficiency, we only have to discard corner solution for η. However, an easy inspection shows that the optimum ηmust be interior.

As an immediate corollary of the above result, we have:

Proposition 4.3. *(Optimal full bundling) A mechanism that entails sales of all goods in the same proportion (full bundling) attains the same worst-case expected profits for the monopolist as a mechanism that sells the goods in a fully separable fashion. Both mechanisms are optimal.*

Proof of Proposition 4.3. Follows from noticing that the expected revenue generated by the mechanism in Proposition 4.2, $M\tilde{k}$, is the same as the one he obtains by selling the goods separately. ☐

In the symmetric case, to insure against uncertainty, the seller can rely on two quite different mechanisms.[14] It can either ignore the fact that all goods are consumed by the same consumer and proceed as if he was dealing with M different consumers, and sell each good in a fully separable fashion, or he can bundle the different goods in fixed proportions and, de facto, sell a single (bundled) good to the consumer. In this latter mechanism, the seller constructs and aggregate measure of willingness to pay, $\frac{\sum_{i=1}^{M}\theta_i}{M\theta^*}$, and sells amounts of the bundled good that depend fully on $\frac{\sum_{i=1}^{M}\theta_i}{M\theta^*}$. What is common between these two sales strategies is their simplicity: rather than bundling goods or selling them separately as a function of type announcements (as in Bayesian multidimensional mechanisms), sell them in a fully separable or fully bundled way.

4.2 Asymmetric case

We now move to the more general case where the means are not necessarily the same. Proposition 4.1 below shows that nature can limit the seller's expected profit to $\tilde{k}_A$. That is, the minmax of the zero-sum game played by the seller and nature is at most the payoff the seller can guarantee selling goods separately (see Proposition 4.1). As a consequence, the value of the zero-sum game is exactly $\tilde{k}_A$, that can be attained by a mechanism that sells the goods separately.

The proof of this result follows very closely the approach adopted by Carroll (2015b), in which a comonotonic distribution that bounds the seller's expected payoff is explicitly constructed. For the sake of completeness, we provide a detailed of proof of such construction in our framework.

Proposition 4.4. *Nature can guarantee at least $\tilde{k}_A$ in the zero sum game played against the seller (i.e., $\tilde{k}_A$ is an upper bound for the minmax value of the zero-sum game). Therefore, a mechanism that sells the goods separately is optimal.*

Proof of Proposition 4.4. For each $i = 1, ..., M$, define the function $\varphi : [0,1] \to [0,1]^M$ by

$$\varphi^i(x) := \min\left\{\frac{\tilde{k}^i}{1-x}, 1\right\}, \text{ for all } i = 1, ..., M.$$

[14] In fact, it can combine the two mechanisms in whatever way he likes.

Notice that φ^i is increasing, Lipschitz and differentiable except at $x = 1-\tilde{k}^i$.[15] Also, $\varphi^i\left([0,1-\tilde{k}^i)\right) = [\tilde{k}^i, 1)$ and $\varphi^i([0,1]) = [\tilde{k}^i, 1]$.

Define the distribution $F \in \mathcal{F}$ as the pushforward of the Lesbegue measure with respect to the function φ:

$$F(\theta) = \text{ Pr } \left(\left\{x \in [0,1]; \varphi^i(x) \leq \theta^i, \text{ for all } i = 1, ..., M\right\}\right).$$

We have that

$$F(\theta^i, \mathbf{1}^{-i}) = \left(1 - \frac{\tilde{k}^i}{\theta^i}\right)^+ + \tilde{k}^i \delta_1\left(\theta^i\right),$$

for all $\theta^i \in [0,1]$, where δ_1 is the Dirac measure concentrated at 1. Indeed, we have that $F\left(\theta^i, \mathbf{1}^{-i}\right) = \Pr\left(\left\{x \in [0,1]; \varphi^i(x) \leq \theta^i\right\}\right)$ since $\varphi^i \leq 1$. It is clear that $F(\mathbf{1}) = 1$. Since $\varphi^i \geq \tilde{k}^i$, we conclude that $F\left(\theta^i, \mathbf{1}^{-i}\right) = 0$ when $\theta^i < \tilde{k}^i$. Now, if $\tilde{k}^i \leq \theta^i < 1$, then $\varphi^i(x) \leq \theta^i$ if and only if $0 \leq x \leq 1 - \frac{\tilde{k}^i}{\theta^i} = F(\theta^i, \mathbf{1}^{-i})$.

Let $\Pi(\theta) = \theta \cdot \boldsymbol{q}(\theta) - \mathcal{U}(\theta)$ be the any seller's profit function. Since the support of F is contained in $\times_{i=1}^M \left[\tilde{k}^i, 1\right]$ and therefore, without loss of generality, $\mathcal{U}\left(\tilde{\boldsymbol{k}}\right) = 0$. Define the projected allocation and rent: $\bar{\boldsymbol{q}}(x) = \boldsymbol{q}(\varphi(x))$ and $U(x) = \mathcal{U}(\varphi(x))$. From the change of variable's theorem we get

$$\int \Pi(\theta)\, dF(\theta) = \int_0^1 \left[\varphi(x) \cdot \bar{\boldsymbol{q}}(x) - U(x)\right] dx. \tag{20}$$

Suppose first that $\mathcal{U}$ is differentiable. Then, the derivative of $U(x)$ is given by

$$U'(x) = \nabla\mathcal{U}(\varphi(x)) \cdot D\varphi(x) = \bar{\boldsymbol{q}}(x) \cdot D\varphi(x), \tag{21}$$

where $D\varphi$ is the differential of φ. Since $U(x) = \int_0^x \bar{\boldsymbol{q}}(\tau) \cdot D\varphi(\tau) d\tau + U(0)$ and $U(0) = \mathcal{U}(\tilde{k}) = 0$, we have that

$$\begin{aligned} \textstyle\int_0^1 U(x)\, dx &= \textstyle\int_0^1 \left(\int_0^x U'(\tau)\, d\tau\right) dx \\ &= \textstyle\int_0^1 (1-x)\, U'(x)\, dx \\ &= \textstyle\int_0^1 (1-x)\, \bar{\boldsymbol{q}}(x) \cdot D\varphi(x) dx. \end{aligned}$$

[15] Indeed, for all $x \in [0, 1-\tilde{k}^i)$, $\varphi^i(x) = \frac{\tilde{k}^i}{1-x}$ and then $\varphi^{i\prime}(x) = \frac{\tilde{k}^i}{(1-x)^2} \leq 1$. For $x > 1 - \tilde{k}^i$, $\varphi^{i\prime}(x) = 0$.

Therefore,

$$\begin{aligned}\int \Pi(\theta)\, dF(\theta) &= \sum_{i=1}^{M} \left(\int_0^1 \varphi^i(x)\, \bar{q}^i(x)\, dx - \int_0^1 (1-x)\, \bar{q}^i(x)\, \varphi^{i\prime}(x)\, dx \right) \\ &= \sum_{i=1}^{M} \int_0^1 \left(\varphi^i(x) - (1-x)\, \varphi^{i\prime}(x) \right) \bar{q}^i(x)\, dx \\ &= \sum_{i=1}^{M} \int_{1-\tilde{k}^i}^1 \bar{q}^i(x)\, dx \le \tilde{k}_A,\end{aligned}$$

where we used that $\varphi^i(x) - (1-x)\,\varphi^{i\prime}(x)$ in $[0, 1-\tilde{k}^i)$ and $\varphi^i(x) - (1-x)\,\varphi^{i\prime}(x) = 1$ in $[1-\tilde{k}^i, 1]$.

For the general case, considering $x', x'' \in [0,1]$, since $\mathcal{U}$ is a convex function

$$\begin{aligned} U(x') - U(x'') &\le (\varphi(x') - \varphi(x'')) \cdot \bar{\boldsymbol{q}}(x'') \text{ and} \\ U(x'') - U(x') &\le (\varphi(x'') - \varphi(x')) \cdot \bar{\boldsymbol{q}}(x')\,. \end{aligned}$$

Hence, $|U(x') - U(x'')| \le 2\,|x' - x''|$, because $\varphi^{i\prime}, \bar{q}^i \le 1$. Hence, U is differentiable and

$$U'(x) = \sum_{i=1}^{M} \varphi^{i\prime}(x)\, \bar{q}^i(x)\,,$$

i.e., we get the same expression of (21) and the same result goes through. Therefore, minmax value of the zero-sum game is $\tilde{k}_A$. Combining this with Proposition 4.1, we obtain that selling the goods separately is optimal. □

5 Multiple periods

In this section, we study how robust the solution in Section 3 is to repeated interactions. There are two main reasons to consider the role of dynamics in our model. First, even if the seller has little information about the buyer's value, extra information might become available in future periods, leading to better revenue extraction. However, ambiguity on the side of the seller regarding this additional information makes the result non-trivial. In our dynamic model this is incorporated by looking at pricing rules that potentially depend on previous consumption behavior by the buyer.

Second, knowledge of average valuations is naturally connected to learning and information acquisition, as conditional expectations with increasing information sets follow a martingale. However, information acquisition is a dynamic phenomenon that should be discussed in a dynamic model.

The dynamic mechanism design literature on revenue maximization (Courty and Li (2000); Battaglini (2005); Pavan et al. (2015)) has described the optimal pricing schedule with repeated interactions and known type distributions. One common feature of these papers is dependence of the optimal dynamic pricing on fine details of the joint distribution.[16]

In contrast, our main result is the irrelevance of dynamics in the presence of large ambiguity, i.e., repeated static optimal pricing is optimal. The intuition for our result is as follows. Optimal static pricing provides the best revenue guarantee that only depends on knowledge of the average valuation. By using the repetition of the static pricing rule, the seller completely separates the multiple periods and obtains a guarantee that only depends on the ex-ante average valuation of each period. The martingale property implies this revenue is the sum of the static revenue over multiple periods.

Certainly the seller cannot obtain more than the repeated static revenue as long as the case of permanent types is in the set of possible processes. For this process the optimal revenue without ambiguity is equal to the repetition of the static revenue.

A central part of our analysis deals with the key distinction between the case of multiple goods and multiple periods: the sequential revelation of information. We assume that the buyer observes in each period his realized valuation. A consequence of the sequential revelation of information is a failure of the revelation principle. For a given (indirect) mechanism, the optimal reporting strategy is dependent on the type process. Hence, the implied direct mechanism connected with a specific outcome is dependent on the type distribution chosen by the infimum operator. One methodological contribution of this paper is to present a way of dealing with dynamics in a world with ambiguity. The seller chooses from the larger set of indirect mechanisms and, in the face of large ambiguity, considers all possible type distribution as well the whole set of optimal reporting strategies induced by this distribution.

We now introduce the formal elements present in the dynamic analysis. Similarly to the static model, the small information held by the seller is described by a set of type distributions that are considered possible. Instead of treating each period independently, we consider type processes that have the martingale property. Let F^* denote the critical distribution, R_1^* denote the robust revenue level and $m_1^* = (q^*, p^*)$ denote the optimal selling mechanism in the static model with average condition $k \in (0,1)$. The set of possible type distributions is $\mathcal{F} \subseteq \Delta\left([0,1]^T\right)$ is assumed to have the following properties.

Assumption. The set $\mathcal{F} \subseteq \Delta\left([0,1]^T\right)$ satisfies:

[16]Specially, Pavan et al. (2015) highlight the role of impulse-response functions as determinants of the optimal distortions in the optimal contract. The calculation of these objects requires knowledge of the entire joint distribution.

(i) Martingale property: $F \in \mathcal{F} \Rightarrow \int \theta_{t+1} dF\left(\theta^T\right) = \theta_t$ for any t satisfying $T-1 \geq t \geq 1$.

(ii) Possibility of permanent types: $F^{*,T} \in \mathcal{F}$, where it is defined by $F^{*,T}\left(\theta^T\right) = F^*\left(\min_t \theta_t\right)$.

The first statement describes types that are derived from a learning process.[17] The second one only states that a specific constant types distribution is contained in the feasible set. The distribution $F^{*,T}$ has perfect correlation across periods, with valuations distributed according to F^*, the critical distribution in the static case. A special case of $\mathcal{F}$ is given by all martingale processes with the initial type distribution satisfying the static restriction $\mathbb{E}\left[\theta_1\right] = k$.

As discussed above, the analysis of the dynamic case cannot make use of the revelation principle. Hence, we need to explicitly consider the optimal reporting strategies induced by a given mechanism and a type distribution. These are defined below.

Definition 5.1. For a fixed distribution $F \in \mathcal{F}$ and mechanism $m \in \overline{\mathcal{M}}$, a reporting strategy $\sigma_0 \in \Sigma_m$ is optimal if

$$\mathbb{E}\left[\mathcal{U}_m\left(\sigma_0 \mid \tilde{\theta}^T\right)\right] \geq \max\left\{\mathbb{E}\left[\mathcal{U}_m\left(\sigma \mid \tilde{\theta}^T\right)\right], 0\right\}$$

for all $\sigma \in \Sigma_m$. The set of optimal strategies is denoted as $\Sigma_{m,F}$.

A direct mechanism is interim-incentive compatible, given a distribution $F \in \mathcal{F}$, if strategy $\sigma^{TT} \in \Sigma_{m,F}$. The set of interim incentive-compatible direct mechanisms is given by$\mathcal{M}_F^I$. We highlight once again that the set of optimal strategies potentially depends both on the mechanism in place and the type distribution F.

Now we are able to describe the decision problem faced by the seller. He decides what (potentially indirect) mechanism to offer to the agent. However, the seller does not know which actual type distribution $F \in \mathcal{F}$ he is facing, as well as what optimal reporting strategy $\sigma \in \Sigma_{m,F}$ is followed by the buyer. On the face of this uncertainty once again the seller tries to obtain the highest possible revenue guarantee. Formally:

$$R_T^* \equiv \sup_{m \in \overline{\mathcal{M}}} \inf_{\left(F \in \mathcal{F}, \sigma \in \Sigma_{m,F}\right)} \int \Pi_m\left(\sigma \mid \theta^T\right) dF\left(\theta^T\right), \tag{22}$$

[17] In our model the buyer is not completely informed about his valuation after the first period. Our model can incorporate the following scenario: the utility generated by consumption in each period $t \geq 1$ is $v + \varepsilon_t$, where $v \sim F$ and $\varepsilon_t \sim G_t\left(\mid v, \varepsilon_1, \ldots, \varepsilon_{t-1}\right)$ satisfying $\int v dF\left(v\right) = k$ and $\int s dG_t\left(s \mid v, \varepsilon_1, \ldots, \varepsilon_{t-1}\right) = 0$ for all $t \geq 1$ and $\left(v, \varepsilon_1, \ldots, \varepsilon_{t-1}\right)$. In the beginning of each period t the buyer observes $\theta_t \equiv v + \varepsilon_t$. Ignorance of the joint distribution of θ^T is generated by ignorance of objects $\left(F, \left(G_t\right)_{t \geq 1}\right)$, and our optimal pricing result applies as long as the learning process $\left(F^*, \left(\delta_0\right)_{t \geq 1}\right)$ where there is no learning is a possibility for the seller. The shock $\left(\varepsilon_t\right)_t$ determines the transitory effects that affect one's utility from consumption. For example, the satisfaction derived from watching a movie can be generated by the underlying taste for movies or from watching it in good company.

where, for any indirect mechanism $m = (q_t, p_t)_{t\in T}$, the notation $m \circ \sigma$ denotes the direct mechanism $(\tilde{q}_t, \tilde{p}_t)_{t\in T}$ defined as $\left(\tilde{q}_t\left(\theta^T\right), \tilde{p}_t\left(\theta^T\right)\right) \equiv \left(q_t\left(\sigma^T\left(\theta^T\right)\right), p_t\left(\sigma^T\left(\theta^T\right)\right)\right)$ for all $\theta^T \in [0,1]^T$ and $t \geq 1$.

We start by describing the revenue guarantee that the seller has from choosing the repetition of the static optimal mechanism. This pricing rule guarantees at least the sum of the static revenue period by period, which is equal to

$$R_T \equiv \frac{1-\delta^{T+1}}{1-\delta} R_1^*.$$

Let m_T^* denote the independent repetition of the static mechanism $m_1^* = (q^*, p^*)$, i.e., it is given by $\left(q_t^*\left(\theta^T\right), p_t^*\left(\theta^T\right)\right) = \left(q^*\left(\theta_t\right), p^*\left(\theta_t\right)\right)$ for all θ^T and $t \geq 1$. Independent pricing leads to incentive constraints that are completely separable over periods. As a consequence agents have incentives to truthfully report their types period by period for any type distribution. In fact, we show that the buyer loses a strictly positive amount by choosing a reporting strategy that leads to a different allocation than truth-telling. This separability leads to the following revenue guarantee.

Lemma 5.1. (Revenue guarantee) The mechanism m_T^* guarantees revenue R_T, i.e.,

$$\inf_{\left(F\in\mathcal{F},\sigma\in\Sigma_{m,F}\right)} \int \Pi_{m_T^*}\left(\sigma \mid \theta^T\right) dF\left(\theta^T\right) = R_T.$$

Proof of Lemma 5.1. First consider any $F \in \mathcal{F}$. Any optimal reporting strategy $\sigma \in \Sigma_{m^*,F}$ has the property that

$$\int \mathbf{1}\left\{\theta^T \in [0,1]^T; \left(q_t\left(\sigma_t\left(\theta^T\right)\right), p_t\left(\sigma_t\left(\theta^T\right)\right)\right) = \left(q_t\left(\theta^T\right), p_t\left(\theta^T\right)\right), \text{for all } t \in T\right\} dF\left(\theta^T\right) = 1.$$

This expression states that any optimal reporting strategy is equivalent to the truth-full reporting strategy. This occurs because mechanism m^* is ex-post incentive compatible. This means that all agents have strict incentives to report truthfully, except when their types is in the exclusion region and they are indifferent among several announcements that lead to the same allocation. We start by showing this property.

Since in mechanism m^* the allocation in period t only depends on the announcements in that period, it is necessarily the case that for $\theta^T \in supp\left(F\right)$:

$$\sigma_t\left(\theta^T\right) \in \arg\max_{\theta'} \theta_t q^*\left(\theta'\right) - p^*\left(\theta'\right),$$

where

$$q^*(\theta') \equiv \max\left\{1-\frac{\ln \theta'}{\ln \tilde{k}}, 0\right\}, \tag{23}$$

$$p^*(\theta') \equiv \max\left\{\frac{\theta'-\tilde{k}}{\ln \tilde{k}}, 0\right\},$$

where $\tilde{k}$ is the solution of (6). Since the objective function is strictly concave for $\theta' > \tilde{k}$ and constant for $\theta' \in \left[0, \tilde{k}\right]$:

$$\arg\max_{\theta'} \theta_t q^*(\theta') - p^*(\theta') = \begin{cases} \theta_t, & \text{if } \theta_t > \tilde{k}, \\ \left[0, \tilde{k}\right], & \text{if } \theta_t \in \left[0, \tilde{k}\right]. \end{cases}$$

This implies that $\left(q_t\left(\sigma_t\left(\theta^T\right)\right), p_t\left(\sigma_t\left(\theta^T\right)\right)\right) = \left(q_t\left(\theta^T\right), p_t\left(\theta^T\right)\right)$ with probability one.

As a consequence,

$$\inf_{F\in\mathcal{F},\sigma\in\Sigma_{m^*,F}} \int \Pi_{m^*}\left(\sigma \mid \theta^T\right) dF\left(\theta^T\right) = \inf_{F\in\mathcal{F}} \int \Pi_{m^*}\left(\sigma^{TT} \mid \theta^T\right) dF\left(\theta^T\right).$$

But $\Pi_{m^*}\left(\sigma^{TT} \mid \theta^T\right)$ is given by $\sum_{t=1}^{T}\left[\delta^t \max\left\{\frac{\theta_t-\tilde{k}}{\ln \tilde{k}}, 0\right\}\right]$. As a consequence it follows that

$$\inf_{F\in\mathcal{F}} \int \Pi_{m^*}\left(\sigma^{TT} \mid \theta^T\right) dF\left(\theta^T\right) \geq \sum_{t=1}^{T}\left[\delta^t \max\left\{\frac{\mathbb{E}\left[\theta_t\right]-\tilde{k}}{\ln \tilde{k}}, 0\right\}\right] = R_T.$$

And this holds as an equality if the marginal distribution $marg_t F \subseteq \left[\tilde{k}, 1\right]$, which is true for distribution $F^{*,T} \in \mathcal{F}$. □

A direct implication of this lemma is that the optimal robust revenue is at least equal to this guarantee.

Corollary 5.1. *(Lower bound) The optimal robust revenue is at least equal to R_T, i.e., $R_T^* \geq R_T$.*

Proof of Corollary 5.1. By definition the following inequality holds

$$R_T^* \geq \inf_{F\in\mathcal{F},\sigma\in\Sigma_{m^*,F}} \int \Pi_{m^*\circ\sigma}\left(\theta^T\right) dF\left(\theta^T\right) = R_T.$$

□

We now show that, in fact, the seller cannot improve upon the revenue guarantee described above. The argument is very simple: the distribution with permanent types following the static critical distribution F^* generates the revenue guarantee R_T as the optimal revenue with known type distribution. As a consequence, by considering the worst possible distribution, the seller can only obtain lower revenue. This gives us an upper bound on the minimax value $\bar{w}$.

Lemma 5.2. *(Upper bound) The (known) type distribution $F^{*,T}$ leads to optimal revenue R_T:*

$$R_T \equiv \sup_{m \in \mathcal{M}^I_{F^*,T}} \int \Pi_m \left(\theta^T\right) dF^{*,T} \left(\theta^T\right),$$

and hence the minimax value $\bar{w}$ satisfies

$$\bar{w} \equiv \inf_{F \in \mathcal{F}} \sup_{\left(m \in \overline{\mathcal{M}}, \sigma \in \Sigma_{m,F}\right)} \int \Pi_m \left(\sigma \mid \theta^T\right) dF \left(\theta^T\right) \leq R_T.$$

Proof of Lemma 5.2. The mechanism m^*_T is in $\mathcal{M}^I_{F^*,T}$ since it is ex-post incentive compatible. We briefly show that it solves the revenue maximization problem with known distribution $F^{*,T}$. Consider an arbitrary mechanism $m \in \mathcal{M}^I_F$ and define $\mathcal{U}^E_m(\theta_1) \equiv \mathbb{E}\left[\mathcal{U}_m\left(\tilde{\theta}^T\right) \mid \theta_1\right]$. Any incentive compatible mechanism $m = (q_t, p_t)_{t \in T}$ satisfies:

$$\mathcal{U}^E_m(\theta_1) = \sum_{t=1}^{T} \delta^{t-1} \left[q_t\left([\theta_1]^T\right)\theta_1 - p_t\left([\theta_1]^T\right)\right] = \max_{\theta' \in [0,1]} \sum_{t=1}^{T} \delta^{t-1} \left[q_t\left([\theta']^T\right)\theta_1 - p_t\left([\theta']^T\right)\right],$$

which implies that, using $Q(\theta) \equiv \sum_{t=1}^{T} \delta^{t-1} q_t\left([\theta]^T\right)$,

$$\mathcal{U}^E_m(\theta_1) = \mathcal{U}^E_m(0) + \int_0^{\theta_1} Q(s)\, ds.$$

Then expected revenue, according to F^{**}, satisfies

$$\begin{aligned}
\int \Pi_m\left(\theta^T\right) dF^{**}\left(\theta^T\right) &= \int \left[\theta Q(\theta) - \int_0^{\theta_1} Q(s)\, ds\right] dF^*(\theta) - \mathcal{U}^E_m(0) \\
&= \int_0^1 Q(\theta)\left[\theta - \frac{1 - F^*(\theta)}{f^*(\theta)}\right] d\theta + \left[F^*(1) - F^*_-(1)\right] Q(1) \\
&= \left[F^*(1) - F^*_-(1)\right] Q(1) \\
&\leq \left[F^*(1) - F^*_-(1)\right] \sum_{t \in T} \delta^{t-1},
\end{aligned}$$

where the third inequality uses the definition of F^* and the last inequality follows from resource constraints $q_t\left(\theta^T\right) \leq 1$. The last term in this sequence of inequalities is the expected revenue obtained by m^*_T since it always sells maximal quantity following an announcement $\theta_t = 1$. This concludes the first part of the statement.

The second part follows from

$$\begin{aligned}\bar{w} &\equiv \inf_{F\in\mathcal{F}} \sup_{\left(m\in\overline{\mathcal{M}},\sigma\in\Sigma_{m,F}\right)} \int \Pi_m\left(\sigma \mid \theta^T\right) dF\left(\theta^T\right) \\ &\leq \sup_{\left(m\in\overline{\mathcal{M}},\sigma\in\Sigma_{m,F}\right)} \int \Pi_m\left(\sigma \mid \theta^T\right) dF^{*,T}\left(\theta^T\right) \\ &= \sup_{m\in\mathcal{M}} \int \Pi_m\left(\theta^T\right) dF^{*,T}\left(\theta^T\right) = R_T.\end{aligned}$$

Where the first inequality uses $F^{*,T} \in \mathcal{F}$ and the second inequality is an application of the revelation principle. □

Concluding the argument, lemmas 5.1 and 5.2 present upper and lower bounds that coincide. Hence the optimal revenue is equal to this level.

Proposition 5.1. *(Optimal revenue) The optimal interim robust revenue, as defined in (22), is equal to R_T and the mechanism m_T^* achieves it.*

Proof of Proposition 5.1. Our proof consists of the following system of inequalities: $R_T \geq \bar{w} \geq R_T^* \geq R_T$.

Since the first and the third inequalities follow from lemmas 5.1 and 5.2, we only need to show the second inequality.

Consider any $F_0 \in \mathcal{F}$ then

$$\begin{aligned}R_T^* = \sup_{m\in\overline{\mathcal{M}}}\left[\inf_{\left(F\in\mathcal{F},\sigma\in\Sigma_{m,F}\right)} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF\left(\theta^T\right)\right] &\leq \sup_{m\in\overline{\mathcal{M}}}\left[\inf_{\sigma\in\Sigma_{m,F_0}} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF_0\left(\theta^T\right)\right] \\ &\leq \sup_{m\in\overline{\mathcal{M}}}\left[\sup_{\sigma\in\Sigma_{m,F_0}} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF_0\left(\theta^T\right)\right] \\ &= \sup_{\left(m\in\overline{\mathcal{M}},\sigma\in\Sigma_{m,F_0}\right)} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF_0\left(\theta^T\right).\end{aligned}$$

The first inequality follows from the removal of one degree of freedom in the infimum operator. The second inequality follows by substituting an infimum by a supremum. The final equality follows from the revelation principle. But now taking the infimum over all possible distributions $F_0 \in \mathcal{F}$ leads to the result. □

The above results are easily extended to the case of multiple goods, combining the arguments presented here with the ones in Section 4. For completeness we present here the general result, with an omitted proof for briefness.

Proposition 5.2. *The optimal dynamic, multidimensional, robust mechanism is the period by period repetition of the static mechanism, which can either entail sales in a fully separated fashion, or full bundling.*

6 Conclusion

We have considered a seller's problem in designing a worst-case mechanism when facing, for T periods, a privately informed buyer and having knowledge, at each period, of a single moment of the distribution from which consumer's multidimensional values are drawn. The results and their interpretations have been extensively discussed in the introduction and the main text, so we conclude with avenues for future research. Before doing so, however, we point out that most of the results we derived in this paper go through in a setting in which payoffs display curvature. This happens, for instance, in settings in which production can be larger than one and the consumer's marginal utility decays with consumption (or the seller's marginal cost of production is increasing). In a companion paper (Carrasco et al. (2015)), we use the zero-sum game interpretation of the robust design problem to derive optimal robust mechanisms in quasi-linear settings with payoffs that are non-linear in the allocation.

Regarding other extensions, it would be nice, in the spirit of what Carroll (2013) does, to consider robust design for more general priors than the degenerate ones we consider here. Dealing with multiple buyers would also be a natural extension of what we have done in this paper. The main difficulties in tackling such extensions are technical. Regarding the former, extending the zero-sum game approach that we use to verify optimality for the case of M goods and T periods is not straightforward. Even for the single good case with general priors (the case considered by Carroll (2013)), it is not quite clear how to compute Nash equilibria of the zero-sum game played by the seller and nature. The case in which there are more than one buyer is even more challenging. In fact, if we were to use Myerson's trick of substituting the incentive compatible representation of the consumer's payoff into the seller's objective, and proceed as we did in this paper, we would end up with a system of Partial Differential Equations (PDE), whose solution is hard to obtain. We could, instead, assume symmetric buyers and combine Myerson's trick with Border (1991)'s conditions to solve for the optimal reduced form robust auction. The difficulty is then to solve a single buyer robust selling mechanism adding the constraints implied by Border (1991). Although challenging, we hope future research addresses these questions.

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