Central Bank Balance Sheet, Liquidity Trap, and Quantitative Easing

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Abstract

We show that when a central bank is not fully financially backed by the treasury and faces a solvency constraint, an increase in the size or a change in the composition of its balance sheet (quantitative easing) can serve as a commitment device in a liquidity trap scenario. In particular, when the short-term interest rate is at the zero lower bound, open market operations by the central bank that involve purchases of long-term bonds can help mitigate deflation and recession under a discretionary policy equilibrium. This change in central bank balance sheet, which increases its size and duration, provides an incentive to the central bank to keep interest rates low in the future in order to avoid losses and satisfy its solvency constraints, approximating its full commitment policy.

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1 Introduction

Since the financial crisis of 2008, many central banks have been forced to change their main policy tool away from the short-term interest rates. As the policy rates reached the zero lower bound (ZLB), they lost their suitability as instruments to stimulate the economy. In a sluggish recovery, there has been a search for alternative expansionary monetary policies. Central bank balance sheet expansion has been the most common choice. In the United States, the Federal Reserve (Fed) purchased a total of $1.75 trillion in agency debt, mortgage-backed securities (MBS) and Treasuries in the "QE1", followed by a second Treasury-only program of $600 billion in the fall of 2010. In September 2011, the Fed introduced QE3, increasing the amount of long-term bonds in its balance sheet. Other countries also followed similar strategies. In March 2009, the Bank of England (BoE) announced it would purchase a total of £75 billion of U.K. gilts, which, after subsequent increases, was expanded to £375 billion in July 2012. On 4 April 2013, the Bank of Japan (BoJ) announced a plan to purchase ¥7.5 trillion of bonds a month and double its monetary base. More recently, on January 2, 2015, the European Central Bank (ECB) announced monthly asset purchases of 60 billion euros to be carried out until at least September 2016.

The stimulative role of QE has since been the focus of intensive debate. Empirically, many studies have demonstrated the effects of these programs on asset prices and interest rates.[1] However, the precise theoretical channel through which these programs affect real variables is unclear and is still under the scrutiny of the academic debate. Most recent mechanisms rely on segmented markets or other sources of financial frictions in order to generate real effects.[2] In this paper, we provide an alternative mechanism in which changes in central bank balance sheet have real effects. Specifically, when the central bank is restricted from incurring in huge financial losses, these programs act as a credible restriction on future monetary policy actions.

In addition, we show that central banks that face solvency constraints can use their balance sheets to mitigate the credibility issues that arise in optimal policy in a liquidity trap. In other words, a central bank that is restricted in the losses it can have is subject to a possible commitment mechanism: if its balance sheet is large or shows long enough duration, possible unfavorable asset price movements coming from interest rate hikes are going to be avoided, restricting upward shifts in the policy rates and leading to a credible higher inflation path. This commitment mechanism allows a discretionary central bank to approximate optimal commitment policies and provides a theoretical justification for the recent adoption of QE programs by several central banks as their short-term interest rates have reached the ZLB.

Identifying channels through which large purchase programs, such as QEs, have real effects is no trivial task. It has been well known since Wallace (1981) that changes in the size or the composition of the central bank's balance sheet have no effect on equilibrium allocations within the framework of general equilibrium models: in a representative agent-based model, a mere shuffling of assets between the central bank and the private sector should not change any asset price in the economy. Instead, macroeconomic theory prescribes a rather different policy in the

[1] See Gagnon et al. (2011), Hamilton and Wu (2011), Krishnamurthy and Vissing-Jorgensen (2011), and Williams (2011) and references therein.

[2] Among others, we refer to Gertler and Kiyotaki (2010), Gertler and Karadi (2013), Vayanos and Vila (2009), and Curdia and Woodford (2011)

liquidity trap scenario. As first noted by Krugman (1998), optimal monetary policy at the ZLB entails a commitment to keep short-term interest rates low for a long period in the future. This policy generates higher level of expected real income and inflation in the future and provides the economy with the necessary incentives for greater real expenditure and larger price increases in the present. The problem, also emphasized in Krugman (1998), is how to make low interest rates in the future credible: the central bank may renege ex post on its promises to pursue its goal of price stability. In fact, why would the central bank generate undesired inflation simply because of a binding constraint in the past?

Addressing this credibility problem, Woodford (2012) suggests the use of explicit statements by central banks about the outlook for future policy in addition to their announcements about the immediate policy actions that are in course. This type of policy, or *forward guidance*, is intended to facilitate the implementation of the optimal policy, as it makes it unambiguously clear that the central bank intends to maintain the benchmark rate at its lower bound for extended periods. Despite all the discussion of its effectiveness in practice, these announcements only constitute a commitment device if associated with costs of reneging (either moral or pecuniary).

Instead of relying on hidden reneging costs, we design a mechanism through which the credibility problem in a liquidity trap scenario can be mitigated if central banks face solvency constraints. More specifically, this mechanism allows this type of central bank to commit to lower future interest rate through a large-scale purchase of long-term securities that creates an incentive not to raise interest rates in the future and thus, avoid losses on its balance sheet.

This result relies on two basic assumptions: (i) central banks are not financially backed by the treasury in all possible states of nature, and (ii) central banks cannot become insolvent. The first observation limits transfers between these authorities and adds a budget constraint to central banks. The second implies that central banks of this type cannot run unlimited losses.[3] We view these assumptions as a consequence of a self-imposed behavior motivated by the political embarrassment caused by large financial bail-outs. Together they provide an additional restriction to monetary policymakers: they cannot undertake actions that lead to excessive losses in their balance sheets. Accordingly, a current large-scale purchase of long-term securities can credibly lock the central bank into low interest rates in the future because interest rate hikes may threaten the central bank's solvency.[4]

This work is closely related to Jeanne and Svensson (2007) (JE07 hereafter). They showed that if central banks in small open economies have capital concerns, then it is possible to create a commitment mechanism that allows independent central banks to achieve a higher future price level through a present currency depreciation. This paper differs from JE07 in two important aspects. First, the commitment mechanism we designed does not rely on the small open economy assumption and hence is more suitable for the U.S. economy. Second, in JE07 capital concerns are modeled as ad-hoc preferences against low levels of capital that are difficult to assess and interpret in practice. Instead, we rely on the more realistic assumption that central banks will not undertake any actions that may undermine their capacity or independence to carry out monetary policy in the future. This is in line with Sims and Negro (2014), where low levels

[3] This is directly related to the literature that assumes balance sheet concerns on the part of the central bank, such as Sims (2004), Berriel and Bhattarai (2009), and Jeanne and Svensson (2007).

[4] For further reference on how interest rates affect central bank's balance sheets, see Reis and Hall (2013).

of capital can prevent a central bank from avoiding self-fulfilling hyper-inflationary equilibria, and Buiter (2008), where the scale of the recourse to seigniorage required to safeguard central bank solvency may undermine price stability. Bhattarai et al. (2014) focus on the implications of joint monetary and fiscal policy to a similar problem, while here we focus on the implications of limited losses of the central bank.

The rest of the paper is organized as follows. Section 2 describes a simple endowment economy model with a central bank and two assets of different maturities. Section 3 revisits the literature on the simple model described in section 2 and characterizes (*i*) the liquidity trap (equilibrium under discretion), (*ii*) the optimal escape from the liquidity trap (equilibrium under commitment) and (*iii*) the credibility problem. In section 4 we show how a long-term security purchase program can serve as a commitment mechanism in the liquidity trap. Section 5 discusses and compares the results derived in sections 3 and 4. In section 6 we set up a quantitative model with production, calibrated to the U.S. economy, and answer two questions: (*i*) Can QE1, QE2 and QE3 serve as a commitment mechanism to stimulate and inflate the U.S. economy? and (*ii*) What are the optimal size and duration of QE programs?

2 A Simple Endowment Economy Model

2.1 The Model Overview

We consider a one-good, representative agent economy. The household consumes and saves by buying riskless government bonds of different maturities. In this simple economy we abstract from production and assume that consumption each period is restricted to an exogenous endowment process. The central bank is not fully financed by the treasury and conducts an price-level targeting by minimizing a quadratic loss function of the price level. We introduce money in this economy by imposing a cash-in-advance constraint: in the beginning of each period, individuals trade cash for bonds, with nominal interest rate i_t. Their consumption during the period is constrained by the cash with which they emerge from this trading. We show in section 3 that the economy falls into a liquidity trap in period 1, with excessively low price level, as a result of an unanticipated fall in expected endowment growth. The same scenario might arise in period 2, depending on the realization of the endowment process.

2.2 The Household

The household's utility function is assumed to take the form,

$$U_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i log(C_{t+i})$$

where C_t is consumption in period t, $\mathbb{E}_t$ is the expectation operator conditional to available information in period t, and β is the discount factor. The household seeks to maximize its utility subject to the budget constraint,

$$Y_t + B_{t-1}^{s,hh} + (1+Q_t)B_{t-1}^{hh} + M_{t-1} = P_t C_t + Z_t + \frac{1}{1+i_t} B_t^{s,hh} + Q_t B_t^{hh} + M_t \tag{1}$$

where Y_t is a stochastic endowment process, M_t, $B_t^{s,hh}$ and B_t^{hh} are respectively the total of money, short and long-term claims on the government debt held by the household. The short-term bond costs $\frac{1}{1+i_t}$ in period t and pays a dollar in period $t+1$ - so that i_t is the nominal interest rate. The long-term bond costs Q_t dollars in period t and pays a dollar in perpetuity, i.e., it is a nominal perpetuity bond. Z_t is a lump-sum tax collected by the government. The household's first order conditions with respect to short and long-term bonds and the cash-in-advance constraint are,

$$1 = \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \frac{1+i_t}{P_{t+1}/P_t} \right] \tag{2}$$

$$1 = \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \frac{1+Q_{t+1}}{Q_t} \frac{P_t}{P_{t+1}} \right] \tag{3}$$

$$M_t = P_t C_t \tag{4}$$

2.3 The Endowment Process

As mentioned before, there is no production and each period consumption equates the exogenous income process, Y_t. We assume that, from indefinitely long before period 1, the agent has been receiving a certain income $y^* e^{\bar{y}}$. In period 1, however, the agent is informed that from period 2 onwards income will follow the process described in (5), and that the realization of this process will become available information to the agent only in period 2,

$$(Y_2, Y_3, ..., Y_{N-1}, Y_N, Y_{N+1}, ...) = \begin{cases} (y^*, y^*, ..., y^*, y^*, y^*, ...) & \text{with probability } 1-\mu \\ (y^*, y^* e^{\underline{y}}, ..., y^* e^{\underline{y}}, y^*, y^*, ...) & \text{with probability } \mu \end{cases} \tag{5}$$

where y^* is the income of the upcoming steady state, $\bar{y} > 0$ and $\underline{y} < 0$. In section 3 we show that in period 1 the unexpected fall in income growth pushes the economy into a liquidity trap as a result of the agent's excess savings. This liquid trap scenario continues in period 2 with probability μ in the low-income realization of process (5), and reverts with probability $1-\mu$ in the high-income realization of (5).

As Figure 2 shows, after N periods the income returns to steady state level, y^*, independently of the realization of process (5), so we have a well defined non-stochastic steady state.

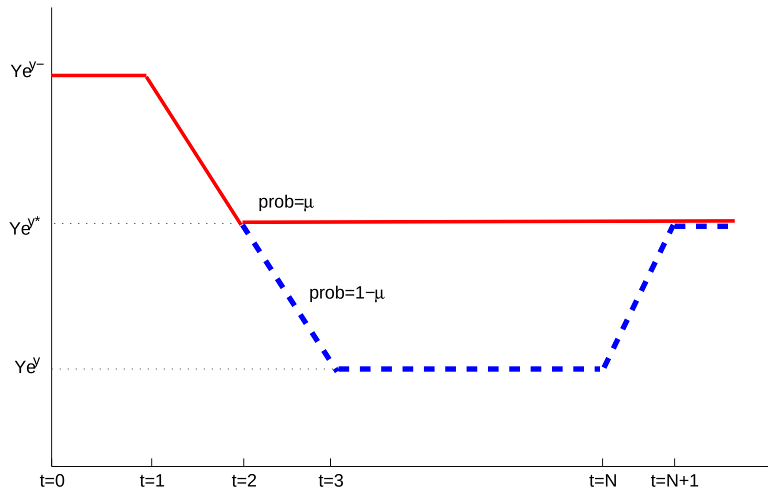

Figure 1: The Endowment Process

2.4 The Public Sector

2.4.1 The central bank

Growing evidence points to the fact that central banks are not fully financed by the treasury in all contingencies. This is more evident in cases where the central bank faces or risks showing unusually large losses in its balance sheet. Following these concerns, we introduce a central bank that is not fully financially backed by the treasury. Since the central bank cannot rely on the treasury for all its financial needs, it is subject to a period-by-period budget constraint,

$$K_t + T_t + M_t = B_t^s + Q_t B_t \tag{6}$$

where B_t^s and B_t are respectively the total of short and long-term bonds held by the central bank in period t. The variable M_t is the outstanding central bank monetary liabilities, T_t denotes the transfers from the central bank to the treasury and K_t is the central bank's capital, residually determined as the excess of the value of its assets over the value of its monetary liabilities,

$$K_t = (1 + i_{t-1}) B_{t-1}^s + (Q_t + 1) B_{t-1} - M_{t-1}$$

In order to obtain the law of motion of capital we rewrite the last equation recursively as,

$$K_t = K_{t-1} - T_{t-1} + i_{t-1} B_{t-1}^s + (1 + Q_t - Q_{t-1}) B_{t-1} \tag{7}$$

One important assumption is that, while considering its budget constraint, all assets of the central bank are marked-to-market. This is a trivially appropriate assumption for modeling the ECB or the Bank of England, which are obliged by law to report this type of pricing. However, if one considers the Fed, this assumption is debatable. In principle, the Fed *can* and actually has adopted historical prices in calculating gains or losses in his balance sheet. We argue that if one is worried about the political implications of a possible recapitalization or decrease in revenues from the Fed to the Treasury, a large gap between historical and market valuations would be an embarrassment for the Fed. So, even without reporting it, the Fed would care about marked-to-market gains and losses in its balance sheet.

The marked-to-market assumption is crucial for the relevance of possible losses in central bank balance sheet. Carpenter et al. (2013), for example, show that losses, considering only losses if assets are sold, are limited and allow for continuous remittances from the central bank to the treasury in an interest rate normalization event. Reis and Hall (2013) and Greenlaw et al. (2013), when analyzing a case similar to ours, show that both remittances have to be zero for a long period and inflation dynamics are severely affected.

The rule T_t is key in this paper. Usually central banks transfer a share of its net income to the treasury in terms of seigniorage revenues. It is important to note that these transfers paid by the central bank to the treasury could be negative. Such transfer payment from the treasury to the central bank can be viewed as the mechanism through which the treasury can inject capital into the central bank, that is, transfer resources to the central bank in order to recapitalize it.

In normal times, the assets and liabilities of a central bank are nearly riskless and net income is usually positive. When the central bank holds other types of assets, especially private debt and assets subject to nominal losses, net income is much more likely to be negative. Negative net income requires fiscal backing to the central bank. The act of capitalizing the central bank would have to be approved by fiscal authorities, subject to the underlying political process.

Even if feasible in economic terms, a fiscal bailout of the central bank is not necessarily politically implementable. In many occasions the tax payer is simply not willing to give up real resources (and, thus, consumption), in order to support the central bank's balance sheet. An interesting example is the ECB, where it is not clear how losses would be split among different fiscal authorities. We include these considerations in the model by assuming the following transfers rule between the central bank and the treasury,

$$T_t = \begin{cases} K_t & \text{if } K_t \geq -\underline{\mathrm{K}} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where $\underline{\mathrm{K}} > 0$. That is, when the central bank's capital is positive it is transferred to the treasury. Fiscal baking, however, has an upper bound: if the capital falls below $-\underline{\mathrm{K}}$, fiscal backing is not allowed by fiscal authorities and the central bank is insolvent.

Central bank insolvency is a issue of considerable controversy since the vast majority of its liabilities is irredeemable. As pointed out by Sims (2005), while an central bank can always pay all its home-currency denominated expenses (financial or operational) through the issuance of base money it may not be optimal or even acceptable, even though it is feasible: it may generate inadmissible high rates of inflation. In addition, there are limits to the amount of real resources the central bank can appropriate by increasing the issuance of nominal base money.[5] Hence, despite the central bank's special ability to issue not just non-interest-bearing but also irredeemable liabilities, central bank's solvency is questioned if its capital falls below some specified level. We rule out central bank insolvency in the model by imposing an lower bound for the central bank's capital[6],

[5]See Buiter (2008).

[6]Note that we are imposing this solvency restriction in nominal terms. This simplifying assumption is dropped in the quantitative model when we take this restriction in real terms.

$$K_t \geq -\underline{\mathrm{K}} \tag{9}$$

where the parameter $\underline{\mathrm{K}}$ can be interpreted as a physical limit imposed by fiscal authorities or a self-imposed restriction in light of the uncertainties of a bail-out. We are assuming here is that policymakers are forbidden to undertake policies that lead to insolvency or that severely compromise monetary policy.

This solvency constraint is related to the literature that assumes balance-sheet concerns on the part of the central bank. Isard (1994) presented a model of currency crises in which the central bank cares about the value of its foreign exchange reserves. More recently, Jeanne and Svensson (2007) assumed that the central bank has an objective function with a fixed loss suffered if the capital of the central bank falls below a critical level. Berriel and Bhattarai (2009) modeled balance sheet concerns by including a target for real capital in the central bank's loss function. These works assume ad-hoc preferences of the central bank against negative or even low levels of capital. Note that the solvency constraint (9) simply prevents the central banker from taking certain policy actions in certain situations, and says nothing about central bankers' preferences about capital adequacy. This is in line with Sims and Negro (2014), where low levels of capital may prevent a central bank from avoiding self-fulfilling hyperinflationary equilibria.

It is important to note that in equilibrium (8) and (9) result in

$$K_t = T_t \tag{10}$$

The nominal interest rate on the short-term bonds is subject to the zero lower bound,

$$i_t \geq 0 \tag{11}$$

It remains to specify which type of assets should be traded by the central bank when it changes monetary supply. The quantity of long-term bonds is determined by a policy rule of the form,

$$B_t = B(S_t) \tag{12}$$

where S_t summarizes the state of the economy. In the next sections we analyze how different specifications of (12) affect equilibrium allocations. Note that, given policy rule (12), equation (6) determines the size of short-term bonds, B_t^s, that results in the desired money supply, M_t.

We assume that the central bank has an objective function corresponding to a price-level targeting regime. The central bank's intertemporal loss function can be written as

$$L_t = \mathbb{E} \sum_{i=0}^{\infty} \beta^i l_{t+i} \tag{13}$$

where $l_t = \hat{p}_t^2$ and $\hat{p}_t$ is the log deviation of the price level P_t from the target $p^* = 1$. In the quantitative model below, we drop the price-level target and assume a more realistic loss function as weighted squared deviations of output gap and inflation from the respective efficient levels.

2.4.2 The treasury and fiscal policy

Instead of specifying a rule that determines the composition of outstanding debt - between the two different types of securities that might be issued - we simply assume that the treasury will supply the required quantity of securities necessary to clear both bond markets. Also, for simplicity we abstract from government spending. The treasury's budget constraint can be written as,

$$T_t + Z_t + \frac{1}{1+i_t} B_t^s + Q_t(B_t + B_t^{hh}) = (B_{t-1}^s + B_{t-1}^{hh,s}) + (1+Q_t)(B_{t-1} + B_{t-1}^{hh}) \tag{14}$$

We specify fiscal policy in terms of a rule that determines the evolution of lump-sum taxes collected by the treasury, Z_t,

$$Z_t = \phi(B_t^{hh} + B_t + B_t^{hh,s} + B_t^s) \tag{15}$$

and choose ϕ so that the fiscal policy is passive.[7]

2.5 Equilibrium

Consider the set of equations[8],

$$\hat{y}_t = \hat{y}_{t+1|t} - [i_t - (\hat{p}_{t+1|t} - \hat{p}_t) - \rho] \tag{16}$$

$$\hat{q}_t = \beta \hat{q}_{t+1|t} - (i_t - \rho) \tag{17}$$

$$\hat{m}_t \begin{cases} = \hat{p}_t + \hat{y}_t & \text{if } i_t > 0 \\ \geq \delta & \text{if } i_t = 0 \end{cases} \tag{18}$$

$$\hat{t}_t = \begin{cases} \hat{k} & \text{if } \hat{k}_t \geq \underline{\mathrm{k}} \\ -1 & \text{if } \hat{k}_t < 0 \end{cases} \tag{19}$$

$$\hat{m}_t = \rho(\hat{t}_t - \hat{k}_t) + \bar{b}^s \hat{b}_t^s + \bar{b}(\hat{q}_t + \hat{b}_t) \tag{20}$$

$$\hat{b}_t = b(\hat{s}_t) \tag{21}$$

$$\hat{k}_t = \hat{k}_{t-1} - \hat{t}_{t-1} + \bar{b}^s \hat{i}_{t-1} + \bar{b}^s \hat{b}^s{}_{t-1} + \bar{b}\hat{b}_{t-1} + \rho^{-1}\bar{b}\,(\hat{q}_t - \hat{q}_{t-1}) \tag{22}$$

$$\hat{k}_t \geq -\underline{\mathrm{k}} \tag{23}$$

$$i_t \geq 0 \tag{24}$$

where $\bar{b} = \frac{b^*}{y^*}\frac{\beta}{1-\beta}$, $\bar{b}^s = \frac{b^{s*}}{y^*}$.

Equations (16) and (17) are the log-linearized around the zero-inflation steady state versions of the household's first-order condition with respect to the short and the long-term bonds.

[7] We use the terminology of Leeper (1991).

[8] Where $\hat{x}_t$ is the log-deviation of variable X around its zero-inflation steady state, i_t is the nominal interest rate ($log(1+i_t)$) and $\rho \equiv log(\beta^{-1})$. The Appendix provides a detailed derivation of the zero-inflation steady and log-linearized equations

Equations (18) and (19) are the money demand induced by the cash-in-advance constraint and the transfers rule, respectively. Equations (20) and (21) determine which type of asset the central bank is acquiring or disposing of when it changes the money supply. Equation (22) is the log-linearized law of motion of the central bank's capital and (23) and (24), are the non-linear restrictions of our model, the solvency constraint and the zero lower bound.

The log-linearized endowment process,

$$(\hat{y}_1, \hat{y}_2, \hat{y}_3, ..., \hat{y}_{N-1}, \hat{y}_N, \hat{y}_{N+1}, ...) = \begin{cases} (\bar{y}, 0, 0, ...) & \text{with probability } 1-\mu \\ (\bar{y}, 0, \underline{y}, ..., \underline{y}, 0, 0, ...) & \text{with probability } \mu \end{cases} \tag{25}$$

Definition 1 *We define a discretion equilibrium as a sequence for prices* $\{\hat{p}_t, i_t, \hat{q}_t\}$ *and quantities* $\{\hat{m}_t, \hat{k}_t, \hat{b}_t^s, \hat{b}_t\}$ *as functions of the stochastic process* $\{\hat{y}_t\}$ *such that the central bank's intertemporal loss function* (13) *is minimized every period subject to* (16)-(24) *when the central bank cannot commit to future policies.*

Definition 2 *We define a commitment equilibrium as a sequence for prices* $\{\hat{p}_t, i_t, \hat{q}_t\}$ *and quantities* $\{\hat{m}_t, \hat{k}_t, \hat{b}_t^s, \hat{b}_t\}$ *as functions of the stochastic process* $\{\hat{y}_t\}$ *such that the central bank's intertemporal loss function* (13) *is minimized in period 1 subject to* (16)-(24) *when the central bank can commit to future policies.*

3 Revisiting the Literature

In this section we assume that the central bank is perfectly backed by the treasury ($\underline{k} \to \infty$) and hence there is no solvency constraint. We then derive equilibrium allocations under discretion and commitment. We show that under discretion the economy falls into a liquidity trap in period 1 since the central bank cannot credibly commit to a higher price level target in period 2. This result closely relates to Krugman (1998): a fall in expected income can lead to deflation even with zero nominal interest rate and despite the size of the money supply because people want to save more than the economy can absorb. Since the central bank cannot commit to a higher price level in period 2, it is forced to deflate in period 1 to inflate in the next period, providing the necessary negative real interest rate. Then we revisit the result in Eggertsson and Woodford (2003), in which deflation in period 1 can be avoided because the central bank is able to commit to a larger future money supply.

Remark 1 *If* $\underline{k} \to \infty$*, the set of relevant restrictions to the central bank's minimization problem reduces to* (16) *and* (24)*.*

Notation 1 *Let* s_t^i *denote the nodes of the exogenous income process* (25)*, for* $i \in \{l, h\}$*. Where "h" indicates the realization of the high-income path while "l" represents the low-income.*

3.1 Equilibrium under Discretion

In this endowment economy, it is intuitive to think in terms of an equilibrium real interest rate, which the economy will deliver whatever the behavior of nominal prices. In "normal" times, when expected income growth is non-negative, the equilibrium real interest rate is positive and policymakers have no trouble in implementing the interest rate required by the price-level target. According to the income process, this will be the case from period 2 onward if the high-income state occurs and from period 3 onward, otherwise,

$$r_t = i_t - (\hat{p}_{t+1|t} - \hat{p}_t) = \rho > 0 \qquad \text{for } s_t^i \text{ for all } 3 \leq t < N \text{ and } i \in \{l, h\}, \text{ and } s_2^h,$$

and one can immediately guess at the solution: the price level, expectations and the nominal interest rate will remain constant at $\hat{p}_{t+1|t} = \hat{p}_t = 0$ and $i_t = \rho$.

However, when expected income growth is negative, we have a version of the credibility problem as in Krugman (1998). This is the case in the low-income state of period 2 if $\underline{y} < -\rho$ and in period 1 if $\bar{y} > \rho$. First consider the former case,

$$r_2^l = i_2^l - (\underbrace{\hat{p}_{3|2}^l}_{=0} - \hat{p}_2^l) = \rho + \underline{y}$$

so if the central bank cannot commit to a higher price level target in period 3, the price-level falls regardless of the current money supply, because any excess money will simply be kept rather then added to spending. This happens because once the nominal rate reaches zero, money and other riskless assets in the economy become perfect substitutes and no matter how much liquidity the central bank injects in the economy, it cannot affect these assets' prices, as in Wallace (1981). Therefore, the central bank can no longer affect the nominal interest rate and hence cannot provide further incentive to spending. To achieve the negative equilibrium real interest rate, the economy must deflate now in order to provide inflation later. As a result, if $\underline{y} < -\rho$, $i_2^l = 0$ and $\hat{p}_2^l = \rho + \underline{y} < 0$.

In period 1 the deflationary scenario repeats if the expected fall in income is substantial,

$$r_1 = i_1 - (\underbrace{\hat{p}_{2|1}}_{=\mu(\rho+\underline{y})} - \hat{p}_1) = \rho - \bar{y}$$

If $\bar{y} > \rho$, the equilibrium real interest rate is negative and the zero lower bound binds, $i_1 = 0$. In addition to the negative equilibrium real interest rate, in period 1 the economy expects low (below the target) price level in the low-income state of period 2, which creates the need for an even lower price level in period 1 to achieve the equilibrium real interest rate, $\hat{p}_1 = \rho - \bar{y} + \mu(\rho + \underline{y}) < 0$.

Deflation in this endowment economy is costless. In a sticky price production economy, deflation increases real wages and leads to production inefficiencies. In this case, the central bank is better off if it could commit to raise the price level in period 2 to avoid deflation in period 1. However, if expectations are rational, the private sector anticipates the central bank's lack of incentives to keep the inflationary commitment. In this awkward situation, the monetary authority has no conventional tools to fight deflation.

We illustrate these results by calibrating the model and plotting the state-contingent path of the nominal interest rate and the price level from period 0 to period 4. In Figure 2, the dashed red line shows the evolution of these variables in the high-income state of the income process ($s = s^h$) and the blue dashed line in the low-income state ($s = s^l$). The inability of the central bank to set a negative nominal interest rate results in deflation in period 1. Since there is 50 percent chance that the equilibrium real interest rate will remain negative in the next period, this creates expectation of future deflation - as shown by the dashed green line - which creates even more deflation in period 1. Even if the central bank lowers the short-term nominal interest rate to zero, the real rate of return is positive because the private sector expects deflation.

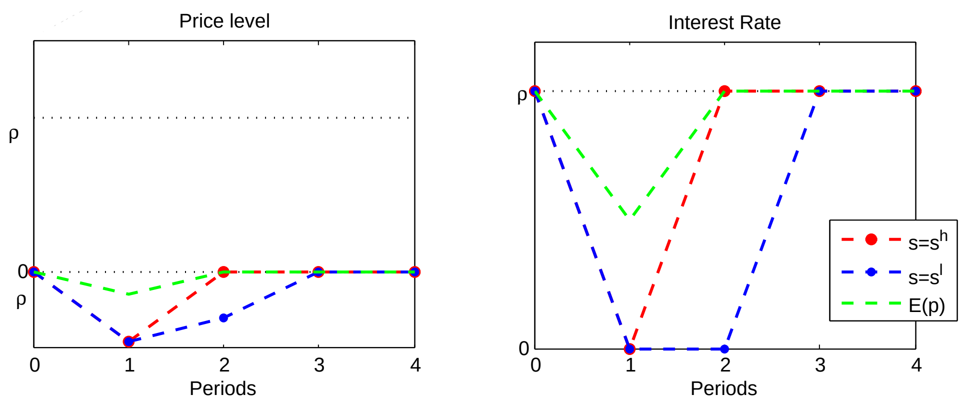

Figure 2: The Discretion Equilibrium. In the calibrated model we interpret periods as quarters, and assume coefficient values of $\beta = 0.99$, $\rho = log(\beta^{-1})$, $\underline{y} = -1.3\rho$, $\bar{y} = 1.3\rho$, $y^* = \rho$ and $\mu = 1/2$.

In the Appendix we provide the detailed derivation of these results.

3.2 Equilibrium under Commitment

Eggertsson and Woodford (2003) designed the optimal monetary policy under commitment when the natural interest rate becomes unexpectedly negative in period 1 and reverts back to steady state with fixed probability every period. This policy involves committing to the creation of an output boom once the natural rate again becomes positive, and hence to the creation of future inflation. In their numerical experiment, the state contingent nominal interest rate falls to zero immediately after the natural rate turns negative and stays low a few periods after the natural rate becomes positive again. Hence the central bank delivers the inflation and output boom with which it had promised - so as to make real interest rates negative - when the economy entered the liquidity trap.

We derive an analytical (comparable to Eggertsson and Woodford (2003)) solution in a simplified version of the commitment problem: we assume that the central bank is able to commit to a price level only in period 2. Despite its simplicity, this version of the commitment problem captures all the action behind the optimal policy in the liquidity trap in Eggertsson and Woodford (2003). The simplified problem is,[9]

$$\begin{aligned} \underset{i_1, i_2^l, i_2^h \geq 0}{\text{minimize}} \quad & \frac{1}{2}\left[(\hat{p}_1)^2 + \beta\mu(\hat{p}_2^l)^2 + \beta(1-\mu)(\hat{p}_2^h)^2\right] \\ \text{s.t.} \quad & i_1 + \hat{p}_1 = \mu\hat{p}_2^l + (1-\mu)\hat{p}_2^h + \rho - \bar{y} \\ & i_2^l + \hat{p}_2^l = \rho + \underline{y} \\ & i_2^h + \hat{p}_2^h = \rho \end{aligned} \tag{26}$$

In the Appendix we provide the analytical solution for this problem. Here, to illustrate the results we calibrate the model and plot the commitment equilibrium for the nominal interest rate and the price level.[10]

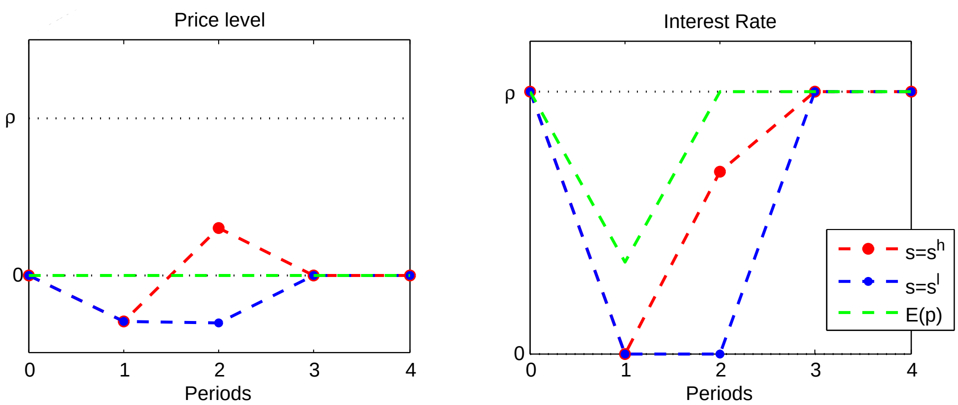

Figure 3: The Commitment Equilibrium. In the calibrated model we interpret periods as quarters, and assume coefficient values of $\beta = 0.99$, $\rho = log(\beta^{-1})$, $\underline{y} = -1.3\rho$, $\bar{y} = 1.3\rho$, $y^* = \rho$ and $\mu = 1/2$.

Figure 3 shows the optimal price level from period 0 to period 4. Observe that the optimal policy involves committing to the creation of inflation once the equilibrium real rate again becomes positive in the high-income state of period 2. Such a commitment stimulates spending and reduces deflationary pressures while the economy remains in the liquidity trap. Inflation expectations lower the interest rate, even when the nominal interest rate cannot be reduced. In contrast with the discretionary equilibrium, the dashed green line shows that expected period 2

[9] Because the central bank cannot commit to particular price levels for periods $3 \leq t < N$, the public will simply expect $\hat{p}_{t|1} = 0$ for all $3 \leq t < N$ and the central bank has no reason to deviate from these expectations. As a result we can disregard periods $3 \leq t < N$, and write the commitment problem in this simple form. Moreover, because expectations are fulfilled in a commitment equilibrium, we replace expected period 2 price level by their actual values.

[10] In the calibrated model we interpret periods as quarters, and assume coefficient values of $\beta = 0.99$, $\rho = log(\beta^{-1})$, $\underline{y} = -1.3\rho$, $\bar{y} = 1.3\rho$ and $\mu = 1/2$

price level is not negative in period 1. As a result, deflation in period 1 is mitigated.

This figure also shows the corresponding state-contingent nominal interest rate under the optimal commitment, and contrasts it to the evolution of the nominal interest rate under a discretionary equilibrium. To increase inflation expectations in the trap, the central bank commits to keeping the nominal interest rates below the zero-inflation steady state in s_2^h. As in Eggertsson and Woodford (2003), this is an example of history-dependent policy, in which the central bank commits to raise the interest rates slowly at the time the equilibrium real interest rate becomes positive in order to affect expectations when the zero bound is binding.

3.3 The Credibility Problem

The commitment equilibrium is time-inconsistent. Despite the central bank's inclination in period 1 to commit to a higher price-level target in the high-income state of period 2, it will have no incentives to keep this commitment when it is called to do so. The reason is very simple. Denote by "c" and "d" the values of the period loss function (13) in the commitment and discretion equilibriums, respectively. Evidently, $L_1^d > L_1^c$ and $L_2^c > L_2^d = 0$. Because of the pure forward looking nature of this model, in period 2 the central bank faces the same restrictions regardless of its actions in period 1. Hence, the central bank will be tempted to implement the commitment outcome in period 1 and the discretion in period 2. Because expectations are rational, the private sector anticipates the central bank's lack of incentives to keep its word, expects the zero-inflation target for period 2 and the commitment equilibrium can not be achieved.

4 Fiscally Constrained Central Bank and Quantitative Easing

In this section we assume that $\underline{\mathrm{k}} < \infty$ so that the solvency constraint is a relevant restriction to the economy's equilibrium. We show that a long-term bond purchasing program (or QE) can help mitigate the deflation even under a discretionary equilibrium. This change in balance sheet composition provides an incentive to the central bank to keep interest rates low in period 2 to avoid losses in its balance sheet.

More specifically, we show that for a given loss limit, $\underline{\mathrm{k}}$, there is a level of steady state long-term bond holdings $b_k^* > 0$ such that for all $b^* \geq b_k^*$, if the zero lower bound is binding in period 1 then the solvency constraint is binding, at least, in the high-endowment state of period 2. As a consequence, binding solvency constraints in states in which the equilibrium real interest rate is positive imply price levels above the target. Inflation in the high-endowment state of period 2 increases expected inflation and lowers the real interest rate in period 1.

Forward iteration of (17) and substitution in (22) results in

$$\hat{k}_t = \bar{b}\hat{b}_{t-1} + \bar{b}\rho^{-1} \sum_{i=0}^{\infty} \beta^i \left(i_{t-1+i|t-1} - i_{t+i|t} \right) \tag{27}$$

Equation (27) expresses how the short-run interest rate path and private sector's expectations about it affect the central bank's capital. The central bank suffers capital losses every time it sets the interest rate above what was expected by the private sector in the previous period. This fact together with the solvency constraint will be a useful mechanism to shape private sector expectations.

In addition, we specify the simplest possible policy rule for open market operations with long-term bonds (21)

$$\hat{b}_t = 0 \quad \text{for all } t \tag{28}$$

Under process (28), the central bank holds (in level) b^* units of long-term bonds in its balance sheet for all periods. We interpret (28) as an LSAP program, or QE, in which the central bank establishes a target for long-term bond holdings and conducts monthly purchases to achieve this target. We assume $0 < b^* < \rho y^*$. This implies (i) $\bar{b}^s > 0$, so the policy rule (28) does not constrain the central bank's ability to control the money supply (see equation (20)); (ii) $\bar{b} = \frac{b^*}{y^*}\frac{\beta}{1-\beta} > 0$, so that the central bank's capital level depends crucially on short-term interest rates and expectations. Note that higher $\bar{b}$ means higher holdings of long-term bonds and hence higher exposure to interest-rate risk.

We solve the model from backwards to assure that expectations are consistent.

4.1 Third Period Onward

For all periods $3 \leq t < N$, the real interest rate is given by $\rho > 0$. In this case the central bank sets $i_t = \rho$ to peg $\hat{p}_t = 0$ unless the solvency constraint prevents it from doing so. We show that this is not the case if $\bar{b} \leq \underline{\mathrm{k}}$. To see this, assume $i_t = \rho$ for all $t \geq 3$ and note that from period $t = 2$ onward all uncertainty in the model has been settled and hence perfect foresight applies. In this case $i_{t|3} = i_{t|2} = \rho$ for all $t \geq 3$. Hence,

$$\begin{aligned}
\hat{k}_3 &= \bar{b}\rho^{-1}\sum_{i=0}^{N}\beta^i\left(i_{2+i|2} - i_{3+i|3}\right) \\
&\simeq \bar{b}\rho^{-1}(i_2 - \rho) \qquad \text{if } N \text{ is large} \\
&\geq -\bar{b} \geq -\underline{\mathrm{k}}
\end{aligned}$$

Proposition 1 *Assume that $\bar{b} \leq \underline{k} < \infty$, N is large and the central bank adopts a price-level targeting regime and conducts long-term bond purchases according to policy rule* (28)*. Under discretion, for all $t \geq 3$, $\{\hat{p}_t, i_t\} = \{0, \rho\}$ independently of the realization of the income process.*[11]

4.2 Low-Income State of the Second Period

Proposition (1) implies $i_{t|2} = i_{t|1} = \rho$ for all $3 \leq t < N$. Again, we combine this with the central bank's capital equation (27), the solvency constraint (23) and policy rule (28) to yield,

[11] Why do we impose $\bar{b} \leq \underline{\mathrm{k}}$? It is just a simplifying assumption; when it holds, we are sure that the solvency constraint is never binding for $3 \leq t < N$ and it is very useful because the state variables - interest rate expectations - will no longer influence equilibrium allocations. Naturally, this does not affect the qualitative meaning of our results. In fact, as the calibrated economy of Figure 8 shows, if $\bar{b} > \underline{\mathrm{k}}$ the capital solvency is binding at least in period 3 and the QE becomes even more effective to fight deflation.

$$i_2^l \leq i_1 + \beta(i_{2|1} - \rho) + \frac{\rho \underline{\mathrm{k}}}{\bar{b}} \tag{29}$$

The central bank's solvency constraint is satisfied if and only if (29) holds. The intuition behind this is clear. All sources of capital variation come from $\hat{q}_2 - \hat{q}_1$. Because bond prices and interest rates move in opposite directions, the solvency constraint imposes an upper bound to i_2^l. This upper bound is loose when i_1 and $i_{2|1}$ are high because it means low $\hat{q}_1$. High $\underline{\mathrm{k}}$ means that the treasury is allowed to back the central bank even after high capital losses and hence the upper bound on i_2^l is relaxed. High $\bar{b}$ means a risky balance sheet exposure to interest rates and hence a smaller range for i_2^l.

Because the price levels from period 3 onward have already been determined and do not depend on particular realizations of state $(i_2, i_{3|2})$, minimization of intertemporal loss function (13) is equivalent to minimizing the period loss function l_2. Hence, the central bank's problem is to choose i_2^l so as to minimize l_2 subject to the economy's constraints, given state variables $i_{2|1}$ and i_1, and expectation of next period price level, $\hat{p}_{3|2} = 0$.

$$\begin{aligned}
&\text{minimize} && \frac{1}{2}(\hat{p}_2^l)^2 \\
&\text{s.t.} && r_2 = i_2^l - (\hat{p}_{3|2} - \hat{p}_2^l) = \rho + \underline{\mathrm{y}} \\
& && i_2^l \leq i_1 + \beta(i_{2|1} - \rho) + \frac{\rho k}{\bar{b}} \\
& && i_2^l \geq 0 \\
& && \text{given } i_1, i_{2|1} \geq 0 \text{ and } \hat{p}_{3|2} = 0
\end{aligned}$$

Proposition 2 *Assume that $\underline{y} < -\rho$, $\bar{b} \leq \underline{k} < \infty$, N is large and that the central bank adopts a price-level targeting regime and conducts long-term bond purchases according to policy rule* (28)*. Under discretion, the central bank's policy functions are,*

$$i_2^l(i_{2|1}, i_1) = 0 \tag{30}$$

$$\hat{p}_2^l(i_{2|1}, i_1) = \rho + \underline{y} \tag{31}$$

A negative real interest rate and predetermined price level expectation pushes the economy against the zero lower bound if the solvency constraint does not prevent the central bank from doing so. Note however that if $\bar{b} \leq \underline{\mathrm{k}}$, then $i_2^l = 0$ satisfies the solvency constraint for any value of the state variables $(i_1, i_{2|1})$.

4.3 High-Income State of the Second Period

In the high-income state of period 2, the central bank solves the same problem but now it faces a positive equilibrium real interest rate,

$$
\begin{aligned}
&\text{minimize} && \frac{1}{2}(\hat{p}_2^h)^2 && (32)\\
&\text{s.t.} && r_2 = i_2^h - (\hat{p}_{3|2} - \hat{p}_2^h) = \rho && (33)\\
& && i_2^h \leq i_1 + \beta(i_{2|1} - \rho) + \frac{\rho \underline{k}}{\bar{b}} && (34)\\
& && i_2^h \geq 0 && (35)\\
& && \text{given } i_1, i_{2|1} \geq 0 \text{ and } \hat{p}_{3|2} = 0 && (36)
\end{aligned}
$$

Proposition 3 *Assume $\bar{b} \leq \underline{k} < \infty$, N is large and that the central bank adopts a price-level targeting regime, conducts long-term bond purchases according to policy rule* (28)*. Under discretion, the central bank's policy functions are,*

$$
i_2^h(i_1, i_{2|1}) = \begin{cases} i_1 + \beta(i_{2|1} - \rho) + \frac{\rho k}{b} & \text{if } i_1 + \beta i_{2|1} \leq \rho(1 + \beta - \frac{k}{b}) \\ \rho & \text{if } i_1 + \beta i_{2|1} > \rho(1 + \beta - \frac{k}{b}) \end{cases} \tag{37}
$$

$$
\hat{p}_2^h(i_1, i_{2|1}) = \begin{cases} \rho(1 - \frac{k}{b}) - i_1 - \beta(i_{2|1} - \rho) & \text{if } i_1 + \beta i_{2|1} \leq \rho(1 + \beta - \frac{k}{b}) \\ 0 & \text{if } i_1 + \beta i_{2|1} > \rho(1 + \beta - \frac{k}{b}) \end{cases} \tag{38}
$$

The proof in the Appendix.

In the high-income state of period 2 the relevant non-linear constraint is the solvency constraint. The positive real interest rate in this state provides the central bank the incentive to rise the interest rate to reach the price level target. The optimal policy is $i_2^h = \rho$ and $\hat{p}_2^h = 0$ but it is only feasible for sufficiently high values of state variables i_1 and $i_{2|1}$,

$$
\hat{k}_2^h(i_1, i_{2|1}) \begin{cases} = -\underline{k} & \text{if } i_1 + \beta i_{2|1} \leq \rho(1 + \beta - \frac{k}{b}) \\ > -\underline{k} & \text{if } i_1 + \beta i_{2|1} > \rho(1 + \beta - \frac{k}{b}) \end{cases} \tag{39}
$$

As (39) indicates, when $i_1 + \beta i_{2|1} \leq \rho(1 + \beta - \frac{k}{b})$ the solvency constraint binds and prevents the central bank from raising the nominal interest rate all the way, resulting in an undesired high price level.

4.4 First Period

In the first period agents condition their expectations about period 2 interest rate and price level according to,

$$
i_{2|1} = \mu i_2^l(i_1, i_{2|1}) + (1 - \mu) i_2^h(i_1, i_{2|1}) \tag{40}
$$

$$
\hat{p}_{2|1} = \mu \hat{p}_2^l(i_1, i_{2|1}) + (1 - \mu) \hat{p}_2^h(i_1, i_{2|1}) \tag{41}
$$

Since equations (40) and (41) always hold in equilibrium, we use them to eliminate $i_{2|1}$ from the state space in the second period and rewrite (37) and (38) in the simpler form,

$$
i_2^h(i_1) = \begin{cases} \frac{1}{1-\beta(1-\mu)} \left[i_1 + \rho\left(\frac{k}{b} - \beta\right)\right] & \text{if } i_1 \leq \rho(1 + \mu\beta - \frac{k}{b}) \\ \rho & \text{if } i_1 > \rho(1 + \mu\beta - \frac{k}{b}) \end{cases} \tag{42}
$$

$$\hat{p}_2^h(i_1) = \begin{cases} \rho - \frac{1}{1-\beta(1-\mu)}\left[i_1 + \rho\left(\frac{k}{b} - \beta\right)\right] & \text{if } i_1 \le \rho(1+\mu\beta - \frac{k}{b}) \\ 0 & \text{if } i_1 > \rho(1+\mu\beta - \frac{k}{b}) \end{cases} \tag{43}$$

$$\hat{k}_2^h(i_1) \begin{cases} = -\underline{\mathrm{k}} & i_1 \le \rho(1+\mu\beta - \frac{k}{b}) \\ > -\underline{\mathrm{k}} & i_1 > \rho(1+\mu\beta - \frac{k}{b}) \end{cases}$$

Since agents are informed about the drop in endowment only in period 1, the economy is in steady state from period 0 backwards, it implies $i_{t|0} = \rho$ for $t \ge 0$. This fact helps to write the central bank's capital in period 1 as,

$$\begin{aligned} \hat{k}_1 &= \bar{b}\rho^{-1}\left[i_0 - i_1 + \beta(i_{1|0} - i_{2|1}) + \beta^2(i_{2|0} - \rho) + ...\right] \\ &= \bar{b}\rho^{-1}\left[\rho - i_1 + \beta(\rho - i_{2|1})\right] \\ &\ge \bar{b}\rho^{-1}(\rho - i_1) \\ &\ge 0 > -\underline{\mathrm{k}} \qquad \text{if } i_1 \le \rho \end{aligned}$$

Since in period 0 the private sector expects steady state interest rates for a long period, capital losses can only occur if $i_1 > \rho$, which is clearly suboptimal in this set up. Hence, the solvency constraint can be ignored. The central bank then sets i_1 to minimize the intertemporal loss function taking into to account that its actions in period 1 affect the state of the economy in period 2 and thus affect the private sector expectations according to policy (43),

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\left[(\hat{p}_1)^2 + \beta(1-\mu)(\hat{p}_2^h)^2\right] \\ &\text{s.t.} && r_1 = i_1 - \left[\mu\hat{p}_2^l(i_1) + (1-\mu)\hat{p}_2^h(i_1) - \hat{p}_1\right] = \rho - \bar{y} \\ & && i_1 \ge 0 \text{ and } (43) \end{aligned}$$

Proposition 4 *Assume that $\underline{k} \ge \bar{b}$, N is large and that the central bank adopts a price-level targeting regime and conducts long-term bond purchases according to policy rule* (28). *Under discretion, the central bank's policy rate in the first period is,*

$$i_1 = \begin{cases} (1+\mu)\rho - \Delta y^e & \textit{if } \Delta y^e \le \rho\left[\frac{k}{b} + (1-\beta)\mu\right] \\ \frac{1-\beta(1-\mu)}{1+(1-\beta)(1-\mu)}\left[2\rho - \Delta y^e - \frac{\rho(1-\mu)}{1-\beta(1-\mu)}\left(\frac{k}{b} - \beta\right)\right] & \\ \qquad \textit{if } \rho\left[\frac{k}{b} + (1-\beta)\mu\right] \le \Delta y^e \le \rho\left[2 - \frac{1-\mu}{1-\beta(1-\mu)}\left(\frac{k}{b} - \beta\right)\right] & \\ 0 & \textit{if } \Delta y^e > \rho\left[2 - \frac{1-\mu}{1-\beta(1-\mu)}\left(\frac{k}{b} - \beta\right)\right] \end{cases} \tag{44}$$

where $\Delta y^e \equiv \bar{y} - \mu\underline{y}$. The proof is in the Appendix.

Proposition 4 makes clear that, even without major losses in period 1, interest rates may be restricted. This is explained by a precautionary action of the central bank to protect its balance sheet against losses in period 2 onwards since these losses may drive the central bank to its solvency constraint.

5 Results

We illustrate the results derived in sections 3 and 4 by calibrating the model economy for three different compositions of the central bank balance sheet. For each calibration we compare the equilibrium outcome of key variables of these specifications with the usual discretion and commitment outcomes. In the calibrated model we interpret periods as quarters, and assume coefficient values of $\beta = 0.99$, $\rho = log(\beta^{-1})$, $\underline{y} = -1.3\rho$, $\bar{y} = 1.3\rho$, $y^* = \rho$, $\mu = 1/2$ and $k = 0.9$.

In the first experiment we chose $\bar{b}$ so that the steady-state ratio between short and long-term bonds held by the central bank, b^*/b^{s*}, is equal to 0.09. Figure 4 plots the state-contingent equilibrium paths for the price level, interest rate, long-term bond price and the central bank's capital level: the red dashed line shows the evolution of these variables in the high-income state of the income process while the blue dashed line represents the low-income states. Observe that deflation in period 1 is lower here in comparison with both the discretion and commitment equilibrium. This happens because the solvency constraint prevents the central bank from raising the interest rate in the high-income state of period 2, as shown by the dashed red line in the bottom-right plot of Figure 4. In this state the positive equilibrium real interest rate provides the central bank with the incentive to raise the nominal interest rate. However, because increases in the nominal rate entail declines in long-term bond price and, thus, losses in the central bank balance sheet, the central bank is only able to raise the nominal rate up to the point that the solvency constraint binds (roughly half way through). As a result, the price-level stays substantially above the target providing the required inflation expectation in period 1 without the need for a large price fall.

Note that this effect is quite strong even though the ratio $b^*/b^{s*} = 0.09$ is relatively small. This happens because long-term bonds in this model have infinite duration and hence even small variations in the nominal interest rate have substantial impact on bonds' prices.[12]

Despite low deflation in period 1, the equilibrium allocation in Figure 4 differs from the commitment outcome and thus is not optimal. The reason is that the marginal cost of inflation in the high-income state of period 2 exceeds the marginal benefit that it generates in reducing deflation in period 1. Taking this into account, we ask if there is a specific composition of the central bank balance sheet that generates, under discretion, the exact, or at least approximate, optimal commitment solution. The answer is yes. In Figure 5 we repeat the exercise but now we calibrate the steady-state ratio between short and long-term bonds held by the central bank, b^*/b^{s*}, to equal 2%. The equilibrium results of this calibration almost exactly replicate the commitment equilibrium. The central bank is carrying the precise risk on its balance sheet so that the solvency constraint in the high-state of period 2 binds exactly at the optimal interest rate. This generates the optimal level of inflation in period 2 and hence the optimal deflation in period 1.

[12]In the quantitative model of section 6, with perpetuities of finite duration, the required ratio between long and short-term bonds is significantly higher.

Lastly, Figure 6 plots the discretion equilibrium when the steady-state ratio between short and long-term bonds held by the central bank is less than 1%. In this case, because the central bank's assets are nearly riskless, the solvency constraint does not prevent the bank from raising the interest rate all the way in the high-income state of the period 2. As a consequence, the economy suffers high deflation in period 1 just as in the conventional discretion equilibrium.

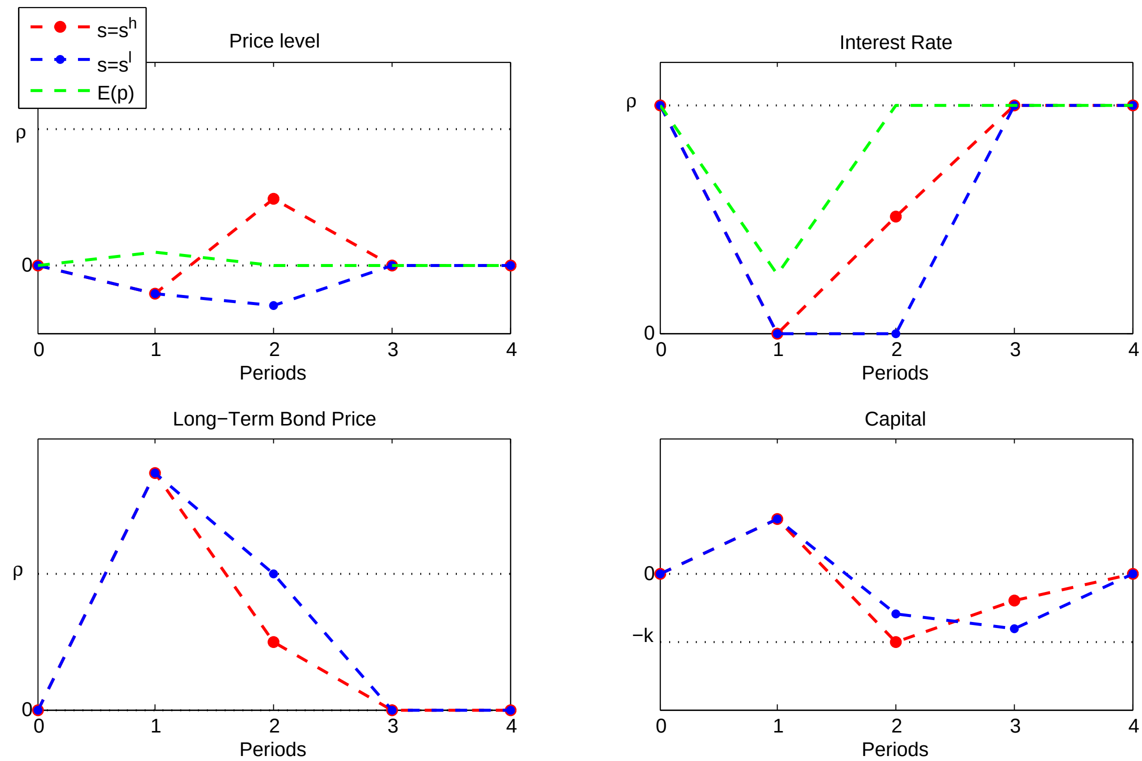

Figure 4: The Discretion Equilibrium with Quantitative Easing. In the calibrated model we interpret periods as quarters, and assume coefficient values of $\beta = 0.99$, $\rho = log(\beta^{-1})$, $\underline{y} = -1.3\rho$, $\bar{y} = 1.3\rho$, $y^* = \rho$, $\mu = 1/2$, $\underline{k} = 0.9$ and $b^*/b^{s*} = 0.09$.

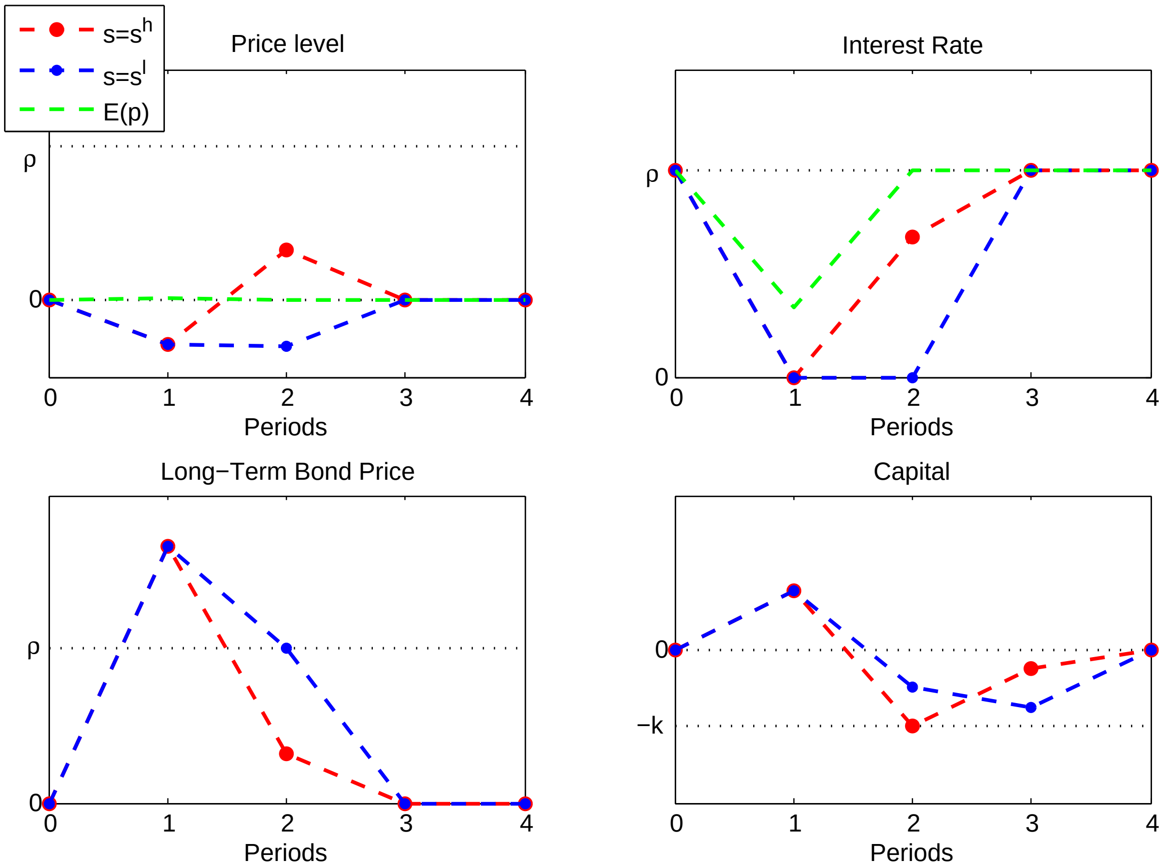

Figure 5: The Discretion Equilibrium with Quantitative Easing In the calibrated model we interpret periods as quarters, and assume coefficients values of $\beta = 0.99$, $\rho = log(\beta^{-1})$, $\underline{y} = -1.3\rho$, $\bar{y} = 1.3\rho$, $y^* = \rho$, $\mu = 1/2$, $\underline{k} = 0.9$ and $b^*/b^{s*} = 0.02$.

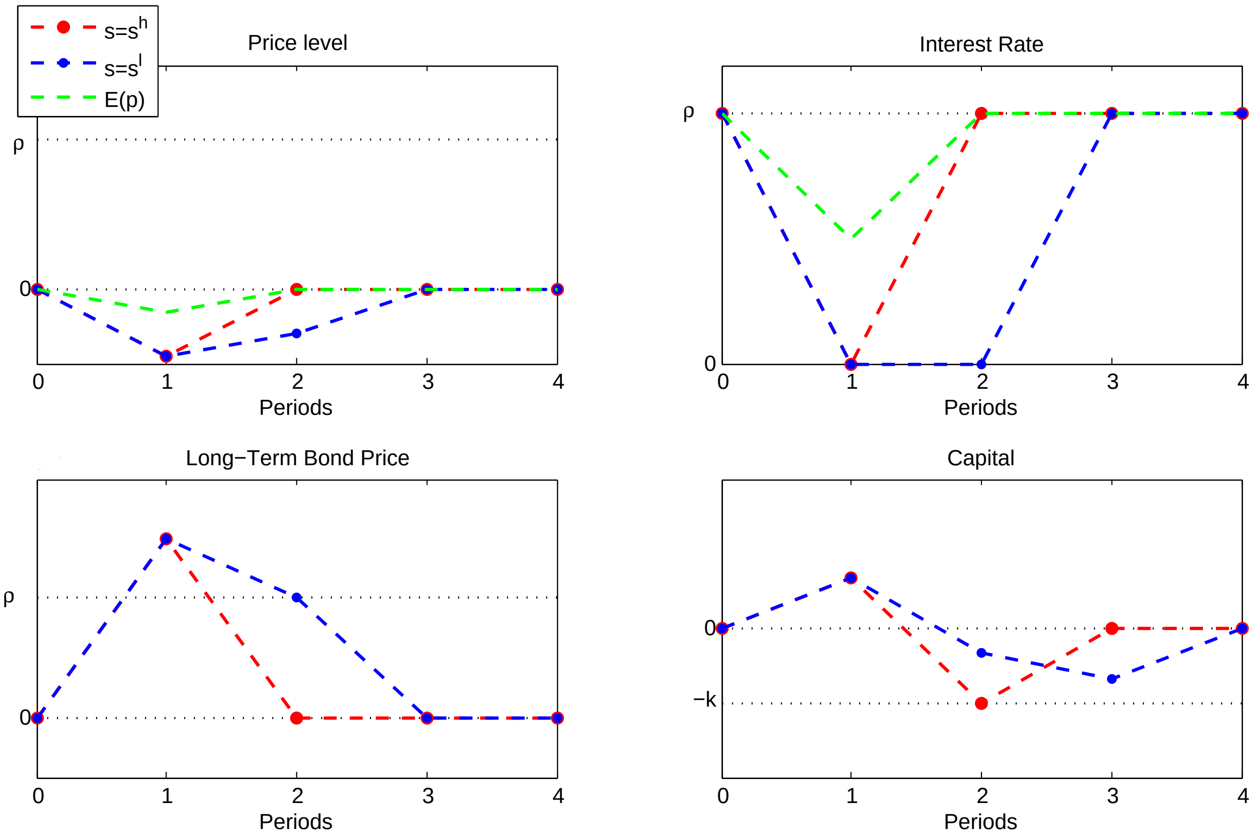

Figure 6: The Discretion Equilibrium with Quantitative Easing In the calibrated model we interpret periods as quarters, and assume coefficient values of $\beta = 0.99$, $\rho = log(\beta^{-1})$, $\underline{y} = -1.3\rho$, $\bar{y} = 1.3\rho$, $y^* = \rho$, $\mu = 1/2$, $\underline{k} = 0.9$ and $b^*/b^{s*} = 0.01$.

6 Quantitative Model

In this section we consider a closed economy with production. In this setup, we allow for a more realistic calibration of the central bank balance sheet and are able to analyze the impact of a central bank solvency constraint on real variables such as output and real interest rates in a liquidity trap.

As before, households consume and save by buying riskless claims on government debt. In this more general setup, however, the central bank conducts monetary policy by minimizing a standard quadratic loss function of inflation and the output gap.

Here, our focus is to show that our main insight applies in a fully dynamic model with production and sticky prices. In order to do that, we choose the basic workhorse model of the new-Keynesian literature, which is known to have limited quantitative implications. A full quantitative approach to the question of the role of central bank balance sheet restriction at the zero lower bound would require a more detailed model that, while interesting, would change the focus on the mechanism highlighted in this paper.

6.1 Model

In this section we consider the standard new-Keynesian closed-economy model as in Gali (2008), augmented in three dimensions: (i) the monetary authority is not completely financially backed by treasury; (ii) both consumers and central bank are allowed to trade with treasury-issued securities of different maturities; and (iii) nominal interest rates are subject to the zero lower bound. Since it has recently appeared extensively in the literature, we simply present the framework and do not derive all the structural equations.

6.1.1 Household and asset markets

Time is separated into discrete periods, $t = 0, 1, ...$ The economy has a private sector, consisting of a household and firms, and public sector, consisting of a central bank and a government. The household consumes and saves according the utility function,

$$\mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left[\frac{C_{t+1}^{1-\sigma}}{1-\sigma} + \frac{\theta}{1-b} \left(\frac{M_{t+i}}{P_{t+i}} \right)^{1-b} - \frac{N_{t+i}^{\varphi}}{1+\varphi} \right], \qquad \sigma, \varphi, \theta, b > 0$$

where $\mathbb{E}_t$ denotes expectation conditional on information available in period t, β is the discount factor, C_t denotes consumption of goods in period t, N_t denotes supply of labor and σ is the coefficient of risk aversion.

The consumption good, C_t, is a Dixit-Stiglitz composite of an infinity of varieties of mass one, each of them produced by a monopolist firm,

$$C_t \equiv \left[\int_0^1 c_{jt}^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}$$

ϵ_t is the consumer's elasticity of substitution over the varieties. The corresponding price index satisfies:

$$P_t \equiv \left[\int_0^1 p_{jt}^{1-\epsilon} dj \right]^{\frac{1}{1-\epsilon}}$$

The budget constraint in period t for the household is

$$C_t + \frac{Z_t}{P_t} + \sum_{s \in S} Q_t^s \frac{B_t^{hh,s}}{P_t} \leq \sum_{s \in S} \sum_{j=1}^{t} \left(\delta_s^{t-(j-1)} + Q_{t|t-j}^s \right) \frac{B_{t|t-j}^{hh,s}}{P_t} + \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}$$

where Z_t is a nominal lump-sum tax, N_t is aggregate labor supply, $Q_{t-j|t}^s$ and $B_{t-j|t}^{hh,s}$ are respectively the nominal price and the agent's holdings in period t of a perpetuity of type s issued by the treasury in period $t-j$. A perpetuity of type $s \in S = \{1, ..., S\}$ issued in period t pays δ_s^j dollars $j+1$ periods later, for each $j \geq 0$ and some decay factor $0 \leq \delta_s < 1$. The implied steady state duration of this bond is then $(1 - \beta\delta_s)^{-1}$. We fix $\delta_1 = 0$ to resemble a short-term bond that costs $Q_t^1 = \frac{1}{1+i_t}$ in period 1 and pays off one dollar in period $t+1$ where i_t is the implied short-term riskless interest rate. Finally, Π_t is aggregate firm nominal profits.

As in Woodford (2001), we can write $Q^s_{t+1|t-j} = \delta_s^{j+1} Q_t$ for all $j \geq 1$ and $s \in S$. This is very convenient since it implies that one needs to keep track, at each point in time, of the equilibrium price of only one type of bond. Then we can rewrite the household budget constraint as,

$$\frac{Z_t}{P_t} + C_t + \sum_{s \in S} \frac{B_t^{hh,s}}{P_t} Q_t^s \leq \sum_{s \in S} (1 + \delta_s Q_t^s) \frac{B_{t-1}^{hh,s}}{P_t} + \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}$$

We assume a non-Ponzi condition, where the real value of net wealth of private agent, NW_t, does not become arbitrarily negative. Adding a transversality condition, we get a boundary condition that the rate of growth of private wealth cannot exceed β^{-1},

$$lim_{i \to \infty} \mathbb{E}_t \beta^i \left[C_{t+i}^{-\sigma} NW_{t+i} \right] = 0$$

6.1.2 The firm

Each intermediate product j is produced by a single firm j with a technology that is linear in labor input with a exogenous stochastic process A_t:

$$Y_t(j) = A_t N_t(j)$$

where $N_t(j)$ denotes labor input in the production of intermediate good j. There is hence a continuum of firms producing intermediate goods. Aggregate labor supply and demand will be given by

$$N_t \equiv \int_0^1 N_t(j) dj$$

Prices are set as in Calvo (1983). Every period a firm is able to revise its price with probability $1 - \alpha$. The lottery that assigns rights to change prices is $i.i.d$ over time and across firms. Firm j's price, $p(j)$, is chosen so as to maximize the expected utility value of profits.

6.1.3 The central bank

The central bank trades securities issued by the treasury of all maturities. As clear below, the central bank does not rely completely on the treasury for its fiscal needs and hence it is subject to a period-by-period budget constraint:

$$K_t + M_t = \sum_{s \in S} Q_t^s B_t^s + T_t \tag{45}$$

$$K_t \equiv \sum_{s \in S} (1 + \delta_s Q_t^s) B_{t-1}^s - M_{t-1} \tag{46}$$

where B_t^s is the central bank's holdings in period t of a perpetuity of type s, T_t is the transfers from the central bank to the treasury, M_t is the outstanding monetary liabilities and K_t the capital level.

It is convenient to write (45) and (46) recursively as,

$$k_t = k_{t-1}\frac{P_{t-1}}{P_t} - t_{t-1}\frac{P_{t-1}}{P_t} + \sum_{s\in S}(1+\delta_s Q_t^s - Q_{t-1}^s)b_{t-1}^s\frac{P_{t-1}}{P_t} \tag{47}$$

here $k_t = \frac{K_t}{P_t}$, $b_t^s = \frac{B_t^s}{P_t}$ and $t_t = \frac{T_t}{P_t}$. Assume that the central bank transfers to the treasury are dictated by the rule,

$$t_t = \begin{cases} \tau k_t & \text{if } k_t > \underline{\text{k}} \\ 0 & \text{if } k_t < -\underline{\text{k}} \end{cases} \tag{48}$$

We can break down (48) into three cases, (i) the central bank transfers a fraction $0 < \tau \leq 1$ of its capital to the treasury when it is positive; (ii) the treasury capitalizes the central bank, restoring its solvency, when the capital is negative but does not violate the specified lower bound, $-\underline{\text{k}}$; and (iii) no transfer takes place and the central bank becomes insolvent when the its capital falls below $-\underline{\text{k}}$.

As discussed in section 2, we add the capital solvency constraint[13],

$$k_t \geq -\underline{\text{k}}$$

Finally, we specify a rule that determines which type of securities the central bank acquires (or sells) when it varies the money supply,

$$Q_t^s b_t^s = \eta_s(m_t + k_t - t_t) \quad \text{for } s \in S$$

$m_t = \frac{M_t}{P_t}$ and $\sum_{s\in S}\eta_s = 1$. Note that η's must add to one to assure that equation (46) holds in equilibrium.

6.1.4 The treasury

We specify fiscal policy in terms of a rule that determines the evolution of lump-sum taxes Z_t as a function of total government liabilities, D_t, here defined to be the outstanding non-monetary liabilities among the different types of securities that might be issued by the government. As in Davig and Leeper (2006), we suppose that the government adjusts lump-sum taxes - or primary fiscal surplus since we abstract from public consumption - in response to the gross value of nominal debt,

$$Z_t = \Omega D_{t-1}^{\zeta}$$

[13]Note that we are imposing the solvency constraint in real terms.

and $\zeta > 0$.

$$D_t = \sum_{s \in S} Q_t^s \left(B_t^{s,hh} + B_t^s \right)$$

We then write the fiscal budget constraint,

$$\sum_{s \in S} (1 + \delta_s Q_t^s)(B_{t-1}^{s,hh} + B_{t-1}^s) = D_t + T_t + Z_t$$

Here again, fiscal policy is passive and does not restrict monetary policy.

6.2 Equilibrium

In this section we log-linearize the household's first-order-condition, all budget constraints, policy rules and the non-linear constraints around the zero-inflation steady state.[14] In the Appendix, we provide detailed description of the equations that characterize the equilibrium. In short, we add to the standard model asset price relations for bonds with different maturities, central bank balance sheet rules and budget constraints, and two inequality restrictions, the zero lower bound on interest rates and the solvency constraint.

More specifically, by log-linearizing the household's first-order conditions with respect to the other types of securities we develop additional $S-1$ forward-looking equilibrium relations,

$$\hat{q}_t = \beta \delta_s \hat{q}_{t+1|t} - (i_t - \rho) \qquad \text{for } s = 2, ..., S \tag{49}$$

where these are asset pricing relations. The price of each security is the discounted pay-off plus the expected value of the security in the next period. At these prices, the household is willing to buy and sell any quantity of these assets.

We also present here the log-linearized policy rules that specify the quantity of each asset that should be acquired by the central bank each period,

$$\hat{b}_t^s = \left(\frac{\eta_s}{m^* + k^* - t^*} \right) \left[m^* \hat{m}_t + k^* \hat{k}_t - t^* \hat{t}_t \right] - \hat{q}_t^s \qquad \text{for all } s \in S \tag{50}$$

and the central bank's capital,

$$\hat{k}_t = \hat{k}_{t-1} - \hat{t}_{t-1} - (k^* - t^*)\pi_t + \sum_{s \in S} \omega_s \left[(\delta_s \hat{q}_t^s - \hat{q}_{t-1}^s) + \rho(\hat{b}_{t-1}^s - \pi_t) \right] \tag{51}$$

where starred variables denote steady-state values and $\omega_s = \frac{q_s^* b_s^*}{k^*}$. Note that $0 \leq w_s \leq 1$ denotes the steady-state relative importance of security of type s on the central bank's balance sheet

[14]The zero-inflation steady state and the linearization derivations are presented in detail in the Apppendix

composition and ρ is the steady-state net return on these securities. Also, we specify a rule for transfer from the treasury, which is zero if capital is excessively low:

$$\hat{t}_t = \begin{cases} \hat{k}_t & \text{if } \hat{k}_t \geq -\underline{\mathrm{k}} \\ 0 & \text{if } \hat{k}_t < -\underline{\mathrm{k}} \end{cases}$$

Finally, we show the two non-linear constraints, the zero lower bound on nominal interest rates and the solvency constraint:

$$i_t \geq 0 \tag{52}$$

$$\hat{k}_t \geq -\underline{\mathrm{k}} \tag{53}$$

As standard in this local equilibrium analysis, our economy can be represented in matrix notation as

$$H \begin{bmatrix} X_{t+1} \\ x_{t+1|t} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + Bi_t + C\epsilon_t. \tag{54}$$

along with inequalities (52) and (53). In this notation, X_t is the vector of the predetermined variables and x_t is the vector of forward-looking variables. Vector X_t defines the state of the economy in period t while x_t collects the non-predetermined variables in the model. They summarize the forward-looking aspect of private agents' behavior - the private sector's best response to the government's sequence of actions.

6.2.1 The central bank's loss function

The period welfare losses experienced by the household is given by the central bank's period loss function,

$$L_t = \frac{1}{2}\left[\pi_t^2 + \lambda x^2\right]$$

where $\lambda = \frac{\kappa}{\epsilon}$ is the weight the central bank attributes to output deviation from the target relative to inflation deviations. In matrix notation,

$$L_t = \frac{1}{2} x_t' W x_t \tag{55}$$

6.2.2 The discretion equilibrium

Here we consider an equilibrium that occurs when policy is conducted under discretion so that the government is unable to commit to any future actions. The idea behind this equilibrium concept is to define a set of state variables that directly affect market conditions and assume that the strategies of the two authorities as well as the private-sector expectations depend only on this state. Under this concept of equilibrium, the central bank problem is to choose a sequence $\{i_t\}_{t\geq 0}$ as a function of the exogenous process $\{r_t^n\}_{t\geq 0}$ so as to minimize the period-loss function, L_t, subject to the system (54), the zero lower bound, the solvency constraint and initial conditions X_0. The solution to this problem satisfies the Bellman Equation,

$$
\begin{aligned}
V_t\left(X_t, \epsilon_t, r_t^n\right) = \min_{i_t} \quad & L_t + \beta \mathbb{E}_t V_{t+1}\left(X_{t+1}, \epsilon_{t+1}, r_{t+1}^n\right) \\
\text{s.t.} \quad & (52) \text{ - } (55) \\
& X_0 \text{ is given}
\end{aligned}
\tag{56}
$$

6.2.3 The commitment equilibrium

Here we consider an equilibrium that occurs when policy is conducted under commitment so that the government is able to commit to future actions. Consider minimizing once and for all the intertemporal loss function, subject to (54), the zero lower bound and the solvency constraint, where the initial condition X_0 is given. That is,

$$
\begin{aligned}
\min \quad & \mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^\tau L_{t+\tau} \\
\text{s.t.} \quad & (52) \text{ - } (55) \\
& X_0 \text{ is given}
\end{aligned}
$$

6.3 The Experiment

We follow Eggertsson and Woodford (2003) in considering the following experiment: suppose the natural rate of interest is unexpectedly negative in period 0 and reverts back to the steady state value with a fixed probability in every period. We investigate how central bank balance sheet composition affects inflation and output in the aftermath of a liquidity trap.

6.4 Calibration

In the numerical analysis we follow Eggertsson and Woodford (2003) and interpret periods as quarters, and assume coefficient values of $\sigma = 0.5$ and $\beta = 0.99$, and, choose φ and α so that κ $= 0.02$. The assumed value of the discount factor implies a long-run real interest rate equal to four percent per year ($\rho = 0.01$). We assume in period 0 that the natural interest rate becomes -2 percent per year ($r^l = -0.005$) and then reverts back to the steady-state value of 4 percent per year with probability ($\gamma = 0.25$) each quarter. Thus the natural interest rate is expected to be negative for three quarters on average at the time the shock occurs. We choose $b = 2$ so that long-run interest elasticity of money demand is in line with estimates in Hoffman and Rasche (1991). Also we set $\theta = 1/100$ so that the steady-state ratio of nominal GDP to real money balances equals one, which is roughly in line with the data if we consider money by the broad definition of M2.[15] Lastly, we assume the central bank gives five times more weight to inflation

[15] Money and quasi money (M2) amounts to 90% of U.S. GDP in the World Bank's 2014 estimates.

($\lambda = 0.2$).

We calibrate the public sector of the model by allowing the central bank to hold treasury-issued bonds of eight different maturities. We then have the freedom to choose 28 parameters: $\{\delta_1, ...\delta_8\}$, $\{\eta_1, ..., \eta_8\}$, $\{\Phi_1, ..., \Phi_8\}$, $\underline{\text{k}}$, ρ_b, ρ_m and τ. These parameters together determine the steady-state values, q_s^*, b_s^*, m^*, t^* and k^*. We set $\delta_1 = 0$ to represent the conventional short-term bond that is sold for $1/(1 + i_t)$ in period t and delivers one dollar in period $t + 1$ - so that i_t is the short-term interest rate. The remaining 27 parameters are chosen to replicate key features of the U.S. Federal Reserve's balance sheet.[16]

The System Open Market Account (SOMA), managed by the Federal Reserve of New York, provides dollar-denominated assets acquired by the Federal Reserve via open market operations. From this source we collected information on the size and the composition of three types of securities held as of August 28, 2014. These securities are: Treasury Bills (T-Bills), Treasury Notes (T-Notes) and Treasury Bonds (T-bonds). Additional information about these assets - such as issue and maturity dates, coupon and principal payments, duration and market value - were gathered from a Bloomberg Terminal in August 29, 2014. Table (1) summarizes the data - Tables (3) to (10) present the data in detail.

Table 1: Summary of U.S. Treasury Notes and Bonds held by the Federal Reserve as of August 28, 2014

Maturity (within)	**Yield*** (quarterly)	**Average Duration*** (quarterly)	**Portfolio Market Value** (in $ millions)	**PMV/NGDP****
6mo	0,019	0,80	$ 100,2	0.000
1y	0,021	3,14	$ 3.328,3	0.000
2y	0,123	6,49	$ 164.998,7	0.038
3y	0,232	9,78	$ 206.086,2	0.047
5y	0,404	15,62	$ 732.958,9	0.169
10y	0,581	25,03	$ 903.620,6	0.200
20y	0,667	39,21	$ 151.628,9	0.035
30y	0,761	68,62	$ 978.906,4	0.226
Total			$ 3.141.628,1	0.725

Source: http://www.ny.frb.org/markets/soma.html
*Collected from a Bloomberg Terminal August, 29, 2014
**Ratio of Portfolio Market Value to 2014 Quarterly Nominal GDP

We separate the Treasuries held by the Fed into eight groups: Treasuries maturing within 6 months, 1 year, 2 years and so on, as indicated by the first column of Table (1). Columns (3) and (4) show the average Macauley duration of the securities and the market value of the portfolio held by the Fed for each category.

Our strategy to calibrate $\delta' s$ and $\eta' s$ is based on Table (1). For each maturity group, we target the average duration and the ratio of the Fed's portfolio market value to 2014 quarterly nominal GDP. In our model economy, the steady-state duration of a perpetuity is given by $(1 - \beta\delta_s)^{-1}$, so we can map each duration in column (3) of Table 1 into a unique value of δ_s. By choosing δ_s, one can immediately derive the steady-state price of the security $q_s^* = \beta/(1 - \beta\delta_s)$. Given δ's and q^*'s, we chose η's to minimize the sum of squared difference from the steady-state portfolio market value to GDP ratio implied by the model, $q_s^* b_s^* / p^* y^*$, and the data, column (5)

[16] Because we are not interested in the fiscal side of the economy, we choose ζ so that the fiscal policy is passive and the fiscal side of the economy becomes irrelevant for equilibrium of non-fiscal variables.

in Table 1, subject to $\sum_{s \in S} \eta_s = 1$. Also, we set $\rho_b = 0.9$. Table (2) presents the calibration results and Figure 7 plots the portfolio market value in the data and in the model,

Table 2: Steady State Calibration

δ_s	η_s	**Duration** $1/(1-\beta\delta_s)$ (quarterly)	**Price** $q_s^* = \beta/(1-\beta\delta_s)$	**Holdings** $b_s^* = \eta_s \left(\frac{1-\beta\delta_s}{\beta}\right)\left(\frac{1}{\theta}(1-\beta)\right)^{-1/b}$ (in millions)	**PMV/NGDP** $q_s^* b_s^* / p^* y^*$
0,000	0,034	1,00	$ 0,99	0,034	0,034
0,689	0,035	3,14	$ 3,11	0,011	0,035
0,854	0,072	6,49	$ 6,42	0,011	0,072
0,907	0,081	9,78	$ 9,68	0,008	0,081
0,945	0,203	15,62	$ 15,46	0,013	0,203
0,970	0,243	25,03	$ 24,78	0,009	0,243
0,984	0,069	39,21	$ 38,82	0,001	0,069
0,995	0,260	68,62	$ 67,93	0,003	0,260
Total	1				1
Implied $k^*/p^*y^* = \rho m^*/\tau = 0.01$					

We choose $\tau = 1$ so that the central bank's steady state capital ratio to quarterly nominal GDP is 1% what is in line with data.[17]

Note that the Fed's portfolio market value in Table 1 (0.725) is less than unity because we disregarded certain types of assets held by the Fed such as mortgage-backed securities, federal agency debt securities and other types of loans. This explains the differences between portfolio market value to GDP ratio implied by the model and in the data, as shown below. We focus only on government bonds, since we do not model explicitly any risky debt. Moreover, increasing the size of the balance sheet by incorporating long duration assets (such as MBS and the other excluded asstes) would only increase the quantitative implications of our model, as will be clear below.

[17] See http://www.federalreserve.gov/releases/h41/current/h41.htm#h41tab9. In March 2014, the Fed's consolidated capital was $57 bilions against $4 trillions of quartely GDP

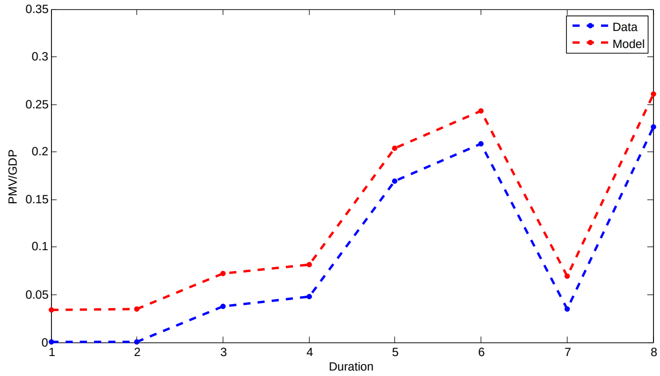

Figure 7: Calibration Results

6.5 The Experiment - The Solution Algorithm

We consider the following experiment: we assume that in period 0 the natural interest rate becomes unexpectedly negative and then reverts back to the steady-state with a probability γ each quarter. We characterize optimal policy under discretion within this set up. The non-usual part of this solution is the presence of occasionally binding zero lower bound and solvency constraints that introduce nonlinear restrictions to equilibrium. Our strategy is to consider it as a model with four regimes.[18]

$R1$. shock is present, ZLB is binding and SC is slack

$R2$. shock is not present, ZLB is binding and SC is slack

$R3$. shock is not present, ZLB is slack and SC is binding

$R4$. shock is not present, ZLB and SC are slack

The model was linearized around the stationary Regime 4 in which Blanchard and Kahn (1980) conditions apply. The advantage of this approach is that in each regime the system of necessary conditions for equilibrium is linear so we can use standard methods to characterize the solution. One has to be careful to deal with expectations when transitioning from one regime to another. We deal with that with a guess-and-verify approach. First, we guess the period in which each regime applies. Second, we proceed and verify, and if necessary update, the initial guess.[19]

Notice that we calibrate the probability of the return of normal levels of the natural real interest rate, so that the expected duration of $R1$ is four quarters. The duration of the other regimes is contingent on policies implemented and size and composition of central bank balance sheet.

[18] This is an adaptation of the solution method used in Eggertsson and Woodford (2003), which was generalized in Guerrieri and Iacoviello (2015). This method adapt a first-order perturbation approach and applies it to handle occasionally binding constraints in dynamic models. However, endogenous states prevent us from applying this method directly to (56). This is because each of the endogenous variables depends on the mapping between the endogenous state (i.e., bond prices and holdings) and the unknown functions $v(.)$, $\mathbb{E}_t x_{t+1}(.)$, $\mathbb{E}_t \pi_{t+1}(.)$ and $\mathbb{E}_t \hat{q}_{t+1}(.)$ so that one needs to know the derivative of these functions with respect to the endogenous policy state variable to calculate the first order-conditions. Eggertsson (2006) suggests a matching coefficients approach to handle the unknown functions.

[19] We present all details in the Appendix.

6.6 Results

6.6.1 Does a large central bank balance sheet reduce the effects of a liquidity trap?

In this section we compare the results of the discretion equilibrium described in 6.2.2 when $\underline{k} < \infty$, i.e., when the central banks is not fully backed by the treasury and faces a solvency constraint, with the conventional discretion solution when $\underline{k} = \infty$ and the conventional commitment solution when $\underline{k} < \infty$. We use the quantitative model calibrated for the U.S economy as described in section 6.4 to analyze how a central bank expansion in balance sheet impacts the dynamics of output, inflation and interest rates during a liquidity trap. In addition, we show that an expanded central bank balance sheet in our model can approximate the unconstrained commitment solution, i.e., quantitative easing programs act as a commitment to approximate first-best policy responses in the zero-lower bound.

In the Figure 8 below, we show the results with the size and average duration of the central bank balance sheet that we see in the data. It is hard to calibrate the amount of losses allowed for the central bank balance sheet. We allow for losses up to 144 billion dollars (around 3 times the Fed's contributions to the treasury after the crises) and experiment with this value later.

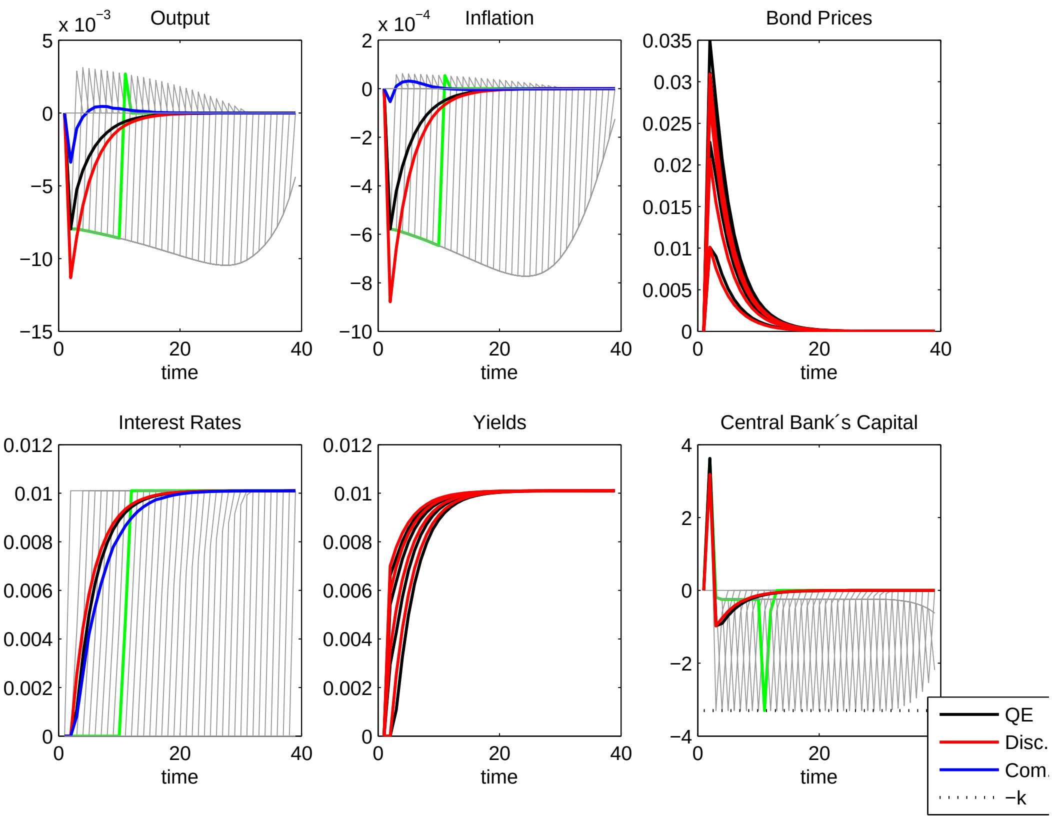

Figure 8: Conventional and unconventional discretion and commitment for the calibrated U.S. economy - fiscal support of $ 144 billions.

Figure 8 shows the response-impulse function of the shock to the natural interest rate for key variables. The gray lines show state-contingent evolution of these variables. The first gray line shows the equilibrium if the natural interest rate returns to steady state in period 1, the next line if it returns in period 2, and so on. The blue, black and red lines show expected evolution of the same variables at the time the shock hits the economy for the commitment, unconventional (QE) discretion and conventional discretion equilibriums, respectively.

Its known that a negative shock to the natural interest rate leads to deflation and recession. As is clear in the impulse response to this shock, these negative effects are mitigated in the discretionary case with central bank balance sheet restrictions. More specifically, the effects of the shock lie in between the discretion and commitment solution (both without any balance sheet restriction), indicating that a large central bank balance sheet can serve as a device to approximate the commitment solution, if the central bank faces limits to its losses.

What is the mechanism behind this result? In general, when the natural interest rate is negative, current and expected future interest rates are low and bond prices are high. The central bank profits with the favorable movements in the price of assets it holds and its net worth is also high. When the shock reverts the economy starts the transition to return to steady state. The rate of convergence of interest rate and bond prices to steady state will determine the behavior of the Fed's net worth. If interest rates and bond prices jump to steady state immediately after the shock reverts, the Fed's net worth is deeply affected. Without fiscal backing and facing limits to its losses, the central bank restricts interest rate movements and, thus, bond price movements, in order to smooth net worth losses.

The overall lesson of this section is that a large central bank as observed after the crisis is consistent with a commitment device for policymakers to move away from the discretion solution to the commitment one, if the central bank cannot incur losses larger than three times its usual profit. In our model, this mechanism is also reflected in the welfare calculations. In the discretion, losses are 1.8486e-06, while in commitment they are 9.4242e-08. With losses restricted to \$ 144 billions our welfare losses sum to 9.0125e-07.

6.6.2 What is the optimal central bank balance sheet?

A natural question that follows the mechanism in the previous section is: how far can the central bank balance sheet go as a commitment device? There are two dimensions within our model to search for the optimal central bank balance sheet: average duration of assets and its overall size. As one can see in (51), the central bank budget constraint losses, and thus our results, depend on the total size of the balance sheet in the same way they depend on asset duration. So here, in order to conserve space, we focus on increasing the duration of the assets in the central bank's balance sheet instead of the completely analogous exercise of expanding balance sheet by the same proportion.

We see in Figure 9 that, the more we increment the duration of the balance sheet, the closer we get to the commitment solution. In this case, a longer duration of the balance sheet allows for larger losses when the natural rate shock reverts. Given the same loss limit, the central bank smoothes the interest rate path more intensively in order reduce the losses in its net worth. The very same logic applies to an increase in the balance sheet. As a conclusion, the longer the duration or the larger the balance sheet, the closer the discretion solution with balance sheet constraint is to the commitment solution.

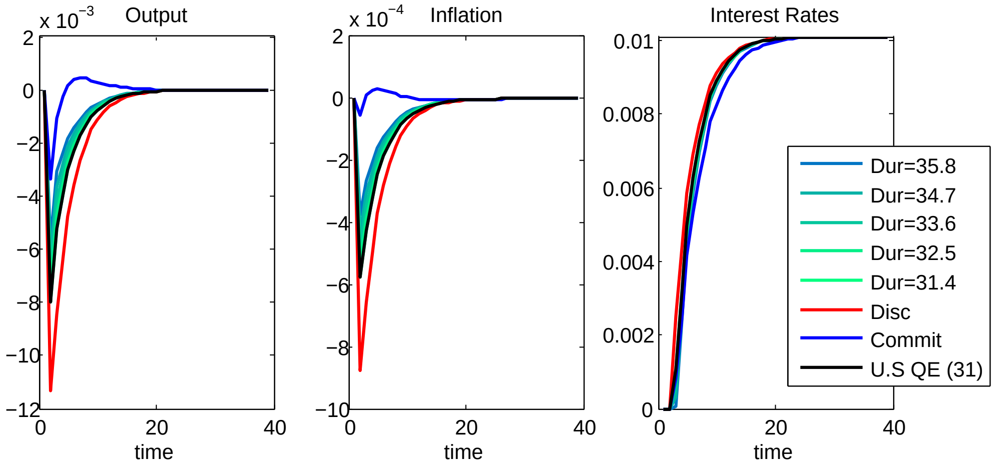

Figure 9: Conventional and unconventional discretion and commitment equilibriums for different average portfolio durations - fiscal support of $ 144 billion ($\underline{k}$=3.3).

One caveat to this result is that there are limits to the losses the central bank can suffer and still achieve an equilibrium satisfying both its solvency constraint and the zero-lower bound restriction to interest rates. Given the calibrated loss limit, the closest one can get to the commitment solution is with the duration of 35.8 quarters. After this limit, losses are so large that either one of the two restrictions are violated.

6.6.3 The role of loss limit

Naturally, our results depend a lot on the specific calibrated loss limit. Since there is no crystal clear way to calibrate this parameter, we perform sensitivity analysis with this specific value.

In Figure 10, we show that the tighter the loss limit, the better we approximate the commitment solution. This is intuitive: the tighter the loss limit, the more the Fed has to smooth the impact of an interest rate hike on its net worth.

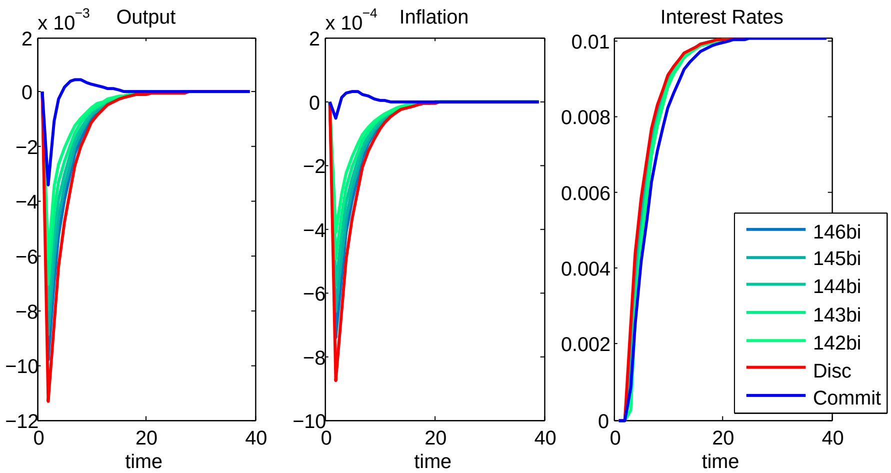

Figure 10: Conventional and unconventional discretion and commitment equilibriums for the calibrated U.S. eeconomy and different levels of fiscal support ($\underline{k}$).

7 Conclusion

We conclude that an asset-purchase program that changes the composition of assets in the central bank balance sheet (quantitative easing) can serve as a commitment device in a liquidity trap scenario. This is because such an open market operation provides an incentive to the central bank to keep interest rates low in future to avoid losses in its balance sheet.

First, we do that in a simple model where we explicitly show that if the treasury does not back the central bank and if it faces a solvency constraint, a quantitative easing program and a consequent change in central bank balance sheet composition affects the way an endowment shock influences inflation dynamics in a liquidity trap. More specifically, we show that the balance sheet discretionary equilibrium goes in the direction of the standard commitment equilibrium.

Second, in a production economy, we show that the current size and duration of the Fed's balance sheet moves the economy reaction to a liquidity trap shock closer to the reaction of an economy under full-commitment policy. Moreover, a larger or longer duration balance sheet would approximate the commitment solution even better.

Overall, this article points to an alternative channel through which quantitative easing is important and highlights non-standard channels of monetary policy when the fiscal backing of the central bank by the treasury is imperfect.

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8 Appendix - For Online Publication

8.1 The Endowment Economy

8.1.1 The discretion equilibrium

$\{i_2^l = 0, \hat{p}_2^l = \rho + \underline{\mathrm{y}}, i_2^h = \rho, \hat{p}_2^h = 0, i_1 = 0, \hat{p}_1 = \rho - \bar{y} + \mu(\rho + \underline{\mathrm{y}})\}$

8.1.2 The commitment equilibrium

The commitment problem is,

$$\begin{aligned} &\underset{i_1, i_2^l, i_2^h \geq 0}{\text{minimize}} && \frac{1}{2}\left[(\hat{p}_1)^2 + \beta\mu(\hat{p}_2^l)^2 + \beta(1-\mu)(\hat{p}_2^h)^2\right] \qquad (57) \\ &\text{s.t.} && i_1 + \hat{p}_1 = \mu\hat{p}_2^l + (1-\mu)\hat{p}_2^h + \rho - \bar{y} \\ & && i_2^l + \hat{p}_2^l = \rho + \underline{\mathrm{y}} \\ & && i_2^l + \hat{p}_2^h = \rho \end{aligned}$$

Set up the Lagrangian,

$$\mathbb{L} = \frac{1}{2}\left[(\hat{p}_1)^2 + \beta\mu(\hat{p}_2^l)^2 + \beta(1-\mu)(\hat{p}_2^h)^2\right] + \lambda_1\left[\hat{p}_1 - \mu\hat{p}_2^l - (1-\mu)\hat{p}_2^h - \rho + \bar{y}\right] + \\ + \lambda_l\left[\hat{p}_2^l - \rho - \underline{\mathrm{y}}\right] + \lambda_h\left[\hat{p}_2^h - \rho\right]$$

First-order conditions,

$$\hat{p}_1 + \lambda_1 = 0 \qquad (58)$$

$$\mu\hat{p}_2^l - \lambda_1\mu + \lambda_l = 0 \qquad (59)$$

$$\beta(1-\mu)\hat{p}_2^h - \lambda_1(1-\mu) + \lambda_h = 0 \qquad (60)$$

Slackness conditions

$$\lambda_1\left[\hat{p}_1 - \mu\hat{p}_2^l - (1-\mu)\hat{p}_2^h - \rho + \bar{y}\right] = 0 \qquad (61)$$

$$\lambda_l\left[\hat{p}_2^l - \rho - \underline{\mathrm{y}}\right] = 0 \qquad (62)$$

$$\lambda_h\left[\hat{p}_2^h - \rho\right] = 0 \qquad (63)$$

The values of $\{\lambda_1, \lambda_l, \lambda_h, i_1, i_2^l, i_2^h, \hat{p}_1, \hat{p}_2^l, \hat{p}_2^h\}$ that satisfy (58) - (63) are,

1. If $\bar{y} < \rho$, $\underline{y} < -\rho$.

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_l &= 0 \\
\lambda_h &= 0 \\
i_1 &= \rho - \bar{y} \\
i_2^l &= \rho + \underline{y} \\
i_2^h &= \rho \\
\hat{p}_1 &= 0 \\
\hat{p}_2^l &= 0 \\
\hat{p}_2^h &= 0
\end{aligned}$$

2. If $\Delta y^e < (1-\mu)\rho$ and $\underline{y} > -\rho$

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_l &= -\beta\mu(\underline{y} + \rho) > 0 \\
\lambda_h &= 0 \\
i_1 &= (1+\mu)\rho - (\bar{y} - \mu\underline{y}) \\
i_2^l &= 0 \\
i_2^h &= \rho \\
\hat{p}_1 &= 0 \\
\hat{p}_2^l &= \rho + \underline{y} \\
\hat{p}_2^h &= 0
\end{aligned}$$

3. If $\bar{y} > \rho$ and $\bar{y} < (2+\beta)\rho$,

$$
\begin{aligned}
\lambda_1 &= \frac{\beta}{1+\beta}\left(\bar{y}-\rho\right) > 0\\
\lambda_l &= 0\\
\lambda_h &= 0\\
i_1 &= 0\\
i_2^l &= \rho - \left(\frac{1}{1+\beta}\right)\left(\bar{y}-\rho\right)\\
i_2^h &= \rho\\
\hat{p}_1 &= -\left(\frac{\beta}{1+\beta}\right)\left(\bar{y}-\rho\right)\\
\hat{p}_2^l &= \left(\frac{1}{1+\beta}\right)\left(\bar{y}-\rho\right)\\
\hat{p}_2^h &= \left(\frac{1}{1+\beta}\right)\left(\bar{y}-\rho\right)
\end{aligned}
$$

4. $(1+\mu)\rho < \Delta y^e < (2+\beta)\rho$ and $\bar{y} - (1+\mu)\underline{y} > (2+\mu)\rho$

$$
\begin{aligned}
\lambda_1 &= \left(\frac{\beta}{1+\beta-\mu}\right)\left(\bar{y}-\mu\underline{y}-(1+\mu)\rho\right) > 0\\
\lambda_l &= \left(\frac{\beta}{1+\beta-\mu}\right)\left(\bar{y}-\mu\underline{y}-(1+\mu)\rho-\rho-\underline{y}\right) > 0\\
\lambda_h &= 0\\
i_1 &= 0\\
i_2^l &= 0\\
i_2^h &= \left(\frac{1}{1+\beta-\mu}\right)\left[(2+\beta)\rho-(\bar{y}-\mu\underline{y})\right]\\
\hat{p}_1 &= \left(\frac{\beta}{1+\beta-\mu}\right)\left[\rho(1+\mu)-(\bar{y}-\mu\underline{y})\right]\\
\hat{p}_2^l &= \rho+\underline{y}\\
\hat{p}_2^h &= \left(\frac{1}{1+\beta-\mu}\right)\left[\bar{y}-\mu\underline{y}-(1+\mu)\rho\right]
\end{aligned}
$$

5. $\Delta y^e > (2+\beta)\rho$

$$
\begin{aligned}
\lambda_1 &= \bar{y} - q\underline{\mathrm{y}} - 2\rho > 0 \\
\lambda_l &= \mu(\bar{y} - \mu\underline{\mathrm{y}} - 2\rho) - \beta\mu(\rho + \underline{\mathrm{y}}) \\
\lambda_h &= (1-\mu)(\bar{y} - \mu\underline{\mathrm{y}} - (2+\beta)\rho) \\
i_1 &= 0 \\
i_2^l &= 0 \\
i_2^h &= 0 \\
\hat{p}_1 &= 2\rho - (\bar{y} - \mu\underline{\mathrm{y}}) \\
\hat{p}_2^l &= \rho + \underline{\mathrm{y}} \\
\hat{p}_2^h\rho &=
\end{aligned}
$$

In summary,

- $\bar{y} < \rho$ and $\underline{\mathrm{y}} > -\rho \Rightarrow i_1, i_2^h, i_2^l > 0$,
- $\Delta y^e < (1+\mu)\rho$ and $\underline{\mathrm{y}} < -\rho \Rightarrow i_1, i_2^h > 0$ and $i_2^l = 0$,
- $\rho < \bar{y} < (2+\beta)\rho \Rightarrow i_1, i_2^l = 0$ and $i_2^h = 0$,
- $\Delta y^e > (2+\beta)\rho \Rightarrow i_1 = i_2^l = i_2^h = 0$.

8.1.3 Fiscally constrained central bank

Proposition 1 *Assume that $\bar{b} \leq \underline{\mathrm{k}} < \infty$, N is large and the central bank adopts a price-level targeting regime and conducts long-term bond purchases according to policy rule* (28). *Under discretion, for all $t \geq 3$, $\{\hat{p}_t, i_t\} = \{0, \rho\}$ independent of the realization of the income process.*
Proof: Let $i_t = \rho$ for all $t \geq 3$ and note that from period $t = 2$ onward all uncertainty in the model has been settled and hence perfect foresight applies. In this case $i_{t|3} = i_{t|2} = \rho$ for all $t \geq 3$. Hence,

$$
\begin{aligned}
\hat{k}_3 &= \bar{b}\rho^{-1} \sum_{i=0}^{N} \beta^i \left(i_{2+i|2} - i_{3+i|3}\right) \\
&\simeq \bar{b}\rho^{-1}(i_2 - \rho) \qquad \text{if } N \text{ is large} \\
&\geq -\bar{b} \geq -\underline{\mathrm{k}}
\end{aligned}
$$

But $l_t \geq 0$ and $l_t = 0$ when $i_t = \rho$ for $3 \leq t < N$. ■

Proposition 2 *Assume that $\underline{\mathrm{y}} < -\rho$, $\bar{b} \leq \underline{\mathrm{k}} < \infty$, N is large and that the central bank adopts a price-level targeting regime and conducts long-term bond purchases according to policy rule* (28). *Under discretion, the central bank's policy functions are,*

$$i_2^l(i_{2|1}, i_1) = 0 \tag{64}$$

$$\hat{p}_2^l(i_{2|1}, i_1) = \rho + \underline{\mathrm{y}} \tag{65}$$

Proof: The problem in the low-income state of the period is to choose i_2^l given i_1 and $i_{2|1}$,

$$
\begin{aligned}
&\text{minimize} && \frac{1}{2}(\hat{p}_2^l)^2 \\
&\text{s.t.} && r_2 = i_2^l - (\hat{p}_{3|2} - \hat{p}_2^l) = \rho + \underline{\mathrm{y}} \\
& && i_2^l \leq i_1 + \beta(i_{2|1} - \rho) + \frac{\rho \underline{\mathrm{k}}}{\bar{b}} \\
& && i_2^l \geq 0 \\
& && \text{given } i_1, i_{2|1} \geq 0 \text{ and } \hat{p}_{3|2} = 0
\end{aligned}
$$

Set up the Lagrangian

$$
\mathbb{L} = \frac{1}{2}(\hat{p}_2^l)^2 + \lambda_r[i_2^l + \hat{p}_2^l - \rho - \underline{\mathrm{y}}] - \lambda_l i_2^l + \lambda_k \left[i_2^l - i_1 - \beta(i_{2|1} - \rho) - \frac{\rho \underline{\mathrm{k}}}{\bar{b}} \right]
$$

The first order condition with respect to i_2^l and $\hat{p}_2^l$,

$$
\lambda_r - \lambda_l + \lambda_k = 0 \tag{66}
$$

$$
\hat{p}_2^l + \lambda_r = 0 \tag{67}
$$

slackness conditions

$$
\lambda_l i_2^l = 0 \tag{68}
$$

$$
\lambda_k \left[i_2^l - i_1 - \beta(i_{2|1} - \rho) - \frac{\rho \underline{\mathrm{k}}}{\bar{b}} \right] = 0 \tag{69}
$$

$$
\lambda_r[i_2^l + \hat{p}_2^l - \rho - \underline{\mathrm{y}}] = 0 \tag{70}
$$

Note that $i_2^l = 0$, $\hat{p}_2^l = \rho - \underline{\mathrm{y}}$, $\lambda_k = 0$, $\lambda_r = \lambda_j = -(\rho + \underline{\mathrm{y}}) > 0$ satisfy conditions (66) - (70) and $i_1 + \beta(i_{2|1} - \rho) + \frac{\rho \underline{\mathrm{k}}}{\bar{b}} \geq -\beta\rho + \frac{\rho \underline{\mathrm{k}}}{\bar{b}} \geq (1-\beta)\rho > 0 = i_2^l$. ∎

Proposition 3 *Assume $\bar{b} \leq \underline{k} < \infty$, N is large and that the central bank adopts a price-level targeting regime, conducts long-term bond purchases according to policy rule* (28) *and $\underline{k} \geq \bar{b}$. Under discretion, the central bank's policy functions are,*

$$
i_2^h(i_1, i_{2|1}) = \begin{cases} i_1 + \beta(i_{2|1} - \rho) + \frac{\rho k}{b} & \text{if } i_1 + \beta i_{2|1} \leq \rho(1 + \beta - \frac{k}{b}) \\ \rho & \text{if } i_1 + \beta i_{2|1} > \rho(1 + \beta - \frac{k}{b}) \end{cases}
$$

$$
\hat{p}_2^h(i_1, i_{2|1}) = \begin{cases} \rho(1 - \frac{k}{b}) - i_1 - \beta(i_{2|1} - \rho) & \text{if } i_1 + \beta i_{2|1} \leq \rho(1 + \beta - \frac{k}{b}) \\ 0 & \text{if } i_1 + \beta i_{2|1} > \rho(1 + \beta - \frac{k}{b}) \end{cases}
$$

Proof: The problem in the low-income state of period is to choose i_2^l given i_1 and $i_{2|1}$,

$$\begin{aligned} &\text{minimize} && \frac{1}{2}(\hat{p}_2^h)^2 \\ &\text{s.t.} && r_2 = i_2^h - (\hat{p}_{3|2} - \hat{p}_2^h) = \rho \\ &&& i_2^h \leq i_1 + \beta(i_{2|1} - \rho) + \frac{\rho \underline{\text{k}}}{\bar{b}} \\ &&& i_2^h \geq 0 \\ &&& \text{given } i_1, i_{2|1} \geq 0 \text{ and } \hat{p}_{3|2} = 0 \end{aligned}$$

Set up the Lagrangian

$$\mathbb{L} = \frac{1}{2}(\hat{p}_2^h)^2 + \lambda_r[i_2^h + \hat{p}_2^h - \rho] - \lambda_h i_2^h + \lambda_k \left[i_2^h - i_1 - \beta(i_{2|1} - \rho) - \frac{\rho \underline{\text{k}}}{\bar{b}} \right]$$

The first order condition with respect to i_2^h and $\hat{p}_2^h$,

$$\lambda_r - \lambda_h + \lambda_k = 0 \tag{71}$$

$$\hat{p}_2^h + \lambda_r = 0 \tag{72}$$

slackness conditions

$$\lambda_h i_2^h = 0 \tag{73}$$

$$\lambda_k \left[i_2^h - i_1 - \beta(i_{2|1} - \rho) - \frac{\rho \underline{\text{k}}}{\bar{b}} \right] = 0 \tag{74}$$

$$\lambda_r[i_2^h + \hat{p}_2^h - \rho] = 0 \tag{75}$$

1. If $i_1 + \beta i_{2|1} > \rho(1 + \beta - \frac{\underline{\text{k}}}{\bar{b}})$. Then $i_2^h = \rho$, $\hat{p}_2^h = 0$ and $\lambda_k = \lambda_r = \lambda_j = 0$ satisfy conditions (66) - (70), and $i_1 + \beta(i_{2|1} - \rho) + \frac{\rho \underline{\text{k}}}{\bar{b}} > \rho = i_2^h$.

2. If $i_1 + \beta i_{2|1} \leq \rho(1 + \beta - \frac{\underline{\text{k}}}{\bar{b}})$. Then

$$\begin{aligned} i_2^h &= i_1 + \beta(i_{2|1} - \rho) + \rho \frac{\underline{\text{k}}}{\bar{b}} \\ \hat{p}_2^h &= \rho \left(1 - \frac{\underline{\text{k}}}{\bar{b}}\right) - i_1 - \beta(i_{2|1} - \rho) \\ \lambda_r &= -\rho \left(1 - \frac{\underline{\text{k}}}{\bar{b}}\right) + i_1 + \beta(i_{2|1} - \rho) \\ \lambda_h &= 0 \\ \lambda_k &= \rho \left(1 - \frac{\underline{\text{k}}}{\bar{b}}\right) - i_1 - \beta(i_{2|1} - \rho) \end{aligned}$$

   satisfy conditions (66) - (70).

$\blacksquare$

Proposition 4 *Assume that* $(1+\mu\beta)^{-1}\bar{b} \le \underline{k} \le \bar{b}$*, N is large and that the central bank adopts a price-level targeting regime, conducts long-term bond purchases according to policy rule* (28). *Under discretion, the central bank's policy rate in the first period is,*

$$i_1 = \begin{cases} (1+\mu)\rho - \Delta y^e & \text{if } \Delta y^e \le \rho\left[\frac{k}{\bar{b}} + (1-\beta)\mu\right] \\ \frac{1-\beta(1-\mu)}{1+(1-\beta)(1-\mu)}\left[2\rho - \Delta y^e - \frac{\rho(1-\mu)}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\right)\right] & \\ \qquad \text{if } \rho\left[\frac{k}{\bar{b}} + (1-\beta)\mu\right] \le \Delta y^e \le \rho\left[2 - \frac{1-\mu}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\right)\right] & \\ 0 & \text{if } \Delta y^e > \rho\left[2 - \frac{1-\mu}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\right)\right] \end{cases} \tag{76}$$

Proof: The problem in period 1 is,

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\left[(\hat{p}_1)^2 + \beta(1-\mu)(\hat{p}_2^h)^2\right] \\ &\text{s.t.} && r_1 = i_1 - \left[\mu\hat{p}_2^l(i_1) + (1-\mu)\hat{p}_2^h(i_1) - \hat{p}_1\right] = \rho - \bar{y} \\ & && i_1 \ge 0 \text{ and } (43) \end{aligned}$$

1. If $\Delta y^e \le \rho\left[\frac{k}{\bar{b}} + (1-\beta)\mu\right]$, the solution to the central bank's problem is $\bar{i}_1 = (1+\mu)\rho - \Delta y^e$.

   Define $f(i_1) = \frac{1}{2}\left[(1+\mu)\rho - \Delta y^e + (1-\mu)\hat{p}_2^h(i_1) - i_1\right]^2$. Guess and verify: $i_1^* = (1+\mu)\rho - \Delta y^e \ge (1+\beta\mu)\rho \Rightarrow \hat{p}_2^h(i_1^*) = 0$ and hence $f(i^*) = 0$. Since $f(i_1) \ge 0$ for all $i_1 \ge 0$, i_1^* achieves the minimum.

2. If $\rho\left[\frac{k}{\bar{b}} + (1-\beta)\mu\right] \le \Delta y^e \le \rho\left[2 - \frac{1-\mu}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\right)\right]$, the solution to the central's bank problem is

   $$\bar{i}_1 = \frac{1-\beta(1-\mu)}{1+(1-\beta)(1-\mu)}\left[2\rho - \Delta y^e - \frac{\rho(1-\mu)}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\rho\right)\right]$$

   Guess and verify:

   $$i_1^* \equiv \frac{1-\beta(1-\mu)}{1+(1-\beta)(1-\mu)}\left[2\rho - \Delta y^e - \frac{\rho(1-\mu)}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\rho\right)\right]$$

   Note that $\Delta y^e \ge \rho\frac{k}{\bar{b}} + (1-\beta)\mu\rho \Rightarrow i_1^* \le (1+\mu\beta)\rho \Rightarrow \hat{p}_2^h(i_1^*) = \rho - \frac{1}{1-\beta(1-\mu)}\left[i_1^* + \rho\frac{k}{\bar{b}} - \beta\rho\right]$.

Then,

$$
\begin{aligned}
f(i_1^*) &= \frac{1}{2}\left[(1+\mu)\rho - \Delta y^e + (1-\mu)\hat{p}_2^h(i_1^*) - i_1^*\right]^2 \\
&= \frac{1}{2}\left[2\rho - \Delta y^e - \frac{\rho}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\right) - \left(\frac{1+(1-\beta)(1-\mu)}{1-\beta(1-\mu)}\right) i_1^*\right]^2 \\
&= \frac{1}{2}\left[\left(\frac{1+(1-\beta)(1-\mu)}{1-\beta(1-\mu)}\right) i_1^* - \left(\frac{1+(1-\beta)(1-\mu)}{1-\beta(1-\mu)}\right) i_1^*\right]^2 \\
&= 0
\end{aligned}
$$

Since $f(i_1) \geq 0$ for all $i_1 \geq 0$, i_1^* achieves the minimum.

3. If $\Delta y^e > 2\rho - \frac{1-q}{1-\beta(1-q)}\left(\frac{k}{b} - \beta\rho\right)$, the solution to the central bank problem is $\bar{i}_1 = 0$.

   If $i_1 < \rho\left[(1+\mu\beta) - \frac{k}{b}\right]$

$$
\begin{aligned}
f'(i_1) &= \underbrace{(1-(1-\mu)\partial_{i_1}\hat{p}_2^h(i_1))}_{>0}\left[i_1 + \Delta y^e - (1+\mu)\rho - (1-\mu)\hat{p}_2^h(i_1)\right] \\
&= (1-(1-\mu)\partial_{i_1}\hat{p}_2^h(i_1))\left[i_1 + \Delta y^e - \rho\left[2 - \frac{1-\mu}{1-\beta(1-\mu)}\left(\frac{k}{\bar{b}} - \beta\right)\right]\right] \\
&> 0
\end{aligned}
$$

   If $i_1 > \rho\left[(1+\mu\beta) - \frac{k}{b}\right]$

$$
f'(i_1) = \underbrace{(1-(1-\mu)\partial_{i_1}\hat{p}_2^h(i_1))}_{>0}\left[i_1 + \Delta y^e - (1+\mu)\rho\right] > 0
$$

   Hence, $f'(i_1) > 0$ for all $i_1 \in [0, (1+\mu\beta)\rho - \frac{k}{b}) \cup ((1+\mu\beta)\rho - \frac{k}{b}, +\infty) \Rightarrow \bar{i}_1 = 0$ achieves the minimum.

■

8.1.4 The zero-inflation steady state

We find a zero-inflation steady state for prices $\{i, q^*, p^*\}$ and quantities $\{m^*, k^*, t^*, b^*, b^*_{hh}, t^*_{hh}\}$ by solving the system of equations formed by the first order conditions of the agent's problem, the central bank balance sheet, the treasury budget constraint, the fiscal rule, policy rule (12) and the transfers between the two authorities as functions of an unitary price level, $p^* = 1$, the steady state endowment level, y^*, and an arbitrary b^{s*}_{hh},.

Euler equations,

$$
\begin{aligned}
i^* &= (1-\beta)/\beta \\
Q^* &= \beta/(1-\beta)
\end{aligned}
$$

Cash in advance

$$m^* = y^*$$

Transfers

$$t^* = k^*$$

The Balance Sheet equations

$$k^* = t^* + b^{s*} + qb^* - m^* \tag{77}$$
$$k^* = (b^{s*} + q^*b^*) + (i^*b^s + b^*) - m^* \tag{78}$$

implies,

$$b^* = i^*(y^* - b^{s*}) \tag{79}$$
$$k^* = i^*y^* \tag{80}$$
$$t^* = i^*y^* \tag{81}$$

It is left to the treasury budget and the fiscal rule to determine t^{hh*} and b_{hh*},

$$i^*(b^{s*} + b^{s*}_{hh}) + b^* + b^*_{hh} = t^*_{hh} + t^*$$
$$t^{hh*} = \phi(b^*_{hh} + b^* + b^{s*}_{hh} + b^{s*})$$

Adding them together yields

$$b^*_{hh} = \left(\frac{\phi - i^*}{1-\phi}\right)(b^{s*} + b^{s*}_{hh}) - b^* + \left(\frac{1}{1-\phi}\right)t^*$$

Using (79) and (81),

$$\begin{aligned} b^*_{hh} &= \left(\frac{\phi - i}{1-\phi}\right)(b^{s*} + b^{s*}_{hh}) - i^*(y^* - b^{s*}) + \left(\frac{1}{1-\phi}\right)i^*y^* \\ &= \left(\frac{\phi(1-i)}{1-\phi}\right)b^{s*} + \left(\frac{i^*\phi}{1-\phi}\right)y^* + \left(\frac{\phi - i^*}{1-\phi}\right)b^{s*}_{hh} \end{aligned} \tag{82}$$

We now check if the goods market clears in this steady state. From the HH budget constraint,

$$
\begin{aligned}
c^* &= y^* + i^* b^{s*}_{hh} + b^*_{hh} - t^{hh*} \\
&= y^* + i^* b^{s*}_{hh} + \left[\left(\frac{\phi(1-i)}{1-\phi} \right) b^{s*} + \left(\frac{i\phi}{1-\phi} \right) y^* + \left(\frac{\phi - i}{1-\phi} \right) b^{s*}_{hh} \right] - \phi(b^{hh*} + b^* + b^{s*}_{hh} + B^{s*}) \\
&= (1-\phi) B^{s*}_{hh} + (1-\phi) B^*_{hh} - \phi(B^{s*} + B^*) + y^* \\
&= B^{s*}[\underbrace{\phi - i^* + i^*(1-\phi) - \phi + i^*\phi}_{=0}] + y^*[\underbrace{1 - i^*\phi + i^* - i^*(1-\phi)}_{=1}] \qquad \text{using (82) and (79)} \\
&= y^*
\end{aligned}
$$

The last relation clears the goods market. Hence we can define the zero-inflation steady state for prices $\{i, Q, P\}$ and quantities $\{m^*, k^*, t^*, b^*, b^*_{hh}, t^*_{hh}\}$ as functions of $\{b^*, b^*_{hh}\}$ and steady state income y^* as

$$
\begin{aligned}
p^* &= 1 \\
i^* &= (1-\beta)/\beta \\
q^* &= \beta/(1-\beta) \\
c^* &= y^* \\
m^* &= y^* \\
k^* &= \left(\frac{1-\beta}{\beta} \right) y^* \\
t^* &= \left(\frac{1-\beta}{\beta} \right) y^* \\
b^* &= B(s^*, y^*) \\
b^{s*} &= y^* - \left(\frac{\beta}{1-\beta} \right) b^* \\
b^*_{hh} &= \left(\frac{\phi}{1-\phi} \frac{2\beta - 1}{\beta} \right) b^{s*} + \left(\frac{\beta\phi}{(1-\beta)(1-\phi)} \right) y^* - \left(\frac{\beta - \phi(1-\beta)}{(1-\beta)(1-\phi)} \right) b^{s*}_{hh} \\
t^*_{hh} &= \phi(b^*_{hh} + b^* + b^{s*}_{hh} + b^{s*})
\end{aligned}
$$

8.1.5 Linear model

In this section we present the set of linearized equations related to the equilibrium definitions. First,

$$
\hat{q}_t = \beta \hat{q}_{t+1|t} - (i_t - \rho) \tag{83}
$$

$$
\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - [i_t - \mathbb{E}_t(\hat{p}_{t+1} - \hat{p}_t) - \rho] \tag{84}
$$

$$
\hat{m}_t \begin{cases} = \hat{p}_t + \hat{y}_t & \text{if } i_t > 0 \\ \geq \delta & \text{if } i_t = 0 \end{cases} \tag{85}
$$

where $\hat{p}_t$, $\hat{c}_t$, $\hat{q}_t$ and $\hat{m}_t$ are the log-deviations of the price level, consumption, long-term bond price and money balances from their zero-inflation steady states, $\pi_{t+1} = log(P_{t+1}/P_t)$, i_t is the nominal interest rate ($log(1 + i_t)$) and $\rho \equiv log(\beta^{-1}) \sim i^*$. Equations (83) and (84) are usual asset pricing relations with respect to the long-term and the short-term bonds respectively, and equation (85) is the money demand.

Log-linearization of (7) around the same steady state yields [20],

$$\hat{k}_t = \hat{k}_{t-1} + \hat{t}_{t-1} + \bar{b}^s \hat{i}_{t-1} + \bar{b}^s \hat{b^s}_{t-1} + \bar{b}\hat{b}_{t-1} + q^* \bar{b}(\hat{q}_t - \hat{q}_{t-1})$$

It will be useful to rewrite this equation: forward iteration of (83) and substitution in the above equation results in

$$\hat{k}_t = \hat{k}_{t-1} + \hat{t}_{t-1} + \bar{b}^s \hat{i}_{t-1} + \bar{b}^s \hat{b^s}_{t-1} + \bar{b}\hat{b}_{t-1} + q^* \bar{b} \sum_{i=0}^{\infty} \beta^i \left(i_{t-1+i|t-1} - i_{t+i|t} \right) \tag{86}$$

Then we log-linearize the household's and the treasury's budget constraints, (1) and (14) to get

$$\begin{aligned} &\hat{e}_t + \bar{b}^s_{hh} \hat{b}^{s,hh}_{t-1} + \bar{b}^{hh} \hat{b}^{hh}_{t-1} + \bar{b}_{hh} \hat{q}_t = \\ &= \hat{p}_t + \hat{c}_t + i^* \hat{t}^{hh}_t + i^* \bar{b}_{hh} \hat{i}_t + \beta i^* \bar{b}_{hh} \hat{b}^{hh}_t + \bar{b}_{hh} \delta \hat{q}_t + \bar{b}_{hh} \hat{b}_t \end{aligned} \tag{87}$$

$$\begin{aligned} &q^* \hat{t}_t + t^{hh} \hat{t}^{hh}_t + (\bar{b}^s + i^* \bar{b}^s_{hh}) \beta (1-\beta) \hat{i}_t + \beta \bar{b}^s \hat{b}^s_t + \beta \bar{b}^s_{hh} \hat{b}^{s,hh}_t + (\bar{b} + \bar{b}^{hh}) \hat{q}_t + \bar{b}\hat{b}_t + \\ &+ \bar{b}^{hh} \hat{b}^{hh}_t = \bar{b}^s \hat{b}^s_{t-1} + \bar{b}^s_{hh} \hat{b}^{s,hh}_{t-1} + \bar{b}\hat{b}_{t-1} + (\bar{b} + \bar{b}^{hh}) \hat{q}_t + \bar{b}_{hh} \hat{b}^{hh}_{t-1} \end{aligned} \tag{88}$$

The fiscal policy,

$$\hat{t}^{hh}_t = \bar{b}_{hh} \hat{b}^{hh}_t + i^* \bar{b}^s_{hh} \hat{b}^{s,hh}_t + \bar{b}\hat{b}_t + i^* \bar{b} \hat{b}^s_t \tag{89}$$

equation (20)

$$\rho \hat{k}_t + \hat{m}_t = \rho \hat{t}_t + \bar{b}^s \hat{b}^s_t + \bar{b}(\hat{q}_t + \hat{b}_t) \tag{90}$$

and long policy rule (12)

$$\hat{b}_t = b(\hat{s}_t, \hat{y}_t) \tag{91}$$

The first non-linear restriction in the model, (24), is the zero lower bound for the nominal interest rate. The second non-linear restriction, (23), is the log-linearized version of (9), which simply says that the central bank capital cannot go below the specified lower bound.

$$i_t \geq 0 \tag{92}$$

$$\hat{k} \geq -\underline{\mathrm{k}} \tag{93}$$

[20] Where $\bar{b} = \frac{b^*}{y^*} \frac{\beta}{1-\beta}$, $\bar{b}_{hh} = \frac{b_{hh}}{y^*} \frac{\beta}{1-\beta}$, $\bar{b}^s = \frac{b^{s*}}{y^*}$ and $\bar{b}^s_{hh} = \frac{b^{s*}_{hh}}{y^*}$.

Log-linearizing (8) and using (23) results in,

$$\hat{t}_t = -\hat{k}_t \tag{94}$$

Lastly we add the market clearing condition,

$$\hat{c}_t = \hat{y}_t \tag{95}$$

8.2 Recursive Balance sheet

It will be convenient to further develop the side of the central bank balance sheet. We first use the fact that the central bank capital plus its monetary liabilities and the transfers must equal the value of its asset purchases each period,

$$K_t + T_t + M_t = B_t^s + Q_t B_t \tag{96}$$

use (96) to rewrite the central bank's capital in the recursive form:

$$\begin{aligned} K_t &= (1 + i_{t-1})B_{t-1}^s + (Q_t + 1)B_{t-1} - M_{t-1} \\ &= (1 + i_{t-1})B_{t-1}^s + (Q_t + 1)B_{t-1} - \left(B_{t-1}^s + Q_{t-1}B_{t-1} - K_{t-1} - T_{t-1}\right) \\ &= K_{t-1} + T_{t-1} + i_{t-1}B_{t-1}^s + \left(1 + Q_t - Q_{t-1}\right) B_{t-1} \end{aligned} \tag{97}$$

8.3 The Quantitative Model

8.3.1 The solution algorithm

We consider the following experiment: we assume that in period 0 the natural rate of interest becomes unexpectedly negative and then reverts back to the steady-state positive value with a probability γ each quarter. We characterize optimal policy under discretion within this set-up. The tricky part of the solution is the presence of occasionally binding ZLB and CS constraints that entail nonlinear restrictions to equilibrium. Our strategy is to consider it as a model with 4 regimes:[21]

$R1$. shock is present, ZLB is binding and SC is slack

$R2$. shock is not present, ZLB is binding and SC is slack

$R3$. shock is not present, ZLB is slack and SC is binding

$R4$. shock is not present, ZLB and SC are slack

[21] This is an adaptation of the solution method used in Eggertsson and Woodford (2003), which was further generalized in Guerrieri and Iacoviello (2015). This method adapt a first-order perturbation approach and applies it to handle occasionally binding constraints in dynamic models. However, endogenous states prevent us from applying this method directly to our discretion problem. This is because each of the endogenous variables depend on the mapping between the endogenous state (i.e. bond prices and holdings) and the unknown functions $v(.)$, $\mathbb{E}_t x_{t+1}(.)$, $\mathbb{E}_t \pi_{t+1}(.)$ and $\mathbb{E}_t \hat{q}_{t+1}(.)$ so that one needs to know the derivative of these functions with respect to the endogenous policy state variable to calculate the first order conditions. Under assumptions that we consider fairly restrictive, Eggertsson (2006) suggests a matching coefficients approach to handle the unknown functions.

Note that the model was linearized around the stationary regime 4 in which Blanchard and Kahn (1980) conditions apply. The advantage of this approach is that in each regime the system of necessary conditions for equilibrium is linear so we can use standard methods to characterize the solution. The tricky part is to deal with expectations when transitioning from one regime to another. To deal with that, we employ a guess-and-verify approach. First, we guess the period in which each regime applies. Second, we proceed and verify, and if necessary update, the initial guess. Let τ be the date when the current guess implies that the model will return to the stationary regime 4. Then for any $t \geq \tau$, and given X_τ, transition matrices must satisfy the following Bellman equation,

$$\begin{aligned} V_t\left(X_t, \epsilon_t\right) = &\min \quad L_t + \beta \mathbb{E}_t V_{t+1}\left(X_{t+1}, \epsilon_{t+1}\right) \\ &\text{s.t} \quad (55) \text{ and } (54) \end{aligned} \tag{98}$$

Since the loss function is quadratic and the constraints are linear, it follows that the optimal value of the problem will be quadratic. In period $t+1$, the optimal values will depend on X_{t+1} and ϵ_{t+1} and can hence be written as $[X_{t+1}, \epsilon_{t+1}]' V_{t+1} [X_{t+1}, \epsilon_{t+1}] + \frac{\beta}{1-\beta} w_{t+1}$, where V_{t+1} is a positive semi-definite matrix and w_{t+1} is a scalar independent of X_{t+1} and ϵ_{t+1}. Ljungqvist and Sargent (2004) describe a method to characterize the solution to (98) and show that the resulting transition matrices and value functions turn out to be time independent. Hence, we can find matrices G, G^s, M, M^s and V, such that for all $t \geq \tau$, given V, the sequences formed by the following systems satisfy the Bellman equation (98).

$$\begin{aligned} X_{t+1} &= M X_t + M^s \epsilon_t \\ x_t &= G X_t + G^s \epsilon_t \end{aligned}$$

Note that using $x_{\tau|\tau-1} = G M X_{\tau-1}$ we can switch from the rational expectation system (54) to simpler differential equation system,

$$\tilde{H} X_\tau = A \begin{bmatrix} X_{\tau-1} \\ x_{\tau-1} \end{bmatrix} + B i_{\tau-1} + C \epsilon_{\tau-1} \tag{99}$$

The solution in period $\tau - 1$ must satisfy the bellman equation,

$$\begin{aligned} [X_{\tau-1}, \epsilon_{\tau-1}]' V_{\tau-1} [X_{\tau-1}, \epsilon_{\tau-1}] = &\min \quad L_{\tau-1} + \beta \mathbb{E}_{\tau-1} [X_\tau, \epsilon_\tau]' V [X_\tau, \epsilon_\tau] \\ &\text{s.t} \quad (55),\ (99),\ i_t \geq 0 \text{ and } \hat{k} \geq -\underline{\mathrm{k}} \end{aligned}$$

Since V is known and (99) involves no expectation operators, one can simply set-up the Lagrangian and take first-order and slackness conditions. Coupled with the current guess of regime results in the linear system,

$$
\Gamma_0^i \begin{bmatrix} X_\tau \\ x_{\tau-1} \\ \Phi_{\tau-1} \\ i_{\tau-1} \end{bmatrix} = \Gamma_1^i X_\tau + \Gamma_2^i \epsilon_{\tau-1}
$$

Where Φ_t is the vector of lagrange multipliers and $i \in \{1, ..., 6\}$ indexes the current regime. We solve the above system and find matrices $G_{i,\tau-1}$, $M_{i,\tau-1}$, $G^s_{i,\tau-1}$ and $M^s_{i,\tau-1}$. Moreover, from the transition matrices we can recover the problem's value function $V_{i,\tau-1}$ which will be necessary to solve the model in period $\tau - 2$.

Iterate back in this fashion until X_0 is reached, applying regime 1 to 4 at each iteration, as implied by the current guess of regimes. Taking into account the guess for the solution obtained from this process, we compute paths for i_t and $\hat{k}_t$ to verify the current guess of regimes. If the guess is verified we stop. Otherwise, we update the guess for when regimes 1 to 4 apply and repeat the process.

8.3.2 The Zero Inflation Steady State

We find a zero-inflation steady state for prices $\{i, q^*, p^*\}$ and quantities $\{m^*, k^*, t^*, b^*, b^*_{hh}, t^*_{hh}\}$ by solving the system of equations formed by the first order conditions of the agent's problem, the central bank balance sheet, the treasure budget constraint, the fiscal rule, policy rule (12) and the transfers between the two authorities as functions of an unitary price level, $p^* = 1$, the steady state endowment level, y^*, and an arbitrary b^{s*}_{hh},.

$$
\begin{aligned}
&y^* = 1 \\
&c^* = 1 \\
&i^* = (1-\beta)/\beta \equiv \rho \\
&m^* = \left(\frac{1}{\theta}(1-\beta)\right)^{-1/b} = 1 \\
&q^*_s = \beta/(1-\beta\delta_s) \qquad \text{for all } s \in S \\
&k^* = \frac{1}{\alpha}\left[\beta^{-1}\sum_{s\in S}\eta_s - 1\right]\left(\frac{1}{\theta}(1-\beta)\right)^{-1/b} m^* = \frac{\rho}{\alpha} m^* \\
&t^* = \left[\beta^{-1}\sum_{s\in S}\eta_s - 1\right]\left(\frac{1}{\theta}(1-\beta)\right)^{-1/b} m^* = \rho m^* \\
&b^*_s = \eta_s\left(\frac{1-\beta\delta_s}{\beta}\right)\left(1+\rho(\alpha^{-1}-1)\right) m^* \qquad \text{for all } s \in S
\end{aligned}
$$