

Optimal Monetary Interventions in Credit Markets

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Abstract

We characterize optimal credit market interventions in a monetary economy with limited commitment and limited monitoring. Under the optimal trading mechanism, both money and debt circulate and the optimal inflation rate is positive. We show that the optimal intervention among all feasible policies is to purchase privately issued debt with newly created money. This policy highlights the trade off between the benefits of increased lending and its harmful inflationary effects. In particular, the welfare impact of these policies differs significantly from lump-sum injections of money.

JEL Codes: E00; E50; E52

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1 Introduction

Credit market frictions are generally viewed as a key factor that impacts economic activity, and the mitigation of these frictions has always been an important objective of policy interventions. In particular, after the Great Recession, central banks have become more active in these interventions with the introduction of various lending facilities and the purchase of private debt, and more upfront about their objective of increasing lending in the private sector. At the same time, since these interventions are conducted by *monetary* authorities, policymakers and economists alike are concerned about their medium or long-term potential inflationary impact.¹ From a theoretical perspective, though, the literature has mostly focused on two extreme scenarios, making it difficult to understand the impact of monetary interventions in the credit market. One extreme considers credit policies

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¹For example, Spencer Dale (2013), chief economist of the Bank of England, has said that “with little slack in the economy, businesses would put up prices if extra quantitative easing found its way into consumers’ pockets”. In turn, John Taylor (2012), in his testimony before the Congress, argued that “This large expansion of reserve balances creates risks. If it is not undone, then the bank reserves will eventually pour out into the economy, causing inflation.”

in cashless economies, as in Gertler and Kiyotaki (2010), thus ignoring the effects of credit policies on markets that use money as means-of-payments. The other extreme considers monetary policies in economies without credit, as in Williamson and Wright (2010), thus lacking an explicit consideration of interventions in the credit market by monetary authorities.

In our view, a coherent model of optimal monetary interventions in the credit market requires two key elements. First, we need a friction which renders the credit market imperfect, and thus make liquidity (or its lack thereof) relevant. In pure credit economies, Kehoe and Levine (1993) and Alvarez and Jermann (2000) have shown that the inability of individuals to commit introduces an endogenous borrowing constraint which captures such credit market imperfections. Second, we need a setting where money plays a meaningful role. As pointed out by Wallace (2001, 2014), this requires some degree of imperfect monitoring, so credit cannot be used everywhere. However, this is not a trivial matter as the limitations on monitoring necessary to make money relevant may render credit impossible.

In this paper, we propose a model with limited commitment and limited monitoring, where both money and debt can be used in transactions, and study optimal interventions in the credit market. We contribute in two aspects. First, we obtain that both money and debt are necessary to implement desirable allocations. Second, we study the welfare implications of all feasible monetary policies, i.e., all policies which respect voluntary participation and are consistent with the monitoring technology. In particular, we show that only two classes of policies are relevant: the first class, labeled expansionary policies, purchases private debt with newly created money; the second class, labeled deflationary policies, gives interest on money financed by taxes in the credit sector. The first class resembles debt-purchase interventions implemented by central banks, and highlight a new trade-off: they produce inflation but improve lending by relaxing the borrowing constraint of buyers. In contrast, the second class involves the opposite trade-off. These trade-offs differ drastically from the usual approach of modeling monetary injections through lump-sum transfers, which we also analyze.

Our model is based on the Lagos-Wright (2005) environment (LW henceforth), but we introduce two key modifications. First, we allow for the use of debt by assuming an imperfect monitoring technology which records promises-to-pay from buyers and through which sellers can access past records in a fraction of meetings. Second, we adopt a mechanism-design approach. This approach makes it transparent that the benefits of monetary interventions are driven by the frictions coming from limited commitment and limited monitoring, the frictions that we are explicit about, and not by implicit frictions underlying an inefficient use of the monitoring technology, or an inefficient determination of the terms of trade. In particular, an implication of using mechanism design is that the standard LW model with one round of decentralized trade is not suitable for our purposes. In fact, Hu, Kennan, and Wallace (2009) (HKW henceforth) show that, under a constant money supply and no monitoring, the optimal trading mechanism implements any allocation implementable under perfect monitoring. We depart from this special case by assuming that agents interact in three stages in each period: the first two stages

correspond to the decentralized market (DM) in LW, and the last stage corresponds to the centralized market (CM). We focus on the model with two DM rounds for simplicity, but most of our results generalize to models with many DM rounds.

In contrast to the literature, our formulation of the monitoring technology requires a minimum amount of information. In particular, it only records the identities of the agents involved in the transaction and the debt incurred by the buyer. These recorded histories are updated periodically into credit records and can be accessed only by the future partners of the buyer with access to the technology. This formulation is thus quite different from the one in Cavalcanti and Wallace (1999) and Cavalcanti, Erosa, and Temzelides (1999), which requires the public observation of recorded histories due to their focus on inside money. Our monitoring technology is also limited in that not all sellers have access to it. Our specifications capture the typical operations of unsecured loans, including credit cards and commercial papers.

Due to the presence of limited monitoring, our use of mechanism design has some novel features. First, on the extensive margin, the mechanism chooses which meetings will be monitored, subject to the overall constraint on the availability of the monitoring technology. This choice endogenously determines the money sector and the credit sector in the economy with respect to the economic fundamentals. Second, on the intensive margin, it chooses the means of trade and the terms of trade in a meeting, subject to participation constraints, which include individual rationality and pairwise core. Thus, the use of money or debt as the means-of-payment is endogenously determined according to the characteristics of the meeting, and the terms of trades depend on the the agents' money holdings and credit records.

We provide a complete characterization of implementable allocations. Our results apply to three different scenarios depending on the availability of the monitoring technology. In one extreme we have unlimited monitoring, where both DM rounds are monitored. In the other extreme we have no monitoring and a constant money supply, where neither DM round is monitored. We are interested in the limited monitoring case, where only one DM round is monitored and monetary interventions are allowed, but the extreme scenarios are useful benchmarks.

Consider, first, the extreme scenarios. Under unlimited monitoring, implementability requires that buyers are willing to repay the debt accumulated in the two DM rounds, given the expected gains from trade in all future periods. This constraint endogenously determines the debt limit that buyers face and hence the level of productions in the DM rounds. Under no monitoring and a constant money supply, the same overall participation constraint applies, if we replace debt repayments with accumulation of real balances. However, without monitoring, buyers with different private histories who enter the second DM round with the same money holdings must be treated equally. This information friction gives rise to an additional constraint to prevent buyers from only accumulating real balances to participate in the second DM round.²

²In contrast, in the LW model with only one DM round, since all buyers enter that round with the same money holding, the optimal mechanism under no monitoring can use this feature to achieve the constrained efficient allocations under perfect monitoring, as shown in HKW.

Consider, now, the limited monitoring case. First, we obtain that, with a constant money supply, debt is irrelevant in the sense that the set of implementable allocations is the same as the one implemented under no monitoring.³ This result sets a benchmark against which we study the impact of monetary interventions. Indeed, we show that debt regains its relevance when we consider expansionary policies which purchase private debt with newly created money. We obtain three results regarding this class of interventions.

First, we show that debt-purchase interventions allow a redistribution of liquidity across the DM rounds, which expands the set of implementable allocations at least up to the unlimited monitoring case. Consider an allocation where, without interventions or monitoring, the buyer does not want to participate in the first DM round but the overall participation constraint is satisfied. Such an allocation is implementable with an expansionary policy by having the first DM round monitored. The constructed expansionary policy relaxes the participation constraint in the first round by purchasing debts there, and, since the second round has relatively abundant liquidity, the inflation it creates does not deter buyers' participation in that round.

Second, we obtain that the optimal debt-purchase intervention generically improves welfare against no intervention. On the one hand, when the best allocation under no intervention does not satisfy the participation constraint for the first DM round, we can use expansionary policy to relax that constraint and obtain a better allocation. On the other hand, when such allocation does satisfy the participation constraint for the first DM round, the optimal debt-purchase intervention implements a better allocation than the one achieved under unlimited monitoring. We do so by having the second DM round monitored. Since some buyers are “unlucky” and do not have an opportunity to participate in that round, the inflation tax required to purchase the debt necessary to induce participation of the “lucky” buyers is relatively small. This allows an efficient reallocation of liquidity which cannot be accomplished absent interventions, even under unlimited monitoring.

Finally, we show that expansionary policies dominate all other interventions which respect voluntary participation, and which condition on the information released by the monitoring technology. In particular, debt-purchase interventions dominate deflationary policies that tax monitored trades and dominate inflationary policies with lump-sum injections of money in terms of social welfare.⁴ However, while inflation through lump-sum transfers can only be harmful in our environment, we provide conditions under which a deflationary policy can be optimal if we introduce heterogeneity in the cost of monitoring.

We also provide a full characterization of the optimal debt-purchase policy, establishing a link between the fundamentals of the economy and the optimal inflation rate. When the economy is subject to persistent shocks, we show that not only the overall

³This result is not robust to the introduction of more than two rounds of DM trade in every period. In this case, we show that the need for both money and debt re-emerges under limited monitoring and a constant money supply. See Section 5.1 for more discussions.

⁴By assuming quasi-linearity, as in LW, our model shuts down the potential benefits of lump-sum inflation that operates through the extensive margin, as in Deviatov and Wallace (2013) and Wallace (2014).

technology/preference shock matters, but also the relative impact of shocks to the credit sector (monitored round) and to the money sector (the non-monitored round). We provide numerical examples where beneficial shocks to the credit sector increase the optimal inflation rate while beneficial shocks to the money sector decrease it. Thus, to be optimal, the debt-purchase program has to target the sector where the liquidity is tight. Our results then suggest that a fully anticipated debt purchase intervention can indeed mitigate the limited commitment friction in the credit market and hence increase lending in the private sector. Because of this benefit, the optimal long-term inflation rate is positive, and its precise level depends on the liquidity needs in the money and the credit sectors.

Now we turn to some related literature. Kehoe and Levine (1993) and Alvarez and Jermann (2001) introduce endogenous borrowing constraints in pure credit economies, and this concept was applied to the LW setting by a few papers to obtain coexistence of money and debt. In a model with one DM round and with unlimited monitoring, Gu, Mattesini, and Wright (2014) give a negative result and show that money and credit cannot both play meaningful transactional roles. In contrast, Sanches and Williamson (2010) obtain coexistence between money and credit by introducing an exogenous cost of using money, while Liu, Wang, and Wright (2015) assume perfect enforcement of debts but introduce an exogenous cost of using credit. Gomis-Porqueras and Sanches (2013) and Lotz and Zhang (2013) obtain coexistence under limited monitoring, but the interventions considered and the coexistence results are specific to the suboptimal trading mechanism adopted.⁵ None of these papers consider debt-purchase interventions. In contrast, by having two DM rounds, we show the optimality of debt-purchase interventions and that both money and credit are used in the optimal trading mechanism. There are other papers that also study the LW setting with two DM rounds.⁶ Guerrieri and Lorenzoni (2009) study the amplification mechanism and Telyukova and Wright (2008) explain the credit card debt puzzle. In contrast to ours, both papers assume perfect enforcement and hence have no endogenous borrowing constraints.

The paper proceeds as follows. In the next section we present the environment, define trading mechanisms, strategies and equilibrium. In section 3, we consider implementation without policy interventions. In section 4 we introduce expansionary monetary policies and characterize the set of implementable allocations under such policies. We then characterize the constrained efficient allocation under expansionary monetary policies and the optimal interventions. We also consider alternative monetary policies. Section 5 presents extensions and concludes. All proofs are in the Appendix.

⁵Some papers assume that debts need to be settled in cash, which also leads to the coexistence between money and credit. This coexistence, though, is driven by their complementarity, and not by their substitutability, as considered here and in the papers mentioned above. References are Berentsen, Camera and Waller (2007), Ferraris and Watanabe (2008), and Li and Li (2013).

⁶Some study pure currency economies, including Berentsen, Camera, and Waller (2005) on the short-run neutrality of money, and Ennis (2009) on the hot potato effect.

2 Model

This section begins with the description of the environment. We then define trading mechanisms, strategies, and equilibrium.

2.1 Environment

Time is discrete and the horizon is infinite. The economy is populated by three types of agents, labeled as buyers, type-1 sellers, and type-2 sellers. The set of buyers is denoted \mathbb{B} , the set of type- i sellers is denoted \mathbb{S}_i , $i = 1, 2$, and each type has measure one. Each period is divided into three stages. Buyers randomly meet type- i sellers in stage i , and the probability of a successful meeting is σ_i , where $i = 1, 2$. There are three goods, one for each stage. At stage $i = 1, 2$, a type- i seller can produce x_i units of stage i good for a buyer at cost $c_i(x_i)$ and the buyer's utility is $u_i(x_i)$. We assume that both u_i 's are strictly concave and increasing, continuously differentiable, and satisfy the Inada conditions: $u_i'(0) = \infty$, $u_i'(\infty) = 0$; similarly, we assume that both c_i 's are convex, strictly increasing, and continuously differentiable. Let x_i^* be the solution to $u_i'(x) = c_i'(x)$, the quantity that maximizes the surplus. In the last stage, agents meet in a centralized market. In this market, they can all consume and produce, and the utility is linear, represented by z (negative values are interpreted as disutility for production). Agents maximize their life-time expected utility with discount factor δ . We let $\rho = \frac{1-\delta}{\delta}$. We call the first two stages DM rounds and the last stage CM round. See also Figure ??.

There exists a technology which keeps track of buyers' trading histories and is available to a fraction of sellers. We call a meeting a monitored meeting if the technology is accessible, and a non-monitored meeting otherwise. This technology works as follows. For each buyer $b \in \mathbb{B}$, a recorded history at period t is a triple, $h = (h_1, h_2, h_3) \in H$, such that for $i = 1, 2$, $h_i = (b, s_i, z_{i,c})$ keeps track of the buyer's round i DM promise to the seller, where b is the identity of the buyer, s_i is the identity of the seller, and $z_{i,c}$ is the promise-to-pay in terms of CM good (debt), and $h_3 \in \mathbb{R}_+$ keeps track of the total repayment. We assume that the repayment is first used to repay the seller from \mathbb{S}_1 , if any, before used to repay the seller from \mathbb{S}_2 .⁷ For $i = 1, 2$, if the buyer does not meet a seller in round- i , or if the buyer meets a seller but there is no trade, h_i is empty. The recorded history h_i is also empty in non-monitored meetings. There also exists a technology, comprised of a set of records R and a function, $\omega : R \times H \rightarrow R$, which updates the record of the buyer based on his recorded history. This technology is only accessible to sellers in monitored meetings and allows the seller to observe the record $r \in R$ of the buyer.

The total measure of monitored meetings is constrained by $\ell \in \{0, 1, 2\}$. We say that monitoring is unlimited if $\ell = 2$, monitoring is limited if $\ell = 1$, and there is no monitoring if $\ell = 0$. When monitoring is limited, we assume that either all type-1 sellers have the

⁷Thus, we are assuming that buyers must settle all debt in the same period. Since the utility function in the CM is linear, this assumption is without loss of generality. Moreover, we could allow the buyer to choose whom to repay to but this would only complicate the notation without adding any insight.

technology or all type-2 sellers have the technology. We are mainly interested in the limited monitoring case, which is meant to capture the idea that monitoring is costly and the society can only afford monitoring a proportion of transactions.⁸

Our monitoring technology resembles the typical operations of unsecured loans, such as commercial papers. It only records the identities of the agents involved in the transaction and the amount of debt. In particular, it does not record agents' money holdings or refusal to trade. In that sense, even unlimited monitoring differs from the usual perfect monitoring assumption in repeated games. It is also much weaker than the notion of memory in Kocherlakota (1998), which includes all actions of all direct and indirect partners of an agent. The debt is unsecured due to the limited commitment friction. The credit records correspond to credit scores (FICO scores, for example) or agency ratings.

Lastly, there is an intrinsically useless, divisible, and storable object, called money. The money supply at the end of period t is denoted by M_t . We assume that agents' money holdings are observable in a match.⁹

2.2 Trading mechanisms

Instead of imposing a particular trading mechanism, we allow arbitrary trading mechanisms that are incentive compatible subject to the frictions in the environment. Taking a mechanism-design approach, we consider a proposal consisting of the following objects:

(P1) A subset $C \subseteq \{1, 2\}$ of seller types with the monitoring technology. When $i \in C$, we say that round- i DM is monitored.

(P2) A sequence of debt limits, $\{D_t\}_{t=0}^{\infty}$, two records, G (Good) and B (Bad); and a sequence of updating functions $\{\omega_t\}_{t=0}^{\infty}$ such that:

- (i) $\omega_t(r, \emptyset) = r$ for $r \in \{G, B\}$;
- (ii) $\omega_t(B, h) = B$ for all $h \in H$;
- (ii) $\omega_t(G, h) = G$ iff $h_3 \geq \min\{D_t, z_{1,c} + z_{2,c}\}$.

We also assume that, if round-2 DM is monitored, the seller observes the buyer's available debt limit, $D_t - z_{1,c}$, in the second DM round. Intuitively, D_t sets the maximum amount of debt a buyer can credibly incur in a given period.

(P3) The proposed trades are given by a function o_i^t defined as follows: if round- i DM is monitored, then

$$o_i^t(m, r, d) = (x, z_{i,c}, z_{i,m}),$$

where m is the buyer's real balance holdings, r is his record, d is his available debt limit,

⁸We restrict the values of ℓ to be in the set $\{0, 1, 2\}$ for simplicity, but our results do not depend on this restriction. Similarly, the assumption that we give the technology to all sellers of the same type is without loss of generality as well. See Section 5.1 for more discussions.

⁹This assumption is common in the literature based on LW, especially those that allow generalized Nash bargaining, as in Gu, Mattesini, and Wright (2014). Relaxing this assumption would introduce additional private information. See also discussions in Section 5.2.

and $(x, z_{i,c}, z_{i,m})$ is the proposed trade— x is the quantity to be produced by the seller, $z_{i,c}$ is the promise the buyer makes to the seller, and $z_{i,m}$ is the transfer of real balances from the buyer to the seller; if round- i DM is not monitored, then

$$o_i^t(m) = (x, z_{i,m}),$$

where m is the buyer's real balance holdings and $(x, z_{i,m})$ is the trade.

(P4) The price for money ϕ_t in the CM, which implies aggregate real balances $Z_t = \phi_t M_t$, and an initial distribution of money holdings μ .

In the proposal, (P1) chooses the meetings to be monitored, which will be the ones with access to credit, only subject to the overall constraint on the number of DM rounds to be monitored given by ℓ . Thus, in contrast to the previous literature on money and credit, here access to credit is endogenously determined.

In (P2), we implicitly assume that $|R| = 2$ and a particular updating rule that uses debt limits, even though our technology allows for an arbitrary finite set of credit records and an arbitrary updating rule. As argued in Bethune, Hu, and Rocheteau (2014), this is without loss of generality.

The functions o_i^t in (P3) map the credit records (if applicable) and the money holding of the buyer to a proposed trade. To implement this proposed trade in a decentralized manner, we use the following trading protocol in meetings in the DM. First both the buyer and the seller respond with *yes* or *no* to the corresponding proposed trade. If both respond with *yes* then they move to the next stage; otherwise, the meeting is autarkic. If they move to the next stage, the buyer makes a take-it-or-leave-it offer, which is implemented if the seller responds with *yes* while the originally proposed trade by the mechanism is carried out otherwise. In turn, the trading mechanism in the CM stage is as follows. Each buyer chooses how much (if any) to repay of his promises, and agents trade competitively against ϕ_t to rebalance their money holdings.

Our trading protocol is in the spirit of a direct mechanism. In particular, we allow arbitrary ways to split the trading surpluses only subject to individual rationality and coalition-proofness. This trading mechanism generalizes the trading protocols considered in Zhu (2008) and HKW to our setting with monitored meetings. As in those papers, the first stage ensures that the mechanism satisfies individual rationality, and the second stage ensures that it satisfies the pairwise core requirement and hence is coalition-proof.¹⁰

In what follows, we focus only on stationary proposals, which can be written as

$$\mathcal{P} = [C, D, (o_1, o_2), (Z, \mu)].$$

¹⁰Note that, since we allow the buyer to make a take it or leave it offer which may differ from the trade proposed by the mechanism, the restriction that the updating function ω only gives a bad record to a buyer who does not repay the promise (up to the debt limit) is with loss of generality. In particular, one could prevent a buyer from making a different offer by conferring a bad record in case he does so. We do not allow such punishments for two reasons. First, punishing a buyer from making a different promise, even if he repays the promise, seems implausible. Second, our results on constrained efficient allocations do not depend on this restriction, and it renders the analysis more tractable.

2.3 Strategies and equilibrium

We denote by s_b the strategy of a buyer $b \in \mathbb{B}$. In each DM meeting at period t , s_b maps the buyer's private history up to that meeting, and his real balance holdings, his record, and the available debt limit upon entering the meeting, to his response $\{yes, no\}$. Note that in our environment a buyer's trading history is his private information, and his matched partner can only observe his current real balance holdings, and, if the meeting is monitored, his record and available debt limit. In turn, conditional on both the buyer and the seller responding with *yes*, s_b gives the buyer offer to the seller. In the CM round s_b maps the buyer's trading history up to that round to his repayment decisions and to his final money holdings when leaving the CM.

We denote by s_{s_i} the strategy of a type- i seller s_i , where $i \in \{1, 2\}$. In the i th DM round, the strategy s_{s_i} maps the buyer's real balance holdings and, if the meeting is monitored, his record and available debt limit, to the seller's response $\{yes, no\}$, and, conditional on both responding *yes*, another function that maps the buyer's offer to $\{yes, no\}$. We assume, without loss of generality, that sellers do not carry money across periods.

We use symmetric Perfect Bayesian Equilibrium as our solution concept, and hence a strategy profile may be denoted (s_0, s_1, s_2) , where s_0 is the buyer strategy for all buyers b , and s_i is the strategy for all sellers from \mathbb{S}_i . We define an equilibrium, consisting of a proposal \mathcal{P} and a strategy profile s as follows.

Definition 2.1. An equilibrium is a list

$$\langle (s_0, s_1, s_2), [C, D, (o_1, o_2), (Z, \mu)] \rangle,$$

such that, given the price of money, each strategy is sequentially rational conditional on other players' strategies; and the centralized market for money clears at every date.¹¹

Throughout the paper we restrict attention to equilibria with the following characteristics: (1) buyers and sellers respond with *yes* in all DM meetings and buyers always offer the trades proposed by the mechanism; (2) the initial distribution of money across buyers is degenerate—all buyers hold M_0 units of money; (3) buyers in state G always repay their debt. We call such equilibria simple equilibria.

In what follows, to simplify notations and to convey our main insights, we restrict attention to the case where $\sigma_1 = 1$. Our results are robust to the case where $\sigma_1 < 1$ and we give a more detailed discussion in Section 5.1. Under the assumption that $\sigma_1 = 1$, an allocation associated with a simple equilibrium can be denoted by

$$\mathcal{L} = [(x_1, x_2), (z_1, z_2)],$$

¹¹Technically speaking, a Perfect Bayesian Equilibrium also requires a belief system for each agent that is consistent with his observations. In our proofs, we construct belief-free equilibria in the sense that the equilibrium strategies are optimal against *any* belief system that is consistent with the observed behavior (and credit records) and with others' equilibrium strategies.

where x_i denotes a buyer's consumption in round- i DM and z_i denotes CM consumption of a round- i seller. Moreover, we restrict our attention to allocations that satisfy $z_1 \leq u_1(x_1) \leq u_1(x_1^*)$ and $z_2 \leq u_2(x_2) \leq u_2(x_2^*)$. This restriction is without loss of generality as far as constrained optimal allocations are concerned, but it avoids many uninteresting cases.

3 Implementation without monetary intervention

In this section, we characterize the set of implementable allocations with a constant money supply. We first consider two benchmark cases: a pure credit economy with unlimited monitoring, and a pure currency economy with no monitoring. We then consider limited monitoring with a constant money supply.

3.1 Pure credit economy with unlimited monitoring

Given an allocation, $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, the buyer's ex ante expected discounted lifetime payoff is given by

$$\sum_{t=0}^{\infty} \delta^t \{ [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - z_2] \} = \frac{1}{1 - \delta} \{ [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - z_2] \}.$$

For \mathcal{L} to be implementable under unlimited monitoring, it is then necessary that repaying the maximum equilibrium debt, which would amount to $z_1 + z_2$, and continuing with the equilibrium future payoffs, is preferred to repaying nothing and receiving no trade in all future periods, that is,

$$-(z_1 + z_2) + \frac{\delta}{1 - \delta} \{ [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - z_2] \} \geq 0,$$

which, by the definition of $\rho = (1 - \delta)/\delta$, can be rewritten as

$$-\rho(z_1 + z_2) + [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - z_2] \geq 0. \quad (1)$$

For a seller from \mathbb{S}_i to be willing to produce in the DM, his production cost must be covered by his payoffs in the CM, that is,

$$z_1 \geq c_1(x_1) \text{ and } z_2 \geq c_2(x_2). \quad (2)$$

Finally, to ensure that the pairwise core requirement is satisfied, the proposed pairwise round-1 surplus plus the buyer's round-2 surplus should be higher than the pairwise round-1 surplus for the output level \hat{x}_1 given by

$$\hat{x}_1 = \min\{x_1^*, c_1^{-1}(z_1 + z_2)\}.$$

Note that the buyer has enough liquidity to induce the seller to produce \hat{x}_1 because $c_1(\hat{x}_1) \leq z_1 + z_2$. Formally, the condition is given by

$$u_1(x_1) - c_1(x_1) + \sigma_2[u_2(x_2) - z_2] \geq u_1(\hat{x}_1) - c_1(\hat{x}_1). \quad (3)$$

Indeed, if the condition (3) does not hold, then the buyer would deviate to offering (\hat{x}_1, \hat{z}_1) for some promise \hat{z}_1 to make both agents better off than the proposed trade.

The following theorem shows that these three necessary conditions are also sufficient.

Theorem 3.1 (Implementability under Unlimited Monitoring). *Let $\ell = 2$ and assume that money has no value. An allocation $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$ is implementable if and only if it satisfies (1), (2), and (3).*

To prove sufficiency, we take $D = z_1 + z_2$ to be the debt limit, and we let the buyer keep a good record as long as he either repays in full or at least D from obligations made in the two stages. Conditions (1) and (2) ensure that buyers are willing to repay D in the CM and sellers are willing to produce. To ensure that the buyer has no profitable deviating offers other than the one involving \hat{x}_1 , we use the HKW mechanism to implement the round-2 allocation so that the buyer can receive a positive round-2 surplus only if his available debt limit when entering round-2 DM is at least z_2 . Then, we show that the condition (3) is sufficient to ensure that the deviating offer with \hat{x}_1 is not profitable.

3.2 Pure currency economy with no monitoring

In the absence of monitoring, money is necessary to implement any positive production. Hence, to implement an allocation $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, buyers have to hold at least $z_1 + z_2$ units of real balances upon leaving the CM. First we remark that conditions (1) and (2) are still necessary for individual rationality: without (1) buyers are better off holding zero real balances when leaving the CM and not participating in any trades; without (2) sellers will not produce. Similarly, the condition (3) is still necessary for the pairwise core requirement. Indeed, if it does not hold, the buyer can deviate and request \hat{x}_1 as output and some monetary payment to make both agents better off.

However, with money alone, one more condition is necessary, because the buyer can hold real balances that are only sufficient for round-2 DM trade but skip round 1. Without the monitoring technology such deviation is not detectable. This leads to the following condition:

$$-\rho z_1 + [u_1(x_1) - z_1] \geq 0. \quad (4)$$

According to condition (4), the surplus for the buyer in round-1 DM, $u_1(x_1) - z_1$, has to be sufficiently large to compensate for his cost of holding z_1 units of real balances across periods.

The following theorem shows that these necessary conditions are also sufficient.¹²

¹²We extend the HKW mechanism to show the sufficiency. However, while in HKW the implementation

Theorem 3.2 (Implementability under No Monitoring). *Let $\ell = 0$ and assume that the money supply is constant. An allocation $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$ is implementable if and only if (1), (2), (3), and (4) are satisfied.*

Compared to Theorem 3.1, Theorem 3.2 requires an additional condition, (4). This additional constraint reflects the fact that in a pure currency economy, buyers with different private histories who enter the round-2 DM with the same money holding must be treated equally. In particular, all buyers who enter round-2 DM with the equilibrium amount of real balances must enjoy the equilibrium surplus in that round, even though they may have defected in their CM trades or in their round-1 DM trades. This additional constraint makes implementation in a pure currency economy more restrictive than implementation in a pure credit economy. In contrast, in the baseline LW model with only one DM round, since the optimal money holding to leave the CM is independent of the buyer's money holding upon entering the CM, the buyer's private history leading to the DM round has no bite. Accordingly, as shown in HKW, the optimal mechanism under no monitoring can use this feature to achieve any allocation implementable under perfect monitoring. Here we show that such results only hold for the LW model with one DM round but would fail generically in models with many DM rounds.

3.3 Constant money supply with limited monitoring

In the presence of limited monitoring, money is necessary to sustain positive production in non-monitored meetings.¹³ It turns out that, if monitoring is limited, credit trades cannot meaningfully expand the set of implementable outcomes than money alone. In particular, condition (4) is still necessary in the presence of limited monitoring, irrespective of whether the first DM round or the second DM round is monitored. When round-1 DM is monitored, (4) is necessary to ensure that the buyers are willing to repay their debts. Indeed, in the CM the minimum repayment is z_1 and the future surpluses from monitored trades is

$$\sum_{t=1}^{\infty} \delta^t [u_1(x_1) - z_1] = \frac{1}{\rho} [u_1(x_1) - z_1],$$

and this implies that condition (4) is necessary for repayment to be individually rational. Similarly, if the second DM round is monitored, then money is necessary to finance the trades in round-1 DM meetings. Hence, the condition (4) is necessary for otherwise the buyer would prefer to skip round-1 meetings. We have the following lemma.

is achieved by using a mechanism according to which the buyer surplus may be discontinuous in his real balances, here in round-2 trades we use a mechanism that ensures the continuation value is continuous in real balances at the round-1 DM. This is useful because continuity guarantees existence of the proposed trades as the solution to a maximization problem.

¹³This necessity follows from our formulation of the monitoring technology and the assumption that there is a continuum of agents. In particular, the arguments used by **Araujo and Carmago (2014)** to sustain trades without money in the Cavalcanti-Wallace setting do not apply here as the records are only accessible by sellers with the monitoring technology.

Lemma 3.1. *Let $\ell = 1$ and assume that the money supply is constant. An allocation $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$ is implementable under a constant money supply only if it satisfies (1), (2), and (4).*

When round-1 DM is monitored, the condition (3) is also necessary because in round-1 DM, the buyer can use both money and debt, and hence the previous logic applies. When round-2 DM is monitored, however, this condition may not be necessary because the buyer cannot transfer debt from round-2 DM to round-1 DM.¹⁴ Nevertheless, as shown below, (3) is not binding when we look for constrained efficient allocations and hence relaxing it does not help.

Lemma 3.1, compared against Theorem 3.2, shows that **under limited monitoring and a constant money supply**, the use of debt does not expand the set of implementable outcomes. However, it turns out that this result is overturned when we consider monetary interventions, particularly the ones that directly purchase private debts.

4 Monetary intervention under limited monitoring

In this section, we introduce a class of monetary interventions which expand the set of implementable allocations under limited monitoring. This class of interventions is purely monetary in the sense that no taxation other than inflation is used. We first characterize the set of implementable allocations and then look at the optimal monetary policy within this class. Finally, we discuss alternative monetary interventions, for which fiscal actions are allowed, as long as they are consistent with our monitoring technology, and show that they are suboptimal.

4.1 Expansionary monetary policies (EMP)

We consider interventions that use the seigniorage revenue from money creation to purchase privately issued debt. We label these interventions expansionary monetary policies (henceforth EMP). In our stationary environment, we are mainly concerned with the long-run implications of these policies. In particular, the EMP is fully anticipated and the policymaker has full commitment power.

Consider an environment with limited monitoring and a proposal with $C = \{i\}$, that is, round- i DM has monitored meetings. The EMP sets a maximum amount k of private debt that the mechanism will redeem. That is, for any recorded promise at period t , $(b, s_i, z_{i,c}^b)$, the mechanism will print new money and use it to purchase $\min\{k, z_{i,c}^b\}$ of the

¹⁴It may be surprising that (3) is necessary under unlimited monitoring but not under limited monitoring. This has to do with our restriction to updating rules which only give a bad record to a buyer who does not repay his debt. Thus, if debt is accepted in both rounds, a buyer can deviate by moving debt from round-2 DM to round-1 DM. He cannot do that if monitoring is limited and $C = \{2\}$ because debt is not accepted in round-1 DM.

debt from the seller s_i .¹⁵ Feasibility then implies that the inflation rate π must satisfy

$$\int_{b \in \mathbb{B}} \min\{z_{i,c}^b, k\} db = \pi \phi_t M_{t-1}, \quad (5)$$

where M_{t-1} is the amount of money in the economy in period t , right before the monetary intervention. Thus, we require that, at every period t , the inflation rate is consistent with the amount of debt which is purchased by the EMP.

Our EMP resembles some credit policies implemented by central banks. In particular, the left-hand side of (5) captures the direct purchase of private debts, such as commercial papers or other assets, and the right-hand side corresponds to its inflationary implications. The EMP here is assumed to be fully anticipated by the agents, and hence the purchase by the EMP is an implicit subsidy to the credit sector, funded by an inflation tax in the monetary sector.

Formally, a proposal now includes $\mathcal{P} = [C, D, (o_1, o_2), (Z, \mu)]$ and an EMP (k, π) . We say that an EMP is active if $k > 0$. An allocation, $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, is implementable with EMP if it is implementable under a proposal \mathcal{P} and an EMP (k, π) .

Theorem 4.1 provides a full characterization of implementable allocations with EMP under limited monitoring. We distinguish two cases: the first uses $C = \{1\}$ (round-1 DM has monitored meetings) while the second uses $C = \{2\}$ (round-2 DM has monitored meetings).

Theorem 4.1 (Expansionary Monetary Policies). *Assume limited monitoring.*

(i) *An allocation, $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, is implementable with EMP and $C = \{1\}$ if and only if (1), (2), and (3).*

(ii) *An allocation, $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, is implementable with EMP and $C = \{2\}$ if and only if (2), (4), and*

$$\{-\rho z_1 + [u_1(x_1) - z_1]\} + \frac{(\rho + 1)\sigma_2}{\rho + \sigma_2} \{-\rho z_2 + \sigma_2 [u_2(x_2) - z_2]\} \geq 0. \quad (6)$$

Theorem 4.1 (i) shows that, when the first DM round is monitored, the EMP under limited monitoring implements exactly the same set of allocations as the pure credit economy under unlimited monitoring. If the condition (4) holds, the EMP is inactive and, as shown in Theorem 3.2, a constant money supply suffices even if debt is not used. However, if (4) does not hold, an active EMP (and hence the use of debt) is necessary to implement the allocations achieved with unlimited monitoring. Under EMP, the buyer has incentive to repay his debt if and only if

$$-\rho(z_1 - k) + [u_1(x_1) - z_1 + k] \geq 0.$$

¹⁵ Alternatively, we can formulate the expansionary monetary policy as a proportional subsidy. More precisely, the policy sets a fraction κ for the monitored stage i , and the mechanism commits to purchase κ fraction of the debts issued by buyers to stage- i sellers. One issue with this scheme is that it may induce buyers to overissue their debts. In particular, this scheme may implement production above the first-best level. In any case, it can be shown that, in terms of constrained efficient allocations, this scheme does not do better than ours.

To satisfy the above inequality, the minimal amount of debt to be purchased by the EMP and the corresponding inflation rate are given by

$$k_1 = z_1 - \frac{u_1(x_1)}{1 + \rho} \text{ and } \pi_1 = \frac{k_1}{z_2}, \quad (7)$$

where z_2 is the amount of real balances needed to finance round-2 DM trades. It turns out that, by (1), this inflation rate is consistent with participation in the round-2 monetary trade.

Theorem 4.1 (ii) gives different conditions from (i) when round-2 DM is monitored. Because there is no monitoring in round-1 DM, the condition (4) is necessary to ensure that buyers are willing to hold money and participate in round-1 trades. Now, if $-\rho z_2 + \sigma_2[u_2(x_2) - z_2] \geq 0$, then a constant money supply suffices and debt is not needed. When that inequality fails, however, EMP is necessary, and under the EMP, the buyer has incentive to repay his debt incurred in round-2 trades if and only if

$$-\rho(z_2 - k) + \sigma_2[u_2(x_2) - z_2 + k] \geq 0.$$

To satisfy the above inequality, the minimal amount of debt to be purchased by the EMP and the corresponding inflation rate are given by

$$k_2 = z_2 - \sigma_2 \frac{u_2(x_2)}{\sigma_2 + \rho} \text{ and } \pi_2 = \sigma_2 \frac{k_2}{z_1}. \quad (8)$$

Note that in this case the fraction of buyers issuing debts in round-2 DM is σ_2 in equilibrium. Different from the case with first DM round monitored, however, when round-2 DM has monitored meetings, an active EMP can implement allocations which cannot be implemented with unlimited monitoring.¹⁶ Indeed, since, under (4) and $\sigma_2 < 1$, (6) is less restrictive than (1), there are allocations which do not satisfy (1) but satisfy (2), (4), and (6). As we shall see later, these may include constrained-efficient allocations. The intuition for this result runs as follows. Under unlimited monitoring but without the EMP, a buyer who participates in the two DM rounds incurs a cost $z_1 + z_2$ in the CM round in order to redeem the debts issued in exchange for the DM goods. Under limited monitoring but with EMP and $C = \{2\}$, the cost associated with obtaining the same amount of DM goods is given by $(1 + \pi_2)z_1 + z_2 - k_2$, i.e., the direct cost of redeeming part of the debts issued by the buyer himself, and the indirect cost of holding z_1 real balances to participate in the first DM round when the inflation rate is given by π_2 . Feasibility of the EMP implies $\pi_2 z_1 = \sigma_2 k_2$, and we can rewrite the difference in the costs with and without EMP, $-k_2 + \pi_2 z_1$, as $-(1 - \sigma_2)k_2$, which is negative whenever $\sigma_2 < 1$.

4.2 Optimal monetary policy

We now characterize the set of optimal allocations. Our main focus is on the necessity of the EMP to achieve such allocations. For a given allocation, $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, social

¹⁶This result does not contradict Kocherlakota's (1998) result that credit is superior to money as our monitoring technology is less demanding than Kocherlakota. Besides, Kocherlakota did not consider monetary interventions.

welfare is given by

$$\mathcal{W}(\mathcal{L}) = \frac{1 + \rho}{\rho} \{ [u_1(x_1) - c_1(x_1)] + \sigma_2 [u_2(x_2) - c_2(x_2)] \}. \quad (9)$$

We say that an allocation is *constrained efficient* if it maximizes (9) among all implementable allocations. Note that, without taking implementability into account, the optimal allocation is given by $x_1 = x_1^*$ and $x_2 = x_2^*$. It does not depend on z_1 and z_2 as the utility in the CM is linear.

To maximize social welfare, it is without loss of generality to have the constraint (2) binding, i.e., to consider only allocations of the form $\mathcal{L} = [(x_1, x_2), (c_1(x_1), c_2(x_2))]$.¹⁷ We say that a pair (x_1, x_2) is a constrained-efficient allocation if $[(x_1, x_2), (c_1(x_1), c_2(x_2))]$ is a constrained-efficient allocation. Note that, although we are interested in the case of limited monitoring, this result applies irrespective of the degree of monitoring.

If the first-best allocation is implementable under no monitoring and a constant money supply, then debt is not essential. Now, by Theorem 3.2, to determine whether a first-best allocation, (x_1^*, x_2^*) , is implementable under no monitoring and a constant money supply amounts to check whether conditions (1) and (4) hold under that allocation, and we have the following corollary. Note that (3) is trivially satisfied for any first-best allocation.

Corollary 4.1. *The first-best allocation, (x_1^*, x_2^*) , is implementable under no monitoring and with a constant money supply if and only if*

$$\rho \leq \rho^M \equiv \min \left\{ \frac{[u_1(x_1^*) - c_1(x_1^*)] + \sigma_2 [u_2(x_2^*) - c_2(x_2^*)]}{c_1(x_1^*) + c_2(x_2^*)}, \frac{u_1(x_1^*) - c_1(x_1^*)}{c_1(x_1^*)} \right\}. \quad (10)$$

The first term inside the min operator corresponds to (1), and the second term corresponds to (4). By Corollary 4.1, when $\rho \leq \rho^M$, debt is not necessary to implement the first-best, and we are only interested in the case where $\rho > \rho^M$. Note that, by Lemma 3.1, the constrained efficient allocation under limited monitoring with a constant money supply cannot do better than the optimal allocation subject to (1) and (4). Against this allocation, below we show that for $\rho > \rho^M$, the constrained efficient allocation with EMP has higher welfare.

For the constrained-efficient allocation with EMP, we consider two candidates depending on whether $C = \{1\}$ or $C = \{2\}$. According to Theorem 4.1 (i) for the case $C = \{1\}$, the relevant constraints include (1) and (3), and according to Theorem 4.1 (ii) for the case $C = \{2\}$, the relevant constraints include (4) and (6). These two sets of constraints correspond to two candidates for the constrained efficient allocations. The first, denoted by (x_1^{C1}, x_2^{C1}) , maximizes (9) subject only to (1), as the lemma below shows that (3) is not binding. The second, denoted by (x_1^{C2}, x_2^{C2}) , maximizes (9) subject to (4) and (6).

Lemma 4.1. *Assume limited monitoring. Both (x_1^{C1}, x_2^{C1}) and (x_1^{C2}, x_2^{C2}) are implementable with EMP, and either one of them is the constrained-efficient allocation.*

¹⁷To be more precise, as in HKW, when the first best is implementable, it can be implemented with a mechanism under which (2) is binding but it does not need to be. When the first best is not implementable, however, (2) has to be binding to implement the constrained efficient allocation.

As said above, since both (1) and (4) are necessary under a constant money supply, the constrained efficient allocation without intervention cannot do better than (x_1^{C1}, x_2^{C1}) . Thus, by Lemma 4.1, if (4) fails for (x_1^{C1}, x_2^{C1}) , then an active EMP is necessary. Our next theorem shows that, at least generically, even when (4) holds for (x_1^{C1}, x_2^{C1}) , EMP is still necessary to implement the constrained-efficient allocation. The genericity condition first requires $\sigma_2 < 1$. Second, it rules out the knife-edge case where the three conditions, (1), (4), and (6), are all binding for the constrained efficient allocation. Formally, this amounts to ruling out the case where the constrained-efficient allocation is equal to (\bar{x}_1, \bar{x}_2) , defined as the unique positive solution to

$$-\rho c_1(\bar{x}_1) + [u_1(\bar{x}_1) - c_1(\bar{x}_1)] = -\rho c_2(\bar{x}_2) + \sigma_2[u_2(\bar{x}_2) - c_2(\bar{x}_2)] = 0. \quad (11)$$

Generically, $(\bar{x}_1, \bar{x}_2) \neq (x_1^{C1}, x_2^{C1})$, as the latter has to satisfy the FOC's implied by the maximization problem as well. We have the following theorem.

Theorem 4.2. *Suppose that $\rho > \rho^M$, $\sigma_2 < 1$, and that $(\bar{x}_1, \bar{x}_2) \neq (x_1^{C1}, x_2^{C1})$. Then, the constrained efficient allocation with EMP cannot be implemented with a constant money supply.*

As noted earlier, if (4) fails for (x_1^{C1}, x_2^{C1}) , then an active EMP is necessary. To prove Theorem 4.2, we show that when (4) holds for (x_1^{C1}, x_2^{C1}) , we can construct another allocation that satisfies (4) and (6) and that gives a higher welfare than that of (x_1^{C1}, x_2^{C1}) , and hence (x_1^{C2}, x_2^{C2}) is the constrained-efficient allocation and EMP is necessary. The crucial observation is that when $\sigma_2 < 1$ and when (4) is slack, (6) allows for better allocations than (1).

The optimal EMP to implement the constrained efficient allocations can be inferred from Theorem 4.1. Depending upon which of the two candidates is the constrained-efficient allocation, the optimal EMP is either given by (7) with $(z_1, z_2) = (c_1(x_1^{C1}), c_2(x_2^{C1}))$ or given by (8) with $(z_1, z_2) = (c_1(x_1^{C2}), c_2(x_2^{C2}))$. However, the optimal EMP is not uniquely determined when the first-best allocation is implementable, and the above two formulas give the policies that correspond to the minimal optimal interventions.

Note that the two candidates for the constrained efficient allocation require different monitored rounds. Thus, in our theory, not only money and debt are necessary, but the determination of how the monitoring technology should be allocated across DM rounds is endogenous and depends on economic fundamentals. Moreover, although the EMP is essential, the nature of the optimal policy will depend largely on the fundamentals as well.

We illustrate these results by some numerical examples, using the following functional forms: for $i = 1, 2$, $u_i(x_i) = x_i^{a_i}$, $a_i \in (0, 1)$, and $c_i(x_i) = x_i$. Here we set $a_1 = 0.99$ and $a_2 = 0.97$. When $\sigma_1 = 1$, and $\sigma_2 = 0.95$, $\rho^M = 1.01\%$, and the first-best is implementable with EMP if and only if $\rho \leq 1.97\%$. Moreover, the constrained-efficient allocation is (x_1^{C1}, x_2^{C1}) for $\rho \in (1.01\%, 15\%]$, and the optimal EMP, (k, π) , is depicted in Figures ?? and ?. For such ρ 's, the formula (7) applies.

The optimal intervention is not monotonic in the discount rate. When the first-best is implementable, $x_1^{C1} = x_1^*$ and hence, by (7), both optimal k and π increase with ρ , since

we need more subsidies but the output level is kept constant. For larger ρ 's, however, x_1^{C1} decreases with ρ and hence we have two opposite effects on k : as ρ gets larger, the output level gets lower, but for any given level of round-1 output, a higher subsidy is necessary to maintain incentive compatibility. In our example, the first effect overcomes the second and k decreases with ρ . Similarly, there are two opposite effects on π as well: as ρ gets larger, equilibrium real balances decrease and this tends to increase π , but for any given level of real balances, a lower k decreases the necessary inflation rate. The second effect overcomes the first and hence π decreases with ρ .

We also remark that, for any $\rho > \rho^M$, the constrained-efficient allocation is given by (x_1^{C2}, x_2^{C2}) if σ_2 is sufficiently small. In the above example, for $\rho \in (1.01\%, 2\%]$, the constrained-efficient allocation is given by (x_1^{C2}, x_2^{C2}) if $\sigma_2 < 0.29$. Thus, the fundamentals also matter for the choice of which sector should be endowed with the monitoring technology under the optimal trading mechanism.

Aggregate Shocks

Here we give some examples to illustrate how the optimal EMP responds to productivity shocks. We only focus on comparative statics across steady states, but our model can be extended to allow for persistent shocks and the results there would be similar to findings reported here. We use the above functional forms, but introduce shocks to both DM rounds: for $i = 1, 2$, $u_i(x_i) = \theta_i x_i^{a_i}$, $a_i \in (0, 1)$, and $c_i(x_i) = x_i$, and we set $a_1 = 0.99$, $a_2 = 0.97$, and $\rho = 2\%$. In this case, the optimal EMP not only depends on the magnitude of θ_1 and θ_2 , but it also depends on the relative size of the two. Figure ?? shows the optimal inflation rates for $(\theta_1, \theta_2) \in [0.9.1.1] \times [0.9.1.1]$. We remark that under this range of parameters, the optimal mechanism has $C = \{1\}$ and the optimal policies are given by (7). In Figure ??, the optimal inflation rate increases with θ_1 but decreases with θ_2 . This implies that, to determine the optimal monetary policy, not only how the shock affects the overall economy matters, but how the shock affects the monitored sector (credit sector) relative to the non-monitored sector (monetary sector) also matters. In particular, if the shock is more beneficial to the credit sector, i.e., if θ_1 increases more, then the optimal inflation rate is pro-cyclical. In contrast, if the shock is more beneficial to the monetary sector, i.e., if θ_2 increases more, then the optimal inflation rate is counter-cyclical.

To illustrate this point, we control θ_1 and θ_2 by a parameter η as follows:

$$\theta_1 = 2 \times (\eta + 0.45) \text{ and } \theta_2 = \frac{0.17}{0.22 - \eta}.$$

Under this parametrization, both θ_1 and θ_2 increase with η , but the relative increase depends on the value of η . The above findings then imply that the optimal inflation rate should increase with η when θ_1 increases more than θ_2 , that is, when η is relatively small, while it should decrease with η when θ_1 increases more than θ_2 , that is, when η is relatively large. Figure ?? illustrates this result: the optimal inflation rate, π , measured in the vertical axis, first increases and then decreases with η , measured in the horizontal axis.

These results demonstrate that liquidity needs are endogenously determined by the fundamentals. Moreover, the optimal policy response in terms of liquidity provision requires a detailed knowledge about how shocks to the fundamentals affect different sectors, especially the distribution between the credit and monetary sectors in the economy.

4.3 Alternative monetary policies

Here we consider alternative monetary policies and show that the EMP dominates in terms of social welfare. We only consider policies that are consistent with our monitoring technology. This has two implications. First, in terms of taxation, inflation is the only possible tax for non-monitored meetings, and taxation in monitored meetings is possible but has to be voluntary in the sense that the only punishment is to give a bad record that is observable in future monitored meetings. In terms of subsidies, the mechanism can subsidize monitored trades, as done in EMP, or it can subsidize non-monitored trades through deflation.

As a result, only two types of policies are relevant: the first subsidizes monitored trades through inflation, which is the EMP, and the second subsidizes non-monitored trades with deflation financed by taxation in monitored trades. The other two combinations, namely, taxing and subsidizing only monitored trades or only non-monitored trades, are either neutral or harmful.¹⁸ Lastly, to compare with the literature, we also consider lump-sum injections of money as a separate case.

Deflationary monetary policy

First we consider interventions that taxes monitored trades. Such taxes can be thought of as a fee to use the monitoring technology. All taxes are paid in the CM. Because the mechanism can only punish agents conditional on records, it is without loss of generality to assume that the only punishment for not paying the fee is to give the individual a bad credit record. Using these fees to subsidize monitored trades will be at best neutral. Because non-monitored trades are anonymous, the only way to subsidize that sector is to use the fees to retire a fraction of the money stock and hence provide interest on money holdings through deflation. We call such interventions deflationary monetary policy (DMP).

Consider a mechanism where round- i DM has monitored meetings. Then buyers may issue debt in those meetings. The DMP sets a fee (in terms of the CM good), η , on the use of the monitoring technology and then retires money with those revenues in the CM. Therefore, if a buyer b chooses to accept a monitored trade at period t , the buyer has to pay extra η to keep his good record. Let τ be the net money contraction rate. Thus, for

¹⁸There are papers that use inflation tax to finance interest on money, but the amount of interest is conditioned on money holdings. Examples of such schemes include the interest-bearing money in Andolfatto (2010) and the progressive and regressive schemes in Wallace (2014). Such schemes are not consistent with our monitoring technology, since there is no record of agents' money holdings. Moreover, it can be shown that such policies always have to respect (1), and hence they cannot do better than the optimal EMP.

each t , $M_{t+1} = (1 - \tau)M_t$ and we focus only on proposals with constant real balances, that is, $\phi_{t+1} = \phi_t/(1 - \tau)$. Then, if a measure β of buyers use monitored trades, feasibility requires a corresponding deflation rate τ such that

$$\beta\eta = \tau\phi_t M_{t-1}. \quad (12)$$

An allocation is implementable with (η, τ) that satisfies (12) if there exists some proposal $\langle \mathcal{P}, (\eta, \tau) \rangle$ such that the allocation is consistent with the simple equilibrium outcome under such proposal. We have the following theorem.

Theorem 4.3 (Deflationary Monetary Policy). *Assume limited monitoring. For any allocation, $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, we have that*

(i) \mathcal{L} is implementable under $C = \{1\}$ only if it satisfies (1), (2), and (4),

(ii) \mathcal{L} is implementable under $C = \{2\}$ only if it satisfies conditions in (i), or it satisfies (2) and

$$\frac{\sigma_2 + \rho}{(1 + \rho)\sigma_2} \{-\rho z_1 + [u_1(x_1) - z_1]\} + \{-\rho z_2 + \sigma_2[u_2(x_2) - z_2]\} \geq 0, \quad (13)$$

$$-\rho z_2 + \sigma_2[u_2(x_2) - z_2] \geq 0. \quad (14)$$

The necessary conditions in Theorem 4.3 are almost sufficient. What is missing is the pairwise core requirement, which is necessary in some cases. However, the cases for which (3) is relevant under EMP differ from those under DMP, and hence, there may be allocations that are not implementable with EMP due to (3) but are implementable with DMP. However, (3) never binds for constrained-efficient allocations.¹⁹ Since any allocation that satisfies (1), (2), and (4), together with (3), is implementable with a constant money supply, these conditions appear in both Theorem 4.3 (i) and (ii). Moreover, (i) shows that under first DM round monitored, DMP cannot do better than constant money supply, while (ii) gives some potentially more relaxed conditions. Ignoring the pairwise core requirement, (3), Theorem 4.3 shows that implementation with DMP is more restrictive than with EMP. In fact, one can show that unless the DMP can implement the first-best, EMP strictly dominates DMP generically.

The crucial factor that leads EMP to dominate DMP is that, by using inflation, EMP taxes all agents while it only subsidizes a subset of agents who engage in monitored trades. In contrast, DMP can only tax agents who engage in monitored trades, but it subsidizes all agents as non-monitored trades are anonymous. Thus, contrary to the suggestion that deflationary policies are optimal in pure currency economies, here we show that policies using inflation to subsidize the credit sector are better than deflationary policies.

The result that DMP is dominated by EMP, however, crucially depends on our freedom to choose which DM round to be monitored. Instead, if the DM round to be monitored is exogenously given, then it can be the case that DMP is necessary to implement the

¹⁹We can only show this result for sufficiently high σ_1 's, and hence, theoretically speaking, DMP may be useful for low σ_1 's. However, in all our numerical examples, the pairwise core requirement never binds, even for fairly small σ_1 's.

constrained efficient allocation. In particular, if we impose the constraint that only the second DM round can be monitored, then by Theorem 4.1 (ii), EMP requires (4) while it is not needed for DMP as shown in Theorem 4.3 (ii) (it can be shown that conditions there are also sufficient), and hence we can find parameters under which the first-best allocation is implementable with DMP but not with EMP or constant money supply.

Lump-sum transfers of money

The most commonly discussed monetary policy in the literature is the lump-sum injection of money. Here we assume that, as typically in the literature, newly created money is given to all buyers with equal amount before the CM opens.²⁰ Let π be the net money growth rate and let M_{t-1} be the average money holdings at the beginning of period t . Then, the policy gives each buyer πM_{t-1} units of money at the beginning of period- t CM in a lump-sum fashion. The following theorem shows that such policy shrinks the set of implementable outcomes even against a constant money supply.

Theorem 4.4 (Implementability under lump-sum transfers). *Assume limited monitoring and let the net money growth rate be π with lump-sum transfers. Let $\zeta = (1 + \pi - \delta)/\delta \geq \rho$. Then, an allocation, $\mathcal{L} = [(x_1, x_2), (z_1, z_2)]$, is implementable under $\pi > 0$ only if it satisfies (2), (4), and*

$$-\rho z_1 - \zeta z_2 + [u_1(x_1) - z_1] + \sigma_2[u_2(x_2) - z_2] \geq 0, \quad (15)$$

or

$$-\zeta z_1 - \rho z_2 + [u_1(x_1) - z_1] + \sigma_2[u_2(x_2) - z_2] \geq 0. \quad (16)$$

Compared against Lemma 3.1, the conditions in Theorem 4.4 are more restrictive: (15) and (16) are more restrictive than (1) while (2) and (4) are the same. As in Lemma 3.1, (3) is necessary under $C = \{1\}$ but may not be necessary under $C = \{2\}$. Because (3) is never binding for constrained-efficient allocations subject to constant money supply, these results imply that such inflation is never optimal even against a constant money supply.

5 Concluding Remarks

In this paper we propose a model of monetary interventions in credit markets that features a liquidity role for both money and debt. We show that expansionary monetary policies, which directly purchase privately issued debt through inflation, can efficiently reallocate liquidity from the monetary sector to the credit sector. In particular, the main channel through which those policies improve welfare is by relaxing the borrowing constraints in

²⁰Because of the lump-sum nature, the assumption that only buyers receive the transfers and the assumption that every buyer receive the same amount are not crucial. For example, the result would be exactly the same if the money transfer is given randomly in a lump-sum fashion. In fact, this random transfer would be more consistent with our environment because it does not require the money to be sent to each agent, which may require some monitoring.

the credit market. Our model highlights that the harmful impacts of inflation on the monetary sector are more than compensated by the benefits in the credit sector, if the policy is implemented optimally. Finally, our results show that the optimal intervention depends on the details of economy, especially how the shocks are distributed across different sectors. To obtain these results we make a number of assumptions. Below we discuss the robustness of our results to relaxing some of our assumptions. We also offer some possible extensions.

5.1 Robustness to simplifying assumptions

General limited monitoring. In the main text we only consider three cases: $\ell = 0$, $\ell = 1$, and $\ell = 2$, and, when $\ell = 1$, we assume that one DM round is monitored. For continuous ℓ , we have the following results. First, generically, there is an open interval $(\underline{\ell}, \bar{\ell})$ around $\ell = 1$ such that for all ℓ in that interval, an active EMP is optimal. For small ℓ 's a constant money supply can be optimal without using debts. For large ℓ 's the use of debt is always optimal but interventions may not be necessary. We also consider the model with one DM round, and we show that the constrained efficient allocation is the same for all $\ell \in [0, 1]$, and can be implemented without using debt or any intervention. This result generalizes HKW. We leave the formal arguments to the Supplemental Material, Section 1.

Meeting probabilities with $\sigma_1 < 1$. Our results generalize to the case where $\sigma_1 < 1$. The main technical difficulty in this extension is dealing with asymmetric allocations. However, since the first-best allocation is independent of σ_1 , we can follow the same logic of Theorem 4.2 to show that for any $\sigma_1 \leq 1$, there exists a $\rho^0 > \rho^M$ such that for all $\rho \in (\rho^M, \rho^0]$, the first-best is implementable only with EMP and hence the optimal inflation rate is positive. We can also extend this result to lower ρ 's as well, and there exists $\rho^1 > \rho^0$ such that for all $\rho \in (\rho^0, \rho^1]$, the first-best is not implementable but the constrained-efficient allocation is only implementable with EMP. As a result, for any σ_1 , the EMP is generically optimal for a range of discount factors. Moreover, for any ρ , the EMP is generically essential for sufficiently high σ_1 's. The formal arguments are contained in Supplemental Material, Section 2.

Multi-stage DM's. We can also introduce many DM rounds, say, I rounds. It is straightforward to derive conditions for implementation of the first-best allocations. In particular, we can show that for $\ell = I - 1$, generically there is a range of discount rates under which both money and debt are used in the optimal trading mechanism and the first best allocation can be implemented with an active EMP. Moreover, for $I \geq 3$ and $\ell < I$, it can be the case that both money and credit are used in any optimal mechanism while the EMP is not necessary to implement the constrained efficient allocation. See the Supplemental Material, Section 3, for more details.

Alternative meeting patterns. One special feature of our model is that buyers can only meet sellers from \mathbb{S}_i at round i . However, our results are robust to other arrangements. In particular, we can accommodate the following setting. In both DM rounds, a buyer may meet a seller from \mathbb{S}_1 or \mathbb{S}_2 or none. We can show that under this setting the main result, Theorem 4.2 still hold in we impose sufficient symmetry; the asymmetric case is similar to the case where $\sigma_1 < 1$ considered above. See the Supplemental Material, Section 4, for details.

5.2 Extensions

Other assets. In our model we only allow unsecured credit arrangements. In reality many credit arrangements involve both collateralized and unsecured elements. To take such arrangements seriously, it is necessary to introduce other assets, such as capital or nominal bonds, that would coexist with money as means-of-payments. This coexistence issue has been difficult to deal with. One possible route is provided by Hu and Rocheteau (2013, 2014), who show that money and assets with higher rate-of-returns can coexist under the optimal trading mechanism. Alternatively, we could follow Williamson (2012) and assume that nominal bonds can only be used in monitored trades. We conjecture that, as long as credit or collateralized trades still include some unsecured elements and hence face endogenous borrowing constraints, policy analogous to our EMP may still be beneficial.

Heterogeneity in credit trades. In our analysis heterogeneity across rounds is allowed but not within each DM round. Our results generalize to the case where such heterogeneity is observable to the monetary authority and hence the debt purchase policy can condition on the type of the meeting. However, if we introduce unobservable heterogeneity in credit trades, then this informational friction may reduce the benefits of the debt purchase interventions as it may encourage inefficient lending. The mechanism design problem then has to deal with this private information as well. We leave these issues for future research.

Unobservable money holdings. In our analysis we assume that buyers' money holdings are observable in the meetings. Under this assumption, the pairwise core is well-defined in any meeting, and we have characterized the optimal trading mechanism under the pairwise core requirement. Alternatively, one may assume that money holdings are buyers' private information, and impose the constraint that buyers cannot overstate them (the "show-me" constraint), as in HKW. Our implementation and constrained-efficiency results under EMP will go through under this alternative assumption if we drop the pairwise core requirement. However, as pointed out by Wallace (2014), the pairwise core requirement under this alternative assumption is problematic, as a general notion of core under asymmetric information is not available, especially in environments without quasi-linearity. In particular, in the LW model with one DM round, buyers are willing to truthfully report their money holdings as long as their DM surpluses are increasing in

their money holdings, as shown in HKW. While the constructed trading mechanism in our proofs satisfy this property, it does not guarantee truth-telling in the first DM round, as the continuation of money is no longer linear there.

6 Appendix: Proofs

Proof of Theorem 3.1

(\Rightarrow) First we prove necessity. We have proved the necessity of (1) and (2) in the text. Now consider (3). Suppose it does not hold and

$$u_1(\hat{x}_1) - c_1(\hat{x}_1) > [u_1(x_1) - z_1] + \sigma_2[u_2(x_2) - z_2] + [z_1 - c_1(x_1)]. \quad (17)$$

The first two terms of the right side of (17) correspond to the expected surplus of the buyer in the two DM rounds and the last term is the surplus of the seller in the first DM round. By (17) there exists $0 \leq \hat{z}_1 \leq z_1 + z_2$ such that

$$u_1(\hat{x}_1) - \hat{z}_1 > [u_1(x_1) - z_1] + \sigma_2[u_2(x_2) - z_2] \text{ and } \hat{z}_1 - c_1(\hat{x}_1) > z_1 - c_1(x_1),$$

and the buyer has a profitable deviation. Note that this deviation does not affect the buyer's credit record as long as he repays \hat{z}_1 in the CM.

(\Leftarrow) We now prove sufficiency. Since money has no value, the proposed trade only consists of output x and promise z and it does not condition on real balances and has no transfer of money. Buyers with record B has zero debt limit and the proposed trades are $o_i(B, 0) = (0, 0)$ for $i = 1, 2$. Since record B is self-absorbing, there are no profitable deviations under record B . Under record G , the debt limit is $D = z_1 + z_2$ and the proposed trade in the first DM round is $o_1(G, D) = (x_1, z_1)$. In the second DM round, we need to specify $o_2(G, d)$ for all $d \in [0, D]$, but implementation requires $o_2(G, z_2) = (x_2, z_2)$.

To specify $o_2(G, d)$, we first construct a function ξ as follows: $\xi(d) = u_2(x_2) - z_2$ if $d \geq z_2$ and let $\xi(d) = 0$ if $d < z_2$. As constructed below, $\xi(d)$ will be the surplus to the buyer in the second DM round if his available debt limit is d . Define $o_2(G, d)$ as the solution to

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}_+ \times [0,d]} & -c_2(x) + y \\ \text{s.t. } & u_2(x) - y \geq \xi(d). \end{aligned} \quad (18)$$

The solution to (18) exists for all $d \in [0, D]$ and is unique, and the constraint is binding at the optimum. Hence, under $o_2(G, d)$, the buyer's round-2 DM surplus is given by $\xi(d)$ if his debt limit is d . Moreover, the program ensures that the proposed trade is on the Pareto frontier given that the buyer cannot make a promise larger than d .

Now we show that agents follow a simple equilibrium; in particular, we need to show that the buyer has no profitable deviating offers at both DM rounds.

We start with the second DM round. The seller is willing to agree with the proposed trade, since he is guaranteed with a nonnegative surplus. Since under $o_2(G, d)$, the proposed trade is always on the Pareto frontier, the buyer cannot make deviating offers that make both parties better off. It then remains to show that $o_2(G, z_2) = (x_2, z_2)$. Suppose, by contradiction, that $(x, y) \neq (x_2, z_2)$ is the solution to (18). This implies that $y \leq z_2$, and that

$$-c_2(x) + y > -c_2(x_2) + z_2 \text{ and } u_2(x) - y \geq u_2(x_2) - z_2,$$

which leads to

$$u_2(x) - c_2(x) > u_2(x_2) - c_2(x_2),$$

and implies $x > x_2$. Since $u_2(x) - y \geq u_2(x_2) - z_2$, we have $y > z_2$, a contradiction. Thus, $o_2(G, z_2) = (x_2, z_2)$.

Now we consider the first DM round. Since $z_1 \geq c_1(x_1)$ the seller is willing to agree to the proposed trade. Here we show that the buyer with record G has no profitable deviating offers to make. Here we shall make use of the constraint (3). Suppose, by contradiction, there exists such a profitable deviating offer, (x, y) . Then,

$$u_1(x) - y + \sigma_2 \xi(D - y) > u_1(x_1) - z_1 + \sigma_2 \xi(z_2) \text{ and } y - c_1(x) \geq z_1 - c_1(x_1),$$

which leads to

$$u_1(x) - c_1(x) + \sigma_2 \xi(D - y) > u_1(x_1) - c_1(x_1) + \sigma_2 [u_2(x_2) - z_2]. \quad (19)$$

Consider two cases.

(a) If $y \leq z_1$, $y - c_1(x) \geq z_1 - c_1(x_1)$ implies $c_1(x) \leq c_1(x_1)$ and $x \leq x_1 \leq x_1^*$. Thus, since $\xi(D - y) = \xi(D - z_1) = [u_2(x_2) - z_2]$ for $y \leq z_1$, we have

$$u_1(x) - c_1(x) + \sigma_2 \xi(D - y) \leq u_1(x_1) - c_1(x_1) + \sigma_2 [u_2(x_2) - z_2],$$

a contradiction with (19).

(b) If $y > z_1$, then $D - y < D - z_1 = z_2$ and $\xi(D - y) = 0$. As a result, if (x, y) is a profitable deviation, we must have

$$u_1(x) - c_1(x) > u_1(x_1) - c_1(x_1) + \sigma_2 [u_2(x_2) - z_2],$$

a contradiction to (3).

Finally, we show that buyers are willing to repay their debt in the CM. Under the proposed trades, the continuation value for a buyer with record G and with record B , upon entering the DM, is given by

$$\begin{aligned} V_G &= \frac{1}{1 - \delta} \{u_1(x_1) - z_1 + \sigma_2 [u_2(x_2) - z_2]\}, \text{ and} \\ V_B &= 0. \end{aligned}$$

Thus, the buyer is willing to repay $z_1 + z_2$ in order to keep a record G if and only if

$$-(z_1 + z_2) + \delta V_G \geq \delta V_B,$$

which is equivalent to (1).

Proof of Theorem 3.2

(\Rightarrow) First we prove necessity. Let z be the amount of real balances that the buyer hold in equilibrium. Note that in equilibrium the buyer has to hold at least $z_1 + z_2$ units of real balances and hence $z \geq z_1 + z_2$. Since holding zero real balances and then returning to equilibrium behavior is a feasible deviation, this implies that

$$-z + \delta \{u_1(x_1) + \sigma_2 u_2(x_2) + (1 - \sigma_2)z_2 + (z - z_1 - z_2)\} \geq 0,$$

which, by $z \geq z_1 + z_2$, implies

$$-(1 - \delta)(z_1 + z_2) + \delta \{[u_1(x_1) - z_1] + \sigma_2[u_2(x_2) - z_2]\} \geq 0,$$

which is equivalent to (1) and hence (1) is necessary. Similarly, the buyer can choose to hold $z - z_1$ units of real balances in the CM and participate only in round-2 DM trades. To deter this deviation, it must be the case that

$$\begin{aligned} & -z + \delta \{u_1(x_1) + \sigma_2 u_2(x_2) + (1 - \sigma_2)z_2 + (z - z_1 - z_2)\} \\ & \geq -(z - z_1) + \delta \{\sigma_2 u_2(x_2) + (1 - \sigma_2)z_2 + (z - z_1 - z_2)\}, \end{aligned}$$

that is,

$$-(1 - \delta)z_1 + \delta[u_1(x_1) - z_1] \geq 0,$$

which is equivalent to (4) and hence (4) is necessary.

Condition (2) is necessary to make sure the seller wants to participate. The necessity of (3) follows the same argument as in the proof of Theorem 3.1, i.e., if (3) does not hold, the buyer can have a profitable deviation if he uses all his real balances in the first DM round.

(\Leftarrow) We now prove sufficiency. The proposal is such that the buyer holds $z_1 + z_2$ real balances at the end of the CM. Since there are no monitored meetings, a proposed trade only consists of output x and a transfer of real balances z . On the equilibrium path, $o_1(z_1 + z_2) = (x_1, z_1)$ and $o_2(z_2) = (x_2, z_2)$. It remains to specify $o_1(m)$ and $o_2(m)$ for all $m \geq 0$, show that the buyer is willing to bring $z_1 + z_2$ real balances into the DM rounds and show that the buyer has no profitable deviating offers in both DM rounds. Standard arguments in the LW environment shows that the continuation value for a buyer entering CM with m units of real balances is given by

$$W(m) = m + W(0).$$

We start with the specification of $o_2(m)$. Let $\epsilon \in (0, z_2)$ be such that

$$2\epsilon < \sigma_2[u_2(x_2) - z_2] \min \left\{ \frac{c'_1(x_1)}{u'_1(x_1) - c'_1(x_1)}, \frac{1}{\rho} \right\}, \quad (20)$$

and $\xi(m)$ be such that

$$\xi(m) = \begin{cases} u_2(x_2) - z_2 & \text{if } m \geq z_2; \\ 0 & \text{if } m \leq z_2 - \epsilon; \\ \left(1 - \frac{z_2 - m}{\epsilon}\right) [u_2(x_2) - z_2] & \text{if } m \in (z_2 - \epsilon, z_2). \end{cases}$$

Note that ξ is a piecewise linear continuous function. As constructed below, $\xi(m)$ will be the surplus given to be a buyer with m units of real balances in round-2 DM. We make it continuous to ensure our construction of round-1 DM trades.

Then, $o_2(m)$ solves

$$\begin{aligned} & \max_{(x,y) \in \mathbb{R}_+ \times [0,m]} -c_2(x) + y \\ & \text{s.t. } u_2(x) - y \geq \xi(m). \end{aligned} \quad (21)$$

The solution to (21) exists for all $m \geq 0$ and is unique, and the constraint is binding for all m . Hence, for $o_2(m) = (x, y)$ and for a buyer with m units of real balances upon entering the round-2 DM, his continuation value by following the equilibrium is given by

$$u_2(x) + W(m - y) = u_2(x) + (m - y) + W(0) = \xi(m) + m + W(0).$$

Since the program selects a point on the Pareto frontier for the pair for the given real balances m , the buyer cannot make a deviating offer that makes both parties better off. In turn, exactly the same argument used in the proof of Theorem 3.1 implies $o_2(z_2) = (x_2, z_2)$. Indeed, if, by construction, there is a profitable deviation (x, y) with $y \leq z_2$, it must be that

$$-c_2(x) + y > -c_2(x_2) + z_2 \text{ and } u_2(x) - y \geq u_2(x_2) - z_2,$$

which leads to

$$u_2(x) - c_2(x) > u_2(x_2) - c_2(x_2),$$

and implies $x > x_2$. Since $u_2(x) - y \geq u_2(x_2) - z_2$, we would have $y > z_2$, a contradiction.

We now consider $o_1(m)$. Let $\eta(m) = u_1(x_1) - z_1 + \sigma_2 \xi(z_2)$ if $m \geq z_1 + z_2$ and let $\eta(m) = \sigma_2 \xi(m)$ otherwise. For each $m \in \mathbb{R}_+$, let $o_1(m)$ be a solution to

$$\begin{aligned} & \max_{(x,y) \in \mathbb{R}_+ \times [0,m]} -c_1(x) + y \\ & \text{s.t. } u_1(x) - y + \sigma_2 \xi(m - y) \geq \eta(m). \end{aligned} \quad (22)$$

Since ξ is continuous, a solution to (22) exists with the constraint binding at the optimum. Moreover, for $o_1(m) = (x, y)$ and for a buyer with m units of real balances upon entering the round-1 DM, his continuation value by following the equilibrium is given by

$$u_1(x) + \sigma_2 [\xi(m - y) + (m - y)] + (1 - \sigma_2)(m - y) + W(0) = \eta(m) + m + W(0). \quad (23)$$

The buyer has no deviating offer to make both parties better off. We now show that $(x_1, z_1) \in o_1(z_1 + z_2)$. We shall use the constraint (3) here. Suppose, by contradiction, that $(x, y) \neq (x_1, z_1)$ gives the seller a higher surplus without violating the constraint. Hence,

$$u_1(x) - y + \sigma_2 \xi(z_1 + z_2 - y) \geq u_1(x_1) - z_1 + \sigma_2 \xi(z_2) \text{ and } y - c_1(x) > z_1 - c_1(x_1),$$

which leads to

$$u_1(x) - c_1(x) + \sigma_2 \xi(z_1 + z_2 - y) > u_1(x_1) - c_1(x_1) + \sigma_2 [u_2(x_2) - z_2]. \quad (24)$$

We need to consider three cases.

(a) If $y \leq z_1$, $y - c_1(x) \geq z_1 - c_1(x_1)$ implies $c_1(x) \leq c_1(x_1)$ and $x \leq x_1 \leq x_1^*$. Moreover, $y \leq z_1$ also implies $z_1 + z_2 - y \geq z_2$ and $\xi(z_1 + z_2 - y) = u_2(x_2) - z_2$. As a result

$$u_1(x) - c_1(x) + \sigma_2 \xi(z_1 + z_2 - y) \leq u_1(x_1) - c_1(x_1) + \sigma_2 [u_2(x_2) - z_2],$$

a contradiction with (24).

(b) If $y \geq z_1 + \epsilon$, then, $z_1 + z_2 - y < z_2$ and $\xi(z_1 + z_2 - y) = 0$. As a result, if (x, y) is a profitable deviation, we must have

$$u_1(x) - c_1(x) > u_1(x_1) - c_1(x_1) + \sigma_2 [u_2(x_2) - z_2],$$

a contradiction with (3).

(c) Let $y = z_1 + \epsilon'$, where $\epsilon' \in (0, \epsilon)$. First, $y = z_1 + \epsilon'$ implies

$$\xi(z_1 + z_2 - y) = \left(1 - \frac{\epsilon'}{\epsilon}\right) [u_2(x_2) - z_2],$$

and

$$c'_1(x_1) (x - x_1) \leq c_1(x) - c_1(x_1) < y - z_1 = \epsilon'.$$

The latter inequality condition comes from the convexity of $c_1(x)$ and the fact that $y - c_1(x) > z_1 - c_1(x_1)$ can be rewritten as $c_1(x) - c_1(x_1) < y - z_1 = \epsilon'$. Now, the concavity of $u_1(x) - c_1(x)$ implies

$$[u_1(x) - c_1(x)] - [u_1(x_1) - c_1(x_1)] \leq [u'_1(x_1) - c'_1(x_1)](x - x_1).$$

Combining this last inequality with the two conditions above and (20), we have

$$[u_1(x) - c_1(x)] - [u_1(x_1) - c_1(x_1)] < \sigma_2 [\xi(z_2) - \xi(z_1 + z_2 - y)],$$

a contradiction to (24).

It remains to show that it is optimal for the buyer to bring $z_1 + z_2$ units of real balances into the DM rounds. Recall that by (23), a buyer's continuation value upon entering DM with m units of real balances is given by

$$V(m) = \eta(m) + m + W(0).$$

Thus, the CM problem is then given by

$$\max_{m \geq 0} -m + \delta V(m),$$

which is equivalent to

$$\max_{m \geq 0} -m + \delta [\eta(m) + m] = \frac{1}{\rho} [-\rho m + \eta(m)].$$

We show that $m = z_1 + z_2$ is the optimal choice by considering three cases.

(a) $m \geq z_1 + z_2$. Since $\eta(m)$ is constant for all $m \geq z_1 + z_2$, any $m > z_1 + z_2$ is strictly dominated by $m = z_1 + z_2$.

(b) $m \in [z_2, z_1 + z_2)$. Then, $\eta(m) = \sigma_2 \xi(m)$. Since $\xi(m)$ is constant for $m \geq z_2$, any $m \in (z_2, z_1 + z_2]$ is strictly dominated by z_2 .

(c) $m \in [0, z_2 - \epsilon)$. Then, $\eta(m) = \sigma_2 \xi(m)$. Since $\xi(m)$ is constant for $m \in [0, z_2 - \epsilon)$, any $m \in [0, z_2 - \epsilon)$ is strictly dominated by 0.

(d) $m = z_2 - \epsilon'$, where $\epsilon' \in (0, \epsilon]$. Then, $\eta(m) = \sigma_2 \xi(m)$. Here we show that m is strictly dominated by z_2 . This will be the case if

$$-\rho m + \sigma_2 \xi(m) < -\rho z_2 + \sigma_2 \xi(z_2),$$

which, using the definition of $\xi(m)$, is equivalent to

$$\rho(z_2 - m) < \sigma_2 \frac{(z_2 - m)}{\epsilon} [u_2(x_2) - z_2],$$

and holds true by (20).

Thus, in order to check for profitable deviations in the CM, we only need to compare the choice of $z_1 + z_2$ real balances against a choice of z_2 or a choice of zero real balances. The latter follows from (1) while the first follows from (4).

Proof of Lemma 3.1

It is straightforward to verify that (2) is necessary and sufficient to ensure participation of sellers. Here we show that (1) is necessary to ensure participation of buyers. By following the equilibrium behavior, the continuation value upon entering DM at any given period is

$$V = \frac{1}{1 - \delta} \{ [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - z_2] \}.$$

Consider the CM decision for a buyer who has the largest amount of debt and the smallest amount of real balances along the equilibrium path. To follow the equilibrium behavior, he needs to produce $z_1 + z_2$ in order to repay his debt or to rebalance his money holdings. However, he has a deviation to produce nothing in the CM and receive no trade afterwards. This deviation is not profitable only if

$$-(z_1 + z_2) + \delta V \geq 0,$$

which is equivalent to (1).

Now we show the necessity of (4). If $C = \{2\}$, since the first DM round has to be financed with a transfer of real balances, (4) is necessary to prevent buyers from skipping the first DM round; to prove this we can use exactly the same arguments as in Theorem 3.2. Suppose then that $C = \{1\}$. Let $z_{1,m}$ and $z_{1,c}$ be the transfer of real balances and promises to pay in the first DM round, and let z be the real balances the buyer has to bring into the first DM round. Note that $z_1 = z_{1,m} + z_{1,c}$. Then, the buyer may deviate and choose

to repay nothing and only hold $z - z_{1,m}$ units of real balances in order to participate in the second DM round. This deviation is profitable if

$$-(z - z_{1,m}) + \frac{\delta \sigma_2 [u_2(x_2) - z_{2,m}]}{1 - \delta} > -(z + z_{1,c}) + \frac{\delta \{ [u_1(x_1) - z_{1,m}] + \sigma_2 [u_2(x_2) - z_{2,m}] - z_{1,c} \}}{1 - \delta}$$

which can be rewritten as

$$-\rho(z_{1,m} + z_{1,c}) + u_1(x_1) - (z_{1,m} + z_{1,c}) < 0.$$

Hence, (4) is necessary to prevent this profitable deviation.

Proof of Theorem 4.1

Proof of (i) (\Rightarrow) First, we prove necessity. To prove the necessity of (1), consider an arbitrary EMP (k, π) . Consider a buyer in the CM. To follow the equilibrium behavior, the buyer in the CM needs to repay $\min\{z_1 - k, 0\}$ and buy $(1 + \pi)z_2$ units of real balances. Note that it has to be case that $z_1 - k \geq 0$; for otherwise they can deviate to a bigger trade without any cost. Alternatively, he can deviate to repay nothing and hold zero real balances. Thus, for him to follow the equilibrium behavior, it must be the case that

$$-(z_1 - k) - (1 + \pi)z_2 + \frac{\delta}{1 - \delta} \{ [u_1(x_1) - (z_1 - k)] + \sigma_2 [u_2(x_2) - z_2] - \pi z_2 \} \geq 0,$$

that is,

$$-\rho z_1 - \rho z_2 + [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - z_2] + (1 + \rho)(k - \pi z_2) \geq 0.$$

Since (5) implies that $k = \pi z_2$, the above inequality is equivalent to (1). We need (2) to ensure the participation of sellers, while the necessity of (3) follows exactly the same argument as in the proof of Theorem 3.2, i.e., if (3) does not hold, the buyer can have a profitable deviation if he diverts all his real balances to the first DM round.

(\Leftarrow) We now prove sufficiency. First we formulate the EMP. If $-\rho z_1 + [u_1(x_1) - z_1] \geq 0$, then Theorem 3.2 implies that the allocation is implementable if we set $k = \pi = 0$ and let agents use only real balances in both DM rounds. If $-\rho z_1 + [u_1(x_1) - z_1] < 0$, the EMP is (7) and the debt limit of the buyer is $D = z_1 - k$ if the buyer has a record G , and $D = 0$ if the buyer has a record B . It remains to specify the proposed trades $o_1(m, r, D)$ and $o_2(m)$ for all $m \geq 0$ and $r \in \{G, B\}$. In equilibrium the buyers always remain with record G . Standard arguments in the LW environment show that the continuation value for a buyer with record G entering CM with m units of real balances and with y_c units of debt is given by

$$W_G(m, y_c) = m - \min\{D, \max\{y_c - k, 0\}\} + W_G(0, 0).$$

We start with $o_2(m)$. Let $\epsilon \in (0, z_2)$ be such that

$$2\epsilon < \sigma_2 [u_2(x_2) - z_2] \min \left\{ \frac{c'_1(x_1)}{u'_1(x_1) - c'_1(x_1)}, \frac{1}{\rho + (1 + \rho)\pi} \right\}. \quad (25)$$

Condition (25) is exactly the same as (20) except the term ρ , which is replaced by $\rho + (1 + \rho)\pi$, reflecting the difference in the cost of holding money. Let

$$\xi(m) = \begin{cases} u_2(x_2) - z_2 & \text{if } m \geq z_2; \\ 0 & \text{if } m \leq z_2 - \epsilon; \\ \left(1 - \frac{z_2 - m}{\epsilon}\right) [u_2(x_2) - z_2] & \text{if } m \in (z_2 - \epsilon, z_2). \end{cases}$$

Note that ξ is a piecewise linear continuous function. Then, $o_2(m)$ solves

$$\begin{aligned} & \max_{(x,y) \in \mathbb{R}_+ \times [0,m]} -c_2(x) + y \\ & \text{s.t. } u_2(x) - y \geq \xi(m). \end{aligned} \tag{26}$$

The solution to (26) exists for all m and is unique, with the constraint binding at the optimum. Hence, for $o_2(m) = (x, y)$ and for a buyer with m units of real balances and y_c units of debts upon entering the round-2 DM, his continuation value by following the equilibrium is given by

$$u_2(x) + W_G(m - y, y_c) = u_2(x) + (m - y) + W_G(0, y_c) = \xi(m) + m + W_G(0, y_c).$$

Our construction also ensures that the proposed trade lies on the Pareto frontier for the pair, taking the amount of real balances of the buyer as given and assuming that the buyer follows the equilibrium behavior. Hence, for any $m \geq 0$, there is no profitable deviating offer in the second DM round. We need to show, however, that $o_2(z_2) = (x_2, z_2)$. Suppose, by contradiction, a different pair $(x, y) \neq (x_2, z_2)$ with $y \leq z_2$ solves (26) for $m = z_2$. Then, we must have

$$-c_2(x) + y > -c_2(x_2) + z_2 \text{ and } u_2(x) - y \geq u_2(x_2) - z_2,$$

which requires

$$u_2(x) - c_2(x) > u_2(x_2) - c_2(x_2),$$

and implies $x > x_2$. Since $u_2(x) - y \geq u_2(x_2) - z_2$, we would have $y > z_2$, a contradiction. This proves that $o_2(z_2) = (x_2, z_2)$.

Now we turn to $o_1(m, r, d)$. Let $\eta(m, G) = u_1(x_1) - (z_1 - k) + \sigma_2 \xi(m)$ and let $\eta(m, B) = \sigma_2 \xi(m)$. η is a continuous function since ξ is. If $r = G$, for each $m \in \mathbb{R}_+$, let $o_1(m, G, z_1 - k)$ be a solution to

$$\begin{aligned} & \max_{(x,y_c,y_m) \in \mathbb{R}_+ \times [0,D+k] \times [0,m]} -c_1(x) + y_c + y_m \\ & \text{s.t. } u_1(x) - \max(y_c - k, 0) - y_m + \sigma_2 \xi(m - y_m) \geq \eta(m, G). \end{aligned} \tag{27}$$

If $r = B$, for each $m \in \mathbb{R}_+$, let $o_1(m, B, 0)$ be a solution to

$$\begin{aligned} & \max_{(x,y_m) \in \mathbb{R}_+ \times [0,m]} -c_1(x) + y_m \\ & \text{s.t. } u_1(x) - y_m + \sigma_2 \xi(m - y_m) \geq \eta(m, B). \end{aligned} \tag{28}$$

Because ξ is continuous, a solution to (27) and (28) exists with the constraints binding at the optimum. Moreover, for $o_1(m, G, D) = (x, y_c, y_m)$ and for a G -buyer with m units of real balances upon entering the round-1 DM, his continuation value by following the equilibrium is given by

$$\begin{aligned} & u_1(x) + \sigma_2[\xi(m - y_m) + (m - y_m)] + (1 - \sigma_2)(m - y_m) + W_G(0, \max(y_c - k, 0)) \\ = & u_1(x) + \sigma_2\xi(m - y_m) + (m - y_m) + W_G(0, 0) - \max(y_c - k, 0) \\ = & \eta(m, G) + m + W_G(0, 0). \end{aligned} \quad (29)$$

Similarly, for $o_1(m, B, 0) = (x, y_m)$ and for a B -buyer with m units of real balances upon entering the round-1 DM, his continuation value by following a simple equilibrium is given by

$$\begin{aligned} & u_1(x) + \sigma_2[\xi(m - y_m) + m - y_m] + (1 - \sigma_2)(m - y_m) + W_B(0) \\ = & \eta(m, B) + m + W_B(0). \end{aligned} \quad (30)$$

Here $W_B(m)$ is the continuation value for a buyer with record B upon entering the CM. Note that buyers under record B cannot issue debt. Since B is self-absorbing, the linearity of $W_B(m)$ also follows from standard arguments.

Our construction also ensures that the proposed trade lies on the Pareto frontier for the pair, taking the amount of real balances and the credit record of the buyer as given. Hence, for any $m \geq 0$, there is no profitable deviating offer in the first DM round, irrespective of the record of the buyer.

We now show that $(x_1, z_1, 0) \in o_1(z_2, G, z_1 - k)$. Here again we need (3). Suppose, by contradiction, that (x, y_c, y_m) gives the seller a higher surplus without violating the constraint. Since the debt paid by the buyer is $y_c - k$, we can assume that $y_c \geq k$, otherwise we could increase y_c and give the seller a higher surplus without changing the surplus of the buyer. Hence,

$$u_1(x) - (y_c - k) - y_m + \sigma_2\xi(z_2 - y_m) \geq u_1(x_1) - (z_1 - k) + \sigma_2\xi(z_2),$$

and

$$y_m + y_c - c_1(x) > z_1 - c_1(x_1),$$

which implies

$$u_1(x) - c_1(x) + \sigma_2\xi(z_2 - y_m) > u_1(x_1) - c_1(x_1) + \sigma_2[u_2(x_2) - z_2]. \quad (31)$$

Consider two cases.

(a) $y_m \geq \epsilon$. Then, $\xi(z_2 - y_m) = 0$. In this case, (31) implies that a buyer has a profitable deviation if he transfers all his real balances to the first DM round, which contradicts (3).

(b) $y_m \in (0, \epsilon)$. Then,

$$\xi(z_2 - y_m) = \left(1 - \frac{y_m}{\epsilon}\right) [u_2(x_2) - z_2].$$

However, since $y_m + y_c - c_1(x) > z_1 - c_1(x_1)$ and $y_c \leq z_1$, we have

$$c'_1(x_1)(x - x_1) \leq c_1(x) - c_1(x_1) < y_m.$$

Now, the concavity of $u_1(x) - c_1(x)$ implies

$$[u_1(x) - c_1(x)] - [u_1(x_1) - c_1(x_1)] \leq [u'_1(x_1) - c'_1(x_1)](x - x_1).$$

Combining this last inequality with the two conditions above and (25), we obtain

$$[u_1(x) - c_1(x)] - [u_1(x_1) - c_1(x_1)] < \sigma_2[\xi(z_2) - \xi(z_2 - y_m)],$$

a contradiction to (31).

Now we show that the following strategies form a simple equilibrium. All agents respond with *yes* to the proposed trades and buyers offer the proposed trades, on both equilibrium and off-equilibrium paths. Buyers under state G always repay their debts up to D , and buyers under state B never repay anything. All buyers leave the CM with z_2 units of real balances, on both equilibrium and off-equilibrium paths. These strategies are consistent with the continuation values given by (29) and (30).

It remains to prove that a buyer wants to keep a good record and that, irrespective of his record, a buyer wants to leave the CM with z_2 real balances. To prove these results, observe that, by (29) and (30), the continuation value for a buyer with m units of real balances and with record G is given by

$$V_G(m) = \eta(m, G) + m + W_G(0, 0), \quad (32)$$

and that for a buyer with m units of real balances and with record B is given by

$$V_B(m) = \eta(m, B) + m + W_B(0). \quad (33)$$

Note that, according to the equilibrium behavior, buyers enter the CM with z_1 units of debts, and hence

$$W_G(0, z_1) = -[(z_1 - k) + (1 + \pi)z_2] + \delta V_G(z_2) = -[(z_1 - k) + (1 + \pi)z_2] + \delta[\eta(z_2, G) + z_2 + W_G(0, z_1)],$$

and hence

$$W_G(0, z_1) = -[(z_1 - k) + (1 + \pi)z_2] + \frac{\delta}{1 - \delta} \{u_1(x_1) - (z_1 - k) + \sigma_2[u_2(x_2) - z_2] - \pi z_2\}.$$

Similarly,

$$W_B(0) = -(1 + \pi)z_2 + \frac{\delta}{1 - \delta} \{\sigma_2[u_2(x_2) - z_2] - \pi z_2\}.$$

We first prove that buyers are willing to leave the CM with z_2 units of real balances. The CM problem for a buyer with record r in the CM is given by

$$\max_m -(1 + \pi)m + \delta [\eta(m, r) + m + W_r(0)] = \delta \{-[\rho + (1 + \rho)\pi]m + \eta(m, r) + W_r(0)\}.$$

Recall that

$$\eta(m, r) = \begin{cases} \mathbf{1}_{r=G}[u_1(x_1) - z_1 + k] + \sigma_2 \xi(z_2) & \text{if } m \geq z_2, \\ \mathbf{1}_{r=G}[u_1(x_1) - z_1 + k] + \sigma_2 \xi(m) & \text{otherwise.} \end{cases}$$

Since $\eta(m, r)$ is constant for all $m \geq z_2$ but the cost of holding money increases with m , any $m > z_2$ is strictly dominated by $m = z_2$. We now show that for any $\epsilon' \in (0, \epsilon]$, $z_2 - \epsilon'$ is strictly dominated by z_2 . This will be the case if

$$-[\rho + (1 + \rho)\pi](z_2 - \epsilon') + \sigma_2 \xi(z_2 - \epsilon') < -[\rho + (1 + \rho)\pi]z_2 + \sigma_2 \xi(z_2),$$

which is equivalent to

$$[\rho + (1 + \rho)\pi]\epsilon' < \sigma_2[\xi(z_2) - \xi(z_2 - \epsilon')] = \sigma_2 \frac{\epsilon'}{\epsilon}[u_2(x_2) - z_2],$$

which holds by (25). Finally, for any $m \leq z_2 - \epsilon$, it is strictly dominated by zero as $\xi(m)$ is constant below $z_2 - \epsilon$. Thus, to show that holding z_2 is optimal, it is sufficient to show that it is better than 0, and this will be the case if and only if

$$-[\rho + (1 + \rho)\pi]z_2 + \sigma_2[u_2(x_2) - z_2] \geq 0.$$

Using $\pi z_2 = k = z_1 - \frac{u_1(x_1)}{1 + \rho}$, we can rewrite this inequality as

$$-\rho(z_1 + z_2) + [u_1(x_1) - z_1] + \sigma_2[u_2(z_2) - z_2] \geq 0,$$

which corresponds to (1).

Lastly, we show that a buyer under state G has incentive to repay $z_1 - k$ whenever his debt is at least z_1 . Consider a buyer who enters the CM with z_1 units of debts. His record will become G if he fails to repay at least $z_1 - k$. Thus, he is willing to repay $z_1 - k$ if and only if

$$-(z_1 - k) + \delta V_G(z_2) \geq \delta V_B(z_2),$$

and by (32) and (33), this is equivalent to

$$-(z_1 - k) + \delta \{\eta(z_2, G) - \eta(z_2, B) + W_G(0, 0) - W_B(0)\} \geq 0,$$

that is,

$$-(z_1 - k) + \frac{\delta}{1 - \delta} \{u_1(x_1) - z_1 + k\} \geq 0.$$

This holds by (7).

Proof of (ii) (\Rightarrow) First, we prove necessity. Since $C = \{2\}$, to finance consumption in the first round the buyer has to bring at least z_1 units of real balances. Moreover, as in the pure currency economy, the buyer may deviate to bringing only sufficient real balances for the round-2 DM (which could be zero in this case). Hence, (4) is necessary as in the pure currency economy. In fact, for the same reason, if the inflation rate is π , then it is necessary that

$$-(1 + \pi)z_1 + \delta u_1(x_1) \geq 0,$$

that is,

$$-(1 + \rho)\pi z_1 - \rho z_1 + [u_1(x_1) - z_1] \geq 0. \quad (34)$$

To prove the necessity of (6), consider an arbitrary EMP (k, π) . Consider a buyer with z_2 units of debts in the CM. To follow the equilibrium behavior, the buyer in the CM needs to repay $\min\{z_2 - k, 0\}$ and buy at least $(1 + \pi)z_1$ units of real balances. Note that it has to be case that $z_2 - k \geq 0$; for otherwise they can deviate to a bigger trade without any cost. Alternatively, he can deviate to repay nothing and hold zero real balances. Thus, for him to follow the equilibrium behavior, it must be the case that

$$-(1 + \pi)z_1 - (z_2 - k) + \frac{\delta}{1 - \delta} \{ [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - (z_2 - k)] - \pi z_1 \} \geq 0,$$

that is,

$$-\rho z_1 - \rho z_2 + [u_1(x_1) - z_1] + \sigma_2 [u_2(x_2) - z_2] + (1 + \rho)(k - \pi z_1) \geq 0.$$

Since (5) implies that $\sigma_2 k = \pi z_2$, the above inequality, together with (34), implies (6). As usual, (2) is necessary, otherwise sellers would not be willing to participate.

(\Leftarrow) We now prove sufficiency. First we formulate the EMP. If $-\rho z_2 + [u_2(x_2) - z_2] \geq 0$, then we set $k = \pi = 0$. In this case, Theorem 3.2 implies that the allocation is implementable if agents use only real balances in both DM rounds. If $-\rho z_2 + [u_2(x_2) - z_2] < 0$, the EMP is given by (8), the debt limit is $D = z_2 - k$ if the buyer has a record G , and is equal to $D = 0$ if the buyer has a record B . In equilibrium the buyers always remain with record G . Standard arguments in the LW environment shows that the continuation value for a buyer with record G entering CM with m units of real balances and with y_c units of debt is given by

$$W_G(m, y_c) = m - \min\{D, \max\{y_c - k, 0\}\} + W_G(0, 0).$$

It remains to specify the proposed trades $o_1(m)$ and $o_2(m, r, D)$ for all $m \geq 0$ and $r \in \{G, B\}$.

We start with $o_2(m, r, D)$. Let $o_2(m, G, D)$ be a solution to

$$\begin{aligned} \max_{(x, y_c, y_m) \in \mathbb{R}_+ \times [0, D+k] \times [0, m]} & -c_2(x) + y_c + y_m \\ \text{s.t.} & u_2(x) - \max(y_c - k, 0) - y_m \geq u_2(x_2) - (z_2 - k), \end{aligned} \quad (35)$$

and let $o_2(m, B, 0)$ be a solution to

$$\begin{aligned} \max_{(x, y_m) \in \mathbb{R}_+ \times [0, m]} & -c_2(x) + y \\ \text{s.t.} & u_2(x) - y_m \geq 0. \end{aligned} \quad (36)$$

The solutions to (35) and (36) exist and are unique, with the constraints binding at the optimum.

Hence, for $o_2(m, G, D) = (x, y_c, y_m)$ and for a G -buyer with m units of real balances upon entering the round-2 DM, his continuation value by following the equilibrium is given by

$$\begin{aligned} u_2(x) + W_G(m - y_m, y_c) &= u_2(x) + (m - y_m) - \max\{0, y_c - k\} + W_G(0, 0) \\ &= u_2(x_2) - (z_2 - k) + m + W_G(0, 0). \end{aligned}$$

Similarly, for $o_2(m, B, 0) = (x, y_m)$ and for a B -buyer with m units of real balances upon entering the round-2 DM, his continuation value by following the equilibrium is given by

$$\begin{aligned} u_2(x) + W_B(m - y_m) &= u_2(x) + (m - y) + W_B(0) \\ &= m + W_B(0). \end{aligned}$$

Our construction also ensures that the proposed trade lies on the Pareto frontier for the pair, taking the amount of real balances of the buyer as given and assuming that the buyer follows equilibrium behavior. Hence, for any $m \geq 0$, there is no profitable deviating offer in the second DM round. Note also that, irrespective of his real balances m , the surplus under $o_2(m, G, D)$ for the buyer is $u_2(x_2) - (z_2 - k)$, which does not depend on his real balances. Finally, we show that $o_2(0, G, z_2 - k) = (x_2, z_2, 0)$. Indeed, if there exists $(x, y_c, 0) \neq (x_2, z_2, 0)$ which give a higher surplus to the seller without violating the constraint, it must be that

$$u_2(x) - (y_c - k) \geq u_2(x_2) - (z_2 - k) \text{ and } -c_2(x) + y_c \geq -c_2(x_2) + z_2,$$

which implies

$$u_2(x) - c_2(x) \geq u_2(x_2) - c_2(x_2),$$

and $x > x_2$. Since $-c_2(x) + y_c \geq -c_2(x_2) + z_2$, we have $y_c - k > z_2 - k = D$. This cannot happen as the buyer cannot credibly promise more than the debt limit.

We now move to $o_1(m)$. Since the first DM round is non-monitored, the payoff of the buyer in this round cannot depend on his record or on the debt limit. Let $\eta(m) = u_1(x_1) - z_1$ if $m \geq z_1$ and let $\eta(m) = 0$ otherwise. In turn, let $o_1(m)$ be a solution to

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}_+ \times [0,m]} & -c_1(x) + y \\ \text{s.t.} & u_1(x) - y \geq \eta(m). \end{aligned} \tag{37}$$

The solution to (37) exist and is unique with the constraints binding at the optimum.

Hence, for $o_1(m) = (x, y)$ and for a G -buyer with m units of real balances upon entering the DM, his continuation value by following the equilibrium is given by

$$\begin{aligned} & u_1(x) + \sigma_2[u_2(x_2) - (z_2 - k) + (m - y)] + (1 - \sigma_2)(m - y) + W_G(0, 0) \\ &= \eta(m) + m + \sigma_2[u_2(x_2) - (z_2 - k)] + W_G(0, 0), \end{aligned}$$

that is,

$$V_G(m) = \eta(m) + m + \sigma_2[u_2(x_2) - (z_2 - k)] + W_G(0, 0). \tag{38}$$

Similarly, for $o_1(m) = (x, y)$ and for a B -buyer with m units of real balances upon entering the DM, his continuation value by following the equilibrium is given by

$$u_1(x) + (m - y) + W_B(0) = \eta(m) + m + W_B(0),$$

that is,

$$V_B(m) = \eta(m) + m + W_B(0). \quad (39)$$

Moreover, the construction of $o_1(m)$ ensures that there is no profitable deviating offer in the first DM. Now we show that $o_1(z_1) = (x_1, z_1)$. If this is not the case, there exists $(x, y) \neq (x_1, z_1)$, with $y < z_1$, which provide a higher surplus to the seller without violating the constraint. Hence

$$u_1(x) - y \geq u_1(x_1) - z_1 \text{ and } -c_1(x) + y \geq -c_1(x_1) + z_1,$$

which implies

$$u_1(x) - c_1(x) \geq u_1(x_1) - c_1(x_1),$$

and $x > x_1$. Since $-c_1(x) + y \geq -c_1(x_1) + z_1$, we have $y > z_1$, a contradiction.

Now we show that the following strategies form a simple equilibrium. All agents respond with *yes* to the proposed trades and buyers offer the proposed trades, on both equilibrium and off-equilibrium paths. Buyers under state G always repay their debts up to D , and buyers under state B never repay anything. All buyers leave the CM with z_1 units of real balances, on both equilibrium and off-equilibrium paths.

First we show that the buyer carry z_1 real balances to the DM, irrespective of his record r . By linearity of W_r , the CM problem is given by

$$\begin{aligned} & \max_{m \geq 0} -(1 + \pi)m + \delta V_r(m) \\ & = \delta \{ -[\rho + (1 + \rho)\pi]m + \eta(m) + \mathbf{1}_{r=G} \{ \sigma_2 [u_2(x_2) - (z_2 - k)] \} + W_r(0, 0) \}. \end{aligned}$$

Note that $\eta(m)$ is constant for all $m \geq z_1$ and is constant for all $m \in [0, z_1]$. Thus, we only need to show that bringing z_1 is better than zero, which will be the case if and only if

$$-[\rho + (1 + \rho)\pi]z_1 + u_1(x_1) - z_1 \geq 0.$$

Using $\pi z_1 = \sigma_2 k = \sigma_2 z_2 - \sigma_2 \frac{\sigma_2}{\sigma_2 + \rho} u_2(x_2)$, we can rewrite this inequality as

$$\{ -\rho z_1 + u_1(x_1) - z_1 \} + \frac{(1 + \rho)\sigma_2}{\sigma_2 + \rho} \{ -\rho z_2 + \sigma_2 [u_2(x_2) - z_2] \} \geq 0,$$

which corresponds to (6).

Now we show that buyers are willing to repay their debts. Note that, according to the equilibrium behavior,

$$\begin{aligned} W_G(0, z_2) & = -[(1 + \pi)z_1 + (z_2 - k)] + \delta V_G(z_1) \\ & = -[(1 + \pi)z_1 + (z_2 - k)] + \delta \{ \eta(z_1) + \sigma_2 [u_2(x_2) - (z_2 - k)] + z_1 + W_G(0, z_2) \}, \end{aligned}$$

and hence

$$W_G(0, z_2) = -[(1 + \pi)z_1 + (z_2 - k)] + \frac{\delta}{1 - \delta} \{[u_1(x_1) - z_1] + \sigma_2[u_2(x_2) - (z_2 - k)] - \pi z_1\}.$$

Similarly,

$$W_B(0) = -(1 + \pi)z_1 + \frac{\delta}{1 - \delta} \{[u_1(x_2) - z_1] - \pi z_1\}.$$

We need to show that the buyer wants to keep the record G . He has incentive to repay $z_2 - k$ in order to keep the record if and only if

$$-(z_2 - k) + \delta V_G(z_1) \geq \delta V_B(z_1),$$

and by (38) and (39), this is equivalent to

$$-(z_2 - k) + \delta \{\sigma_2[u_2(x_2) - (z_2 - k)] + W_G(0, 0) - W_B(0)\} \geq 0,$$

that is,

$$-(z_2 - k) + \frac{\delta}{1 - \delta} \{\sigma_2[u_2(x_2) - (z_2 - k)]\} \geq 0.$$

which holds with equality by (8).

Proof of Lemma 4.1

By definition and by Theorem 4.1 (ii), the allocation (x_1^{C2}, x_2^{C2}) is implementable with EMP under $C = \{2\}$. Next, we show that the allocation (x_1^{C1}, x_2^{C1}) is implementable with EMP under $C = \{1\}$. By Theorem 4.1 (i), it suffices to show that it satisfies (3). Suppose, by contradiction, that for $\hat{x}_1 = \min\{x_1^*, c_1^{-1}(z_1 + z_2)\}$ with $z_i = c_i(x_i^{C1})$, $i = 1, 2$, we have

$$u_1(x_1^{C1}) - c_1(x_1^{C1}) + \sigma_2[u_2(x_2^{C1}) - z_2] < u_1(\hat{x}_1) - c_1(\hat{x}_1). \quad (40)$$

Because (x_1^{C1}, x_2^{C1}) satisfies (1), (40) implies that the pair $(\hat{x}_1, 0)$ satisfies (1) as well. Note that $c_1(\hat{x}_1) \leq z_1 + z_2$. But (40) implies that it generates a higher welfare than (x_1^{C1}, x_2^{C1}) , a contradiction. Thus, (x_1^{C1}, x_2^{C1}) satisfies (3) and hence is implementable with $C = \{1\}$ and with EMP.

Finally, because, by Lemma 3.1 and Theorem 4.1, the set of implementable allocations with EMP includes all allocations implementable under constant money supply, the constrained-efficient allocation is either (x_1^{C1}, x_2^{C1}) or (x_1^{C2}, x_2^{C2}) .

Proof of Theorem 4.2

We consider two cases.

(i) Suppose that (x_1^{C1}, x_2^{C1}) does not satisfy (4), and hence is not implementable with constant money supply. If (x_1^{C1}, x_2^{C1}) is a constrained efficient allocation, then EMP is

necessary to implement it. Otherwise, (x_1^{C2}, x_2^{C2}) achieves a better welfare than (x_1^{C1}, x_2^{C1}) , while a constant money supply can never achieve a welfare higher than that of (x_1^{C1}, x_2^{C1}) . Then, the constrained efficient allocation can only be implemented with EMP but not with constant money supply.

(ii) Suppose that (x_1^{C1}, x_2^{C1}) is implementable with constant money supply and has the maximum welfare among all allocations that can be implemented by a constant money supply. Then, (4) holds, i.e., $-\rho c_1(x_1^{C1}) + [u_1(x_1^{C1}) - c_1(x_1^{C1})] \geq 0$. Also, $\rho > \rho_M$ implies $(x_1^{C1}, x_2^{C1}) \neq (x_1^*, x_2^*)$. This also implies that constraint (1) is binding at (x_1^{C1}, x_2^{C1}) . Thus, if $-\rho c_1(x_1^{C1}) + [u_1(x_1^{C1}) - c_1(x_1^{C1})] = 0$, then $-\rho c_2(x_2^{C1}) + \sigma_2[u_2(x_2^{C1}) - c_2(x_2^{C1})] = 0$. That is, $(x_1^{C1}, x_2^{C1}) = (\bar{x}_1, \bar{x}_2)$, a contradiction. Therefore,

$$u_1(x_1^{C1}) - (1 + \rho)c_1(x_1^{C1}) > 0.$$

This inequality, together with $\sigma_2 < 1$ and (1) for (x_1^{C1}, x_2^{C1}) , implies

$$[u_1(x_1^{C1}) - (1 + \rho)c_1(x_1^{C1})] + \frac{(\rho + 1)\sigma_2}{\rho + \sigma_2} [\sigma_2 u_2(x_2^{C1}) - (\sigma_2 + \rho)c_2(x_2^{C1})] > 0.$$

Because $x_1^{C1} < x_1^*$, there exists $\epsilon > 0$ such that $x'_1 = x_1^{C1} + \epsilon < x_1^*$,

$$u_1(x'_1) - (1 + \rho)c_1(x'_1) > 0,$$

and

$$[u_1(x'_1) - (1 + \rho)c_1(x'_1)] + \frac{(\rho + 1)\sigma_2}{\rho + \sigma_2} [\sigma_2 u_2(x_2^{C1}) - (\sigma_2 + \rho)c_2(x_2^{C1})] > 0.$$

By Theorem 4.1 (ii), (x'_1, x_2^{C1}) is implementable with EMP, but it has a strictly higher welfare than (x_1^{C1}, x_2^{C1}) . Thus, the constrained efficient allocation is (x_1^{C2}, x_2^{C2}) , and it is not implementable without EMP.

Proof of Theorem 4.3

(i) Consider, first, the case where $C = \{1\}$. Let (η, τ) be an arbitrary DMP. Since all agents participate in the first DM round in equilibrium, we have $\beta = 1$ in (12). Let $z_1 = z_{1,c} + z_{1,m}$ be round-1 DM seller's CM consumption levels, where $z_{1,c} > 0$ (for otherwise this is a pure currency economy and the result follows) is financed by debt and $z_{1,m}$ is paid in cash. Hence, in equilibrium, the buyers leave CM with $z_{1,m} + z_2$ units of real balances, and have to repay $z_{1,c}$ units of debts plus the fee η . Consider a buyer's decision in the CM. He could deviate to not repaying his debt, nor carrying any real balances and receiving no trade henceafter. Following the equilibrium behavior is better than this deviation only if

$$-[z_{1,c} + \eta + (1 - \tau)(z_{1,m} + z_2)] + \frac{\delta}{1 - \delta} \{ [u_1(x_1) - z_1 - \eta] + \sigma_2 [u_2(x_2) - z_2] + \tau(z_{1,m} + z_2) \} \geq 0.$$

Since (12) implies $\eta = \tau(z_{1,m} + z_2)$, the above inequality reduces to (1). In turn, (2) is necessary to ensure participation of the sellers. Finally, since buyers can always bring

only z_2 units of real balances without repaying his debts, (4) is necessary to prevent the buyer from skipping the first DM round permanently.

(ii) Consider the case where $C = \{2\}$. Let (η, τ) be an arbitrary DMP. Let $z_2 = z_{2,c} + z_{2,m}$, where $z_{2,c} > 0$ (for otherwise this is a pure currency economy) is financed by debt and $z_{2,m}$ is paid in cash. Hence, in equilibrium, the buyers leave CM with $z_{2,m} + z_1$ units of real balances, and have to repay $z_{2,c}$ units of debts plus the fee η . Consider a buyer's decision in the CM. He could deviate to not repaying his debt, nor carrying any real balances and receiving no trade henceafter. Following the equilibrium behavior is better than this deviation only if

$$-[z_{2,c} + \eta + (1 - \tau)(z_{2,m} + z_1)] + \frac{\delta}{1 - \delta} \{[u_1(x_1) - z_1] + \sigma_2[u_2(x_2) - z_2 - \eta] + \tau(z_{2,m} + z_1)\} \geq 0.$$

Since (12) implies $\sigma_2\eta = \tau(z_1 + z_{2,m})$, we can rewrite the above inequality as

$$\rho(1 - \sigma_2)\eta \leq -\rho(z_1 + z_2) + u_1(x_1) - z_1 + \sigma_2[u_2(x_2) - z_2]. \quad (41)$$

Note that (41) implies (1) since $\eta \geq 0$. Now, we give a lower bound on η . A buyer in the CM may hold only $z_{2,m}$ units of real balances and skip the first DM round. This deviation is not profitable only if

$$-\rho z_1 + (1 + \rho)\tau z_1 + [u_1(x_1) - z_1] \geq 0,$$

that is,

$$\tau z_1 \geq \frac{1}{1 + \rho} \{\rho z_1 - [u_1(x_1) - z_1]\}.$$

This gives a lower bound on τ and hence on η by $\sigma_2\eta = \tau(z_1 + z_{2,m})$, and plugging this lower bound into (41) we obtain (13).

Since (2) is required for seller participation, if (4) holds, then we are done. Finally, note that if (4) does hold, (1) implies (14).

Proof of Theorem 4.4

Assume limited monitoring and let the net money growth rate be π with lump-sum transfers. Let $\zeta = (1 + \pi - \delta)/\delta \geq \rho$. Consider an allocation, $L = [(x_1, x_2), (z_1, z_2)]$. As usual, (2) is required for seller participation, and hence we focus on buyers' incentives.

(i) First, consider the case where $C = \{1\}$. Since the buyer can always skip round-1 DM, the necessity of (4) follows exactly the same proof as in Lemma 3.1. In turn, since only money can be used to finance trades in round-2 DM, buyers have to hold at least z_2 units of real balances in equilibrium. Now consider a buyer in the CM. To follow the equilibrium strategy, he has to buy at least z_2 units of real balances and repay his debt in round-1 DM, and this has to be better than not paying those and staying in autarky afterwards. Hence, since to hold z_2 units of real balances for next DM's requires $(1 + \pi)z_2$ units of CM

goods and the additional money holding or repayment for round-1 DM monitored trades require at least z_1 units of CM good, we have

$$-[z_1 + (1 + \pi)z_2] + \frac{\delta}{1 - \delta} [u_1(x_1) + \sigma_2 u_2(x_2) + (1 - \sigma_2)z_2 - z_1 - \pi z_2] \geq 0,$$

which implies (15). Note that buyers can obtain the lump-sum transfer independent of their behavior.

(ii) Consider now the case where $C = \{2\}$. Since the buyer can always skip round-1 DM and because only money can be used to finance the round-1 consumption, following exactly the same arguments as those in Lemma 3.1, the condition (4) with r replaced by $\zeta > r$ is necessary. Since buyers have to hold at least z_1 units of real balances in equilibrium and finance z_2 either in money or debt, using a similar argument to the case where $C = \{1\}$, it is straightforward to show, using the same arguments as those in (i), that (16) is necessary to avoid buyers from not participating in the whole scheme. Again, note that buyers can obtain the lump-sum transfer independent of their behavior.

References

- [1] Alvarez, F. and U. Jermann (2000). "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica* 68, 775-797.
- [2] Andolfatto, D. (2010). "Essential Interest-bearing Money," *Journal of Economic Theory* 145, 1495-1507.
- [3] Araujo, L., and B. Camargo (2013), "The Essentiality of Money in the Cavalcanti-Wallace Model," *Working Paper*.
- [4] Berentsen, A., G. Camera, and C. Waller (2005), "The Distribution of Money Balances and the Nonneutrality of Money", *International Economic Review*, 46, 456-487.
- [5] Berentsen, A., G. Camera, and C. Waller (2007), "Money, Credit, and Banking", *Journal of Economic Theory*, 135(1), 171-195.
- [6] Bethune, Z., T-W Hu, and G. Rocheteau (2014), "Dynamic Indeterminacy and Welfare in Credit Economies", *Working Paper*.
- [7] Cavalcanti, R., A. Erosa, and T. Temzelides (1999), "Private Money and Reserve Management in a Random Matching Model", *Journal of Political Economy*, 109, 929-945.
- [8] Cavalcanti, R., and N. Wallace (1999), "Inside and Outside Money as Alternative Media of Exchange", *Journal of Money, Credit, and Banking*, 31, 443-457.
- [9] Dale, S. (2013), "Bank of England economist sounds inflation warning on quantitative easing", March 15th, <http://www.theguardian.com/business/2013/mar/15/bank-england-economist-quantitative-easing>.

- [10] Deviatov, A., and N. Wallace (2014), “Optimal inflation in a model of inside money”, *Review of Economic Dynamics*, 17, 287-293.
- [11] Ennis, H. M. (2009), “Avoiding the Inflation Tax.” *International Economic Review*, 5, 607-625.
- [12] Ferraris, L. and M. Watanabe (2008), “Collateral Secured Loans in a Monetary Economy,” *Journal of Economic Theory* 143, 405-424.
- [13] Gertler, M., and N. Kiyotaki (2010), “Financial Intermediation and Credit Policy in Business Cycle Analysis,” in *Handbook of Monetary Economics*, Volume 3, 547-599.
- [14] Gomis-Porqueras, P., and D. Sanches (2013), “Optimal Monetary Policy in a Model of Money and Credit”, *Journal of Money, Credit and Banking*, 45(4), 701-730.
- [15] Gu, C., F. Mattesini, and R. Wright (2014), “Money and Credit Redux”, *Working Paper*.
- [16] Guerrieri, V., and G. Lorenzoni (2009), “Liquidity and Trading Dynamics,” *Econometrica*, 77, 1751-1790.
- [17] Hu, T.-W., J. Kennan, and N. Wallace (2009). “Coalition-Proof Trade and the Friedman Rule in the Lagos-Wright Model,” *Journal of Political Economy*, 117, 116-137.
- [18] Hu, T.-W., and G. Rocheteau (2013). “On the Coexistence of Money and Higher-Return Assets and its Social Role.” *Journal of Economic Theory* 148, 2520-2560.
- [19] Hu, T.-W., and G. Rocheteau (2014). “Monetary Policy and Asset Prices: a Mechanism Design Approach.” *Journal of Money, Credit, and Banking*, forthcoming.
- [20] Kehoe, T. and D. Levine (1993). “Debt Constrained Asset Markets,” *Review of Economic Studies* 60, 865-88.
- [21] Kocherlakota, N. (2009). “Money is Memory.” *Journal of Economic Theory* 81, 232-251.
- [22] Lagos, R., and R. Wright (2005). “A Unified Framework for Monetary Theory and Policy Analysis.” *Journal of Political Economy* 113, 463-484.
- [23] Li, Y. and Y. Li (2013) “Liquidity and Asset Prices: A New Monetarist Approach.” *Journal of Monetary Economics* 60, 426-438.
- [24] Liu, Q., Wang, L., and R. Wright (2015). “Costly Credit and Sticky Prices.” Working Paper.
- [25] Lotz, S., and C. Zhang (2013), “The Coexistence of Money and Credit as Means of Payment,” *Working Paper*.
- [26] Sanches, D., and S. Williamson (2010), “Money and Credit with Limited Commitment and Theft,” *Journal of Economic Theory*, 145, 1525-1549.

- [27] Taylor, John (2012), “Improving the Federal Reserve System: Examining Legislation to Reform the Fed and Other Alternatives”, Testimony before the Subcommittee on Domestic Monetary Policy and Technology of the Committee on Financial Services U.S. House of Representatives, May 8th, <http://financialservices.house.gov>.
- [28] Telyukova, I., and R. Wright (2008), “A Model of Money and Credit with an Application to the Credit Card Debt Puzzle,” *Review of Economic Studies*, 75, 629-647.
- [29] Wallace, N. (2010). “The Mechanism-Design Approach to Monetary Theory.” *Handbook of Monetary Economics*, Volume 3, Friedman, B. M., and Woodford, M., (Eds), Chapter 1, 3-23.
- [30] Wallace, N. (2014). “Optimal Money-creation in ‘Pure-Currency’ Economies: a Conjecture.” *Quarterly Journal of Economics*, 129, 259-274.
- [31] Williamson, S. D. (2012). “Liquidity, Monetary Policy, and the Financial Crisis: A New Monetarist Approach.” *American Economic Review*, 102, 2570-2605.
- [32] Zhu, T. (2008). “Equilibrium Concepts in the Large Household Model.” *Theoretical Economics*, 3, 258-281