Venture Capitalists’ Span of Control and Fund Structure in a Directed Search Model*

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Abstract

I develop a theory of fund size and structure when fund managers add value to the companies they finance, but their time and advice are limited. I propose a matching model where managers span their nurturing activity over more projects, entrepreneurs are privately informed about their projects’ quality, and direct their search to managers who differ in fund size and in the ability to scale up their human capital. I derive necessary and sufficient conditions for positive and negative assortative matching over managerial attention and project quality to emerge. In presence of positive sorting, managers shrink fund size below the efficient level. Entry of unskilled managers feeds back into the equilibrium sorting, and can increase returns at the top of the distribution, which is consistent with empirical findings. When the same trade-offs are incorporated into a dynamic setting, the model provides one reason why we observe closed, finite horizon funds, even when these are socially undesirable. This is due to self selection on the investment side: the best entrepreneurs are those who value the most the exclusive relation that finite-horizon funds guarantee. Managers benefit from committing to a size in the first place.

JEL codes: G24, G31, D82, D83

1 Introduction

Private equity financing, and in particular venture capital, has been undoubtedly a successful model of financing entrepreneurship. The common view among practitioners and academics is

*I am indebted to Francesco Nava and Balázs Szentes for their continuous guidance, support and for many stimulating discussions. This paper has greatly benefited from comments and suggestions by Daniel Ferreira. I thank Michel Azulai, Gianpaolo Caramellino, Anil Dasgupta, Peter Kondor, Marco Pagnozzi, Nicola Persico, Ronny Razin, Antonio Rosato, Emanuele Tarantino and participants at the LSE Theory Work in Progress Seminars, and the Petralia Applied Economics Workshop 2016 for useful comments at various stages. All errors are my own.

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that fund managers add value to the firms they finance through a number of activities such as monitoring, selecting top management, and experimenting innovative business strategies. Venture capitalists (henceforth VCs) are a key input to the firms in their portfolio. The empirical literature has established that: 1) in the cross section, there is a positive size-returns relationship at the fund level and 2) accounting for fund managers fixed effects, average returns to investors are decreasing in fund size. The evidence suggests that there are intrinsic diseconomies of scale in the industry, but also that VCs might differ considerably in their ability to generate returns: if one imagines that better VCs would attract more capital, the negative size-returns relationship can appear reverted in the cross-section. In light of the remarkable role venture capital plays in boosting growth, it is therefore important to understand both how money from investors and projects from entrepreneurs are allocated across VCs, who essentially act as intermediaries between the two. Do money flows follow the underlying heterogeneity in skills across fund managers in an efficient way? Do better firms seek financing from better venture capitalists? And how does this in turn affect VCs’ decisions at the moment a fund is formed?

Given the nature of their business, one of the most significant drivers of such diseconomies of scale in venture capital funds is the limited attention VCs can devote to multiple firms at the same time. If the VCs human capital is the scarce resource, a larger portfolio of investments will inevitably dilute the manager’s contribution to each single investment. This is the first ingredient of my model.

The second and most important feature of the model that I will develop is that entrepreneurs are privately informed about their projects’ quality, and direct their search to different VCs. This is realistic: one major distinction between the activity of VCs compared to that of other fund managers (e.g. buyouts, mutual funds) is that the former invest in target firms that are particularly interested in the managers’ value added to firm growth. The idea that entrepreneurs seeking venture capital money discriminate among VCs based on their reputation and perceived quality is supported by compelling evidence: Hsu (2004) finds that entrepreneurs are willing to accept worse terms in order to affiliate with VCs that can provide greater value added. Recent empirical studies show that there exist sorting in the industry between better VCs and start-up firms with greater potential: Sørensen (2007) finds evidence of positive assortative matching between VCs and the companies in their portfolios; selection is found to have the largest impact on returns, compared to the VCs contribution alone. This evidence brings up the question of whether and how self selection considerations on the investment side (i.e. sorting of different entrepreneurs into different funds) which seem prevalent in the venture capital industry, interact with the determination of fund size in an otherwise standard problem of fund management.

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1For a survey of research in private equity, see Da Rin et al. (2011). The most relevant empirical findings that I will refer to are in Kaplan and Schoar (2005), Harris et al. (2014) and Robinson and Sensoy (2016).

2For direct evidence of this, see for example Cumming and Dai (2011).
Another distinctive feature of the industry is that, differently from, for example, mutual funds, private equity funds are finitely lived. That is, fund managers’ activity is restricted by a clear deadline when their investments must be exited. At the creation of a fund, investors and VCs form a limited partnership. This arrangement can have the negative consequence of forcing VCs to give up investment opportunities that show up too late in the fund’s life. Kandel et al. (2011) find suggestive evidence that being closer to the end of the fund induces myopic behaviour on the side of fund managers. Barrot (2016) finds that the length of the investment horizon is associated to selection of different startups, therefore it has real effects on the VCs’ investment strategy. Why such configuration has become the norm, and what makes it so stable, despite the potential inefficiency that it may generate? Would it be desirable if investors and VCs formed long lasting, firm-like relationship so that the VC’s choice to start a project is not affected by its time of arrival?

I start from these facts and observations to build a matching model where VC and entrepreneurs meet to turn ideas into profitable firms, entrepreneurs are privately informed about their projects’ quality, and can direct their search to agents managing funds of different size and different ability to scale up their skill. The model describes how size emerges as an equilibrium choice by VCs that anticipate the effects this decision has on the type of entrepreneurs that will self select into them. In the model, entrepreneurs trade off matching with better VCs - those who can devote more attention to their projects - versus the lower search frictions associated to worse VCs. This has effects at the initial stage of the game, where VCs contracts with investors over fund size and fees. In particular, when technological complementarities induce positive sorting between VCs’ value added and entrepreneurial quality, VCs’ will tend to shrink the size of their funds below what is efficient. The reason is that VCs don’t internalize the externality associated to their choice on the equilibrium assignment: when too many VCs offer high attention, the relative increase in search frictions is low, hence entrepreneurs’ separation is suboptimal. Moreover, since VCs decisions induce different aggregate sorting outcomes, the model provides novel implications from changes in the VCs skill distribution in the economy. In particular, when more unskilled VCs enter the market, I derive conditions under which investments are both more likely to be at the bottom of the outcome distribution, and more profitable conditional on being at the top. This is consistent with the findings in Nanda and Rhodes-Kropf (2013) who find that investment made in more active periods are more likely to fail and deliver higher returns conditional on not failing. Infact, Kaplan and Schoar (2005) find that, in boom times, capital flows disproportionally to worse funds. Both the inefficiency in fund size and the effect on the shape of the returns distributions would not appear in a benchmark model with random matching, or no heterogeneity among entrepreneurs.

I then move to a dynamic setting where projects don’t realize returns immediately, VCs can match to one entrepreneur every period, and follow the projects until they are ready to produce returns. The interest here is to study the contractual arrangement between the VC
and the investors; specifically, I allow VC to choose between a short-term contract and a long-lasting credit relationship with the investors. In the former case, VCs are forced to wait until the current project has realized its returns before they can get money from a new fund, and go back to the market for entrepreneurs: this means projects under their management won’t overlap. Thus an endogenous form of commitment to the current project emerges. I show that a situation where every VC has the open, long-lasting credit relationship and can hence start new project every period is not sustainable, even when this would be the welfare maximizing solution. The reason is that a deviating VC will be able to skim the market and attract the very best entrepreneurs, that is those entrepreneurs who are willing to pay the highest search friction in order to match to a “committed” VC. This provides a new explanation for the prevalence of closed, finite-horizon funds in private equity, as opposed to the open funds we observe in other contexts where fund managers invest in public securities and are not subject to a two-sided matching problem.

1.1 Relation to the Literature

Despite the considerable variation and the impact it has on returns, little theoretical research has been done on the determinants and consequences of private equity fund size. The paper directly contributes to the literature focusing on size determination in fund management, with particular application to the venture capital asset class. One natural reference is Berk and Green (2004), who derive several predictions concerning fund flows in the mutual fund industry; like in that paper, fund managers in my model possess scarce skills, and therefore receive all the rents from investors by choosing fund size and fees appropriately. However, while in Berk and Green (2004) this results in an efficient allocation of money across managers, adding entrepreneurs self selection in my model produces: 1) a generically inefficient outcome, 2) multiple equilibria that are not welfare equivalent and 3) a feedback effect of entry of unskilled managers on returns at the top of the distribution. Fulghieri and Sevilir (2009) model the optimal investment strategy of a VC who trades off the higher value added from a small portfolio, with the diversification gains from a large one. Inderst et al. (2006) hold portfolio size constant, and model the beneficial effect - through stronger competition among entrepreneurs - of having limited capital at the refinancing stage. I share with the first paper the view that VC’s human capital dilutes with a larger portfolio, and with the second the idea that the amount of capital a VC raises affects the type of projects funded. But in my model the distribution of VCs size and structure affects the sorting; I study the equilibria that result from the interaction among VCs that anticipate this effect.

In terms of the entrepreneur-VC relationship, in my economy matches form between two

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3Notable exceptions are Inderst et al. (2006), Fulghieri and Sevilir (2009) and Silveira and Wright (2015), of which a more detailed discussion will follow.
parties whose payoffs are asymmetrically affected by the current match: while the entrepreneur is solely interested in the return from his project, the VC cares about the total fund’s returns. The VC, faces a typical quality-quantity of matches type of trade-off. This approach to modelling the venture capital environment, and the essential tension implicit to it, is shared with several recent works. In Michelacci and Suarez (2004), the focus is on identifying institutional market characteristics that increase total welfare by alleviating this trade-off and allowing VCs to free up their human capital quicker, without destroying too much of the monitored firm’s value; Jovanovic and Szentes (2013) find conditions under which the optimal contractual arrangement in presence of moral hazard on the entrepreneurs’ side takes the form of an equity contract. They also explain the returns premium to VC-backed firms; Silveira and Wright (2015) study project selection on the VC’s side and optimal fund size when start-up costs are random but committing funds entails opportunity costs. Contrary to mine, none of the aforementioned models analyse sorting of different entrepreneurs with different VCs in presence of these forces. More importantly, while I also assume diseconomies of scale, I don’t restrict intermediaries to run one project at a time. This more realistic assumption allows to study 1) the equilibrium choice of span of control and 2) the choice of how frequently go back to the market and actively search for new investments, possibly before the current one has produced returns.

Like this paper, Marquez et al. (2014) is built on the fundamental observation that investments in venture capital are special in that they are subject to a two-sided matching problem. Marquez et al. (2014) develop a signal-jamming model where VCs with differential ability to produce returns manipulate fund size in order to affect entrepreneurs’ learning; this, coupled with rigidity in fees adjustment ex-post, prevents them from extracting the full surplus from investors. In my model instead, the VCs ability is common knowledge. Moreover, while Marquez et al. (2014) take a reduced form approach to the determination of a fund’s portfolio quality, I study and characterize sorting explicitly; since relative gains from committing higher attention are endogenous, I can derive conditions under which an equilibrium where every VC chooses a certain fund structure might unravel; plus, modeling sorting allows me to study efficiency of the funds allocation across VCs, and study the effects of entry of VCs on the entire distribution of returns.

The paper also contributes to a literature that aims at endogenizing the most observed features and contractual arrangements at the basis of investment funds: in Stein (2005) open-ended fund structure emerges because mutual fund managers compete for money flows and the best ones can credibly signal their ability by offering an open-end structure that can prevent them from fully exploiting arbitrage opportunities; Axelson et al. (2009) explain why buyout funds exhibit a mix of outside debt and equity financing in a setting where the key tension is between imposing discipline to privately informed managers while at the same time making efficient use of their superior screening ability.

On a more abstract level, my paper provides conditions for sorting in a matching environ-
Eeckhout and Kircher (2010) derive general results on the consequences of search frictions in an assignment problem where types on both sides of the market are observable and sellers commit on posted prices. They find that the requirements on the match-value function for positive and negative sorting depend the elasticity of substitution in the matching function. In my model, with directed search from one side and non-transferable utilities, the stronger form of supermodularity (and submodularity) is needed to guarantee sorting, under any specification of the matching function. More results related to my setting are in Eeckhout and Kircher (2016) who study the interaction between the choice of span of control and the sorting pattern in an assignment economy without search frictions.

Roadmap: Section 2 introduces the setup, followed by the characterization of the equilibria and comparative statics on changes to the distribution of skills among venture capitalists; Equilibria are ranked in terms of welfare achieved and compared to a second best solution in Section 3; Section 4 uses results in previous sections to analyse the choice between short and long-term investors-fund manager relationships, in an appropriately accomodated setup; Section 5 concludes; All proofs are relegated to the Appendix.

2 Model

Agents. The economy consists of heterogeneous venture capitalists (henceforth VCs), identical investors and ex-ante identical entrepreneurs. There is a mass of investors, with arbitrarily large measure. Each investor is endowed with one unit of money, which they invest into funds, each managed by a single VC. VCs are exogenously endowed with ability, denoted $x$, that can take values in $[\underline{x}, \bar{x}] \subset \mathbb{R}_+$, according to the distribution $g$. The measure of VCs in the economy is fixed and normalized to one. VC’s will contract with investors on fund size, denoted $w$, possibly depending on $x$. Entrepreneurs are in large supply, and can enter the market upon paying startup cost $c$. If they do, they draw a type $\lambda$, the quality of the project they own, from a distribution $f$ defined over $[\underline{\lambda}, \bar{\lambda}] \subset \mathbb{R}_+$. An higher $\lambda$ is a better project in a way specified in the next paragraph. Entrepreneurs need money and the VCs’ input to make their projects turn into profitable firms $^4$.

Projects. All projects need one unit of money to get started. Call $m$ the measure of projects a given VC is matched to in equilibrium. Define $a$ the attention the VC devotes to each project. Assume $a \in \{a_0, a_1, ..., a_N\}$, with $a_i > a_{i-1}$. VC’s attention, or managerial input, is a function

$^4$A natural interpretation - which fits the common view of the role of venture capitalists - is that young firms need to be constantly monitored, be it because entrepreneurs are unexperienced, or because the lack of collaterals makes it impossible to find alternative sources of financing.
of his ability and the number of firms he is monitoring, $a := a(m, x)$. In particular,

$$a(m, x) = \begin{cases} 
  a_N & \forall m \in [0, m_{N-1}^x) \\
  a_i & \forall m \in [m_i^x, m_{i-1}^x) \\
  a_0 & \forall m > m_0^x
\end{cases}$$

with $m_{i-1}^x - m_i^x = \Delta > 0$ for all $x$ and $i$, and $m_i^x > m_i^{x'}$ for all $i$ and all $(x, x')$ with $x > x'$. In words, VCs’ input gets diluted when working on more projects in parallel and it decreases faster for managers with lower $x$. Essentially, more skilled VCs can manage a larger number of projects for each level of attention they want to guarantee. Each project’s return, $R$, is assumed to be a function of $a$ and of the project’s quality, $\lambda$. Call this function $R(a, \lambda)^6$. It is natural to have $R_a(a, \lambda), R_\lambda(a, \lambda) > 0$. I further assume that $R(a, \lambda)$ is twice continuously differentiable in its arguments.

Matching and Information. While VCs’ size and ability are common knowledge, the entrepreneur’s type, $\lambda$, is his private information. Therefore, I study directed search from the long and informed side of the market, the entrepreneurs. Each VC’s combination of size and ability, $(w, x)$, will therefore form a submarket where entrepreneurs will select into, possibly depending on their type. Finally, assume that as many matches as possible are formed in each submarket; that is, the number of matches as a function of the measure of entrepreneurs searching, $q_e$, and the measure of money available (or “vacancies”), $q_k$, is given by $M(q_k, q_e) = \min\{q_k, q_e\}$.

Payoffs, Strategies and Timing. Investors and VCs contract over a fixed fee $p$ that the VC earns for every dollar invested. The VC chooses the total size of the fund. Investors have access to an alternative investment opportunity, delivering $R_0$. When investing in a certain VC, they will get a fixed share $\alpha \in (0, 1)$ of the VC’s average returns from the fund. Entrepreneurs observe the joint distribution of $(w, x)$ and choose whether or not to pay the startup cost. Those who do, can direct their search towards different VCs. Conditional on being matched, they receive the residual - $(1 - \alpha)$ - share of the returns from their projects. All agents are risk neutral and maximize expected returns.

2.1 Equilibrium Sorting

Market Tightness. Let me first study the subgame where entrepreneurs make the entry decision and direct their search into different VCs. Assume that the allocation of investors’

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5The assumption that attention jumps discontinuously with $m$ is of no consequence in terms of the qualitative results, but allows to guarantee existence of equilibria when size is the VC’s choice.

6The direct implication is that $a$ is all that matters to a given type of entrepreneur. In other words, project’s quality does not interact with VC’s ability or fund size per se. This separability will greatly simplify the analysis.
money generates fund size between \( w \) and \( \overline{w} \) with \( \overline{w} > w \). Denote \( H(w, x) \) the measure of venture capitalists with fund size below \( w \) and ability below \( x \). Upon entry, the search strategy for an entrepreneur is described by a distribution over \([w, \overline{w}] \times [x, \overline{x}]\). Formally, the strategy is a mapping
\[
s : [\Lambda, \overline{\Lambda}] \rightarrow \Delta ([w, \overline{w}] \times [x, \overline{x}])
\]
generating a cdf \( S(w, x; \lambda) \). Calling \( E \) the measure of entrepreneurs who entry, define \( \tilde{S}(w, x, E) \) the measure of entrepreneurs searching in market with size below \( w \) and ability below \( x \). Upon entry, the search strategy for an entrepreneur is described by a distribution over \([w, \overline{w}] \times [x, \overline{x}]\). Formally, the strategy is a mapping
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generating a cdf \( S(w, x; \lambda) \). Calling \( E \) the measure of entrepreneurs who entry, define \( \tilde{S}(w, x, E) \) the measure of entrepreneurs searching in market with size below \( w \) and ability below \( x \).

This is given by summing the search strategy over all the entrepreneurs, so
\[
\tilde{S}(w, x, E) = \int_{\Lambda} E S(w, x; \lambda) dF(\lambda).
\]
On the other side of the market, as a VC managing fund of size \( w \) can follow up to \( w \) projects in parallel, the amount of vacancies in submarket \((w, x)\) is given by
\[
\tilde{H}(w, x) = \int_{-\infty}^{\overline{x}} \int_{-\infty}^{w} \hat{w} dH(\hat{w}, \hat{x}).
\]
To define expected payoffs properly, let \( \theta(w, x, E) \) be the expected ratio of vacancies to entrepreneurs in submarket \((w, x)\), when a measure of \( E \) entrepreneurs has entered. I will refer to \( \theta(w, x, E) \) as market tightness. The function will solve
\[
\int_{-\infty}^{\overline{x}} \int_{-\infty}^{w} \theta(\hat{w}, \hat{x}, E) d\tilde{S}(\hat{w}, \hat{x}, E).
\]
Finally, define \( Q(w, x, E) \) the probability an entrepreneur finds a match when searching in market \((w, x)\). Given the matching function, this is:
\[
Q(w, x, E) := \min \{ \theta(w, x, E), 1 \}
\]

I can now write type-\( \lambda \) entrepreneur’s expected payoff from choosing to search in market \((w, x)\) as:
\[
(1 - \alpha) Q(w, x, E) R(a(x, m(w, x, E)), \lambda)
\]
where \( m(w, x, E) \) is the measure of projects per VC in market \((w, x)\). I can now describe what is an equilibrium of this subgame.

**Definition 1. (Equilibrium in the Subgame).** An equilibrium in the sorting subgame is characterized by a vector \((E, s^*)\) such that:

\[
(i) \quad s^*(\lambda) = \arg \max_s E_{w, x \sim s} [Q(w, x, E; s^*) R(a(x, m(w, x, E; s^*)), \lambda)]
\]
\[
(ii) \quad \int_{\Lambda} E_{w, x \sim s^*} [Q(w, x, E; s^*) R(a(x, m(w, x, E; s^*)), \lambda)] dF(\lambda) = c
\]

Part (i) imposes optimality. Part (ii) from the unlimited number of entrepreneurs: it states that, ex-ante, entrepreneurs must be indifferent between entering the market and staying out. To save on notation, I will denote \( u_\lambda \) the expected equilibrium payoff of type-\( \lambda \), conditional on entry.

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This is endogenous, as it is determined by the investors and VCs equilibrium choice. Hence no assumption on \( H \) is made at this stage.
An immediate observation to make is that in this model, not only the search strategy of entrepreneurs impose an externality to each other through its usual effect on search frictions, it also does by affecting VCs attention. The next result is an implication of this effect.

**Lemma 1.** In any equilibrium, $\tilde{S}(w, x, E) > 0 \forall (w, x)$

Lemma 1 states that in every submarket a positive measure of entrepreneurs must search in equilibrium.

Another consequence of having a large supply of entrepreneurs is that, provided that lower VC’s attention is not too inefficient ex-ante, entrepreneurs will keep entering the market until search frictions kick in. The following assumption and consequent lemma formalize this observation.

**Assumption A1.** $\mathbb{E} R(a_0, \lambda) > c \quad \forall \lambda$

A1 states that, ex-ante, an entrepreneur would strictly benefit from paying the startup cost and match to a VC in absence of search frictions, even when the VC’s attention is fully diluted (at its lowest level it is given by $a_0$).

**Lemma 2.** Under A1, in any equilibrium, in each submarket there are more entrepreneurs than vacancies. That is, $Q(w, x, E) < 1$ and $m(w, x, E) = w, \forall (w, x)$

The implication of Lemma 1 is that all VCs operate at full capacity. If this was not the case, entrepreneurs would find it worthwhile to enter the market and direct their search in the submarket with no search frictions.

The next result is a direct consequence of Lemma 2, and will help characterize the equilibrium strategies in the sorting subgame.

**Corollary 1.** In any equilibrium, $Q(w, x, E)$ is a function of $a$ and $E$ only.

Intuitively, because VCs must operate at full capacity in every equilibrium, attention in market $(x, w)$ is given by $a(x, w)$. As returns are only a function of attention and project’s quality, an entrepreneur must be indifferent between searching in two markets where attention is the same. This suggests that, in essence, the entrepreneur’s strategy reduces to which attention levels $a$ to seek matching with.

**Lemma 3.** For a given $E$, any equilibrium of the sorting subgame is mirrored by one from a game where entrepreneurs can only direct their search to different attention levels, and are then matched with VCs that are at the chosen attention, in proportion to each VC’s size.

In words, because entrepreneurs must be indifferent between searching in any market where attention is the same, any equilibrium can equivalently be represented by one where their strategy is to simply choose to search over different levels of attention, which in this reduced
model is a fixed, predetermined characteristic of the VC. The distribution of vacancies will reflect total size summed across all VCs at a given iso-attention locus in the original model.

Lemma 3 turns useful because it allows to focus on a particular type of sorting equilibrium, where the sole determinant characteristic of a VC, hence what defines a sub-market to search in, is attention. The interest is then to study what requirements should the return function obey to, so that in a general setting, independently of the distribution of types, sorting would emerge. If such conditions are identified, one can conclude that the same sorting pattern would emerge in the original model, once mixed strategies are adjusted accordingly.

Let $\Lambda(a)$ be the set of entrepreneurs applying to market $a$ under strategy $s$, $\Lambda^s(a) := \{\lambda: s(a; \lambda) > 0\}$.

**Definition 2.** An equilibrium exhibits positive (negative) assortative matching if $\forall a, a' \quad a > a' \quad \lambda \in \Lambda^s(a) \land \lambda' \in \Lambda^s(a') > 0 \Rightarrow \lambda > (<) \lambda'$

Intuitively, under positive assortative matching (henceforth PAM), higher attention can not be associated with a worse entrepreneur; however, pooling of more entrepreneurs into a given attention level is allowed. I can now state the main result of this section, that establishes necessary and sufficient conditions for equilibria to exhibit PAM or NAM.

**Proposition 1. (Sorting).** All equilibria exhibit PAM (NAM) if and only if $R(a, \lambda)$ is everywhere logsuper(sub)modular.

Notice that logsuper(sub)modularity implies super(sub)modularity, while the opposite does not hold. To build intuition why a stronger form of supermodularity is necessary for PAM, notice that, as emphasized by Eeckhout and Kircher (2010), when allowing for search frictions in matching models, two forces drive the sorting pattern, in opposite directions: the “trading security motive”, which motivates higher types to select into less crowded markets, and the “match value motive”, which is related to the value of being matched to better types. In this application, the latter motive corresponds to the value of the VC’s attention, which is a bigger concern when $\lambda$ is high.\(^8\) This trade-off becomes evident if one looks at the difference in expected payoff from searching in any two markets, $a$ and $a'$, with $a > a'$, and differentiates it.

\(^8\)It should be noted that the condition in Proposition 1 is particularly strong because search is unilateral. In the framework proposed by Eeckhout and Kircher (2010), where types on both sides of the market are observable, and sellers can commit on posted prices, it is shown that, although supermodularity per se is generally not sufficient, the requirements for PAM to emerge are milder. In particular, the degree of supermodularity depends on the elasticity of substitution in the matching function. Notably, with unilateral directed search, the result that $R$ must be logsupermodular holds true under any specification of the matching function.
with respect to $\lambda$. The gain is increasing in $\lambda$ when:

$$\frac{- (Q(a) - Q(a'))}{Q(a')} R_\lambda(a, \lambda) < \frac{\partial (R(a, \lambda) - R(a', \lambda))}{\partial \lambda}.$$  

To understand why the logsupermodularity is sufficient, notice that the condition implies that for any $(a, a')$ with $a' > a$, the ratio $R(a', \lambda) / R(a, \lambda)$ is strictly increasing in $\lambda$. This means that, if for some type $\tilde{\lambda}, Q(a') R(a', \tilde{\lambda}) > Q(a) R(a, \tilde{\lambda})$, the same would be true for all $\lambda > \tilde{\lambda}$. This ensures separation.

From now, I will consider the case when $R(a, \lambda)$ is logsupermodular.

Assumption A2. $R(a, \lambda)$ is everywhere logsupermodular.

2.2 Endogenous Size

In this section I endogenize the distribution $H(w, x)$, and hence will characterize equilibria of the entire game. I will restrict attention to equilibria where both VCs and entrepreneurs play symmetric, pure strategies.

As in Berk and Green (2004), VCs contract with investors over the fund’s size and a per-dollar fee. Notice that each investor is left with

$$\alpha \mathbb{E}[R(a(x, w), \lambda) | \lambda \in \Lambda^s(a(x, w))] - p$$

Since I assume each VC contracts with a multitude of competing investors, it must be that the net return equals their outside option, $R_0$, as VCs can always push investors to their participation constraint. This gives:

$$w (\alpha \mathbb{E}[R(a(x, w), \lambda) | \lambda \in \Lambda^s(a(x, w))] - R_0) = wp$$

Therefore the VC will choose $w$ to maximize total excess returns (the left hand side of (1)) , and then set $p$ in such a way that investors’ participation constraint binds.

VC Strategy. The VC’s decision can be further simplified by noting that, as entrepreneur’s selection is affected by $w$ only through its effect on $a$, a VC will never set $w$ in $(m_i^x, m_j^x)$. It follows that the relevant strategic choice from a VC is which attention $a_i$ to offer. The VC will consequently ask investors the maximum size conditional on $a_i$, that is $m_i^x$. VC’s strategy is therefore fully described by a mapping $\sigma : [x, \pi] \rightarrow \{a_0, a_1, ..., a_N\}$. Define the set of VCs types choosing to offer $a_i$ given $\sigma$, $X_i^\sigma := \{x : \sigma(x) = a_i\}$. Finally, define the set of attention levels offered in equilibrium $I^* := \{i : X_i^\sigma \neq \emptyset\}$. The amount of vacancies available in market
given a certain strategy profile \( \sigma \), denoted \( W^\sigma_i \), is therefore computed as

\[
W^\sigma_i = \int_{x \in X^\sigma_i} m^\sigma_i dG(x).
\]

It is now possible to define the equilibrium of the entire game. I have so far defined market tightness in a given market, and consequently the probability for an entrepreneur to find a match by searching in it, as the ratio of vacancies to the measure of entrepreneurs who search there in under some strategy profile. Namely, at any \( a_i \) with \( i \in I^* \), and given \( s, E, \) and \( \sigma \), one can compute market tightness as

\[
Q(a_i) = \frac{W^\sigma_i E}{\int_{\lambda \in A^*(a)} dF(\lambda)}.
\]

The equilibrium notion however has to define both market tightness \( Q(a_j) \) and the composition of entrepreneurs searching, \( A^*_j \), for markets where no VC is positioned, that is for any \( j \notin I^* \). The approach I follow is from Guerrieri et al. (2010). Let me first state the equilibrium definition.

**Definition 3. (Equilibrium).** An Equilibrium is a vector \((E, s^*, \sigma^*)\) such that:

1. \( \sigma^* (x) = \arg \max_{a_i \in \{a_0, \ldots, a_N\}} \alpha w^\sigma_i E[R(a_i, \lambda)] \) for \( \lambda \in A^*_i \)
2. \((E, s^*)\) form an equilibrium of the subgame
3. \( u_\lambda \geq Q(a_j) R(a_j, \lambda) \) for all \( j \in I^* \)

with the last condition satisfied with equality whenever \( \lambda \in A^*_j \) and \( Q(a_j) > 0 \).

Part (i) and (ii) state that the VC’s strategy has to be optimal, given the induced distributions and consequent sorting of entrepreneurs, which in turn has to be consistent with the equilibrium in the sorting subgame has I have defined it in the previous section. Consider now part (iii). For \( j \)s that are included in \( I^* \), this is obvious: it says that no entrepreneur is better off by searching in an alternative market. When \( a_j \) is not offered in equilibrium, \( Q(a_j) \) has to be set such that one type is indifferent between getting his equilibrium payoff and moving to \( a_j \), whereas all others weakly prefer not to deviate. \( A^*_j \) will then be defined accordingly. Essentially, what this requirement does is to select out-of-equilibrium beliefs that work as follows: if a VC wants to select a size so that attention is \( a_j \) and, \( a_j \) is not offered in equilibrium, he has to expect to attract the type who is willing to deviate at the lowest \( Q(a_j) \), that is, the type who is willing to pay the highest friction to search in \( a_j \). If \( a_j \) is not attractive to any type \( \lambda \), even when \( Q(a_j) = 1 \), then \( Q(a_j) = 0 \), nobody will search there, and the deviation will not be profitable for any VC.

**Proposition 2. (Characterization).** All equilibria are described by a partition \( \{x, \ldots, x_i, \ldots, \bar{x}\} \) of the set \([x, \bar{x}]\) and a partition \( \{\Lambda_1, \ldots, \Lambda_i, \ldots, \bar{\Lambda}\} \) of \([\Lambda, \bar{\Lambda}]\) such that for any \( i \in I^* \), \( \Lambda^*_i = [\lambda_{i-1}, \lambda_i] \)
and $X_i^{σ^*} = [x_{i-1}, x_i]$. If $i \notin I^*, \lambda_i = \lambda_{i-1}$ and $x_i = x_{i-1}$. For all adjacent $i, j \in I^*$ with $i > j$:

(i) $w_j^{x_i} \left( \mathbb{E} \left[ R(a, \lambda) \mid \lambda \in A_j^{σ^*} \right] - R_0 \right) = w_i^{x_i} \left( \mathbb{E} \left[ R(a, \lambda) \mid \lambda \in A_i^{σ^*} \right] - R_0 \right)$

(ii) $Q(a_j) R(a_j, \lambda_j) = Q(a_i) R(a_i, \lambda_j)$

(iii) For any $j \notin I^*$, 

$$m_i^{x_j} \mathbb{E} \left[ R(a, \lambda_i) \mid \lambda \in A_i \right] \geq m_j^{x_j} R(a_j, \lambda_j) \quad \forall i \in I^*$$

In words the proposition states that all equilibria have the following form: entrepreneurs and VCs select into different attention levels according to their type, with successive subintervals of the equilibrium partitions $A_i^{σ^*}$ and $X_i^{σ^*}$ corresponding to set of VCs and entrepreneurs selecting higher attention. Conditions (i) and (ii) make sure that type at the limit of each subinterval are indifferent between the two adjacent attention levels where types right below and above are assigned to. Condition (iii) is where the refinement kicks in. Notice that, if $j \notin I^*, \lambda_j = \lambda_{j-1}$.

Hence, condition (iii) is requiring that no VC finds it profitable to deviate to an off equilibrium $a_j$, given that this deviation would attract the highest entrepreneur in the set of those who select the closest lower $a_i$ with $i \in I^*$. The example in figure make this easy to visualize.

2.3 Changing VC Skills’ Distribution

The analysis so far has focused on an economy where the measure and distribution of VCs is fixed. There exists evidence though, that money committed in the venture capital industry is highly volatile, that it is subject to booms and busts and that the number of funds dedicated to this asset class vary across time, sometimes in response to the business cycle. Determining the reason why these cycles occur is beyond the scope of this paper. However, the model can offer predictions on how the distribution of returns is affected by the inclusion of new VCs in the economy. Is it possible that a inflow of new VCs has effects on the entire the distribution of outcomes?

Let me restrict the analysis of this section to the case where attention can be either high or low, so that VCs and entrepreneurs can sort into two submarkets only. That is, $a \in \{a_0, a_1\}$. The main advantage is that, for a given equilibrium cutoff $x_0$, the induced equilibrium sorting is unique. This allows to make comparative statics around a candidate equilibrium. It is convenient to define the function

$$\phi \left( a, a', \bar{\lambda} \right) := \frac{\mathbb{E} \left[ R(a, \lambda) \mid \lambda \geq \bar{\lambda} \right] - R_0}{\mathbb{E} \left[ R(a', \lambda) \mid \lambda \leq \bar{\lambda} \right] - R_0}.$$ 

The function $\phi \left( a, a', \bar{\lambda} \right)$ is the expected per dollar excess return from choosing attention, $a$,
and attract entrepreneurs above some $\lambda$, relative to the excess return from choosing attention $a'$ and attract entrepreneurs below the same threshold. The function $\phi (a, a', \lambda)$ need not be monotone in $\lambda$. Below is an example where it is always decreasing.

Example 1. Assume quality $\lambda$ is uniformly distributed over the support $[0, 1]$. Returns are given by $R(a, \lambda) = a + (a - k) \rho(\lambda)$ with $a > k > 0^9$. If $\rho(.)$ is any increasing linear function, it can be verified that the ratio $\phi (a, a', \lambda)$ is decreasing in $\lambda$ for any $a > a'$ and any $k$, $R_0 > 0$.

Generally, equilibria need not be unique. I introduce below one appealing property of a candidate equilibrium, that will help identify the comparative statics of the next section. The property is based on a stability argument and will refine the number of equilibria. Define the adjustment process $\eta : [\bar{x}, \tilde{x}] \rightarrow [\bar{x}, \tilde{x}]$ as follows. $\eta(\tilde{x})$ is the cutoff VC, the one such that VCs above choose $a_1$ and those below choose $a_0$, that will be induced if the starting cutoff is $\tilde{x}$. Notice this is associated to a unique cutoff entrepreneur $\lambda_0(\eta(\tilde{x}))$ as prescribed by equilibrium in the sorting subgame. I impose that the derivative of $\eta$ is positive if

$$w_0^\tilde{x}(E[R(a_0, \lambda) | \lambda \leq \lambda_0(\tilde{x})] - R_0) > w_1^\tilde{x}(E[R(a_1, \lambda) | \lambda \geq \lambda_0(\tilde{x})] - R_0)$$

and negative otherwise. This means intuitively that, if $\tilde{x}$ and $\lambda_0(\tilde{x})$ are such that $\tilde{x}$ strictly prefers the large fund, the new cutoff must go up, so $\eta(\tilde{x}) > \tilde{x}$.

Definition 4. (Stable Equilibria). An Equilibrium vector $(x_0, \lambda_0)$ is stable if, for any $\varepsilon > 0$ there exists a $\beta > 0$ such that, if $|\tilde{x}_0 - x_0| < \beta$, then $|\eta(\tilde{x}_0) - x_0| < \varepsilon$.

A stable equilibrium is one that, after a small perturbation that forces some VCs away from it, will eventually converge back to itself.

I can now conduct comparative statics around a stable equilibrium. In light of the empirical findings about capital flows during booms, one interesting exercise is to study what happens when new unskilled VCs enter the market. More precisely, imagine the distribution of skills $g$ is defined on a support larger than $[\bar{x}, \bar{x}]$. Initially, only VCs in $[\bar{x}, \bar{x}]$ operate. What will happen if some of the worse VCs previously excluded decide to enter? In other words, what are the consequences of a decrease in $\bar{x}$?

Proposition 4. (Entry of unskilled VCs). If $\phi (a_1, a_0, \bar{\lambda})$ is decresing in $\bar{\lambda}$, then, for every stable equilibrium $(x_0, \lambda_0)$, $\frac{\partial x_0}{\partial \bar{z}} \cdot \frac{\partial \lambda_0}{\partial \bar{z}} < 0$. Consequently, there exists a level of returns, $\bar{R}$ such that, for every $R > \bar{R}$, returns are more likely to be below $R$, and are higher conditional on being above $R$.

---

9This function is logsupermodular whenever $\rho' > 0$. 

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The intuition is simple. The first effect is mechanic: the relatively unskilled VCs who enter the market will select $a_0$, hence the number of projects receiving more attention increases. The second effect is a consequence of sorting and would not appear in a benchmark model with no heterogeneity in projects, or random search. The larger number of vacancies in the market for low attention pushes the cutoff $\lambda_0$ up. This implies that those VCs who will keep raising relatively smaller funds will select better projects.

3 Welfare

From an ex-ante perspective, total welfare in the economy amounts to the expected fees VCs receive from the investors. This is due to investors perfectly competing for VCs, and the entrepreneurs’ free entry condition. In expectation, VCs are the only agents extracting rents. Therefore, for a given equilibrium, welfare in the economy is given by:

$$V((E,s^*,\sigma^*)) = \sum_{i \in \{1,..,N\}} W_i^{\sigma^*} \mathbb{E}[R(a_i, \lambda) | \lambda \in A_i^{q^*}]$$

Generally, the equilibrium need not be unique. A question one can ask is whether some equilibria are more desirable than others, from an ex-ante point of view. The next proposition states that some type of equilibria can be unambiguously ranked. Interestingly, the undesirable equilibria are those where markets for higher level of attention are thicker, relatively to those for lower attention.

**Proposition 5. (Ranking Equilibria).** (i). An equilibrium of the game induces higher welfare than any another equilibria where markets for higher attention are thicker, that is $Q_i/Q_j$ is bigger for all $(i,j)$ and $i > j$. (ii). An equilibrium of the game induces higher welfare than any another equilibria where the ratio $W_i/W_j$ is bigger for all $(i,j)$ and $i > j$.

The reason why equilibria where markets for higher level of attention are thicker are Pareto inferior is that, when increases in search frictions for any two adjacent market are small, the resulting assignment is characterized by worse selection at the top, that is, each cutoff $\lambda_i$ is lower, leading to lower average quality at each attention level.

The previous analysis shows the emergence of Pareto dominated equilibria, due to VCs failure to internalize their effect on the aggregate sorting. The natural next step is to study what would be the welfare maximizing allocation of VCs into fund sizes when the induced aggregate effect on sorting is taken into account. Below I define a Second Best Allocation as a solution to this problem.

**Definition 5.** A Second Best Allocation is a mapping $\tilde{\sigma} : [\underline{x}, \overline{x}] \to \{a_0, a_1, ..., a_N\}$ that
solves:
\[
\tilde{\sigma} = \arg \max_{\sigma} \sum_{i \in \{1, \ldots, N\}} W_i^{\sigma} \mathbb{E}[R(a_i, \lambda) | \lambda \in \Lambda_i]
\]
\[s.t. \quad \Lambda_i \in \Lambda_i^{s^*}\]

It is easy to observe that a Second Best allocation must be characterized by a partition of \([\underline{x}, \overline{x}]\), with successive intervals of the partition being assigned to higher levels of attention. Call \(x_{i}^{sb}\) the second best cutoffs. The next result compares the equilibrium with the second best, when \(a \in \{a_0, a_1\}\).

**Proposition 6. (Inefficiently small funds).** When \(a \in \{a_0, a_1\}\), too many VCs choose high attention compared to the second-best solution. That is, \(x_{0}^{sb} > x_{0}\).

## 4 Finite Fund’s Life

Contrary to standard firms and organizations, private equity funds are finitely lived. Fund managers’ activity is restricted by a clear deadline when their investments must be exited so that limited partners get the returns from them. One inefficiency that this can generate is highlighted in Axelson et al. (2009), where fund managers who privately observe project’s quality have the incentive to invest in negative NPV projects towards the end of their fund’s life, in an effort to improve average fund’s returns. However, even in absence of any asymmetric information and conflict of interest with investors, the fact that managers are constrained by capital committed in the vintage year, and don’t approach investors before results from the first investments have materialised, can come at the cost of forcing them to give up new investment opportunities that show up too late in the fund’s life. While it is true that a manager could in principle open new funds in parallel, fundraising is typically time consuming and this strongly limits the extent to which he can put projects “on hold” until enough money is raised; the practice is also limited by contractual restrictions that are meant to protect current investors, so that fund managers can’t form successor funds before the existing one is substantially invested or has completed its investment period.

To keep the model as coherent as possible to the static version, I take this second approach and assume that when managing a fund with finite horizon, the VC can’t actively search for projects to finance before the current one has matured and investors have seen their returns. I will argue that this structure arises in equilibrium due to the incentive that entrepreneurs’ sorting provides.

### 4.1 A Dynamic Setting

Time is divided discretely and continues forever. To make the analysis more accessible, I assume in this section that all fund managers are identical. They are infinitely lived and at each
period in time, indexed by \( t \), maximise the sum of expected fees, with common discount factor \( \delta \in (0, 1) \). Investors are in large supply and are endowed with one unit of money per period. For simplicity, assume now VCs can only find up to one entrepreneur in each round of matching they participate to. It takes two periods for each project, independently on its quality, to develop and produce returns. Returns are again a deterministic function of project’s quality and attention, \( R(a, \lambda) \), which in this section is further assumed to be log-supermodular everywhere. Once a project has produced returns, the match expires. It follows that a VC searching for an entrepreneur can be in either of two states: he can be unmatched, or already be dealing with a project that is currently at its intermediate stage.

**Diseconomies of Scale.** To capture the same quality-quantity trade-off as in the previous sections, I assume VCs attention into a particular project is determined by whether he has dealt with another one in any period of the project’s life span. Denote the levels of \( a \) with and without overlapping projects \( a^h \) and \( a^l \) respectively, with \( a^h > a^l \).

As before, on the other side of the market there are entrepreneurs. Assume at each \( t \) a new generation of entrepreneurs is born. They make the irreversible entry choice, and, conditional on entry, draw a type and direct their search. Those who don’t get matched die. Those who do and get their returns and those who don’t enter the market at all, leave the economy forever. Finally, assume that at any time \( t \) a measure of new VCs enter the market; at random, an equal measure of VCs dies.

### 4.2 Choice of Fund Structure

At the beginning of each period, managers approaching investors can opt for either of two fund structures. In one case, they can choose an open credit line that allows them, at any time \( t \) to have the necessary cash to finance a new project. Alternatively, they can form a closed fund, with a finite, two periods long horizon. In the latter case, a fund consists essentially of a single investment, that matures returns two periods from the initial formation. This structure can be interpreted as a short-term contract between the investor and the VC. One can imagine that an investor writing this type contract will invest its dollar at the intermediate period in the alternative asset. If approached by the VC in the intermediate period, the investor wouldn’t have the liquidity to provide the VC with the money to start a new project. This is realistic: pension funds (representing a large share of investors in venture capital) usually meet capital calls by selling positions in liquid indexes\(^{10}\). Whichever is the interpretation, all that matters is that a short-term contract creates an endogenous commitment not to start a new project before the original has produced its returns.

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\(^{10}\)See Robinson and Sensoy (2016).
Note that, as clarified in Section 2, managers’ choice maximizes the fund’s total returns, because they can set fees so to hold investors to their participation constraint. Therefore a fund manager’s objective is to maximize the discounted sum of expected excess returns.

Summary of the Timing. Let me now summarize the timing of the game:

- At each time \( t \) newborn VCs and pennyless ones approach investors, choose a fund structure, and contract over the fee \( p \) that investors pay them upon realization of each project’s returns. Managers opting for a finite-horizon fund can’t approach investors before the project they are currently financing has produced returns.

- Entrepreneurs observe investors’ strategy and make the entry choice. Those who enter the market privately observe their type \( \lambda \) and direct their search.

- At \( t+1 \) managers who chose a open credit line have the money to search for a new project. Every match formed in \( t \) generates returns \( R(a, \lambda) \) at time \( t + 2 \). By then, \( t \)-generation entrepreneurs who matched leave the market forever.

Assumption A3. \((1 + \delta) R(a^l, \lambda) > R(a^h, \lambda) \quad \forall \lambda \)

The assumption above limits the extent to which a manager’s human capital is destroyed when working on parallel projects. Under A3, keeping quality fixed, it is always optimal to start a new project every period. In turn this means that a manager that expects to attract the same type of projects is always going to search actively for new ones. Effectively, entrepreneurs that are searching for a match will then face the choice between two distinct markets: one where “uncommitted” VCs will provide attention \( a^l \), and the other where attention is at \( a^h \), because agents managing a closed, finite-horizon fund will not be able to search before the original investment has matured.

The Sorting Subgame. I first derive sorting behaviour when a positive measure of vacancies is available in both markets. Normalize the mass of VCs to one and denote \( \gamma_t \) the share of managers running a finite-horizon fund at time \( t \). It is immediate from results in previous sections to observe that equilibrium search behaviour at time \( t \) is then characterized by a threshold \( \lambda^* \) such that entrepreneurs search in the high-attention market if and only if \( \lambda \geq \lambda^* \). This is due to log-supermodularity of \( R(a, \lambda) \). The threshold is implicitly defined by the equation:

\[
\frac{\gamma_t}{1 - \lambda^*} R\left(a^h, \lambda^*\right) = \frac{1 - \gamma_t}{\lambda^*} R\left(a^l, \lambda^*\right)
\]

Lemma 4. The solution to (2) is unique. That is, given a share of managers with finite-horizon funds \( \gamma_t \), there is a unique equilibrium of the sorting subgame.
In principle, VCs might choose different contracts at different times of their lives, and I allow for that. However, for the sake of simplicity, I restrict attention to equilibria where, whenever VCs are indifferent which fund structure to choose, the share of VCs going for either option stays constant. Such equilibria are always possible to construct, provided at a certain time \( t \) agents are indifferent about which contract to choose, thanks to the assumption that new VCs enter the market at each \( t \).

**Definition 6.** A stationary equilibrium is an equilibrium in which the shares of VCs choosing either fund structure is independent on \( t \).

Let me focus on the case when selecting the best entrepreneur is appealing to the VC. Formally, this means imposing the following restriction.

**Assumption A4.** \( R(a^h, \lambda) > (1 + \delta) \mathbb{E}[R(a^l, \lambda)] \)

That is, the returns from following the best entrepreneur exclusively are higher than those from financing two average projects in two subsequent periods. The main result of this section can now be stated.

**Proposition 7. (Equilibrium Fund Structure).** (i). There is no stationary equilibrium where all VC choose the open credit line. (ii). The equilibrium’s measure of VCs with finite-horizon funds, \( \gamma \), is the solution to the equation:

\[
\mathbb{E}[R(a^h, \lambda) \mid \lambda \geq \lambda^*(\gamma)] = (1 + \delta) \mathbb{E}[R(a^l, \lambda) \mid \lambda \leq \lambda^*(\gamma)]
\]

whenever it exists. (iii). When \( \mathbb{E}[R(a^h, \lambda)] > (1 + \delta) R(a^l, \lambda) \), there is a stationary equilibrium where every VC chooses the finite-horizon fund.

Notice that, because of A4, if the function \( \phi(a^h, a^l, \lambda) \) was decreasing in \( \lambda \) - as it is in Example 1 - the condition in (ii) would have no solution, while condition (iii) would always hold. This means that a situation where every VC chooses the finite-horizon fund would be the unique equilibrium.

Similarly to how established in the general static model, the motive to attract better entrepreneurs generates an inefficiency, as VCs don’t internalize the aggregate effect - due to equilibrium sorting - of their choices. In particular, one natural and policy relevant question would be whether allowing these short-term contracts is desirable. It turns out that, under A3, banning the short-term contracts is always beneficial.

**Proposition 8. (Banning finite-horizon funds).** Every equilibrium of the game delivers lower welfare than the case where every VC chooses the open credit line. That is, banning finite-horizon funds improves welfare.
It is easy to see why the “corner” equilibrium where every VC chooses the short-term contract is welfare detrimental. Infact, notice that under A3, the choice of which contract to sign involves a simple trade-off: on the one hand, starting a new project every period allows the VC to make the best use of his human capital, as the dilution in attention is assumed to be small; on the other, committing to an exclusive relation helps the VC attract the best entrepreneurs. However, in equilibrium this commitment confers no benefit at all, since every VC will look alike.

5 Conclusion

I have introduced a matching model of fund management where the two key ingredients are limited attention of fund managers, whose value added decreases with fund size, and directed search from entrepreneurs who are privately informed about their projects’ quality. The two features are inspired by several stylized facts and empirical findings about the venture capital industry. In the model, entrepreneurs trade off matching with better managers - those who opt for relatively smaller funds, and hence can devote more attention to their projects - versus the lower search frictions associated to worse managers. Managers are different in ability to scale up their human capital, and select fund size to maximise total returns. Anticipating that, due to complementarities in the returns function, higher quality entrepreneurs will sort into funds where attention is higher, managers tend to shrink fund size excessively. Since the distribution of fund sizes feeds back into the equilibrium sorting, entry of managers at the bottom of the skills distribution can increase returns at the top. The latter effect, as well as the inefficiency and the emergence of multiple, Pareto-dominated equilibria are a consequence of directed search with unobservable project quality and would not result from a model with random matching or homogenous entrepreneurs. In the second part of the paper, I have used the results from the first part to analyse the equilibrium choice between a closed, finite-horizon fund, versus an open, firm-like investors-manager relationship, in a simple dynamic version of the model. From the managers point of view, finite-horizon funds come at the cost of giving up investment opportunities arriving when the current fund is still ongoing; entrepreneurs value this commitment, because it guarantees exclusive attention. A situation where all managers opt for the open fund unravels, even when this would be the welfare maximising solution, due to the incentive to skim the market and attract the best entrepreneurs, who are the most willing to pay the highest search friction and get exclusive attention. Similarly, one where every managers has the closed fund is sustainable, because a deviation would attract the worse entrepreneurs. This result suggests another reason why venture capitalist raise funds with a finite horizon and with explicit limits on the investments that they can make while the current fund is still ongoing. They might benefit from committing to a size in the first place.
Appendix

Proof of Lemma 1. Assume not and denote \((\tilde{w}, \tilde{x})\) the market where \(\tilde{S}(\tilde{w}, \tilde{x}, E) = 0\). By searching in \((\tilde{w}, \tilde{x})\) an entrepreneur of type \(\lambda\) gets:

\[
R(a_N, \lambda > Q(w, x, E) R(a(x, m(w, x)), \lambda) \quad \forall (w, x)
\]

and hence is better off than searching in any other \((w, x)\). This deviation keeps being profitable until a positive measure of entrepreneurs deviates to \((\tilde{w}, \tilde{x})\), that is, \(\tilde{S}(\tilde{w}, \tilde{x}, E) > 0\). ■

Proof of Lemma 2. Assume not and denote \((\tilde{w}, \tilde{x})\) the market where \(Q(\tilde{w}, \tilde{x}) = 1\). The expected payoff from entry and search in \((\tilde{w}, \tilde{x})\) is given by:

\[
\mathbb{E}R(a(\tilde{x}, m(\tilde{x}, \tilde{w}), E)), \lambda) - c \geq \mathbb{E}R(a_0, \lambda) - c > 0
\]

and hence entrepreneurs whom don’t enter in equilibrium can strictly benefit from a deviation. ■

Proof of Corollary 1. By Lemma 2, returns to type \(\lambda\) in market \((w, x)\) conditional on matching are given by \(R((a(x, w)), \lambda)\). Take two markets \((w, x)\) and \((w', x')\), with associated attention levels \(a\) and \(a'\), with \(a = a'\). Assume that \(Q(w, x, E) > Q(w', x', E)\). Then, any entrepreneur searching in \((w', x')\) could deviate to \((w, x)\) and get:

\[
Q(w, x, E) R(a, \lambda) > Q(w', x', E) R(a', \lambda).
\]

■

Proof of Lemma 3. In the original model entrepreneurs maximize \(Q(w, x, E) R(a(x, w), \lambda)\), and, by Lemma 2, \(Q(w, x, E) = \theta(w, x, E)\). Because \(R(a(x, w), \lambda)\) is constant across an iso-attention locus, and since Lemma 2 must apply to the transformed model, all that remains to show is that market tightness is the same in submarket \(a\) as it is at any point in the iso-attention locus in the original model. That is, formally, \(\theta(a) = \theta(w, x, E) \forall (w, x) : a(x, w) = a\). Call \(\Gamma(a)\) the sum of vacancies across all VCs at a given iso-attention locus. From the definition of \(\theta(a)\), one can write \(\theta(a) = \frac{d\Gamma(a)}{dS(a)}\) with

\[
S(a) := \int_{\tilde{a} \leq a} \int_{\{(w, x) : a(x, w) = \tilde{a}\}} \int_{\lambda} EdS(w, x; \lambda) d\tilde{a}.
\]

Recall that

\[
\Gamma(a) := \int_{\tilde{a} \leq a} \int_{\{(w, x) : a(x, w) = \tilde{a}\}} wdH(w, x) d\tilde{a}.
\]
Therefore,
\[
dΓ (a) = \int_{\{(w,x): a(x,w) = a\}} wdH (w,x)
\]
and
\[
dS (a) = \int_{\lambda} \int_{\{(w,x): a(x,w) = a\}} EdS (w,x;\lambda).
\]
Notice that
\[
d\tilde{S} (w,x,E) \theta (w,x,E) = d\tilde{H} (w,x).
\]
Integrating on both sides over a given iso-attention locus, and taking \(\theta (w,x,E)\) outside of the integral by Corollary 1, it follows that
\[
\theta (w,x,E) \int_{\{(w,x): a(x,w) = \hat{a}\}} d\tilde{S} (w,x,E) = \int_{\{(w,x): a(x,w) = \hat{a}\}} d\tilde{H} (w,x).
\]
Therefore,
\[
\theta (w,x,E) = \left( \int_{\{(w,x): a(x,w) = \hat{a}\}} d\tilde{H} (w,x) / \int_{\{(w,x): a(x,w) = \hat{a}\}} d\tilde{S} (w,x,E) \right) = \theta (a).
\]

Proof of Proposition 1. (Sufficiency). Assume \(R (a,\lambda)\) is logsupermodular everywhere. If there is an equilibrium that does not exhibit PAM everywhere, then there must exist at least two markets \(a_i, a_j\) with \(a_i > a_j\), and two types \(\lambda, \lambda'\) with \(\lambda' > \lambda\) such that \(\lambda \in \Lambda_i\) and \(\lambda' \in \Lambda_j\). Optimality of the search strategy requires that type \(\lambda\) is at least as well off searching in \(a_i\) rather than in \(a_j\) and similarly \(\lambda'\) (weakly) prefers \(a_j\) to \(a_i\), that is:
\[
Q (a_i) R (a_i, \lambda) \geq Q (a_j) R (a_j, \lambda) \quad (3)
\]
\[
Q (a_j) R (a_j, \lambda') \geq Q (a_i) R (a_i, \lambda') \quad (4)
\]
The two inequalities imply
\[
\frac{R (a_i, \lambda)}{R (a_j, \lambda)} \geq \frac{R (a_i, \lambda')}{R (a_j, \lambda')}
\]
which contradicts the fact that \(R (a,\lambda)\) is logsupermodular\(^{11}\).

(Necessity). Assume \(R (a,\lambda)\) is not logsupermodular at some point \((\hat{a}, \hat{\lambda})\). The continuity properties of \(R (a,\lambda)\) (see Section 2) imply that there exists a number \(\varepsilon > 0\), s.t. the function is not logsupermodular anywhere in \([\hat{a} - \varepsilon , \hat{a} + \varepsilon] \times [\hat{\lambda} - \varepsilon , \hat{\lambda} + \varepsilon]\). I construct an economy where

\(^{11}\)Logsupermodularity of \(R (a,\lambda)\) implies that for any \((a, a')\) with \(a' > a\), the ratio \(\frac{R (a',\lambda)}{R (a,\lambda)}\) is strictly increasing in \(\lambda\).
NAM could be supported, hence a contradiction arises. Let $F$ be defined on $[\hat{\lambda} - \varepsilon, \hat{\lambda} + \varepsilon]$, and $a_i \in [\hat{a} - \varepsilon, \hat{a} + \varepsilon]$, for all $i$. By construction, all matches will be in $[\hat{a} - \varepsilon, \hat{a} + \varepsilon] \times [\hat{\lambda} - \varepsilon, \hat{\lambda} + \varepsilon]$. In order for only PAM sorting patterns to emerge, a necessary condition is that for at least two $(\lambda, \lambda')$ with $\lambda > \lambda'$, and two $(a, a')$, with $a > a'$,

$$Q(a) R(a, \lambda) \geq Q(a') R(a', \lambda)$$  \hspace{1cm} (5)$$

$$Q(a') R(a', \lambda') \geq Q(a) R(a, \lambda')$$  \hspace{1cm} (6)$$

and, crucially, at least one of the two inequalities is strict\(^{12}\). When either (5) or (6) or both are satisfied with strict inequality, it holds that

$$\frac{R(a, \lambda)}{R(a', \lambda)} > \frac{R(a, \lambda)}{R(a', \lambda')}$$

which means $R(a, \lambda)$ is logsupermodular somewhere in $[\hat{a} - \varepsilon, \hat{a} + \varepsilon] \times [\hat{\lambda} - \varepsilon, \hat{\lambda} + \varepsilon]$, a contradiction.

**Proof Proposition 2.** By Proposition 1, sorting in the subgame must exhibit PAM. Each point in the sequence that forms the equilibrium partition of the set $[\underline{\lambda}, \bar{\lambda}]$ is a type that must be indifferent between searching in the two markets where types just above and just below are assigned.

Consider a VC with ability $x$ that is choosing between $a_i$ (and associated size, $m_i^x$) and $a_j$ (with associated size $m_j^x$), with $i > j$ and $i, j \in I^*$. He will prefer the first option if and only if:

$$m_i^x E \left[ R(a_i, \lambda) \mid \lambda \in A_i^* \right] > m_j^x E \left[ R(a_j, \lambda) \mid \lambda \in A_j^* \right].$$

Which can be rewritten as:

$$\frac{m_i^x}{m_j^x} > \frac{E \left[ R(a_j, \lambda) \mid \lambda \in A_j^* \right]}{E \left[ R(a_i, \lambda) \mid \lambda \in A_i^* \right].}$$  \hspace{1cm} (7)$$

The right hand side of (7) is independent on $x$. The left hand side is continuous and *increasing* in $x$. Therefore, if (7) holds for some $x$, it will hold for any VC with ability above $x$. Moreover, if the same inequality is reverted for some $x' < x$, then, by the Intermediate Value Theorem, there exist a level of ability $\tilde{x} \in [x', x]$ such that the payoff from $a_i$ and $a_j$ is the same.

Finally, consider a deviation to some $a_j$ with $j \notin I^*$. Condition (iii) from Definition 3 requires that types in $A_j^*$ must be indifferent between searching in $a_j$ and in their assigned equilibrium market, whereas all other types weakly prefer the equilibrium market they are searching in. Take the closest smaller market in $I^*$ to $a_j$, call it $a_i$, and the closest higher

\(^{12}\)Otherwise, it would be possible to support an equilibrium with NAM, and the contradiction would immediately arise.
market in $I^*$ to $a_j$, call it $a_i$. Formally the two are defined as:

$$i := \arg \max_{h \in I^* \setminus \{h \geq j\}} h$$

and

$$\tilde{i} := \arg \min_{h \in I^* \setminus \{h \leq j\}} h.$$

It can be argued that the only type in $\Lambda^*_i$ must be $\lambda_i$, which, since all markets in between $a_i$ and $a_j$ are empty, is also equal to $\lambda_j$. Assume not, and first, assume $Q(a_j)$ is such that some $\lambda \in A_h$ with $h \in I^*$ and $h > j$, is indifferent between $a_h$ and $a_j$. This would mean:

$$Q(a_h) R(a_h, \lambda) = Q(a_j) R(a_j, \lambda)$$

which implies that, $\forall \lambda' \in A_h$ and $\lambda' < \lambda$,

$$Q(a_h) R(a_h, \lambda') < Q(a_j) R(a_j, \lambda').$$

From the fact that $R(a_h, \lambda)/R(a_j, \lambda)$ is increasing in $\lambda$ by logsupermodularity. That is, all types in $A_h$ lower than the indifferent type, would strictly benefit from deviating.

Assume instead some $\lambda \in A_h$ with $h \in I^*$ and $h < j$, is indifferent between $a_h$ and $a_j$. A similar argument applies. In this case, $\forall \lambda' \in A_h$ and $\lambda' > \lambda$,

$$Q(a_h) R(a_h, \lambda') < Q(a_j) R(a_j, \lambda').$$

It remains to show that when $Q(a_j)$ is such that $\lambda_i \in A^*_j$, no contradiction arises. Set $Q(a_j)$ so that $\lambda_i$ is indifferent between deviating to $a_j$ and not. Notice that $\lambda_i$ is indifferent between $a_i$ and $a_\tilde{i}$. That is:

$$Q(a_i) R(a_i, \lambda_i) = Q(a_\tilde{i}) R(a_\tilde{i}, \lambda_i) = Q(a_j) R(a_j, \lambda_i).$$

(8)

The two equalities in (8) imply that all types below $\lambda_i$ strictly prefer $a_i$ to $a_j$ and all types above $\lambda_i$ strictly prefer $a_\tilde{i}$ to $a_j$. Since both $a_i$ and $a_\tilde{i}$ are offered in equilibrium, no type benefits from deviating to $a_j$.

Given what the off-equilibrium deviation attracts, condition (iii) from the Proposition guarantees that no VC offers $a_j$. ■

**Proof Proposition 4.** The proof proceeds in two steps.

(Step 1). Call $I$ the equilibrium exhibiting lower ratio $W_i/W_j$ for any two $(i, j)$ with $i > j$, with $II$ being the other equilibrium. Use the superscripts $I$ and $II$ to denote the limits of the equilibrium partitions $X_i$ and $\Lambda_i$ under equilibrium $I$ and $II$. It can be shown that, for any $i$,
To prove this, assume this is not the case. That is, assume that, for at least some \( i \), \( \lambda_i^I \leq \lambda_i^H \). First focus on the case where the inequality is strict for some \( i \). Take the largest \( i \) such that this holds. Rewrite the indifference condition for the indifferent type, \( \lambda_i \), as:

\[
\frac{R(a_{i+1}, \lambda_i)}{R(a_i, \lambda_i)} = \frac{W_i F(\lambda_{i+1}) - F(\lambda_i)}{W_{i+1} F(\lambda_i) - F(\lambda_{i-1})}.
\]

(9)

Condition (9) has to hold under both equilibrium values \( \lambda_i^I \) and \( \lambda_i^H \). When \( \lambda_i \) is lower, the left hand side decreases (due to logsupermodularity). \( W_i/W_{i+1} \) is higher in equilibrium \( I \) by assumption. Therefore, it must be that \( (F(\lambda_{i+1}) - F(\lambda_i)) / (F(\lambda_i) - F(\lambda_{i-1})) \) is lower under \( I \). The numerator is higher under \( I \), since \( i \) is the largest submarket for which \( \lambda_i^I \leq \lambda_i^H \) (this is also true in case \( i = N - 1 \) and hence \( \lambda_{i+1} = \tilde{\lambda} \)). Therefore, it must be that \( \lambda_{i-1}^I < \lambda_{i-1}^H \), giving a contradiction. It remains to show that it is impossible that \( \lambda_i^I = \lambda_i^H \) for all \( i \). Assume this is the case. This would imply that under the two equilibria, the left hand side of (9) stays constant, as well as the ratio \( (F(\lambda_{i+1}) - F(\lambda_i)) / (F(\lambda_i) - F(\lambda_{i-1})) \). Because \( W_i/W_{i+1} \) is not the same under the two equilibria, the desired contradiction arises.

(Step 2). Given \( \lambda_i^I > \lambda_i^H \), it can be proven that welfare is higher under equilibrium \( I \). Notice that, for any \( i \), average quality is higher under \( I \). That is:

\[
\mathbb{E} \left[ R(a_i, \lambda) \mid \lambda_{i-1}^I \leq \lambda \leq \lambda_{i+1}^I \right] > \mathbb{E} \left[ R(a_i, \lambda) \mid \lambda_{i-1}^H \leq \lambda \leq \lambda_{i+1}^H \right].
\]

Since equilibrium requires that VCs select \( a_i \) to maximise total returns, it must be that each one is strictly better off under \( I \) compared to \( II \). This completes the proof.

Proof of Lemma 4. It is convenient to rewrite (2) as:

\[
\frac{R \left( a^h, \lambda^* \right)}{R \left( a^l, \lambda^* \right)} = \frac{1 - \gamma_t (1 - \lambda^*)}{\gamma_t \lambda^*}
\]

(10)

The left-hand side of (10) is continuous and strictly increasing in \( \lambda^* \) by assumption (as \( R \) is continuous in both arguments and assumed to be log-supermodular in this section). The right hand side is continuous and strictly decreasing in \( \lambda^* \). In particular, notice the left-hand side is positive and finite for all \( \lambda^* \in [0, 1] \). The right hand side is zero as \( \lambda^* \to 1 \) and tends to infinity as \( \lambda^* \to 0 \).

Proof of Proposition 6. I first show that there is no equilibrium where every manager has the open credit line. Take an equilibrium where \( \gamma = 1 \) and consider a manager who deviates to a finite-horizon structure. Out of equilibrium frictions, \( Q \left( a^h \right) \), will be set such that the highest type is indifferent between searching in either market. This means the manager will expect to
attract type $\bar{\lambda}$. This is profitable as long as:

$$R(a^h, \bar{\lambda}) > (1 + \delta) \mathbb{E}[R(a^l, \lambda)]$$

which is true by assumption. ■
References


