Gilded Bubbles*

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Abstract

Excessive credit growth and high asset prices increase systemic risks. Because, in equilibrium, these two variables are jointly determined the analysis of systemic risk and the cost-benefit analysis of macroprudential regulation require a specific framework consistent with the existing empirical evidence. We argue that an overlapping generation model of rational bubbles can explain some of the main features of banking crises, thus providing a microfounded framework for the rigorous analysis of macroprudential policy. We find that credit financed bubbles may have a role as a buffer in channeling excessive credit supply and inefficient investment at the firms’ level. Still, when banks have a risk of going bankrupt a trade-off appears between financial stability and efficiency. When this is the case, macroprudential policy has a key role in improving efficiency while preserving financial stability. Contrarily to common regulatory views, the one size fits all approach where a countercyclical buffer is implemented to counteract an excessive increase in credit growth, the optimal macroprudential policy has to take into account the origin of the shock. As wages, liquidity and productivity shocks determine the equilibrium level of credit, it is not surprising that the optimal macroprudential policy has to adapt to each type of shock.

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“Beautiful credit! The foundation of modern society. Who shall say that this is not the golden age of mutual trust, of unlimited reliance upon human promises?”

–Mark Twain, The Gilded Age

1 Introduction

The recent empirical evidence has confirmed, once more, the existing connections between credit growth, leverage, bubbles and the likelihood of a banking crisis. Intuitively, it is clear that once a bubble starts, it explains both the credit growth and leverage because the bubble creates a demand for credit that is backed by the bubble itself as collateral. Still, to our knowledge, a rigorous modeling of the link between rational bubbles, credit expansion and systemic risk has not been developed yet, in spite of the need to justify macroprudential policy.

There is a need to understand “the critical macroprudential policy tradeoff is between reducing the risks of very costly financial crises and minimizing the costs of macroprudential policies during normal times.” (Benes et al., 2014, p. 4). According to the authors, some progress has been made, but the area requires to focus on the special role played by banks’ balance sheets.

This article builds a simple model that emphasizes the connections between the supply of credit, the equilibrium price of bubbles and systemic risk in the presence of banking. The explicit introduction of financial intermediaries (banks for short) is of interest for three reasons: first, banks allow for levered bubbles; second, banks’ solvency depends upon the behavior of the bubble, and third, a systemic crisis when bubbles are financed by banks may have much more devastating consequences than if bubbles are directly financed by investors, as the anecdotal evidence of the comparison between the dot.com bubble and the housing price bubble of 2007 shows and the empirical analysis of Jordà et al. (2015) rigorously establishes.

There is now a large consensus on the impact of excessive credit growth on financial stability (Jordà et al., 2015; Laeven and Valencia, 2012; Schularick and Taylor, 2012) to the point that Schularick and Taylor (2012) conclude that “credit growth is a powerful predictor of financial crises”. Still, even if the empirical literature points at excessive credit growth as one important correlate to financial crises, it is important to keep in mind that the amount of credit in an economy is endogenously determined, and so the question that arises is: how come there is both a sufficient amount of credit and a correlated amount of positive
net present value projects? One answer is that deregulation and access to wholesale funds suddenly allow to unleash credit and finance projects that were previously unable to obtain funding. The alternative is that the existence of credit may be a necessary condition for asset bubbles to emerge, so that, provided the bubbles expected return is sufficiently high, the demand for the bubble will create a demand for credit. This is an important point, as the supply of credit fuels bubble prices while larger bubbles fuels the demand for credit. So, in this market, the supply of credit creates its own demand.

From an empirical point of view, attributing the credit growth to the existence of bubbles is a complex exercise, as it requires defining a fundamental value and identifying the (positive) residual as the bubble. This will be difficult for real estate prices and may prove impossible for stock markets. It is no wonder that, prior to the crisis, the dominant view was that bubbles, if they ever exist could not be identified. Still, a number of contributions point out the importance of bubbles bursting as a cause of banking crises.\footnote{A first method to identify a bubble consists in detecting housing price deviations of real house prices above some specified threshold relative to trend (Borio and Lowe, 2002; Detken and Smets, 2004; Goodhart and Hofmann, 2008). A second approach focuses on the rate of growth of prices and diagnoses the existence of a bubble by the rate of growth being consistently above some threshold (Bordo and Jeanne, 2002). A third view identifies the bubble by the extent of the peak-trough (Helbling, 2005; Helbling and Terrones, 2003; Claessens et al., 2008). The different approaches can be combined to filter non-bubbles. This is why Jordà et al. (2015), while using the first approach of real increase relative to a trend require also a subsequent price correction larger than 15%. A simpler, completely different strategy in identifying bubbles consists in analyzing the ratio of wealth to GDP, as the numerator, but not the denominator, might reflect the existence of a bubble. This is considered by Carvalho et al. (2012) who motivate their analysis by observing that the ratio of wealth to GDP grows before a crisis.}

It is not surprising that the impact of bubble bursting on banks’ solvency is of paramount importance when these bubbles have been financed by banks’ credit. Using a measure of bubbles that combines the trend of real estate prices with an ulterior price correction, (Jordà et al., 2015, p. S1) show that “when fueled by credit booms, asset price bubbles increase financial crisis risks”. Anundsen et al. (2014) recursively test whether credit and house prices are in a regime characterized by explosive behavior or not, and establish a positive and highly significant effect of exuberant behavior in the housing and credit market on the likelihood of a crisis.

From a theoretical perspective, the analysis of bubbles is particularly appealing because, first, it justifies simultaneously the bubble and the credit boom; second, it relates the bubble to the supply of credit, and, as a consequence, to saving gluts and capital mobility; third, it makes systemic risk endogenous; and last, but not least, it frames the macroprudential policy in a set up where efficiency can be defined, so that the trade-off between economic growth and systemic risk is tractable.

Our research will focus on rational bubbles because this framework allows us to define
the welfare properties of the equilibrium in a straightforward way. As in Martin and Ventura (2012) and Caballero and Krishnamurthy (2006), bubbles are assets that are used as a savings vehicle. In our model, contrarily to Farhi and Tirole (2012) and Martin and Ventura (2016), bubbles are not owned by entrepreneurs but by consumers, as bubbles constitute their best investment opportunity. Consequently, their positive role in resource allocation does not stem from their value in providing additional collateral to credit rationed firms but from preventing inefficient overinvestment; that is, from preserving Solow’s golden rule whereby the efficiency allocation requires the equality between the interest rate and the growth rate.

This paper is not the first to address the issue of rational bubbles in their connection to systemic risk. Indeed, Aoki and Nikolov (2015) consider, as we do, bubbles in a banking economy and their role in generating banking crises. Still, their objective is to compare the impact of household held and bank-held bubbles, and their results show that bank-held bubbles imply a higher systemic risk. As Reinhart and Rogoff (2009) and Jordà et al. (2015) argue, credit-boom-fueled housing price spirals are particularly pernicious. Still, our aim is broader as we would like to explore the mechanism of bubble creation and bursting in connection with credit booms.

The analysis of bubbles require to be specific about the alternative investment vehicles available to savers, as in any overlapping generations (OLG) model the transfer of goods from one generation to the next will be done through these vehicles. Our framework considers three different investment opportunities: acquiring the bubble, depositing in the bank and using the storage technology. Because the storage technology is always available, in equilibrium, the return on deposits and on bubble acquisition should be larger than the return on the storage technology. We show that the existence of equilibria will critically depend on whether the return on the storage technology is positive or negative. In particular, if this return is positive, there will be no bubbles in stationary equilibria. Interestingly, in a fiat money economy, it seems natural to interpret the return on the storage technology as the return on holding cash, which constitutes a second bubble. Our results could then be viewed as the effect of the competition between two bubbles, with the possibility that one is relegated to its transactional role while the other provides a store of value role.

Our concern for bank stability will lead us to consider a set up where the equilibrium allocation depends upon a stochastic element, that could be an endowment, productivity or capital inflows, that will affect both the current allocation as well as future bubble prices. As we will see, some of the intuition for our results can be obtained from the allocation in a riskless steady state economy, but the analysis of bubble bursting and banks’ bankruptcies requires the additional complexity of random shocks. For expositional reasons we take this stochastic shocks to stem not only from productivity shocks but also from the liability side of
banks, that affect their supply of credit. This is a strong simplification, as we would expect the amount of funding a bank is able to attract depends upon the equilibrium interest rate in the economy. Still, we show that endowment and productivity related stochastic shocks could have the same impact, as they both affect interest rates and the overall allocation. Focusing mainly on the liability side has the benefit to simplify the analysis of the impact of capital regulation on the growth rate and riskiness of the resulting allocation. It allows also to consider the role of foreign capital inflows (or sudden stops) as well as Central Banks liquidity injections or withdrawals.

We obtain four regimes, one where households deposit their savings at banks, a second one where households self-finance their purchases of the bubbles, a third one where households borrow from the bank to buy the bubble, and a fourth one where households save in the riskless asset.

We show that, when the storage technology has a negative net return, the efficient equilibrium may be the one characterized by leveraged bubbles. This is the case because, first, when the return on the storage technology is negative, in the absence of a bubble, too low interest rates will lead to excessive capital accumulation. Second, the equilibrium characterized by unlevered bubbles, where the return of the bubble is the same as the return on deposits, is also inefficient, because the spread between deposit and loan rates implies high interest rates and low levels of capital, the opposite of what would happen in the levered economy.

Thus, our results are in line with Jordà et al. (2015) empirical finding and with the conjectures put forward by Mishkin (2008), Mishkin (2009) and other policy-makers after the crisis: bubbles that threaten financial stability are those that are fueled by credit and leverage. More precisely, the coefficient of the interaction between the credit variable and the bubble indicators is highly significant in determining the probability of a crisis, which fits our theoretical model.

Our results coincide with Martin and Ventura (2015) in predicting that bubbles will emerge when interest rates are low and banking liquidity is high. In our framework this results in systemic risk, because the bubbles, even if held by household, will be financed by banks’ credit. Macroprudential policy will then modify the equilibrium as it will change expectations on the future value of the bubble. If macroprudential policy is based on full information and is unconstrained, then it is expectationally robust as in their paper, in the sense that it isolates the economy from both liquidity and productivity shocks.
2 The model

We will consider an overlapping generations economy with households and entrepreneurs that live for two periods. Entrepreneurs and households need financing and banks have a role as providing funds, because of their ability to screen, monitor and enforce contracts (Diamond, 1984). Absent banks, households may provide funding to firms but at a much higher cost that reduces the productivity of the economy. There is a cost of monitoring that implies that, in equilibrium, there is a spread between the bank offered deposit rate, $r_{t+1}^d$ and its lending rate to firms $r_{t+1}^f$ or the households $r_{t+1}^h$.

2.1 Households

We assume that risk neutral households supply one unit of labor when young, and derive utility from their consumption only when old. We denote $c_{i,t}$ the consumption of generation $i$ at time $t$. We make the usual assumption that households receive a labor endowment $N_t$ when young, which will allow them to obtain an equilibrium wage, and receive no endowment when old when they consume. For simplicity we take the labor supply to be inelastic and equal to 1. Consequently, young households at time $t$ will save the equilibrium wage $W_t$ in order to consume when old. They will be able to do this either by depositing in the bank, by buying an asset in fixed supply with no return other than the price at which it will be sold in the future which we refer to as the bubble or by investing in a storage technology. In contrast to Farhi and Tirole (2012) or Martin and Ventura (2016) it is households, not firms, that may invest in the asset that is susceptible of incorporating a bubble. Consequently, any effect that we may capture is completely unrelated to the role of collateral in providing better access to financial market.

The banking system allows households to either borrow an amount $L_t$ at a rate $r_{t+1}^h$ to invest in the bubble or to deposit $D_t$ at the bank with a return of $r_{t+1}^d$. In addition, households can also save by investing $O_t$ in a riskless asset, which offers a return that we denote by $r$. It is important to notice that it is not possible for households to borrow at this rate. We take this rate to be exogenously set.

Consequently, household will have to determine how to allocate their savings, between purchasing $q_t$ units of the bubble at a price $B_t$, depositing in the bank or investing in the riskless asset. The following maximization program (1) corresponds to the problem a household born at $t$ solves.

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\[ \text{If the riskless asset is interpreted as a Treasury bill, only the Treasury can borrow at this rate.} \]
\[
\max_{c_{t,t+1}, q_t, D_t, L_t, O_t} \mathbb{E}_t c_{t,t+1} \tag{1}
\]

\[
s.\ t. \ q_t B_t + D_t + O_t \leq W_t + L_t
\]

\[
c_{t,t+1} \leq \max(0, q_t B_{t+1} - (1 + r_{t+1}^h)L_t + (1 + r_{t+1}^d)D_t + (1 + \tau)O_t)
\]

\[
c_{t,t+1}, q_t, D_t, L_t, O_t \geq 0.
\]

Notice that households are protected by limited liability that ensures \( c_{t,t+1} \geq 0 \).

### 2.2 Entrepreneurs

Each generation of entrepreneurs has a production technology and no other endowment. Similar to households, generation \( t \) of entrepreneurs lives at \( t \) and \( t+1 \) and consumes only in \( t+1 \).

The production process takes one period and, given a productivity level \( A \), allows to produce an output \( Y_{t+1} \) out of its inputs in labor and capital \((N_{t+1}, K_t)\). \( K_t \) is borrowed at \( t \), then labor is hired from generation \( t+1 \) and is paid out of the product \( Y_{t+1} \), without requiring additional borrowing. The production process is simplified as we assume it is riskless, with an exogenous level of labor productivity, \( \gamma > 1 \); i.e. \( Y_{t+1} = F(A, \gamma^t N_{t+1}, K_t) \), where \( F \) satisfies the Inada conditions, and, consequently, the standard result whereby the marginal product of an input equals its cost applies. The equilibrium in the labor market will therefore determine the wage \( W_{t+1} \) at time \( t+1 \) as a function of the previous period capital \( K_t \) and the previous productivity shock, \( A \). Because entrepreneurs live only two periods they will fully consume their profits.

The problem entrepreneurs solve is the following:

\[
\max_{c_{t,t+1}, N_{t+1}, K_t} \mathbb{E}_t c_{t,t+1}^E \tag{2}
\]

\[
c_{t,t+1}^E \leq F(A, \gamma^t N_{t+1}, K_t) - W_{t+1} N_{t+1} - (1 + r_{t+1}^f)K_t
\]

\[
K_t \geq 0, N_t \geq 0, c_{t,t+1}^E \geq 0.
\]

We assume, for the sake of simplification, that capital fully depreciates in production, that \( F(\cdot, \cdot) \) is homogeneous of degree one and that the productivity of labor grows at a constant rate of \( \gamma \). The homogeneity property allows to rewrite the firms profits as:

\[\text{In order to guarantee that entrepreneurs always have a positive consumption, we assume that the distri-}\]
\[ \pi^F_{t+1} = \gamma^t N_{t+1} F \left( A, 1, \frac{K_t}{\gamma^t N_{t+1}} \right) - \frac{W_{t+1}}{\gamma^t} (\gamma^t N_{t+1}) - (1 + r^f_{t+1}) (\gamma^t N_{t+1}) \frac{K_t}{\gamma^t N_{t+1}}. \]

In a Solow economy, with a supply of labor equal to 1, the output will grow at the rate \( \gamma \), and the capital to labor ratio will decrease at the rate \( \gamma \), so that it will be constant in term of efficient units, \( \frac{K_t}{\gamma^t} \).

In what follows we will assume \( \gamma = 1 \) for the sake of exposition and analyze the stationary steady state as our reference, but the reinterpretation of the different variables as being measured in efficient units imply that our results stand for any \( \gamma \)-homothetic economy with a rate of growth \( \gamma \).

### 2.3 Bankers

The role of banks in the economy is to screen, monitor and enforce payment on loans. This entails a cost that implies a spread between deposit and loan rates. We simplify the analysis and assume there exists a representative bank that lives for as long as it is solvent. It starts with equity \( E_0 \) and has equity \( E_t \) at time \( t \). It lends to households that buy the bubble and to entrepreneurs that borrow in order to invest in their firm. The bank obtains an exogenous amount of deposits \( S_t \) that could be interpreted as foreign investment in the country or central bank injection. These funds that depend on the overall financial structure of the economy, as they are related to the amount of private liquidity and of existing financial innovations, are remunerated at the same rate as the riskless asset, \( r \).

The variable \( S_t \) will play a key role in the determination of the equilibrium as it is the basic determinant of banks’ supply of credit. From that perspective, our reference to liquidity determined by exogenous foreign investment is a somewhat restrictive interpretation. Indeed, the amount of credit in the economy may also depend upon the type of collateral that banks accept in their credit operations. Buying stock on margin, as it was common before the 29 crisis, does increase banks supply of credit. Credit default swaps that decrease the regulatory assessment of banks’ credit risk and allow a higher banks’ leverage, also increase the credit supply. Finally, securitization will also allow for an increase in credit supply. The introduction of a stock or bond market that finance a fraction of firms’ investment would not affect the equilibrium, as it implies both a reduction of the demand for loans and the supply of savings.

\textsuperscript{8} button of shocks in the economy has bounded support and that entrepreneurs have a constant endowment each period that is enough to guarantee that \( cE > 0 \).
The equality of assets and liabilities implies

\[ K_t + L_t + O_t^B = E_t + S_t + D_t \]  \hspace{1cm} (3)  

where \( O_t^B \) is the bank’s investment in the riskless asset and \( E_t \) is the bank’s equity before the \( t \) period profits \( \Pi_t \) are realized.

For simplicity we take the bank intermediation costs as a proportion \( 1 - \phi \) of the banks assets, as well as a fraction of the income derived from assets. We assume there is no intermediation associated to investing in the riskless asset.

The bank’s profits \( \Pi_t \), that are generated by its operations at time \( t \) and accrue at the beginning of the next period, once \( B_{t+1} \) is known, are determined by:

\[
\Pi_t = \min \left\{ B_{t+1}, (1 + r_{t+1}^h)L_t \right\} - L_t + r_{t+1}^f K_t + r(O_t^B - S_t) - r_{t+1}^d D_t \\
- (1 - \phi) \left[ (1 + r_{t+1}^f)K_t + (1 + r_{t+1}^h)L_t \right]  \hspace{1cm} (4)  
\]

and the banks participation constraint will be that the expected return on equity is at least equal to the deposit rate.

We will assume banks are price takers and, as a consequence, they will offer any quantity of loans or deposits at the market rates. For the sake of simplicity we also assume depositors are insured at a zero insurance premium and that the random supply of liquidity \( S_t \) is inelastic.

The three interest rates the bank faces are related. First, the bank will lend both to households and firms provided the expected return on the two types of loans the bank offers is the same. This implies that in any equilibrium where credit to households and firms is non zero, we have

\[
(1 + r_{t+1}^f)B_t = \Pr \left( B_{t+1} \geq L_t(1 + r_{t+1}^h) \mid L_t > 0 \right) (1 + r_{t+1}^h)B_t \\
+ \Pr \left( B_{t+1} < L_t(1 + r_{t+1}^h) \mid L_t > 0 \right) \mathbb{E} \left( B_{t+1} \mid B_{t+1} < L_t(1 + r_{t+1}^h) \text{ and } L_t > 0 \right).  \hspace{1cm} (5)  
\]

Equality (5) relates \( r_{t+1}^f \) to \( r_{t+1}^h \) and, as we will see, will allow to use only \( r_{t+1}^f \). Notice that probabilities are conditional on \( L_t > 0 \), since otherwise \( r_{t+1}^h \) is not defined.

Second, the relationship between \( r_{t+1}^f \) and \( r_{t+1}^d \) is given by the cost of financial intermediation, and, for the sake of simplicity, we will assume it is linear:

\[
\phi \left( 1 + r_{t+1}^f \right) = 1 + r_{t+1}^d, \hspace{0.5cm} \phi \in (0,1)  \hspace{1cm} (6)  
\]
This implies that $r^d_{t+1} < r^h_{t+1}$, so households will never borrow and deposit at the same time.

The process for equity will determine whether the bank is solvent or bankrupt. It will depend upon the previous period’s profits, that are only realized once the price of the bubble $B_t$ is obtained. If no dividend is distributed, then:

$$E_{t+1} = \max \{0, E_t + \Pi_t\}.$$  \hspace{1cm} (7)

This means that the bank will lend on the basis of the equity it has before profits, $E_t$, and will only then incorporate its profits.\footnote{Interestingly, if we consider the alternative formulation where banks lend before knowing their own capital and then find the required liquidity in the market, the supply of bank credit will determine the price of the bubble and make banks’ profits indeterminate. This is in line with the joint determination of bank credit and bubbles’ prices. Still, it means pushing the hypothesis of a representative bank too far, as the bank will then determine its own solvency.}

The bank goes bankrupt at the end of time $t$ whenever its losses are larger than its equity: $-\Pi_t > E_t$. The profit equation (4) shows that when banks’ loans are repaid, profits are obviously positive, since $r^h_{t+1} > r^f_{t+1} > r^d_{t+1}$ implies $\Pi_t > E_t r^d_{t+1}$. In our simplified context, only the bursting of a bubble triggers a bank bankruptcy, so that only systemic risk matters. A bank will go bankrupt whenever

$$\Pi_t = B_{t+1} - (1 + (1 - \varphi)(1 + r^h))L_t + \left(\varphi(1 + r^f_{t+1}) - 1 \right)K_t + \varphi(O^B_t - S_t) - r^d_{t+1}D_t < -E_t,$$

so the necessary condition, as intuition suggests is that the bank gets huge losses on its bubble financing, $B_{t+1} - L_t$.

2.4 Equilibrium

For any given distribution on $(S_t, A_t)$, an equilibrium is characterized by a path

$$\{r^f_{t+1}, r^h_{t+1}, r^d_{t+1}, q_t, N_t, W_t, K_t, L_t, D_t, B_t, E_t, q_t, c_{t-1,t}, c^E_{t-1,t}\}.$$

This is simplified as, at each period $t$, the supply of bubbles and labor is fixed, so that $q_t = 1$ and $N_t = 1$ and the variables $E_t, c_{t-1,t}, c^E_{t-1,t}$ and $W_t$, are determined by the decisions from the previous period, so we are left with six key variables, $(r^f_{t+1}, K_t, L_t, D_t, B_t)$, and $r^h_{t+1}$ and $r^d_{t+1}$ deriving from $r^f_{t+1}$ through the cost of financial intermediation and credit risk.

Because of the existence of three different regimes, the equilibrium should also specify under what circumstances a regime switch occurs. In our context it seems natural to hypothesize the hysteresis of decisions so that a regime switch will only occur when driven by changes in parameters that forces it out of its trajectory. We will therefore assume that the
probability of staying in the same equilibrium is either 1 if the equilibrium is feasible or 0 otherwise. This means that the determination of the equilibrium and the expectations have to be solved simultaneously, which implies an additional complexity.

The equivalent of the classical multiplicity of equilibria comes here from the multiplicity of distributions on \((S_t, \mathcal{A}_t)\). Any equilibrium in a riskless bubbly economy would be supported by a single point \((S^*, \mathcal{A}^*)\).

The bubble dynamics is here given by the state of nature \((S_t, \mathcal{A}_t)\), that appear as external shocks in terms of systemic risk. Liquidity and productivity determine both the price of bubbles and the interest rate that will determine the amount of capital and, therefore, the productive efficiency of the economy. This is in contrast for instance with Martin and Ventura (2016), where interest rate is determined either by marginal productivity or by the amount of collateral. From our set up it is impossible to state that many bubbles are instigated by cheap credit (Reinhart and Rogoff, 2009).

3 Equilibrium properties

We will consider here the laws of motion of the different variables and abstract from the issue of banks’ dividend distribution and capital accumulation that will be discussed later on. This allow us to consider the bank’s supply of funds, which equals the sum of external funds deposited, \(S_t\), and the bank accumulated equity, \(E_t\), as the realization of a random variable whose distribution is known. We will denote this liquidity by \(S_t + E_t = \mathcal{L}_t\), so that the state of nature\(^5\) that defines the equilibrium will be \((\mathcal{A}_t, \mathcal{L}_t)\).\(^6\)

To begin with, in equilibrium, the demand for the bubble is derived from the first order condition of (1) with respect to \(q_t\), which implies that the expected return from buying the bubble equals the interest rate (a condition sometimes referred to as “no arbitrage”, a term that does not apply stricto sensu in our framework).

In our framework, because \(r^h > r^d\) and the fact that households cannot deposit at \(r^h\) or borrow at \(r^d\), households have to be on the borrowing or on the deposit side. Therefore the expected return on the bubble may equal either the lending rate or the deposit rate. This gives rise to, at least, two different regimes. In addition, as there is always the option of investing in the riskless asset, in some cases the deposit rate will equal the riskless rate. Finally, given that \(\varphi \in (0, 1)\), there might be states where households find \(r^f\) too high to

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\(^5\)Introducing an endowment shock as part of the state of nature is an obvious extension.

\(^6\)This could be justified if the bank management chooses to keep a level of equity that depends upon its assets, \(E_t = \phi(K_t + \mathcal{L}_t)\), in which case it is determined, under conditions that suffice to have unicity, as the solution to \(E_t = \phi(S_t + E_t)\). Thus, for instance, if the equity is set as the sum of a minimum fixed level \(\bar{E}\) plus a proportion \(\alpha\) of the assets, then \(E_t = \frac{1}{1-\alpha}(\alpha S_t + \bar{E})\).
borrow, but \( r^d \) too low to deposit.

As a consequence, four different regimes appear: an unlevered \( U \)–regime, where investors are indifferent between buying the bubble or depositing, a self financing regime (\( SF \)-regime) where household buy the bubble and neither borrow nor deposit, a levered \( L \)–regime, where households borrow to buy the bubble, and a \( r \) regime, where the interest rate on deposits is determined by the return on the riskless asset and depositors are indifferent between depositing in the bank or investing in the riskless asset.\(^7\) Denoting the expected future price of the bubble by \( \mathbb{E}^U_{t+1} = \mathbb{E}(B_{t+1} \mid U) \), by \( \mathbb{E}^{SF}_{t+1} = \mathbb{E}(B_{t+1} \mid SF) \), by \( \mathbb{E}^L_{t+1} = \mathbb{E}(B_{t+1} \mid L) \), and by \( \mathbb{E}^{r}_{t+1} = \mathbb{E}(B_{t+1} \mid r) \), requires that \( \frac{\mathbb{E}^r_{t+1}}{B_t} = 1 + r^d_{t+1} \) and \( \frac{\mathbb{E}^U_{t+1}}{B_t} = 1 + r^d_{t+1} \), in the \( r \) and \( U \) regimes respectively an equivalent expression, though complicated by the household limited liability,

\[
\mathbb{E} \left( B_{t+1} \mid B_{t+1} \geq L_t(1 + r^h_{t+1}) \right) = (1 + r^h_{t+1})B_t
\]

for the \( L \)–regime and the following to hold for the \( SF \)-regime:\(^8\)

\[
1 + r^d_{t+1} > \frac{\mathbb{E}^{SF}_{t+1}}{B_t} > 1 + r^d_{t+1}
\]

This condition is quite similar to the “no-arbitrage” condition obtained in Farhi and Tirole (2012) as well as in Martin and Ventura (2016). Still, there is a critical difference, in that market segmentation, implying that households cannot borrow either at the deposit rate nor at the Treasury rate makes this condition simply the result of households’ maximization. It is therefore a marginal condition and in more general frameworks, e.g. with heterogeneous household some of which have a utility for the bubble, the only requirement is that the first order condition holds for the **marginal investor**.

It is easy to show that, in equilibrium, the above expression simplifies to \( \mathbb{E}^L_{t+1} = (1 + r^f_{t+1})B_t \), simply by multiplying by \( \Pr(B_{t+1} \mid B_{t+1} \geq L_t(1 + r^h_{t+1})) \) and using (5). In the \( SF \) regime,

\[
1 + r^d_{t+1} > \frac{\mathbb{E}^{SF}_{t+1}}{B_t} > 1 + r^d_{t+1}.
\]

Consequently we can establish the characterization of the four different regimes:

In the \( L \)–regime the laws of motion are defined by the following equations:

\(^7\)Notice that balance sheets shrink when we go from the \( L \)–regime to the \( r \) regime. This is in line with what happened in the US from 2001 to 2006 where banks’ balance sheet expanded but shrank afterwards (Acharya and Naqvi, 2012).

\(^8\)An unexpected subsidy on interest rate resulting from a more favorable taxation or to cross-subsidization due to banking competition to attract clients will result in a higher price for the bubble. If the subsidy is permanent, it will then affect the expected value of the bubble as well as the current value.
\[ E_{t+1}^L = B_t(1 + r_t^f) \]  
\[ K_t + L_t = L_t \]  
\[ B_t = W_t + L_t \]  
\[ D_t = 0 \]  
\[ \frac{\partial F(A, N_{t+1}, K_t)}{\partial K_t} = 1 + r_{t+1}^f \]  
\[ \frac{\partial F(A, N_t, K_{t-1})}{\partial N_t} = W_t \]  
\[ N_t = N_{t+1} = 1, \]

provided \( L_t \geq 0, \) \( r_{t+1}^d \geq r. \)

In the \( SF \)–regime the laws of motion are defined by the following equations:

\[ 1 + r_{t+1}^d > \frac{E_{t+1}^{SF}}{B_t} > 1 + r_{t+1}^d \]  
\[ K_t = L_t \]  
\[ B_t = W_t \]  
\[ D_t = L_t = 0 \]  
\[ \frac{\partial F(A, N_{t+1}, K_t)}{\partial K_t} = 1 + r_{t+1}^f \]  
\[ \frac{\partial F(A, N_t, K_{t-1})}{\partial N_t} = W_t \]  
\[ N_t = N_{t+1} = 1. \]

In the \( U \)–regime the laws of motion are similar to the previous case, but now the no arbitrage condition refers to the deposit rate and \( L_t \) becomes \( -D_t. \) The equilibrium is characterized by the following equations:
\[ \mathcal{E}_{t+1}^U = B_t(1 + r_t^d) \]  
\[ K_t = L_t + D_t \]  
\[ B_t + D_t = W_t \]  
\[ L_t = 0 \]  
\[ \frac{\partial F(A, N_{t+1}, K_t)}{\partial K_t} = 1 + r_{t+1}^f \]  
\[ \frac{\partial F(A, N_t, K_{t-1})}{\partial N_t} = W_t \]  
\[ N_t = N_{t+1} = 1, \]

provided \( D_t \geq 0, r_{t+1}^d \geq r^f. \)

Finally, in the \( r^- \) regime the laws of motion are similar but now \( \varphi(1 + r_{t+1}^f) = 1 + r \). This immediately determines now the no arbitrage condition refers to the loan rate. The equilibrium is characterized by the following equations:

\[ \mathcal{E}_{t+1}^r = B(r)(1 + r_{t+1}^f) \]  
\[ K_t + L_t + O_t^B = L_t \]  
\[ B(r) = W_t + L_t \]  
\[ D_t = 0 \]  
\[ \frac{\partial F(A, N_{t+1}, K_t)}{\partial K_t} = 1 + r_{t+1}^f \]  
\[ \frac{\partial F(A, N_t, K_{t-1})}{\partial N_t} = W_t \]  
\[ N_t = N_{t+1} = 1, \]

provided \( L_t \geq 0. \)

Notice that a bubbleless equilibrium exists with \( B_t = 0 \) and households investing in the riskless asset. Such an equilibrium will imply \( B_{t+1} = 0 \), as otherwise there would be a demand for the bubble and a positive price. The equilibrium will then be akin to the one of the \( r^- \) regime.

\[ ^9 \text{In the case where the bubble is in real estate, a more cumbersome yet realistic model can be built, by assuming two classes of consumers, residential and financiers, with the former enjoying some utility from real estate while the later not. This would imply that the price of real estate is larger than } W_t \text{ even in the } U \text{-regime. Still, the no arbitrage condition holds only for the deposit rates.} \]
3.1 The stationary case

We now prove that an equilibrium exists when \( E^U_{t+1}, E^L_{t+1}, E^r_{t+1} \) and \( E^{SF}_{t+1} \) are time invariant assuming only liquidity shocks and then we show there is a general equivalence property between liquidity and productivity shocks. To prove existence in the stationary case we first show properties that \( B \) and \( r^f \) satisfy in equilibrium (Lemma 1). Then we derive thresholds on \( L \) for which the different regimes exist (Lemma 2). We use the results in these lemmas to prove the existence of an equilibrium in Proposition 1.

**Lemma 1.** Let \( B^L(L, A; E^L) \) and \( B^U(L, A; E^U) \) denote the equilibrium price of the bubble for the \( L \) and \( U \) regimes for given values of \( E^L \) and \( E^U \) and a given realization of \( (L, A) \). Similarly let \( r^L(L, A; E^L) \) and \( r^U(L, A; E^U) \) denote the equilibrium interest rates that the entrepreneurs pay to borrow in these regimes. Then

\[
\begin{align*}
\frac{\partial r^L}{\partial L} &< 0; \quad \frac{\partial r^L}{\partial A} > 0; \quad \frac{\partial B^L}{\partial L} > 0; \quad \frac{\partial B^L}{\partial A} < 0 \\
\frac{\partial r^U}{\partial L} &< 0; \quad \frac{\partial r^L}{\partial A} > 0; \quad \frac{\partial B^U}{\partial L} > 0; \quad \frac{\partial B^U}{\partial A} < 0
\end{align*}
\]

*Proof.* See Appendix A.

The lemma allows to envision how bubbles evolve as a function of the state of nature \( S = (L, A) \). Higher liquidity implies a lower interest rate and a higher equilibrium value for the bubble. Productivity shocks have the opposite effect: they increase interest rates and decrease the value of the bubble. When liquidity is fixed, the buffer effect of the bubble comes into play: an increase in productivity liberates capital from the investment in the bubble through its price decrease to accommodate the firms’ higher demand for capital.

In addition, the lemma shows that systemic risk is high when in the economy high liquidity combines with low productivity, a point of interest for macroprudential policy.

**Lemma 2.** There exist \( L^U, L^L, L^{L-r} \) such that (i) for \( L \leq L^U \) a U-regime equilibrium exists; (ii) for \( L \in (L^U, L^L] \) an SF-regime equilibrium exists; (iii) for \( L \in (L^L, L^{L-r}] \) an L-regime equilibrium exists; (iv) for \( L > L^{L-r} \) an \( r^- \)-regime equilibrium exists.

*Proof.* See Appendix A.

**Proposition 1.** Let \( G \) denote the probability distribution of \( L \). Assume that the support of \( G \) is \([0, \bar{L}]\) for \( \bar{L} < \infty \). Then an equilibrium exists in this economy.

*Proof.* See Appendix A.
3.2 The steady state riskless benchmark

The steady state certainty case provides an interesting benchmark to examine some of the characteristics of the equilibrium, even if in the certainty steady state the economy will stay indefinitely in one of the three regimes. These properties will match the welfare properties of the equilibrium in the stochastic OLG equilibrium.

Because the price of the bubble is constant, this implies that, in a bubbly regime, the return on holding the bubble is zero. Still in an \(L\)-regime, the interest rate on the bubble is the lending interest rate, so that \(r^f = 0\), and the golden rule of equality between interest and growth rates holds. On the other hand, because in the \(U\)-regime the no-arbitrage condition holds for a deposit interest rate, \(r^d = 0\), implying firms face a lending rate of \(r^f = \frac{1}{\varphi} - 1 > 0\) that is higher than the growth rate. In the \(SF\)-regime, since the bubbles is self-financed, it must be the case that \(r^f > 0 > r^d\). Finally in the \(r\) case, the interest rate on loans is reduced to \((1 + r)/\varphi - 1\) and if the bank has excess liquidity it invests itself in the riskless asset.

3.3 The Equivalent Shocks Property

So far our equilibrium depends exclusively upon the realization of the capital inflow/outflow shock. Still, it is easy to see that productivity shocks could play the same role, as stated and proven in the following proposition.

**Proposition 2.** Consider a production function that is subject to a productivity shock, \(\mathcal{A}\). Then for every \(L\) there is an \(\mathcal{A}\) such that the economy attains the same equilibrium.

**Proof.** See Appendix A.

This equivalence will have interesting implications regarding the design of macroprudential policy.

3.4 Welfare Properties

In defining welfare we have to take into account the welfare of the current as well as all future generations and, in particular, the risks they inherit of a bubble bursting. Because each generation consumes only when old, we cannot discount the different utilities, so we assume, as, for instance Allen and Gale (1997), a planner that wants to maximize the long run average of the expected utilities of the different generations, given the starting capital \(K_0\). This corresponds to the ex ante preferences of agents under the veil of ignorance whereby agents neither know ex ante whether they will be consumers, entrepreneurs or bankers and
at what time they will be born. At any point $t$, expectations are taken given the probability distribution of $(A_t, L_t)$ conditional on all information relative to $t - 1$.

$$V(K_0) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \left[ \mathbb{E}_t c_{t,t+1} + \mathbb{E}_t c_{t,t+1}^E + \mathbb{E}_t c_{t,t+1}^B \mid K_0 \right].$$

The utilitarian solution combined with risk neutrality allows to simplify the problem. This is the case because the welfare of each generation is the sum of the consumption of households, entrepreneurs and bankers, allowing us to disregard redistribution between the three classes of agents.

At any time $t + 1$, because repayment of loans and deposits are transfers between the bank and the agents, the aggregate goods available, $Y_{t+1} + S_{t+1}$, are either consumed ($c_{t,t+1}^E + c_{t,t+1} + c_{t,t+1}^B$), invested in the production function, $K_{t+1}$ as capital or used to repay capital and interest to foreign investors, $(1 + r)S_t$. Consequently, we have:

$$c_{t,t+1} + c_{t,t+1}^E + c_{t,t+1}^B = Y_{t+1} - ((1 + r)S_t - S_{t+1}) - K_{t+1}. \quad (12)$$

only occur when the bubble burst and the bank equity is not sufficient to absorb the losses, $\delta = \Pr(E_{t+1} < 0)$. We assume that, when a banking crisis occurs (i.e. when $E_{t+1} < 0$), banks’ assets are put to a different use and the banks’ balance sheet shrinks. In order to model the cost of a banking crises we make the simplifying assumption that a crisis implies a loss in output, denoted $\Delta Y$, during the period where it occurs. Laeven and Valencia (2012) offer an estimation of $\Delta Y$: they calculate the decrease in GDP due to banking crises from 1970 to 2011 across the world. On average this number reaches 23% of GDP.

Consequently, the ex ante welfare associated to a random consumption $(c_{t,t+1}, c_{t,t+1}^E, c_{t,t+1})$ is equivalent, under our utilitarian assumptions, to the maximization of

$$\mathbb{E} [Y_{t+1} - ((1 + r)S_t - S_{t+1}) - K_{t+1}],$$

with $Y_{t+1} = F(N_{t+1}, K_t)$ when banks are solvent and $Y_{t+1} = F(N_{t+1}, K_t) - \Delta Y$ in a banking crisis. Denote by $\tilde{\Delta Y}$ the random variable that takes the value 0 in the absence of a crisis and $\Delta Y$ when there is a crisis, then the equilibrium in each period will be characterized by the realization of a random variable $\tilde{S}_t$, which, in turn will determine a realization for the
equilibrium value of capital, \( \widetilde{K}_t \).

Consequently, if we define \( W_T(K_0) \) as

\[
W_T(K_0) = \sum_{t=0}^{T-1} \mathbb{E} \left[ F(A_t, N_{t+1}, \widetilde{K}_t) - \widetilde{K}_t - \Delta Y_t - (1 + \tau) \widetilde{S}_t + \widetilde{S}_{t+1} \right],
\]

the welfare function can be rewritten as:

\[
V(K_0) = \lim_{T \to \infty} \frac{1}{T} W_T(K_0).
\]

Reordering the summations in \( W_T(K_0) \), we obtain

\[
W_T(K_0) = F(A_t, N_1, K_0) - (1 + \tau) S_0 + \\
+ \sum_{t=1}^{T-1} \mathbb{E} \left[ F(A_t, N_{t+1}, \widetilde{K}_t) - \widetilde{K}_t - \Delta Y_{t+1} - \tau \widetilde{S}_t \right] + \mathbb{E} \left[ \widetilde{S}_{t+1} \right].
\]

This expression is interesting as it reflects the trade-offs between optimal capital allocation that would equal the marginal product of capital to its opportunity cost and the systemic risk it generates. With this expression, it is then possible to assess the impact of the existence of bubbles on the equilibrium, which depends upon the regime.

**Proposition 3.** When the economy is in the \( U \)-regime, bubbles have a negative impact on efficiency. When it is in the \( L \)-regime, bubbles have a positive impact on productive efficiency, provided the risk on interest rates is not too high.

**Proof.** In a \( U \)-regime there is crowding out because of the bubble while interest rates are high. In an \( L \)-regime we have \( \mathbb{E}(\frac{1}{1+\tau}) = 1 \) in the presence of bubbles, with \( r^f > r^c \), so that the expected interest rate \( \mathbb{E}(r) = 1 + \zeta \) where \( \zeta \) is the risk premium that vanishes when the dispersion of interest rates tends towards zero.

\[
\square
\]

4 **Macroprudential policy**

Although the analysis of macroprudential policy has to address both its time and cross section dimension, the modeling of a bubbly economy is intended to provide insights only on the endogenous building of risk and therefore focuses exclusively on the time dimension. If anything, the cross sectional dimension is reflected in the cost of a banking crisis (\( \Delta Y \)), as the banking industry characteristics, such as the level of interbank connections, complexity and the overall contagion will affect this cost (Allen and Babus, 2009; Shin, 2010).
In our framework, a bank’s bankruptcy can only occur in the \( L \)-regime. This will be the case only when the bubble burst and the equity buffer is insufficient to cope with the losses it implies for the bank.

A macroprudential policy \( m(\cdot) \) will be a function of a vector of observable variables \( \mathbf{X}(\mathcal{A}, \mathcal{L}) \) that affects the equilibrium outcome, so that the equilibrium \( \Theta \) is a function \( \Theta(\mathcal{A}, \mathcal{L}, m(\mathbf{X}(\mathcal{A}, \mathcal{L}))) \). We take \( m(\cdot) \) to be a real function.\(^{13}\)

It is important to remark that \( m(\mathbf{X}(\mathcal{A}, \mathcal{L})) \) affects the current outcome also through the change in expectations, and in particular through the change in the expected future value of the bubble.

Our analysis of the welfare properties of the equilibrium allows to determine the general characteristics of the optimal macroprudential policy, as it is the solution to the maximization of \( W_t(K_0) \). This is simplified because the only variable that matters for the measure of welfare is the equilibrium \( K_t \) that will depend upon the state of nature, \((\mathcal{A}, \mathcal{L})\). A macroprudential policy should maximize

\[
W_T(K_0) = F(A_t, N_{t+1}, K_0) - ((1 + r_d^t)S_0 + \sum_{t=1}^{T-1} \mathbb{E}(F(A_t, N_{t+1}, \tilde{K}_t) - \tilde{K}_t - \Delta \bar{Y} + r\tilde{S}_t)) \]

where

\[
\mathbb{E} \left[ (F(A_t, N_{t+1}, \tilde{K}_t) - \tilde{K}_t - \Delta \bar{Y} + r\tilde{S}_t) \right] = \int_0^\infty \int_0^\infty [F(A_t, N_{t+1}, K_t(m_t, \mathcal{A}, \mathcal{L})) - K_t(m_t, \mathcal{A}, \mathcal{L}) - \Delta Y(m_t, \mathcal{A}, \mathcal{L})]
\]

To illustrate the problem, consider a macroprudential policy that can be measured by a single parameter, \( m_t \).\(^{14}\) Then,

\[
\max_{m_t} \int_0^\infty \int_0^\infty (F(A_t, N_{t+1}, K_t(m_t, \mathcal{A}, \mathcal{L})) - K_t(m_t, \mathcal{A}, \mathcal{L}) - \Delta Y(m_t, \mathcal{A}, \mathcal{L})
\]

\[
+ \bar{L}(\mathcal{L} - E_t - 1))d\mathcal{H}_t(A | \mathcal{L})d\mathcal{G}_t(\mathcal{L}) - \int_0^\infty \Delta Y(\mathcal{L}, A)d\mathcal{H}_t(A | \mathcal{L})d\mathcal{G}_t(\mathcal{L})
\]

To better characterize the optimal macroprudential policy, assume \( K_t(m_t, A, L) \) and \( \Gamma(m_t) \)

\(^{13}\)Thus, for example a loan to value rule could be defined by \( m(\mathbf{X}(\mathcal{A}, \mathcal{L})) = 0 \) if the threshold is not reached and \( m(\mathbf{X}(\mathcal{A}, \mathcal{L})) = 1 \) if the threshold is reached, in which case it affects the equilibrium.

\(^{14}\)Alternatively, and more generally, a value function for \( K_T(\mathcal{L}) \) can be introduced.
are differentiable. The first order condition is then:

\[ \int_0^\infty \left[ \frac{\partial}{\partial K} F(A_t, N_{t+1}, K_t(m_t, A, L)) - 1 \right] \frac{\partial K(m_t, L)}{\partial m} dG_t(L) = -\Delta Y(\Gamma(m_t))G_t(\Gamma(m_t)) \frac{\partial \Gamma(m_t)}{\partial m_t}. \] (14)

The optimal policy implies therefore is a tradeoff between the impact of the macroprudential policy on the expected marginal product, that corresponds to the first term,

\[ \int_0^\infty \left[ \frac{\partial}{\partial K} F(A_t, N_{t+1}, K_t(m_t, A, L)) - 1 \right] \frac{\partial K(m_t, L)}{\partial m} dG_t(L) \]

and the marginal impact on systemic risk, \(-\Delta Y(\Gamma(m_t))G_t(\Gamma(m_t)) \frac{\partial \Gamma(m_t)}{\partial m_t}\). When the latter is non-zero, the existence of systemic risk will have a cost in terms of expected product. In other words, macroprudential policy may be costly in terms of productivity.

Still, contrarily to the well-received view that posits macroprudential policy as a social cost, it may be the case that the optimal policy implies no systemic risk. This will be the case when the regulator has perfect information and is able to freely act upon it, as we develop in the next subsection.

To fix the ideas, consider the case without loss of generality, that the macroprudential policy decreases systemic risk, i.e. \(\Gamma(m_t)\) is an increasing function of \(m_t\) \((\frac{\partial \Gamma(m_t)}{\partial m} > 0)\), but decreases the amount of productive investment, \((\frac{\partial K(m_t, L)}{\partial m} < 0)\).

Using the second mean value theorem expression (14) can be rewritten as:

\[ \left[ \frac{\partial}{\partial K} F(A_t, N_{t+1}, K_t(m_t, \hat{A}, \hat{L})) - 1 \right] \int_0^\infty -\frac{\partial K(m_t, L)}{\partial m_t} dG_t(L) = \Delta Y(\Gamma(m_t))G_t(\Gamma(m_t)) \frac{\partial \Gamma(m_t)}{\partial m_t} \]

\[ \Rightarrow \frac{\partial}{\partial K} F(A_t, N_{t+1}, K_t(m_t, \hat{L})) - 1 = \frac{\Delta Y(\Gamma(m_t))G_t(\Gamma(m_t)) \frac{\partial \Gamma(m_t)}{\partial m_t}}{\int_0^\infty -\frac{\partial K(m_t, L)}{\partial m} dG_t(L)}. \]

The result is quite intuitive: first, if \(\Delta Y(\Gamma(m_t))\) or \(G_t(\Gamma(m_t))\) are zero, so that the probability of a bank bankruptcy is negligible, then the golden rule holds at point \(\hat{L}\), (which corresponds to the \(\frac{\partial K(m_t, L)}{\partial m}\) weighted expected value of \(L\)), and \(\hat{m}_t\) is the optimal level for the control.

Regarding productive efficiency, macroprudential policy, expression (14) shows, as intuition suggests, that when the marginal productivity of capital is higher than 1 the policy should incentivize increased capital investment and vice versa.

Second, if the control \(m_t\) does not change the probability of a bank bankruptcy, \((\frac{\partial \Gamma(m_t)}{\partial u} = 0)\), then again, the golden rule holds at this point. This is so because the control is an
ineffective macroprudential instrument.

The first order condition (14) does not provide any guidance regarding Pareto improving policies, and there is no necessarily any trade-off between marginal expected productivity and systemic risk. Thus, for instance, a cap on total credit available to the economy may both increase the productive efficiency while decreasing systemic risk, as we prove for the Cobb-Douglas case.

It may be tempting to think that macroprudential policy should be developed uniquely in order to reintroduce the externality related to systemic risk (see De Nicolò et al., 2012). This would basically imply that, on average, the equilibrium in a bubbly economy is efficient. In general, this is not the case. In the $L$–regime interest rates satisfy $E\left(1 + r_f \right) = 1$. Using the Jensen inequality, this implies that $E\left(1 + r_f \right) > 1$, so that $E\left(\frac{\partial}{\partial K} F(A_t, N_{t+1}, K_t(m_t, A, \mathcal{L})) - 1 \right) > 0$, and there is productive inefficiency because interest rates are, on average, excessively high.

The maximization problem (13) allow us to establish implicit sufficient conditions for a macroprudential policy to improve productive efficiency. Indeed, it suffices to remark that because $F(A_t, N_{t+1}, K_t(m_t, A, \mathcal{L})) - K_t(m_t, A, \mathcal{L})$ is a concave function of $K_t$, we have the following result:

**Lemma 3.** A macroprudential policy that leads to a mean preserving contraction of $K_t$ improves productive efficiency.

### 4.1 Efficient macroprudential policies

As a starting point in our exploration of macroprudential policies, it is worth it to consider the case where regulatory have perfect information and are unconstrained on the type of policy, so that $X(A, \mathcal{L}) = (A, \mathcal{L})$ without any constraint on $m(\cdot)$. If this is the case it is possible to decentralize the first best, characterized by the golden rule, so that at each point in time, $t$, we have $K_t = K(0)$, provided we consider the $L$–regime. This implies a simple regulatory policy.

**Proposition 4.** The full information unconstrained policy is characterized by the golden rule, where the rate of growth of the bubble is equal to the rate of growth of the economy in the $L$–regime.

As $\mathcal{E}^L_{t+1} = B_t(1 + r)$, and $r = 0$, $E(B_{t+1} | B_t) = \mathcal{E}^L_{t+1}$, and in a stationary economy it implies $\mathcal{E}^L$ and $B$ to be constant. Notice that there is a multiplicity of efficient equilibria, as any $\mathcal{E}^L$ will be compatible with the efficient allocation of capital. The risk of bubble bursting is eliminated and, therefore, no bubble related capital policy is required, as the bank does not need any capital to cover systemic risk.
To implement such a policy, let \( m(\mathcal{L}, A) \) be optimal policy that affects the total volume of lending. If the macroprudential policy allows to fine tune the effective amount of funds \( \mathcal{L} \) the bank will be lending, which may imply setting capital controls, this implies the bank will lend \( \mathcal{L} - m(\mathcal{L}, A) \) and has to invest the difference \( m(\mathcal{L}, A) \) in liquid, less profitable investments, such as sovereign debt. With no productivity shocks in a stationary economy, which corresponds to the pure liquidity shocks case, maintaining a constant bubble implies a constant \( K_t \) and because \( B_t = W_t + \mathcal{L}_t - K_t + m(\mathcal{L}_t) \), it requires \( m(\mathcal{L}_t) = \mathcal{L}^* - \mathcal{L}_t \) and it is reminiscent of a sterilization policy. Notice, nevertheless, that, because of our definition of liquidity as equivalent to the credit supply, this type of macroprudential policy takes into account shadow banking in so far it increases or destroys liquidity.

The result is reminiscent of the results in Jordà et al. (2013) that examine the link between deregulation and credit growth and argue that a strong and sustained credit boom cannot be financed with local increase of deposits and wealth (especially if not driven by very strong fundamentals); foreign liquidity, or liquidity stemming from expansive monetary policy or financial innovation (e.g., securitization), needs to be present and to interact with the credit cycles. The implication is here that regulation should be designed taking also into account the shadow banking activity. This is confirmed by Dell’Ariccia et al. (2012) who show that, during a boom, imbalances build and the current account deteriorates.

In the symmetric case of pure productivity shocks, with a constant \( \mathcal{L} \), maintaining a constant bubble in a stationary economy implies a capital that increases with positive productivity shocks and decreases with negative ones. In this case, \( B_t = W_t + \mathcal{L}_t - K_t + m(A_t) \).

In the general case, the optimal policy will take into account both types of shocks and react to increases in productive capital as well as to wages.

Of course, it could be argued that productivity shocks are only observable with a delay and cannot be used as variable in the macroprudential policy. Still, when this is the case, the first best can be also reached by targeting the constant value of the bubble (if observable) or the real interest rate, which are seldom used as part of the macroprudential policies.

It is not surprising that by sterilizing liquidity shocks it is possible to reach the efficient allocation. What our approach allows to show is that, first, the optimality is reached in the \( L^- \) regime, and second, that it can be reached for any level of \( B \) larger than \( W \).

Interestingly, in the stationary case, the first best macroprudential policy is particularly interventionist and never relies on the market invisible hand, even when there is no systemic risk. This is the case because, in the stationary case, as we have \( \frac{e^L}{1 + r_t} = B_t \), by applying the law of iterated expectations and dividing by \( E^L \) we obtain \( E \left( \frac{1}{1 + r_t} \right) \) = 1. This implies that because of the convexity \( E \left( \frac{1}{1 + r_t} \right) \) Jensen’s law implies \( E(r_t) > 0 \). It also implies, as intuition suggests, that the riskiness of the economy has a cost. Hence macroprudential policy goes
here beyond the eradication of systemic risk and improves resource allocation by reducing uncertainty.

**Proposition 5.** For any given level of productivity, $A_t$, imposing a cap $\mathcal{L}$ and a floor $\mathcal{L}$ on $\mathcal{L}$ that leaves $E(K_t)$ unchanged is welfare improving.

*Proof.* Because of Lemma 1, a cap on $\mathcal{L}$ will imply a cap on bubbles and, therefore, will reduce systemic risk. Regarding productive efficiency, notice, first that the caps imply corresponding caps on interest rates. As a consequence, the maximum and minimum amount of capital will be reduced. Since, by assumption, $E(K_t)$ is unchanged, this implies the imposition of a cap and a floor on $\mathcal{L}$ leads to a mean preserving contraction on capital and Lemma 3 allows to conclude.

The above Proposition is related to solvency regulation, countercyclical buffers and credit to GDP limits, but requires a counterpart to these measures as a liquidity injection (the floor).

It is important to emphasize that the result concerns only liquidity shocks. When it comes to productivity shocks, as we discussed later, the first best policy implies that high productivity shocks will be accommodated by an increase in liquidity. It is therefore intuitive that a cap on the supply of credit will have a negative impact when it concerns productivity shocks.

Notice that the proposition does not concern macroprudential policies that impose a limit on lending to finance the investment in the bubble, $L_t \leq \eta(X(A,L))$, that covers risk loan to value ratios, $L_t \leq \gamma_1 B_t$, with $B_t = W_t + \mathcal{L}_t - K_t$, which is equivalent in our framework to $L_t \leq \gamma_2 W_t$ or loan to income ratio $L_t \leq \gamma_3 W_t$. Indeed, in this case, when the cap $\overline{\mathcal{L}}$ is reached, $B_t$ investment in the bubble is rationed and the bubble is determined by the available funding $B(\overline{\mathcal{L}}) = W_t + \overline{\mathcal{L}}$, while interest rate $r^f(\overline{\mathcal{L}})$ is determined by $K(r^f) + \overline{\mathcal{L}} = \mathcal{L}_t$. A sufficient demand for the bubble implies that $\frac{\mathcal{L}_t}{B(\mathcal{L})} \geq 1 + r^f(\mathcal{L}_t)$. This is the case because we know that $B_t$ is an increasing function of $\mathcal{L}_t$, the limit on $\overline{\mathcal{L}}$ will be reached for $\mathcal{L}_t \geq \overline{\mathcal{L}} + K(\frac{\mathcal{L}_t}{B(\mathcal{L})})$.

It is quite likely that this leads to an inefficient allocation. In order to analyze the impact of this type of policies in the stationary case, it is worth to compare the distribution of interest rates for any value of $\mathcal{L}_t$. We know that, by iterating expectations $E(\frac{1}{1+r_t}) = 1$ in equilibrium, whether a cap $\mathcal{L}_t$ is imposed or not. Now, for $\mathcal{L}_t > \overline{\mathcal{L}} + K(\frac{\mathcal{L}_t}{B(\mathcal{L})})$ the interest rate is lower in the economy with the macroprudential policy, as the bubble cannot play its role of buffer and the whole of the $\mathcal{L}_t$ shocks will be accommodated by the firms’ demand for capital. As this happens when interest rates are at the lower end, it implies that at the margin, firms are overinvesting, which is inefficient.
As a consequence of Proposition 7, in the absence of productivity shocks, a cap on loans will improve the expected welfare and reduce systemic risk associated with high values of $B_t$ at the same time. The drawback is, of course, that a cap on bubbles will lead to an inefficient allocation in case of sufficiently large productivity shocks.

5 A Cobb-Douglas Economy

The Cobb-Douglas production function allows to obtain explicit solutions for the stationary equilibrium and illustrate its main properties.

For the sake of tractability, we will now assume that $Y_{t+1} = F(A, N_{t+1}, K_t) = A_t N_{t+1}^{1-\alpha} K_t^\alpha$ and let’s set $\alpha = \frac{1}{2}$. The first order condition, for $K_t$ is given by $\frac{1}{2} A_t K_t^{-1} N_{t+1}^{\frac{1}{2}} = 1 + r_{t+1}$, so that in equilibrium $K_t = \frac{A_t}{4(1+r_{t+1})^2}$, while the first order condition for $W_{t+1}$ is $\frac{1}{2} A_t N_t^{-\alpha} K_t^\alpha - 1 = W_{t+1}$, implying $W_{t+1} = \frac{A_t}{4(1+r_{t+1})}$.

These specifications allow to provide an explicit solution to the system of equations (8,10) and obtain necessary conditions for the existence of the three different regimes.

The $L$–regime is a function of $A_t, L_t$ and the conditional expectation $E(B | L) = \mathcal{E}^L$. So, as $B_t = \frac{\mathcal{E}^L}{1+r_t^L(A,L)}$ and $K^L(A, L) = \frac{A_t}{4(1+r_t^L(A,L))^2}$, after replacing in $B_t = W_t + L_t - \frac{1}{4(1+r_t^L(A,L))^2}$ it is possible to solve for the equilibrium values of $r_t^L(A, L), B(A, L)$ and $K(A, L)$.

$$r^L(A, L) = \frac{1}{2} \frac{1}{W + L} \left[ \mathcal{E}^L + \sqrt{(\mathcal{E}^L)^2 + A^2 (W + L)} \right] - 1$$

$$K^L(A, L) = \frac{A^2}{4(1+r^L(A, L))^2}$$

$$F(A, 1, K^L(A, L)) = \frac{A}{2(1 + r^L(A, L))}$$

$$B^L(A, L) = \frac{2}{A^2} \mathcal{E}^L \left[ \sqrt{(\mathcal{E}^L)^2 + A^2 (W + L) - \mathcal{E}^L} \right]$$

$$B^L(A, L) \geq W.$$
Using $W = \frac{2}{\varphi} \mathcal{E}^L \left[ \sqrt{(\mathcal{E}^L)^2 + A^2 (W + \mathcal{L}) - \mathcal{E}^L} \right]$, the constraint can be simplified to

\[
A \leq A^L = \mathcal{L}^L - \frac{2W \mathcal{E}^L}{W}
\]

\[
\mathcal{L} \geq \mathcal{L}^L = \left( \frac{AW}{2\mathcal{E}^L} \right)^2.
\]

To satisfy the $\varphi(1 + r^L(A, \mathcal{L})) \geq (1 + r)$ condition, the equivalent conditions are

\[
A \geq A^{L:1} = 2\sqrt{\frac{1 + r}{\varphi} \left[ (W + \mathcal{L}) \frac{1 + r}{\varphi} - \mathcal{E}^L \right]}
\]

\[
\mathcal{L} \leq \mathcal{L}^{L:1} = \left( \frac{\varphi A}{2(1 + r)} \right)^2 + \frac{\varphi \mathcal{E}^L}{1 + r} - W
\]

required for the equilibrium to exist, imposes a limit on the values of the productivity shock.

Symmetrically, the $U$—regime solution as a function of the conditional expectation $E(B \mid U) = \mathcal{E}^U$ is obtained as

\[
r^U(A, \mathcal{L}) = \frac{1}{2\varphi} \frac{1}{W + \mathcal{L}} \left[ \mathcal{E}^U + \sqrt{(\mathcal{E}^U)^2 + \varphi^2 A^2 (W + \mathcal{L})} \right] - 1
\]

\[
K^U(A, \mathcal{L}) = \frac{A^2}{4(1 + r^U(A, \mathcal{L}))^2}
\]

\[
F(A, 1, K^U(A, \mathcal{L})) = \frac{A}{2(1 + r^U(A, \mathcal{L}))}
\]

\[
B^U(A, \mathcal{L}) = \frac{2(W + \mathcal{L}) \mathcal{E}^U}{\mathcal{E}^U + \sqrt{(\mathcal{E}^U)^2 + \varphi^2 A^2 (W + \mathcal{L})}}
\]

\[
= \frac{2\mathcal{E}^U}{A^2 \varphi^2 \left[ \sqrt{(\mathcal{E}^U)^2 + \varphi^2 A^2 (W + \mathcal{L}) - \mathcal{E}^U} \right]}
\]

\[
B^U(A, \mathcal{L}) \leq W.
\]

Similarly as before, we can establish a boundary for $\mathcal{A}_t, \mathcal{L}_t$ so that the constraint $B^U(A, \mathcal{L}) \geq W$ is satisfied. Using $W = \frac{2}{\varphi} \mathcal{E}^U \left[ \sqrt{(\mathcal{E}^U)^2 + A^2 (W + \mathcal{L}) - \mathcal{E}^U} \right]$, the constraint can be sim-
plified to

\[ A \geq A^U = L^2 \frac{2E^U}{\phi W}. \]

\[ L \leq L^U = \left( \frac{\phi AW}{2E^U} \right)^2. \]

In the SF-regime the following holds

\[ K_{SF}(A, L) = L \]
\[ r_{SF}(A, L) = \frac{A}{2L^2} - 1 \]
\[ F(A, 1, K_{SF}(A, L)) = L^{1/2} \]
\[ B_{SF}(A, L) = W. \]

Finally, consider a \( L > L^L \) and let \( r^f \) be the loan rate such that \( \phi(1 + r^f) = 1 + r \). The equivalent of the no arbitrage equation holds:

\[ B(r) = \frac{E_{t+1}}{1 + r^f}. \]

From the household’s budget constraint, we can pin down how much they will borrow:

\[ L = B(r) - W. \]

In this regime banks will have extra liquidity that they will not be able to lend. Therefore they will invest in the riskless asset, \( O^B \). From their balance sheet we can pin down this quantity:

\[ O^B = L - K(r^f) - L. \]

Figures 1 to 3 show different values across \( L \) assuming \( F(0.9, N_t, K_{t+1}) = 0.9K_{t+1}^{1/2}N_t^{1/2} \) and assuming \( L \) follows a uniform distribution.\(^{15}\) Figure 1 illustrates the behavior of interest rate as a function of liquidity, with the line representing \( r^d \) being in blue and below the line showing \( r^f \) (in black). As intuition suggests, a liquidity shortage, will make interest rate soar and the \( U \)-regime will prevail. This has an effect on the equilibrium price, as shown in Figure 2 for the bubble as well as on the amount of credit that is channeled to the unproductive yet efficient investment. Figure 3 shows that both production and welfare are increasing across

\(^{15}\)Equilibrium outcomes may vary, depending on the state in which the economy is. Figures 1 to 3 show outcomes for different values at \( t + 1 \) when the economy is in the \( L \)-regime at \( t \).
most values of $L$. However, for values of $L$ just less than $L^{L_{-L}}$ there is a positive probability that banks go bankrupt, and therefore welfare decreases.

**Figure 1:** $r^f$ and $r^d$

5.1 Second Best Macroprudential Policies: The stationary Cobb-Douglas equilibrium

The optimal full information unconstrained policy in stationary Cobb-Douglas case leads to $K^L(A, L) = \frac{A^2}{2}$, $F(A, 1, K^L(A, L)) = \frac{A^2}{4}$ and $L = E - \frac{1}{4}$.

Now, as $F(A, N_{t+1}, K) - K$ is concave in $K$, any mean preserving spread for $\tilde{K}_t$ is welfare decreasing (Rothschild and Stiglitz, 1976). Still, it is not clear what type of macroprudential policies lead to a reduction in the riskiness of $K$. This is why the equivalent proposition regarding the discount factor may be more easy to implement.

**Proposition 6.** For any level of productivity, in the Cobb-Douglas stationary case, a policy that reduces the riskiness of the discount factor $\frac{1}{1+r}$ in the sense of a mean preserving spread is welfare improving.

**Proof.** It suffices to notice that the function $\mathbb{E}\left(\frac{A_t}{2(1+r)} - \frac{A_t^2}{4(1+r)^2}\right)$ is a concave function of $r$. \hfill $\square$
As intuition suggests, a higher interest rate volatility implies a lower expected welfare. A number of macroprudential policies impose a limit on banks’ credit supply. This may be the case, for solvency ratios, for countercyclical buffers and for limits to the credit to GDP ratio. Our framework allows to consider the welfare properties of this type of policy as it would consist in replacing the initial distribution of $L_t$ with a distribution on $L_t$ truncated on the upper side. The following proposition shows that such a macroprudential policy will be efficient as it limits the uncertainty associated with liquidity shocks.

**Proposition 7.** In the $L$–regime, for any given level of productivity, $A_t$, imposing a cap $L$ on $L$ is Pareto efficient. A reduction of the cap $L$ is Pareto efficient provided the economy stays in the $L$–regime.

**Proof.** Let $G(L)$ be the probability distribution of the $L$ process. A cap $L$ implies a truncated distribution for $L$ with a probability of $L = \int_{L}^{\infty} dG(L)$. The first step of the proof is to show that a cap on $L$ decreases the expected value of the bubble, $E$.

To show this, notice that $E\left(\frac{1}{1+r_t}\right) = 1$ will hold for both distributions, and denote $E\left(\frac{1}{1+r_t}\right)$ by $\varphi(L, E)$. Because $\frac{1}{1+r_t(L)} = \frac{2}{A^2} \left[ \sqrt{E^2 + A^2 (W + L) - E} \right]$, we have
\[ \varphi(\bar{L}, \mathcal{E}) = \frac{2}{A} \left[ \int_0^{\mathcal{L}} \left[ \sqrt{\mathcal{E}^2 + A^2 (W + \mathcal{L})} - \mathcal{E} \right] dG(\mathcal{L}) + \left[ \sqrt{\mathcal{E}^2 + A^2 (W + \mathcal{L})} - \mathcal{E} \right] \int_{\mathcal{L}}^{\infty} dG(\mathcal{L}) \right]. \]

The difference \( \varphi(\bar{L} + \Delta, \mathcal{E}) - \varphi(\bar{L}, \mathcal{E}) \) with \( \Delta > 0 \), \( \int_{\mathcal{L}}^{\mathcal{L}+\Delta} \left( \frac{1}{1+r_t(\mathcal{L})} - \frac{1}{1+r_t(\bar{L})} \right) dG(\mathcal{L}), \) is positive, because as proved in Lemma 1, \( \frac{\partial r_t(\mathcal{L})}{\partial \mathcal{L}} < 0 \), so that the integrand is positive. On the other hand, \( \varphi(\bar{L}, \mathcal{E}) \) is decreasing in \( \mathcal{E} \), as

\[ \sqrt{\mathcal{E}^2 + A^2 (W + \mathcal{L})} - \mathcal{E} \]

is decreasing in \( \mathcal{E} \).

Consequently, the equality \( \varphi(\bar{L}, \mathcal{E}) = 1 \) implies that an increase in \( \mathcal{L} \) has to be compensated by an increase in \( \mathcal{E} \).

We will now prove that imposing a cap \( \mathcal{L} \) will reduce the riskiness of the discount factor \( \frac{1}{1+r_t} \) in the sense of a mean preserving spread and the previous lemma will allow us to conclude.

To show that the distribution associated to a cap \( \mathcal{L} \) is a mean preserving reduction of \( \frac{1}{1+r_t} \), we will use the subscript 1 to refer to the original distribution and the subscript 2 to refer to the capped distribution.
For $t < \overline{L}$, we have $\Pr_1\left(\frac{1}{1+r_1} \leq t\right) = \int_0^t \frac{2}{A^2} \left[\sqrt{\varepsilon_i^2 + A^2(W + L) - \varepsilon_i}\right] dG(L)$ and as we have shown $\varepsilon_2 < \varepsilon_1$, we have $\Pr_2\left(\frac{1}{1+r_2} \leq t\right) < \Pr_1\left(\frac{1}{1+r_1} \leq t\right)$. Now, at point $t = \overline{L}$, we have $\Pr_2\left(\frac{1}{1+r_2} \leq t\right) = 1$, so that the inequality is reversed. Consequently, we have the single crossing condition on the cumulative distribution function that characterize the second order stochastic dominance and the previous lemma applies.

6 Conclusion

Even if risky, rational bubbles constitute a way for households to transfer wealth from one period to another. This has a positive impact on the allocation of resources when the financial markets do not allow to do so as efficiently. Credit, as “unlimited reliance upon human promises” has a role here because the efficient allocation of resources is reached for a unique interest rate, and this means the return on the bubble should be the same as the return on productive investment. Still, in a banking economy, the existence of two interest rates that differ in the banks’ margin makes it efficient that bubbles are financed by credit. The other side of the coin is that if a bubble bursts it leads to bank losses that may wipe out the financial institution capital leading it to bankruptcy.

By modeling bubbles in an overlapping generation set up, it is possible to visualize the impact of credit, expectations formation and shocks on the resulting allocation. In equilibrium, the price of the bubble depends upon credit and upon the expected future price of the bubble. To some extent, the model shows that the supply of credit creates its own demand in so far as expectations react to the current prices.

Our analysis lead us to concentrate on two types of shocks, liquidity and productivity. The equilibrium depends on the realization of these shocks and agents rational expectations. For this reason, systemic risk, that is here simply modeled as a bubble bursting when banks are insufficiently capitalized, will also depend on these shocks.

Interestingly, this approach allows us to characterize macroprudential policy and show that it should consider the type of shock that occurs, as, contrarily to conventional wisdom. In our set up the adequate macroprudential policy depends upon the type of shock or on the combination of shocks.

Of course, for each stochastic process that the exogenous shock follows, martingales or mean reverting, there is a different view of bubbles. Nevertheless, this characteristic of the analysis should not be seen as a limitation, but rather as the necessity for policy makers to be well aware of the nature of these shocks.
Appendix A Mathematical Appendix

Proof of Lemma 1:
To avoid cumbersome notation, in this proof we abstract from time subscripts in this proof and we refer to $r^f$ as $r$. Since $F$ is strictly concave and twice differentiable, the demand for credit, $K(r,A) = \left(\frac{\partial F(A,K)}{\partial K}\right)^{-1}(r)$, is strictly decreasing and differentiable in $r$.

Consider first the $L$-regime. To make notation simpler, let $x \equiv E^L$, and define $B^L$ as the value of the bubble in this regime. Using the second and third equations in (8) we have

$$B^L = W + L = W + L - K(r, A) \geq B.$$  

Additionally, from the first equation in (8) we can derive

$$x = (1 + r)(W + L - K(r, A)).$$

This can be solved for the interest rate $r^L(x, A, L)$ that satisfies:

$$\frac{\partial r^L}{\partial L} = -\frac{1 + r^L}{W + L - K(r^L, A) - (1 + r^L)\frac{\partial K}{\partial r^L}} < 0,$$
$$\frac{\partial r^L}{\partial A} = \frac{(1 + r^L)\frac{\partial K}{\partial A}}{W + L - K(r^L, A) - (1 + r^L)\frac{\partial K}{\partial r^L}} > 0.$$

We can now derive $B^L(x, A, L) = W + L - K(r^L(x, A, L), A)$ which satisfies

$$\frac{\partial B^L}{\partial L} = 1 - \frac{\partial K}{\partial r^L} \frac{\partial r^L}{\partial L} > 0,$$
$$\frac{\partial B^L}{\partial A} = -\frac{\partial K}{\partial A} - \frac{\partial K}{\partial r^L} \frac{\partial r^L}{\partial A} < 0.$$

To see why the second inequality holds, notice that

$$\frac{\partial K}{\partial r^L} \frac{\partial r^L}{\partial L} = \frac{1}{1 - \frac{W + L - K(r^L, A)}{(1 + r^L)\frac{\partial K}{\partial r^L}}} < 1.$$

Similarly, the third equality holds because

$$-\frac{\partial K}{\partial r^L} \frac{\partial r^L}{\partial A} = \frac{\frac{\partial K}{\partial A}}{1 - \frac{W + L - K(r^L, A)}{(1 + r^L)\frac{\partial K}{\partial r^L}}} < \frac{\partial K}{\partial A}.$$

Now consider the $U$-regime. To make notation simpler, let $y \equiv E^U$, and define $B^U$ as the value of the bubble in this regime. Using the second and third equations in (10) we have

$$B^U = W - D = W + L - K(r, A) \geq B.$$
Additionally, from the first equation in (10) and the fact that \(1 + r^d = \varphi(1 + r)\) we can derive
\[
y = \varphi(1 + r)(W + L - K(r, A)),
\]
which implicitly determines \(r^U(y, A, L)\) that satisfies
\[
\frac{\partial r^U}{\partial L} = -\frac{1 + r^U}{W + L - K(r^U, A) - (1 + r^U)\frac{\partial K}{\partial r^U}} < 0.
\]
We can now derive \(B^U(y, A, L) = W + L - K(r^U(y, A, L), A)\) which satisfies
\[
\frac{\partial B^U}{\partial L} = 1 - \frac{\partial K}{\partial r^U} \frac{\partial r^U}{\partial L} > 0.
\]

The rest of the proof is the same as for the \(L\)-regime case.

**Proof of Lemma 2:**

Two conditions are required for the equilibrium in the \(L\)-regime to exist: \(L_t \geq 0\) and \(r^L \leq \underline{r}\) the corresponding conditions for the \(U\)-regime equilibrium are: \(D_t \geq 0\) and \(r^U \leq \overline{r}\).

The corresponding thresholds will be given by these conditions holding with equality.

Consider, first, the threshold \(\mathcal{L}^L(\mathcal{E}^L)\), which occurs for \(L_t = 0\), i.e., \(K_t = \mathcal{L}_t\) for \(\frac{\partial F}{\partial K_t}(N_{t+1}, K_t) = 1 + r^U_{t+1}, \mathcal{E}^L_{t+1} = B_t(1 + r^U_t)\) and \(B_t = W_t\). Such a threshold exists as \(\frac{\mathcal{E}^L_{t+1}}{W_t} > 0\), so that \(r^d_t > -1\) and, under the Inada conditions there exists an \(\mathcal{L}^L(\mathcal{E}^L)\) that satisfies \(\frac{\partial F}{\partial K_t}(1, \mathcal{L}^L(\mathcal{E}^L)) = \frac{\mathcal{E}^L_{t+1}}{W_t}\). Now, because of lemma 1, we know that, for \(\mathcal{L}_t > \mathcal{L}^L(\mathcal{E}^L)\), we have \(L_t > 0\). Regarding the second constraint, again, for sufficiently large liquidity shocks, the interest rate will be lower than \(\underline{r}\). When this is the case, a unique liquidity threshold will exist such that \(r^L = \underline{r}\). Using 1, we obtain \(r^L \geq \underline{r}\) if and only if \(L_t \leq \mathcal{L}^L\).

Second, consider, the threshold \(\mathcal{L}^U(\mathcal{E}^U)\), which occurs for \(D_t = 0\), i.e., \(K_t = \mathcal{L}_t\) for \(\frac{\partial F}{\partial K_t}(N_{t+1}, K_t) = 1 + r^U_{t+1}, \mathcal{E}^U_{t+1} = B_t(1 + r^U_t)\) and \(B_t = W_t\). As before, such a threshold exists as \(\frac{\mathcal{E}^U_{t+1}}{W_t} > 0\), so that \(r^d_t > -1\) and, under the Inada conditions there exists \(\mathcal{L}^U(\mathcal{E}^U)\) that satisfies \(\frac{\partial F}{\partial K_t}(1, \mathcal{L}^U(\mathcal{E}^U)) = \frac{\mathcal{E}^U_{t+1}}{W_t}\). Now, because of lemma 1, \(\frac{\partial B^U}{\partial L} > 0\), and, consequently, for any \(\mathcal{L}_t < \mathcal{L}^U(\mathcal{E}^L)\), we have \(B_t < W_t\), so that \(D_t > 0\).

**Proof of Proposition 1:**

Notice that for all \(\mathcal{E}^L\) and \(\mathcal{E}^U\) and \(\mathcal{G}(\mathcal{L})\) we can solve the problems of the agents in this economy. The proof will therefore consist in finding a fixed point in the mapping from \(\mathcal{E}^L\) and \(\mathcal{E}^U\) into the equilibrium values \(\mathcal{E}^L\) and \(\mathcal{E}^U\) that are generated by \(\mathcal{G}(\mathcal{L})\). This amounts to proving that there exists \(\mathcal{E}^L\) and \(\mathcal{E}^U\) such that \(\mathcal{E}^L = \phi_L(\mathcal{E}^L, \mathcal{E}^U)\) and \(\mathcal{E}^U = \phi_U(\mathcal{E}^L, \mathcal{E}^U)\), where
\[
\phi_L(x, y) = \int_{\mathcal{E}^L \leq x} B^L(x, \mathcal{L}) \mathcal{G}(\mathcal{L}) + \int_{\mathcal{E}^L \leq \mathcal{L} \leq \mathcal{E}^L} B^U(y, \mathcal{L}) \mathcal{G}(\mathcal{L})
\]
\[
\phi_U(x, y) = \int_{\mathcal{E}^U \leq x} B^L(x, \mathcal{L}) \mathcal{G}(\mathcal{L}) + \int_{\mathcal{E}^U \leq \mathcal{L} \leq \mathcal{E}^U} B^U(y, \mathcal{L}) \mathcal{G}(\mathcal{L})
\]
and $B^L(x, L)$ and $B^U(x, L)$ are the expressions for the bubbles that we derive in the proof of Lemma 1.

Notice that the function $\phi(x, y) = (\phi_L(x, y), \phi_U(x, y))$ is a continuous mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$. We will prove that there exists $(x, y) = \phi(x, y)$ by proving that there exists $0 < \varepsilon < b < \infty$ such that $\phi : [\varepsilon, b] \times [\varepsilon, b] \to [\varepsilon, b] \times [\varepsilon, b]$. The proof of existence of an equilibrium then follows from Brouwer’s fixed point theorem.

The existence of $\varepsilon$ follows from the fact that $B^L(x, L), B^U(y, L) \to \infty$ as $x, y \to 0$.

**Proof of Proposition 2:**

It is sufficient to prove that the resulting equilibrium $r$ is monotonic in the shock $A$. The result then follows from the proof of Lemma 1. We’ll restrict the proof to the $L$-regime, since an equivalent result holds for the $U$-regime. Similar to before, to avoid cumbersome notation, in this proof we’ll abstract from time subscripts in this proof and we’ll refer to $r^f$ as $r$. Since $F$ is strictly concave in $K$ and strictly increasing in $A$, then demand for credit, 

$$K(r, A) = \left(\frac{\partial F(A, 1, K)}{\partial K}\right)^{-1}(r),$$

is strictly decreasing in $r$ and strictly increasing in $A$.

In the $L$-regime we have

$$B^L = W + L = W + L - K(r^L, A) \geq B.$$

Additionally, from the first equation in (8) we can derive

$$x = (1 + r)(W + L - K(r^L, A)),$$

which implicitly determines $r^L(x, A)$ that satisfies

$$\frac{\partial r^L}{\partial x} = \frac{1}{W + L - K(r^L, A) - (1 + r^L) \frac{\partial K}{\partial x}} > 0,$$

$$\frac{\partial r^L}{\partial A} = \frac{(1 + r^L) \frac{\partial K}{\partial A}}{W + L - K(r^L, A)} > 0.$$
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