Product Choice and Price Discrimination in Markets with Search Costs*

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Abstract

In a seminal paper, Champsaur and Rochet (1989) showed that competing firms choose non-overlapping qualities so as to soften price competition at the cost of giving up profitable opportunities to price discriminate. In this paper we show that an arbitrarily small amount of search costs is enough to give rise to an equilibrium with overlapping qualities. In markets with search costs, competing firms face the monopolist’s incentive to price discriminate, which induces them to offer the full quality range even if this forces them to compete head-to-head. Hence, even though search costs increase prices and reduce consumers surplus for given quality choices, search costs can also lead to lower prices and higher consumer surplus whenever they induce firms to offer broader and overlapping quality offerings. Our analysis also provides predictions regarding pricing by multi-product firms in markets with search costs under various retail market structures. Quality choices and pricing by online bookstores motivate our findings.

Keywords: second degree price discrimination, search, vertical differentiation, retail competition.

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1 Introduction

Since the classical work of Chamberlin (1933), a well known principle in economics is that firms differentiate their products in order to relax competition. Champsaur and Rochet (1989) (CR, thereafter) formalized this Chamberlinian incentive in a model in which quality choices are followed by price competition.\footnote{Shaked and Sutton (1982) formalized the same idea in a model similar to Champsaur and Rochet (1989)’s, with the difference that firms are allowed to offer one quality only. Thus, in Shaked and Sutton (1982), there is no possibility to discriminate consumers at the firm level.} They showed that firms choose non-overlapping qualities because the incentives to soften price competition dominate over the incentives to better discriminate heterogeneous consumers. Yet, in many markets, competing firms often carry overlapping qualities even when this creates fierce competition among them.

In this paper, we argue that market frictions may make CR’s prediction fail in practice. In particular, we focus attention on one of such frictions, namely, that consumer search is costly.\footnote{There is a large empirical literature analyzing pricing in markets that fit well into CR’s theoretical framework and in which search costs are relevant. See Section 2 for more on this.} When consumers are not perfectly informed about firms’ prices and qualities, they cannot choose their preferred option unless they incur search costs to learn and compare all options. Since the seminal work of Diamond (1971), the search literature has shown that the introduction of search frictions can have substantial effects on competition, no matter how search costs are modeled.\footnote{Search models can essentially be classified as models of either simultaneous search (Burdett and Judd, 1983) or sequential search (Stahl, 1989). De los Santos et al. (2012) test which of the two processes best represents actual search for online books, and conclude in favor of the simultaneous search model, which is the approach we adopt in this paper.} However, unlike CR, this literature has broadly neglected the possibility that firms engage in price discrimination through quality choice.\footnote{Unlike the current paper, in which we model second-degree price discrimination, Fabra and Reguant (2017) allow for third-degree price discrimination in markets with search costs.} Understanding the interaction between search costs and price discrimination, and their effects on product choice and pricing by competing firms, is the main goal of this paper.

By introducing search costs \textit{`a la} Varian (1980) in a simplified version of CR, we show that an arbitrarily small amount of search costs is all it takes for firms to offer overlapping quality choices. More generally, this equilibrium always exists because as long as search costs are present (regardless of whether they are large or small) firms’ incentives to discriminate heterogeneous consumers dominate over the incentives to soften price competition. In particular, the marginal incentives faced by firms mimic those under monopoly, even if profits (which do depend on search costs) are below the monopoly level.
CR’s equilibrium with non-overlapping qualities may exist, but only when the cost of high quality is not too high and search frictions are not very relevant.

We show that the comparative statics of equilibrium prices and consumer surplus with respect to search costs can be biased if quality choices are taken as given. Essentially, search costs affect quality choice, and through that, they end up affecting prices and consumer surplus. Two countervailing effects are at play: on the one hand, an arbitrarily small amount of search costs intensifies competition by giving rise to overlapping quality choices; on the other, further increases in search costs relax competition, eventually leading to prices that exceed those in frictionless markets. In sum, while an increase in search costs is in general anti-competitive, search costs might also lead to lower prices and higher consumer surplus when they induce firms to carry broader and overlapping quality choices.\footnote{In general, search costs are thought to relax competition, thus leading to higher prices, although not as intensively as the Diamond paradox would have anticipated (Diamond, 1971). There are some exceptions to this general prediction. Some recent papers have shown that search costs can lead to lower prices, particularly so when search costs affect the types of consumers who search. For instance, see Moraga-González et al. (2017) and Fabra and Reguant (2017).}

Beyond investigating the effects of search costs on firms’ quality choices, we also aim at understanding their effects on equilibrium pricing in general. We show that the incentive compatibility constraints faced by multi-product firms introduce an important departure from Varian (1980): the prices for the various goods sold within a store cannot be chosen independently from each other. In particular, as it is standard in games with asymmetric information, the price of the high quality good has to be reduced to discourage those consumers with high quality preferences from buying the low quality good. Yet, when competition for the high quality good becomes particularly intense, multi-product firms reduce the price difference between the two goods below the level that is necessary to induce separation of consumers’ types. In other words, during periods of sales à la Varian, the incentives to compete may dominate over the incentives to minimize information rents. Additionally, incentive compatibility considerations imply that on average multi-product firms tend to charge lower prices than single-product firms.

Our paper is related to two strands of the literature: (i) papers that analyze competition with search costs, and (ii) papers that characterize quality choices under imperfect competition. The vast part of the search literature assumes that consumers search for one unit of an homogenous good, with two exceptions. Some search models allow for product differentiation across firms but, unlike ours, assume that each firm carries a single product.\footnote{For models with horizontal product differentiation, see for instance Anderson and Renault (1999)} Other search models allow firms to carry several products but, unlike ours,
typically assume that consumers search for more than one ('multi-product search'). In these models, consumers differ in their preference for buying all goods in the same store ('one-stop shopping') rather than on their preferences for quality. These differences are important. In the first type of search models, the single-product firm assumption leaves no scope for price discrimination within the firm. Hence, pricing is solely driven by competitive forces. In the second type of search models, the multi-product search assumption implies that discrimination is based on heterogeneity in consumers’ shopping costs, which become the main determinant of firms’ product choices (Klemperer, 1992).

Within the ‘multi-product search’ literature, two papers deserve special attention. In line with our results, Zhou (2014) finds that multi-product firms tend to charge lower prices than single-product firms. This is not driven by the interaction between competition and price discrimination, as in our paper, but rather by a ‘joint search’ effect, i.e., multi-product firms charge less because they gain more by discouraging consumers from searching competitors. In Rhodes and Zhou (2016), increases in search costs imply that consumers value one-stop shopping more, thus making it more likely that the equilibrium involves multi-product firms. Unlike us, for small search costs, Rhodes and Zhou (2016) predict asymmetric market structures with single-product and multi-product firms co-existing. The driving force underlying our predictions is quite different: since in our model consumers buy a single good, the multi-product firm equilibrium is not driven by one-stop shopping considerations but rather by firms’ incentives to price discriminate consumers with heterogeneous quality preferences. Despite these differences, our paper has one common element with both Rhodes (2014) and Rhodes and Zhou (2016), which is the fact that search costs can give rise to lower prices through their effect on endogenous product choices.

As far as we are aware of, Garret et al. (2016) is the only paper that, like ours, introduces frictions in a model of price competition in which firms can carry more than

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7There is a recent strand of papers in the ordered search literature that analyze obfuscation by multi-product firms (Gamp, 2016; Petrikaite, 2017). Their emphasis is on the monopoly case. See Armstrong (2016) for a discussion.

8One-stop shopping considerations are also the driving force behind the evidence of price dispersion across stores documented by Kaplan et al. (2016).
one product but in which consumers only buy one. However, there are two important distinctions between their analysis and ours. First, in Garret et al. (2016), firms decide qualities and prices simultaneously, rather than sequentially. The simultaneous timing is appropriate in settings where firms can change product design rather quickly, or alternatively, when firms commit to prices for long periods of time; for example, under long term contracts. In contrast, the sequential timing is better suited to capture the notion that in many markets firms can change prices at will, even daily; in contrast, changes in product line decisions usually involve changes in the production and/or retail facilities and thus occur less often (Brander and Eaton, 1984). This distinction is relevant as in simultaneous settings firms cannot affect competition by pre-committing to quality, which is a fundamental driving force of our results. Furthermore, Garret et al. (2016) focus on symmetric equilibria which necessarily involve overlapping product lines, but do not explore whether asymmetric equilibria with non-overlapping product choices could arise in their setting. Still, given that in our set-up firms are endogenously symmetric, our analysis shares some common predictions with Garret et al.’s concerning the comparative statics of prices and relative prices. Like them, in symmetric settings we find that the relative price of the two goods goes down when competition is particularly intense (in our model, when firms price both goods at the lower bound of the price supports). However, this prediction does not always extend to asymmetric product configurations, in which relative prices remain constant both at the upper as well as at the lower bound of the price supports.

Last, our paper also relates to the literature that analyzes quality choices followed by imperfect competition, either quantity competition (Gal-Or, 1983; Johnson and Myatt 2003, 2006 and 2015) or price competition with horizontal differentiation (Stole, 1995). As already noted by CR (p. 535), one of the main consequences of less competitive pricing is to induce wider and, very likely, overlapping product lines. While one may view search costs as an alternative way to allow for less competitive pricing, the effect of search costs on firms’ quality choices is different as compared to other forms of imperfect competition. In models of imperfect competition, the equilibrium with overlapping (i.e., symmetric) quality choices fails to exist when competition is sufficiently intense, e.g., as shown by Gal-Or (1983), when sufficiently many firms compete à la Cournot. The same insight

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9Another set of related papers analyze pricing for add-ons. Ellison (2005) and Verboven (1999) consider models in which consumers are well informed about base product prices but don’t know the price of the add-ons, unless they search. Critically, in these models the customers that are more likely to buy the add-ons are also less likely to search. Our model is not a model of add-on pricing because shoppers observe all prices and non-shoppers only those of the store they visit, and this applies symmetrically for both products regardless of their quality. Furthermore, our results hold regardless of whether there is correlation or not between consumers’ quality preferences and search cost types.
could also apply to models of price competition with horizontal product differentiation. If there is little (horizontal) product differentiation, the equilibrium with overlapping product choices breaks down because the rents lost when dropping a low quality good are small as compared to the increase in the rents from selling higher quality products. In contrast, the impact of search costs on product choices is different. Even if search costs are arbitrarily small, a firm does not want to deviate from the equilibrium with overlapping product choices. The reason is that search costs restore firms’ monopoly power over those consumers who do not search (the *non-shoppers*), even when competition for those who search (the *shoppers*) is very fierce.

The rest of the paper is organized as follows. Section 2 describes an illustrative example that conveys the main intuition of the model while providing supportive empirical evidence. Section 3 describes the model. Section 4 shows that in the absence of search costs firms escape the Bertrand paradox by carrying non-overlapping product lines. In contrast, Section 5 shows that an arbitrarily small amount of search frictions is enough to induce firms to choose overlapping product lines even if this drives prices close to marginal costs. Section 6 characterizes equilibrium pricing for all potential product choice configurations, as well as the Subgame Perfect Equilibrium product choices for all levels of search costs. Section 7 discusses the robustness of the model to several extensions. Section 8 concludes, and proofs are postponed to the Appendix.

## 2 An Illustrative Example

Price discrimination is pervasive in a wide range of markets in which search costs matter. In gasoline markets, consumers have the choice of paying for full-service or self-service gasoline at the same station, or of searching for competing stations (Shepherd, 1991). In the airline industry, travellers can choose whether to fly in business or in economy class, or just in economy class but with certain restrictions (Borenstein and Rose, 1994; Gerardi and Shapiro, 2009). Other examples in which price discrimination, competition and search coexist include coffee shops (McManus, 2000), cereals (Nevo and Wolfram, 2002), theaters (Leslie, 2004), Yellow Pages advertising (Busse and Rysman, 2005), mobile telephony (Miravete and Röller, 2004), cable TV (Crawford and Shum, 2007), or markets in which competing firms offer advanced-purchase discounts (Möller and Watanabe, 2016; Nocke *et al.*, 2011), among others.

To build intuition on the main forces underlying our model, we focus on a market that fits well our modeling framework: the market for online books. While previous empirical papers have analyzed search in these markets (De Los Santos *et al.*, 2012;
Hong and Shum, 2006), their focus has been on estimating buyers’ search behaviour for given product choices and prices. Rather, our focus here is simply to motivate and illustrate the predictions of the model by exploring firms’ product choices and prices given consumers’ search behaviour. For this purpose, we have collected daily book prices at Amazon and Barnes & Noble, the two leading online booksellers, from December 2016 to March 2017, for each of the 2012-2016 #1 New York Times fiction and non-fiction best-sellers.

2.1 Theoretical intuition

To develop intuition, let us think of two online stores competing to sell books to consumers with heterogenous preferences for quality. Before choosing prices, booksellers must decide whether to offer both the hardcopy and the paperback versions of each book, or just one of the two, if any. Since the hardcover version is generally thought of being of better quality than the paperback, we will sometimes refer to the two as the high and low quality goods, respectively. In turn, we will refer to those consumers who are willing to pay the extra cost of producing the hardcover as the high types, and the remaining consumers as the low types.

In the absence of search costs, CR’s prediction is that one store will offer the hardcover, and the other one the paperback. If one of the bookstores deviated from this equilibrium by carrying an additional format, competition would drive its price down to marginal costs, making such a deviation unprofitable. Furthermore, the store would have to give a discount on the other format to stop consumers from buying the one priced at marginal costs.

Alternatively, suppose that there is an arbitrary small amount of consumers (the so-called ‘non-shoppers’) who visit one of two sites at random without searching any further (since these consumers can be thought of having very large search costs, the mass of non-shoppers is a proxy for the level of search costs). Once in the site, the non-shoppers buy the version of the book that offers them higher utility (if positive) given their quality preferences and the prices of the two versions.

In the presence of non-shoppers, bookstores face a trade-off when setting prices: on the one hand, they want to set high prices to maximize the profits from selling the books to the non-shoppers; on the other, they want to set low prices to compete for the

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10In the context of online books, De los Santos et al. (2012) show that, within a 7 days window, 76% of consumers only visit one store. They also report the presence of loyal consumers: 24% of consumers engage in multiple transactions but only buy from one store, even if it exhibits a higher price, thus suggesting the presence of specific store preferences independent of price.
shoppers. These countervailing incentives imply that the equilibrium must be in mixed strategies, with the bookstores choosing random prices between the monopoly level and somewhere above marginal costs. Therefore, in equilibrium, the bookstores’ profits are the same as if they were monopolists over the non-shoppers, but also as if competition washed all the profits from the shoppers away.

This profit structure gives rise to an equilibrium with overlapping product choices, i.e., both bookstores sell the hardcover and the paperback: since carrying the two formats does not erode their market power (as it is the case in the absence of non-shoppers), and doing so allows the bookstores to better discriminate the non-shoppers, they have no incentives to drop neither format. In other words, in the presence of non-shoppers (no matter how few) there is no trade-off between competition and price discrimination, so the “overlapping” equilibrium arises just as in the monopolist case. However, unlike the monopoly case, the product overlap reduces profit levels, to the extent that these go down to zero as the mass of non-shoppers vanishes out.

2.2 Evidence in the data

The product choice and pricing patterns observed in the data are in stark contrast with the predictions of the CR model, but can be rationalized when accounting for search costs, as explained above. First, we find that the two stores sell both the hardcover and the paperback version whenever these versions exist. Consistent with our model, the presence of non-shoppers induces firms to carry overlapping product lines. Second, the prices of the hardcover and paperback versions of the same title do not remain constant, thus suggesting that books are not priced at marginal cost. Consistent with our model, the presence of non-shoppers is a force that can generate price dispersion, as tension between attracting the shoppers versus exploiting the non-shoppers implies that firms price according to mixed strategies.

Figure 1 provides evidence of price dispersion at the book-format level. In panel 1a one can see that prices fluctuate substantially for each title even after partialling out book-store-format means. As shown in panel 1b, this dispersion is not explained by

\(^{11}\)Out of the 200 books that we consider, some of them are only available in paperback or hardcover. For those available in both versions, all of them are served by Amazon and Barnes & Noble at the same time, except in one instance in which Amazon does no longer offer the hardcover version (which is only sold by other sellers). Given the almost complete overlap, we focus our analysis in cases in which both stores sell both versions.

\(^{12}\)Seim and Sinkinson (2016) provide evidence of mixed strategy pricing in online markets. Arguably, other reasons could also explain price dispersion in these markets, e.g. the use of algorithmic pricing (Chen et al., 2016).

\(^{13}\)The figure shows prices for both Amazon and Barnes & Noble, hardcover and paperback. Separate
Figure 1: Patterns in prices for online bookstores

(a) Residual price dispersion

(b) Residual price differences between stores

Notes: This figure shows patterns in online book prices. Panel (a) shows price dispersion after partialing out book-store-format means. Panel (b) takes the difference in prices at a given date between stores for the same book-format.

common fluctuations, e.g., fluctuations for particular books over time that are common across stores. Indeed, there are price differences between stores, which also fluctuate substantially even after taking out constant mean differences by book.

The interaction between price discrimination and search costs can also explain the dispersion in the relative prices of the two formats. Whereas existing search models cannot capture fluctuations in relative prices because they do not allow for price discrimination, our model predicts that search costs not only lead to price differences across stores, but also to price differences across different versions of the book sold within a store and over time. Figure 2 summarizes patterns in relative prices. Panel 2a shows the distribution of relative prices between the hardcover and the paperback of a given title. One can see that there is substantial variation in relative prices, partly due to differences across different titles, and partly due to variation in such relative prices over time. Panel 2b shows the variation in relative prices after partialing out book-store means. One can see that relative prices between the hardcover and the paperback versions also move over time, and that such variation is not just due to variation across books, but also to variation within book titles.

In sum, the market for online books is characterized a series of stylized facts. First, the norm is that all booksellers offer the hardcopy and the paperback versions of the figures for each book format and store exhibit similar distributions. Note that average prices for hardcover and paperback are roughly $20 and $10, respectively. Therefore, variation of a few dollars can imply substantial variation in prices.
Notes: This figure shows patterns in relative prices. Panel (a) shows the distribution of relative prices, with a peak around 2, i.e., the hardcover version of a book is about twice as expensive as the paperback version on average. Panel (b) shows residual variation in relative prices after partialling out book-store means.

3 The Model

3.1 Model Description

Consider a market served by two competing firms,\(^{14}\) who carry either one or the two goods that are available for sale: one with high quality \(q^H\) and high costs \(c^H\), and another one with lower quality \(q^L\) and lower costs \(c^L\);\(^{15}\) we use \(\Delta q \equiv q^H - q^L > 0\) and \(\Delta c \equiv c^H - c^L > 0\)

\(^{14}\)We will sometimes refer to firms as stores.

\(^{15}\)Without substantial effort, our model could be interpreted as one of quantity discounts, with firms offering the different quantities of the same product to consumers with either low or high demands. Results would go through as long as costs are not linear in the quality; for instance, if bigger bundles
to denote the quality and cost differences across goods.\(^\text{16}\)

There is a unit mass of consumers who buy at most one good. Consumers differ in their preferences over quality. A fraction \(\lambda\) have a low valuation for quality \(\theta^L\), while the remaining \(1 - \lambda\) fraction of consumers have a high quality valuation \(\theta^H\), with \(\Delta \theta \equiv \theta^H - \theta^L > 0\).\(^\text{17}\) As in Mussa and Rosen (1978), a consumer of type \(i = L, H\) who purchases good \(j = L, H\) at price \(p^j\) obtains net utility \(u^i = \theta^i q^j - p^j\). We assume that the gross utility of a low type (high type) from consuming the low (high) quality product always exceeds the costs of producing it, i.e., \(c^i < \theta^i q^i\) for \(i = L, H\).

The timing of the game is as follows. First, firms simultaneously decide which product(s) they offer for sale (or “product line”). Carrying a product entails an arbitrarily small fixed cost \(\epsilon\) for the firm.\(^\text{18}\) Once chosen, firms observe the product line of the rival but consumers don’t. Second, firms simultaneously choose the prices for the product(s) they carry and consumers visit the stores in order to learn firms’ product choices and their respective prices.

Following Varian (1980), we assume that there is a fraction \(\mu \leq 1\) of consumers who always visit the two stores (the shoppers), and hence know where to find the cheapest product of each quality type.\(^\text{19}\) Since the remaining \(1 - \mu\) fraction of consumers only visit one store (the non-shoppers),\(^\text{20}\) they can compare the prices of the goods sold within the store they have visited, but not across stores. We assume that the non-shoppers visit one of the two stores with equal probability.\(^\text{21}\) Once consumers have visited the store(s), require costly product design features, such as packaging.

\(^{16}\)We can think of these costs as the wholesale prices at which retailers buy the products from either competitive manufacturers, or from a monopoly manufacturer. Endogenizing the qualities of the products or the costs faced by the retailers is out of the scope of this paper.

\(^{17}\)It is possible to extend our main results to the case in which there are \(N\) potential qualities, \(N\) consumer types and \(N\) competing firms, as discussed in Section 7.

\(^{18}\)This is only used as an equilibrium selection device in the case of no search costs (Section 4). This is in line with CR, who focus on equilibria at which both firms make strictly positive profits.

\(^{19}\)There is a fundamental distinction between introducing search costs which are equal across consumers, versus introducing a fraction of uninformed consumers. It is well known that in a (single-product, homogeneous good) Bertrand model, the former gives rise to the Diamond Paradox, such that all firms charge the monopoly price and consumers do not engage in search. A similar outcome would arise in our set-up. Varian’s approach, which we adopt here, avoids the Diamond Paradox. Furthermore, as already noted, the empirical evidence reports that a large fraction of consumers are uninformed (De los Santos et al., 2012).

\(^{20}\)An implicit assumption is that the fractions \(\mu\) and \(\lambda\) are uncorrelated. As we discuss in Section 7, our main results do not change if we allow for correlation between \(\mu\) and \(\lambda\).

\(^{21}\)In some settings it may be reasonable to assume that non-shoppers observe product lines but not their prices. Accordingly, we have also considered the case in which non-shoppers visit the store that carries their preferred product (and split randomly between the two stores in case they both carry it). The main results of the paper (Propositions 1 and 2) remain unchanged. The proofs are available form
they buy the product that gives them higher utility, provided it is non-negative. In case of indifference, low (high) type consumers buy the low (high) quality product.

3.2 Preliminaries

We start by characterizing the benchmark solutions of monopoly and marginal-cost pricing. This will serve to introduce some concepts and assumptions to be used in the rest of the analysis.

The monopoly solution A monopolist carrying both products that is able to perfectly discriminate consumer types would extract all their surplus by charging the (unconstrained) monopoly prices $p^i = \theta^i q^i$, for a per unit profit of $\pi^i = \theta^i q^i - c^i$, $i \in \{L, H\}$. This holds true regardless of the number of consumers of each type, and regardless of the relative profitability of serving one type or another. This is no longer true however, as we move to the more relevant case of a multi-product monopolist that cannot perfectly discriminate across consumers.

At the optimal solution, the following incentive compatibility constraints must hold

$$\theta^i q^i - p^i \geq \theta^j q^j - p^j, \quad (IC^i)$$

for $i, j \in \{L, H\}$ and $i \neq j$, which can also be re-written as

$$p^i \leq \theta^i q^i - \left(\theta^i q^j - p^j\right).$$

The second term on the right-hand side of the inequality represents consumers’ information rents, i.e., the minimum surplus a type $i$ consumer needs to obtain to be willing to buy good $i$ instead of good $j \neq i \in \{L, H\}$. This expression already highlights an important trade-off between competition and firms’ incentives to discriminate through quality choices. In particular, firms find it relatively less appealing to carry good $j \in \{L, H\}$ when the other firm is also carrying it, not only because competition reduces profits, but also because a lower $p^j$ increases the information rents on good $i \neq j \in \{L, H\}$. This trade-off will play an important role in the analysis that follows.

Our first assumption is standard in models of second-degree price discrimination by a monopolist (Tirole, 1988): a monopolist carrying both goods finds it optimal to sort consumers out. For the multi-product monopolist, the incentive compatible (i.e., constrained the authors upon request.)
monopoly) prices are thus

\[
\begin{align*}
  p^L &= \theta^L q^L \quad \text{and} \\
  p^H &= \theta^H q^H - \Delta \theta q^L \\
      &= \theta^L q^L + \theta^H \Delta q.
\end{align*}
\]

The alternative for the monopolist is to only sell good \( H \) to the high types at the (unconstrained) monopoly price \( \theta^H q^H \), thus avoiding to leave information rents to the high types but also giving up the profits on good \( L \).\(^{22}\) To guarantee that this alternative is indeed less profitable than selling the two goods requires that the profit from selling good \( L \) to the low types be enough to compensate for the information rents that must be left with the high types:\(^{23}\)

\[(A1) \quad \lambda \pi^L \geq (1 - \lambda) \Delta \theta q^L.\]

Note that \((A1)\) is evaluated at monopoly prices. Assuming that a monopolist prefers to carry all qualities does not necessarily imply that the same holds true when competition drives prices below their monopoly level.

The competitive solution Our second assumption is also standard in models of second-degree price discrimination by competing firms: there is no ‘bunching’ at the competitive solution. This requires marginal-cost pricing to be incentive compatible, which is equivalent to assuming that the high types are willing to pay for the extra cost of high quality whereas the low types are not:

\[(A2) \quad \Delta c \in (\theta^L \Delta q, \theta^H \Delta q).\]

Implicit in \((A2)\) is the standard property (e.g., see CR and Johnson and Myatt 2006) that the cost of providing quality must be strictly convex in quality, i.e., \( c^H / q^H > c^L / q^L \); otherwise, either type would buy the high quality product or nothing at all.\(^{24}\)

\(^{22}\)Note that this alternative assumes that serving the high types with product \( H \) is more profitable than serving all consumers with product \( H \) at price \( \theta^L q^H \). This is guaranteed by our assumption \((A3)\) below.

\(^{23}\)In turn, \((A1)\) also guarantees that the optimal price for a monopolist that only carries good \( L \) is \( \theta^L q^L \). In particular, serving all consumers at this price is more profitable than just serving the high types at \( \theta^H q^L \).

\(^{24}\)Note also that convexity ensures that there is a non-empty region of \( \lambda \) values for which \((A1)\) and \((A2)\) are valid.
**Minmax profits** Assumption (A2) implies that the highest profits that can be made out of product \( i \in \{L, H\} \) when good \( j \neq i \) is priced at marginal costs, i.e., good \( i \)'s minmax profits, are strictly positive. In particular, if good \( L \) is sold at \( c^L \), good \( H \) can at most be sold at the highest price that satisfies the high types' incentive compatibility constraint, i.e., \( c^L + \theta^H \Delta q \), thus giving per unit profits of

\[
\varphi^H \equiv \theta^H \Delta q - \Delta c > 0.
\]

The minmax profits for good \( H \) are always strictly below monopoly profits given that, for all values of \( c^L \), good \( L \) imposes a competitive constraint on good \( H \).

In turn, if good \( H \) is sold at \( c^H \), good \( L \) can at most be sold at the highest price that satisfies the low types' participation and incentive compatibility constraints, i.e., \( \min \{ \theta^L q^L, c^H - \theta^L \Delta q \} \), thus giving per unit profits of

\[
\varphi^L \equiv \min \{ \pi^L, \Delta c - \theta^L \Delta q \} > 0.
\]

For \( c^H \geq \theta^L q^H \), the participation constraint binds first, so that good \( L \) can be sold at the monopoly price even when good \( H \) is priced at marginal cost. Alternatively, for \( c^H < \theta^L q^H \), the incentive compatibility constraint binds first, so that the minmax profits for good \( L \) are strictly below monopoly profits.

In sum, the per unit profits that a firm that monopolizes good \( i \in \{L, H\} \) loses when product \( j \neq i \) is made available at marginal cost equal \( \pi^i - \varphi^i \geq 0 \), with equality only for good \( L \) when \( c^H \geq \theta^L q^H \). These expressions will play an important role in the analysis that follows.\(^{25}\)

Last, in order to reduce the number of cases we need to consider without affecting our results, we assume that \( \lambda \geq 1/2 \) and:

\[\text{(A3)} \quad (1 - \lambda)(\pi^H - \Delta \theta q^L) \geq \theta^L q^H - c^H.\]

Assumption (A3) simply states that a firm prefers to sell product \( H \) to the high types at the constrained monopoly price, rather than reducing it to \( \theta^L q^H \) so as to also serve the low types.\(^{26}\)

We are now ready to solve the game. We start by analyzing the case in which all consumers are shoppers, \( \mu = 1 \) (i.e., no search costs), then move on to introducing an arbitrarily small fraction of non-shoppers, \( \mu \to 1 \), and finish by providing a full equilibrium characterization for all parameter values, \( \mu \in [0, 1) \).

\(^{25}\)We will sometimes express profit expressions as functions of \( \varphi^H \) and \( \varphi^L \). The following equalities will be particularly useful throughout the analysis: \( \pi^H - \Delta \theta q^L = \pi^L + \varphi^H \), and if \( c^H \geq \theta^L q^H \) then \( \theta^L q^H - c^H = \pi^L - \varphi^L \).

\(^{26}\)Note that assuming \( c^H \geq \theta^L q^H \) would make (A3) redundant as the left hand side is always positive.
4 Escaping the Bertrand Paradox

In this section we characterize the Subgame Perfect Equilibrium (SPE) product choices and subsequent pricing decisions given those choices, in the absence of search costs. Specifically, we assume that all consumers are shoppers, i.e., $\mu = 1$. We write $(\phi_i, \phi_j)$ to denote firms’ product choices, with $\phi_i \in \{\emptyset, L, H, LH\}$, and use $\Pi(\phi_i, \phi_j)$ to denote the profits of firm $i$ at the pricing stage given those choices, $i \neq j \in \{1, 2\}$.

First, it is easy to show that there cannot exist a SPE involving overlapping product choices. The reason is that, with no search costs, Bertrand competition drives prices down to marginal costs for each overlapping product. Hence, firms lose no profits when dropping such products, but save on the fixed cost $\epsilon \rightarrow 0$ of carrying a product.

The remaining candidate equilibria are of two types: either firms specialize in selling distinct products: $(L, H)$ (“specialization”); and/or one firm monopolizes the market while the other firm remains inactive: $(LH, \emptyset)$, $(L, \emptyset)$, or $(H, \emptyset)$ (“monopolization”).

Analyzing whether $(L, H)$ constitutes a SPE requires characterizing profits at the subgames to which firms could deviate, i.e., $(L, LH)$ and $(LH, H)$, as well as those at $(L, H)$. At subgame $(L, LH)$, good $L$ is priced at marginal cost $c^L$, so that firm $L$ makes zero profits while firm $LH$ makes minimax profits on product $H$, $(1 - \lambda)\phi^H > 0$. Thus, even though firm $LH$ is a monopolist over product $H$, competition from good $L$ forces the firm to give up strictly profits $(1 - \lambda)(\pi^H - \phi^H) > 0$. Similarly, at subgame $(LH, H)$, good $H$ is priced at marginal cost $c^H$, so that firm $H$ makes zero profits while firm $LH$ makes minimax profits on good $L$, $\lambda\phi^L$. Note that competition from good $H$ forces the firm to give up profits $\lambda(\pi^L - \phi^L) \geq 0$, which are strictly positive only when $c^H < \theta^Lq^H$.

In contrast, at $(L, H)$ a pure strategy equilibrium does not exist. This stems from an important result: at any pure strategy equilibrium, firms’ prices must satisfy incentive compatibility. Otherwise, the firm selling good $H$ would sell nothing and would thus be better off reducing its price until incentive compatibility is achieved. However, if the high types’ incentive compatibility constraint is binding, the firm carrying good $L$ could in turn attract all customers by slightly reducing its own price. Since these opposing forces destroy any candidate pure strategy, the equilibrium has to be in mixed strategies.

In turn, since firms cannot obtain profits below their minmax, and these are strictly positive, it follows that all prices in the support of the mixed strategies must strictly exceed marginal costs.

This has meaningful implications for equilibrium product choices. First, since at $(H, L)$ product $L$ is priced above marginal costs, the firm that carries good $H$ makes profits strictly above its minmax, and thus, strictly above the profits it makes when it also carries good $L$, i.e., at $(LH, L)$. In other words, the price of good $H$ does not have
to be discounted as much at \((H, L)\) as under \((LH, L)\) given that the information rents of the high types are lower the higher the price of good \(L\).

Similarly, at \((L, H)\), product \(H\) is priced \textit{above} marginal costs. Thus, the firm that carries good \(L\) makes profits (weakly) above its minmax, and thus (weakly) above those at \((LH, H)\). If \(c^H \geq \theta^L q^H\), the profit comparison is weak: since good \(H\) does no impose a competitive constraint on good \(H\), firm \(L\) loses no profits by also carrying good \(H\). However, the fixed cost \(\epsilon\) breaks the indifference in favour of \((L, H)\). In sum, since neither firm can gain by deviating from \((L, H)\), the “specialization equilibrium” constitutes a SPE of the game with no search costs.

Last, we also need to consider the subgames in which one firm chooses not to carry any product. Among these, the only relevant one is \((LH, \emptyset)\), as \((A1)\) implies that a monopolist is better off carrying the two products. In line with the analysis of entry games followed by Bertrand competition, it is straightforward to see that \((LH, \emptyset)\) constitutes a SPE of the game with no search costs: entry would drive prices down to marginal costs, not allowing the firm to recover \(\epsilon\).

The following Proposition summarizes our main result of this section:

\textbf{Proposition 1} Assume \(\mu = 1\). All Subgame Perfect Equilibria (SPE) involve non-overlapping product choices. In particular, there exist two (pure-strategy) SPE: the “specialization equilibrium” \((L, H)\), and the “monopolization equilibrium” \((LH, \emptyset)\).

\textbf{Proof.} See the Appendix. \(\blacksquare\)

In equilibrium, either firms avoid competition completely, ending up in the “monopolization”equilibrium; or they mitigate it through product choice, ending up in the “specialization” equilibrium. This result is fully in line with CR,\(^{28}\) who were the first to show that simultaneous quality choices followed by price competition give rise to non-overlapping product lines. In the next section we assess whether this prediction is robust to introducing search frictions.

\(^{27}\)If \(c^H < \theta^L q^H\), there also exists a symmetric mixed strategy equilibrium such that firms choose \(L\) and \(H\) with positive probability. On the contrary, if \(c^H > \theta^L q^H\), this equilibrium does not exist as it is dominated by playing \(LH\).

\(^{28}\)The “monopolization” equilibrium is also present in CR, although their main focus is on equilibria in which both firms make positive profits.
5 Back to the Bertrand Paradox

Before solving the game for \( \mu \in [0,1) \), in this section we show that an arbitrarily small amount of search costs is enough to induce firms to choose overlapping product lines.\(^{29}\)

To explore this in more detail, let us first consider the subgame \((LH, LH)\) (“overlapping”). Search costs, no matter how small, imply that marginal cost pricing is not an equilibrium as firms could make positive profits out of the non-shoppers. Similarly, setting prices at the (constrained) monopoly level is not in equilibrium either as firms would have incentives to charge slightly lower prices so as to attract the shoppers. More generally, search costs rule out any equilibrium in pure strategies as firms face a trade off between charging high prices to exploit the non-shoppers versus charging low prices to attract the shoppers. Since firms must be indifferent between charging any price in the support, expected equilibrium profits can be computed by characterizing profits at the upper bound, where firms serve their share of non-shoppers at (constrained) monopoly prices,

\[
\Pi(LH, LH) = \frac{1 - \mu}{2} \left[ \lambda \pi^L + (1 - \lambda)(\pi^H - \Delta \theta q^L) \right]. \tag{1}
\]

Importantly, each firm’s profits are a fraction \((1 - \mu)/2\) of the multi-product monopolist’s profits because firms only make profits out of the non-shoppers. This is true in expectation only, as for prices below the upper bound firms make profits out of the shoppers too (which compensate for the lower profits made out of the non-shoppers at prices below the upper bound). As \(\mu\) approaches 1 and all customers become shoppers, the equilibrium price distributions concentrate around marginal costs, and firms’ profits are driven down to (almost) zero. The Bertrand outcome is thus restored.

Could firms escape from the Bertrand paradox by having one of them drop one product, either \(L\) or \(H\)?\(^{30}\) Let us first analyze the incentives of moving from \((LH, LH)\) to \((H, LH)\). Since a pure strategy equilibrium does not exist, and firms have to be indifferent across all prices in the support, expected profits for product \(H\) equal those of serving the non-shoppers at the upper bound. Since firm \(H\) is not constrained by incentive compatibility, its optimal price at the upper bound is the (unconstrained) monopoly price. Its expected profits become

\[
\Pi(H, LH) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H. \tag{2}
\]

Since firm \(H\)’s profits are a fraction \((1 - \mu)/2\) of monopoly profits, comparing (1) and (2) is equivalent to assessing the monopolist’s incentives to carry the high quality good

\(^{29}\)Technically speaking \(\mu\) must satisfy \((1 - \mu) \min\{\pi^L, (1 - \lambda)\pi^H\}/2 > \epsilon\), so it can be made arbitrarily close to 1 as \(\epsilon \rightarrow 0\).

\(^{30}\)No firm has incentives to drop both products altogether as they both make positive profits at \((LH, LH)\).
only versus carrying the two goods. Assumption (A1) guarantees that (1) exceeds (2) as the losses from not selling the low quality product exceed the information rents left to the high types. Thus, even though product $L$ erodes the rents made on product $H$, the firm is better off carrying it.

The alternative is for one of the two firms to drop product $H$, thus moving from $(LH, LH)$ to $(L, LH)$. Now, the expected profits of firm $L$ must be equal to the profits of serving all the non-shoppers at the unconstrained monopoly price,

$$\Pi(L, LH) = \frac{1 - \mu}{2} \pi^L,$$

again a fraction $(1 - \mu)/2$ of monopoly profits. This payoff is strictly less than (1) since the firm gives up the extra profit that firm $L$ could make by selling the high quality good to the non-shopper high types.

In sum, the presence of non-shoppers restores firms’ monopoly power over the non-shoppers, while competition for the shoppers implies that firms make no expected profits out of them. Hence, firms’ incentives to price discriminate through product choice mimic those of the monopolist. Consequently, in the presence of infinitesimally small search costs, there exists a SPE with overlapping product lines $(LH, LH)$, in stark contrast with CR’s prediction.

To assess whether this equilibrium is unique or not, first note that the rents created in the presence of search costs destroy the “monopolization” equilibrium characterized above. The “specialization” equilibrium is also ruled out when the cost of providing high quality is sufficiently large, $c^H \geq \theta^L q^H$ (or equivalently, if the cost of providing quality is sufficiently convex). Starting at $(L, H)$, firm $L$ is strictly better off adding product $H$ given that under $(LH, H)$ it can now price discriminate the non-shoppers without eroding its profits on good $L$. Indeed, the firm would be able to increase its profits by $(1 - \mu) (1 - \lambda) \phi^H/2 > 0$ from selling the high rather than the low quality product to the non-shopper high types, while it would still make profits $\lambda \pi^L$ out of the low types.

In contrast, if the costs of high quality are low, the addition of good $H$ erodes the rents of good $L$, making firm $L$ worse off: the rents on product $H$ are infinitesimally small while the profits on good $L$ would go down by $\lambda (\pi^L - \phi^L) > 0$. Similarly, firm $H$ does not want to add product $L$ as its profits would fall by $(1 - \lambda) (\pi^H - \phi^H) > 0$. Thus, the “specialization” equilibrium survives the introduction of infinitesimally small search costs but only when the costs of providing high quality are sufficiently low.

Our second Proposition summarizes these results.

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31Note that in this case the firm would serve both the low and the high-types, since the latter are also willing to buy the low quality product at the unconstrained monopoly price for product $L$. 

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Proposition 2 Assume $\mu \to 1$.

(i) The “overlapping” equilibrium $(LH, LH)$ constitutes a SPE. At $(LH, LH)$, equilibrium prices approximate marginal costs.

(ii) The “specialization” equilibrium $(L, H)$ constitutes a SPE if and only if $c^H < \theta^L q^H$. At $(L, H)$, equilibrium prices are strictly above marginal costs.

(iii) The “overlapping” equilibrium is unique if and only if the cost of providing high quality is sufficiently large, i.e., $c^H \geq \theta^L q^H$.

Proof. See the discussion above. A formal derivation can be found as a particular case of the proof to Proposition 7. ■

Propositions 1 and 2 form a remarkable result: arbitrarily small search frictions (weakly) reduce prices by giving rise to an equilibrium with overlapping product lines.\footnote{Using the terminology of Armstrong (2015), non-shoppers create a positive search externality to the shoppers, who end up paying lower prices.} It is worth pointing out that the existence of the equilibrium with overlapping product lines is not driven by the rents created by search frictions. Indeed, as this section has demonstrated, when search costs are arbitrarily small such rents are close to zero and yet the “overlapping” equilibrium exists.

In contrast, in previous papers analyzing quality choices followed by imperfect competition (Gal-Or, 1983; Johnson and Myatt 2003 and 2015; Stole, 1995), the “overlapping” equilibrium exists only if the rents created by imperfect competition are high enough (e.g., few firms competing à la Cournot, or price competition with sufficiently differentiated products). In those papers, just as in CR, there is a tension between competition and price discrimination: competition reduces the rents on the overlapping products at the same time as it enlarges consumers’ information rents, thus reducing the gains from price discrimination. Under the “overlapping” equilibrium that arises with search costs, such a tension is not present because firms only care about the profits made out of the non-shoppers, out of which they obtain monopoly profits (in expectation). Thus, firms’ product choices are solely driven by their incentives to discriminate consumers, leading them to carry the full product range. This shows that the impact of search costs on product choices, and through these on prices, is fundamentally different as that of other forms of imperfect competition.

6 Equilibrium Product and Price Choices

In this section we characterize equilibrium product and price choices for all values of $\mu < 1$. We show that the “overlapping” equilibrium exists for all $\mu < 1$, but that the
“specialization” equilibrium fails to exist when $\mu$ is sufficiently low or, for all $\mu$, when $c^H$ is sufficiently high. In general, the “overlapping” equilibrium is more likely to be unique the higher the level of search costs and/or the higher the costs of providing high quality.

We again proceed by backwards induction by first analyzing equilibrium pricing behavior and then product line choices. The pricing subgames will also serve to understand pricing decisions for non-overlapping product configurations, which may prove relevant to cases in which product choices may be constrained by factors outside our model.

6.1 Pricing Behavior

We first provide an important property of pricing behavior by multi-product firms.

**Lemma 1** In equilibrium, multi-product firms choose incentive compatible prices for their products, i.e., $\Delta p \in [\theta^L \Delta q, \theta^H \Delta q]$.

**Proof.** See the Appendix. ■

Lemma above shows that it is always optimal for a multi-product firm to choose prices that satisfy incentive compatibility. The intuition is simple. If the price of the high quality product is too high so that all consumers buy the low quality product, it is profitable for the firm to reduce $p^H$, while leaving $p^L$ unchanged, so as to attract the high types and obtain a larger profit margin. Similarly, if the price of the high quality product is too low so that all consumers buy it, it is profitable for the firm to increase $p^H$, while leaving $p^L$ unchanged, so as to extract more surplus from the high types as these are willing to pay more for higher quality. This result constitutes an important departure from Varian (1980), as it implies that the price of one product cannot be picked independently from the price of another product within the same store.\(^{33}\)

We are now ready to characterize equilibrium pricing at every possible subgame.

**Full product overlap** We start by considering subgames with full product overlap: $(LH, LH)$, $(L, L)$, and $(H, H)$. The last two are similar to Varian’s. Since single-product firms selling the same product are not constrained by incentive compatibility, they play a mixed strategy equilibrium with an upper bound equal to the (unconstrained) monopoly price. Under $(L, L)$ all consumers are served, but under $(H, H)$ the low types are left out of the market. We thus focus here on the case of full product overlap among multi-product firms, $(LH, LH)$.

\(^{33}\)This is in contrast to Johnson and Myatt (2015) prediction. In a model of quality choice followed by Cournot competition, they find conditions under which the equilibrium prices chosen by multi-product oligopolists are close to the single-product prices.
Proposition 3 Given product choices \((LH, LH)\), there does not exist a pure strategy equilibrium. The equilibrium must be in mixed strategies, and it must satisfy the following properties:

(i) At the upper bound of the price support, firms choose the (constrained) monopoly prices, \(p^H = \theta^H q^H - \theta^L \Delta q\) and \(p^L = \theta^L q^L\). Thus, at the upper bound, the high types’ incentive compatibility constraint is binding, \(\Delta p = p^H - p^L = \theta^H \Delta q\).

(ii) At the lower bound of the price support, firms choose prices that are strictly above marginal costs, \(p^i > c^i\) for \(i = L, H\), and such that the high types’ incentive compatibility constraint is not binding, \(\Delta p = p^H - p^L < \theta^H \Delta q\).

Proof. See the Appendix. ■

The non-existence of pure strategy equilibria is shared with most models of simultaneous search, starting with Varian (1980). It stems from firms’ countervailing incentives, as on the one hand they want to reduce prices to attract the shoppers, but on the other, they want to extract all rents from the non-shoppers.

Despite this similarity, our analysis shows that equilibrium pricing by multi-product firms has a distinctive feature: it is constrained by incentive compatibility (Lemma 1). This comes up clearly when characterizing the upper bound of the price support: firms are not able to extract all the surplus from the non-shopper high types because firms have to give up information rents \(\Delta \theta q^L\).

Since firms make strictly positive profits at the upper bound, prices at the lower bound must be strictly above marginal costs. The reduction in prices from the upper to the lower bound is more pronounced for the high quality product than for the low quality one. Competition for the high types is fiercer because selling the high quality product is more profitable. In turn, this implies that at the lower bound, the incentive compatibility constraint for the high types is not binding, so that the price wedge between the two products at the upper bound is wider than at the upper bound. We can conclude that high quality products are relatively cheaper during periods of “sales” à la Varian, i.e., when both goods are priced at the lower bounds of the price supports. Even when firms do not price the two goods simultaneously at the lower bound, the relative price difference never exceeds the one under monopoly, \(\theta^H \Delta q\), as otherwise incentive compatibility would not be satisfied (Lemma 1). Thus, competition among multi-product firms reduces the relative prices of the two goods.

Since firms have to be indifferent between charging any price in the support, including the upper bounds, expected equilibrium profits are unambiguously given by

\[
\Pi (LH, LH) = \frac{1 - \mu}{2} \left[ \lambda \pi^L + (1 - \lambda) \left( \pi^H - \Delta \theta q^L \right) \right].
\]  
(3)
Just as we noted in the previous section, these profits are a fraction \((1 - \mu) / 2\) of the (constrained) monopoly profits.

At the lower bound, each firm attracts all the shoppers plus its share of the non-shoppers of each type. Hence, expected profits can also be expressed as a function of the lower bounds,

\[
\Pi (LH, LH) = \frac{1 + \mu}{2} \left[ \lambda (p^L - c^L) + (1 - \lambda) (p^H - c^H) \right].
\]

(4)

Since there are two goods, and only one profit level, as defined in equations (3) and (4), the problem has an extra degree of freedom: there are potentially many price pairs \(p^L > c^L\) and \(p^H > c^H\) satisfying \(\Delta p < \theta^H \Delta q\) that yield the same equilibrium profits. This implies that, even though equilibrium profits are unique and well defined, there might be multiplicity of mixed strategy equilibria.

Because the incentive compatibility constraint of the high types is binding at the monopoly solution, a natural equilibrium to consider is one in which firms keep on pricing the low quality product as if they were just selling that product, but adjust their pricing for the high quality one. The following Lemma characterizes such an equilibrium:

**Lemma 2** Given product choices \((LH, LH)\), there exists a mixed-strategy equilibrium in which firms choose \(p^L\) in \([p^L, \overline{p}^L]\) according to

\[
F^L(p^L) = \frac{1 + \mu}{2\mu} - \frac{1 - \mu}{2\mu} \frac{(\overline{p}^L - c^L)}{(p^L - c^L)}
\]

and such that, for given \(p^L\), the price \(p^H\) is chosen in \([p^H, \overline{p}^H]\) to satisfy

\[
\frac{p^H - c^H}{p^L - c^L} = \frac{\overline{p}^H - c^H}{\overline{p}^L - c^L}
\]

where

\[
p^i = c^i + \frac{1 - \mu}{1 + \mu} (\overline{p}^i - c^i) > c^i,
\]

and \(\overline{p}^i\) are the (constrained) monopoly prices, for \(i = L, H\).

**Proof.** See the Appendix.

The proposed equilibrium has several appealing features. While firms price the low quality product as if they were just selling that product (as in Varian’s model), on average they choose lower prices for the high quality product than when they only sell that product. This is a direct implication of the fact that the firm cannot extract all the surplus of the non-shopper high types. Indeed, the distribution of \(p^H\),

\[
F^H(p^H) = \frac{1 + \mu}{2\mu} - \frac{1 - \mu}{2\mu} \frac{(\overline{p}^H - c^L)}{(p^H - c^L)}
\]

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has the same functional form as in Varian. However, since the upper bound $p^H$ is the constrained monopoly price, the whole distribution puts higher weight on lower prices all along the support than in the independent products case.

Under this equilibrium, the choice of $p^L$ results in a unique choice of $p^H$ such that the relative profit margin of the two products remains constant along the whole support; see equation (5). In particular, the relative markups of the two products are the same as under monopoly. That is, under this equilibrium, competition affects the price levels but not the price structure within the firm.

The price difference that is embodied in this price structure can be expressed as

$$\Delta p = \kappa \theta^H \Delta q + (1 - \kappa) \Delta c.$$  

Consistently with Lemma 1, the price difference is a weighted average between $\theta^H \Delta q$ (i.e., the price difference at the monopoly solution) and $\Delta c$ (i.e., the price difference at the competitive solution), where the weight $\kappa = (p^L - c^L) / (\bar{p}^L - c^L)$ represents the distance to the upper bound. At the upper bound, when the incentive compatibility constraint of the high types is binding, the price difference is maximal, $\Delta \bar{p} = \theta^H \Delta q$. As we move down the support, the incentive compatibility constraint is satisfied with slack and the price difference narrows down. The difference is minimal at the lower bound, when $\kappa = (1 - \mu) / (1 + \mu)$. Importantly, as $\mu$ approaches one, the prices at the lower bound converge to marginal costs, and the price gap approaches $\Delta c$. The equilibrium would thus collapse to the competitive solution. On the other extreme, as $\mu$ approaches zero, the prices at the lower bound converge to monopoly prices so that the price gap approaches $\theta^H \Delta q$. The equilibrium would thus collapse to the monopoly solution.

Partial product overlap Let us now characterize equilibrium pricing in the subgames with partial overlap: $(L, LH)$, $(H, LH)$. Interestingly, even though the single-product firm does not face an incentive compatibility constraint within its store, its pricing is nevertheless affected by incentive compatibility considerations through the effect of competition.

The following Proposition characterizes the price equilibrium at the $(L, LH)$ subgame.

**Proposition 4** Given product choices $(L, LH)$:

(i) A pure strategy equilibrium does not exist.

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34 Clearly, there exists another equilibrium with the same price supports and the same price distribution for good $L$ but in which the firm randomizes the price of good $H$, given the choice of $p^L$, such that the two prices remain incentive compatible. Again, this multiplicity is inconsequential for the purposes of this analysis as all equilibria yield equal expected profits.
(ii) At the unique mixed-strategy equilibrium, firm LH charges \( p^H = p^L + \theta^H \Delta q \), and both firms choose \( p^L \) in \([p^L, \theta^L q^L]\), with firm L putting a probability mass at the upper bound.

**Proof.** See the Appendix. ■

In equilibrium, the two firms choose random prices for the low quality product over a common support. In turn, given its price choice for the low quality good, the multi-product firm prices the high quality product to just comply with incentive compatibility for the high types. Hence, unlike the previous case, the price difference between the two products remains constant at \( \theta^H \Delta q \) over the whole support, and the density of prices for the high quality product is the same as that for the low quality product (just shifted out to the right by \( \theta^H \Delta q \)). It follows that, whenever the multi-product firm has the low price for the low quality product, all the shoppers (both the low or the high types) buy from it. Otherwise, the single-product firm serves all the shoppers, including the low and the high types.

It is worth pointing out that the multi-product firm charges lower prices on average as compared to the single-product firm. The reason is that, when it has the low price, its ability to discriminate between the low and the high types allows the multi-product firm to make extra profits \( \mu (1 - \lambda) \phi^H \) out of the shopper high types. Since the multi-product firm has stronger incentives to undercut its rival’s price, the single product firm has to put a probability mass at the upper-bound. In turn, since the two firms cannot put a mass at the same price, it follows that when the single-product firm is pricing at the upper bound it is only selling to the non-shoppers with probability one. Importantly, this implies that the single-product firm’s profits are a fraction \((1 - \mu)/2\) of the monopoly profits

\[
\Pi (L, LH) = \frac{1 - \mu}{2} \pi^L.
\]

Now we turn to characterizing the price equilibrium at the \((H, LH)\) subgame.

**Proposition 5** Given product choices \((H, LH)\), there exists \( \hat{\mu} \in (0, 1) \) such that:

(i) For \( \mu \leq \hat{\mu} \), there exists a unique pure strategy equilibrium: firm H chooses the (unconstrained) monopoly price \( p^H = \theta^H q^H \), and firm LH chooses the (constrained) monopoly prices, \( p^H = \theta^H q^H - \Delta \theta q^L \) and \( p^L = \theta^L q^L \).

(ii) For \( \mu > \hat{\mu} \), there does not exist a pure strategy equilibrium. In the mixed-strategy equilibrium, firm LH chooses prices \( p^H \) in \([p^H, \theta^H q^H - \Delta \theta q^L]\) with a mass on its upper bound, and \( p^L = \min \{\theta^L q^L, p^H - \theta^L \Delta q^L\} \). Firm H chooses prices \( p^H \) in \([p^H, \theta^H q^H - \Delta \theta q^L], \theta^H q^H\) with a (strictly) positive mass on its upper bound.
Proof. See the Appendix. ■

There now exists a pure strategy equilibrium as long as the fraction of shoppers $\mu$ is small enough. At this equilibrium, the multi-product firm charges the (constrained) monopoly prices, while the single-product firm charges the (unconstrained) monopoly price for the high quality product. The single-product firm does not want to fight for the shoppers as it prefers to just serve the non-shoppers at the unconstrained monopoly price than fighting for the shoppers at the cost of leaving informational rents to the non-shoppers.

When the fraction of shoppers is higher, the above is no longer an equilibrium as it now pays the single-product firm to fight for the shoppers. In this case, the equilibrium must be in mixed strategies. The precise shape of the mixed strategy equilibrium depends on whether it pays firm $H$ to serve the low types or not.

If $c^H \geq \theta^L q^H$, it never pays firm $H$ to serve the low types because the costs of high quality exceed their willingness to pay for it. Thus, the two firms compete for the shopper high types only, while the low quality product is still priced at the monopoly level, $\theta^L q^L$. Since the incentive compatibility constraint of the multi-product firm is not binding, its profits are the same as if the two products were sold independently. In contrast, when $c^H < \theta^L q^H$, the low types might be tempted to buy the high quality good when its price is sufficiently low. In this case, the price of the low quality good has to be reduced below the monopoly price to achieve separation. In particular, this implies that at the lower bound of the price support of firm $LH$, the price difference between the two goods falls down to $\Delta p = \theta^L \Delta q < \Delta c$. In sum, even though the multi-product firm is a monopolist over the low quality good, competition with the rival’s high quality good forces the firm to reduce the prices of both goods.

Regarding the single-product firm, since $\theta^H q^H - \Delta \theta q^L$ is the highest price that the multi-product firm would ever charge for the high quality good, the firm will play either the (unconstrained) monopoly price, $\theta^H q^H$, or something less than the (constrained) monopoly price, $\theta^H q^H - \Delta \theta q^L$. Any price in between is unprofitable, either because it doesn’t extract enough from the non-shopper high types or because it doesn’t attract the shoppers when the multi-product firm happens to price the good at or below $\theta^H q^H - \Delta \theta q^L$. In either case, profits remain as in the pure strategy equilibrium because $\theta^H q^H$ always

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35Interestingly, there is continuity between the pure and the mixed-strategy equilibrium. The two firms charge the upper bounds of their price supports, $\theta^H q^H - \Delta \theta q^L$ and $\theta^H q^H$, with positive and identical mass. This mass fades away as $\mu$ grows larger—from one, when $\mu \to \hat{\mu}$ towards zero, when $\mu \to 1$.

36Unlike in subgame $(LH, LH)$, the equilibrium is now unique: since one firm only has one product, there are no longer two degrees of freedom as in the symmetric two product case.
belongs to the price support. Therefore, for all $\mu$,

$$\Pi(H, LH) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H,$$

again are a fraction $(1 - \mu)/2$ of the monopoly profits.

Last, just as in the previous subgame, the equilibrium price distribution used by the multi-product firm for the high quality good (weakly) first-order stochastically dominates that of the single-product firm. It follows that, on average, the price charged by the single-product firm for the high quality product exceeds the one charged by the multi-product firm.

**Non-overlap** Let us now move to characterizing equilibrium pricing in the subgames with no product overlap: $(\emptyset, L)$, $(\emptyset, H)$, $(\emptyset, LH)$ and $(L, H)$. The first three correspond to the monopoly solution already characterized in Section 2.2 above. Hence, here we turn our attention to the more interesting subgame with specialized firms, $(L, H)$.

**Proposition 6** *Given product choices $(L, H)$, there exists $\hat{\mu} \in (\hat{\mu}, 1)$ such that:*

(i) For $\mu \leq \hat{\mu}$, there exists a unique pure strategy price equilibrium: firms charge the (unconstrained) monopoly prices $p^H = \theta^H q^H$ and $p^L = \theta^L q^L$.

(ii) For $\mu > \hat{\mu}$ there does not exist a pure strategy equilibrium. At the unique mixed-strategy equilibrium, firm $L$ chooses prices $p^L$ in $[p^L, \theta^L q^L]$ with a mass on the upper bound. If $\mu \in (\hat{\mu}, \bar{\mu})$ firm $H$ chooses prices $p^H$ in $\{[p^H, \theta^H q^H - \Delta \theta q^L], \theta^H q^H\}$ with a mass on the upper bound that falls to zero as $\mu \to \bar{\mu}$; if $\mu \geq \bar{\mu}$, $\theta^H q^H$ is not part of firm $H$’s support.

**Proof.** See the Appendix. ■

Equilibrium pricing at subgames $(L, H)$ and $(H, LH)$ share some similarities. In particular, just as in Proposition 5, if the mass of shoppers $\mu$ is small enough, there exists a pure strategy equilibrium as the firm selling the high quality product is better off serving the non-shopper high types at the (unconstrained) monopoly price than competing for the shopper high types.\(^{37}\) Furthermore, there is continuity between the pure and the mixed strategy equilibrium in that the probability mass that the high quality firm puts on the (unconstrained) monopoly price fades away as $\mu$ grows larger.

The main difference between the two subgames is that, under $(L, H)$, the high quality firm chooses not to include the unconstrained monopoly price in the support when $\mu$ is

\(^{37}\)Note that the threshold for the existence of a pure-strategy equilibrium is the same under both subgames.
very large. The reason is that the profits from serving a small fraction of non-shoppers become lower than the profits from fighting for the shoppers.\footnote{At subgame \((H, LH)\), competition for good \(H\) is more intense given that both firms carry it. This explains why in that case firm \(H\) always puts mass at the unconstrained monopoly price, while at subgame \((H, L)\) firm \(H\) eventually decides not to include it in its price support.}

To illustrate this, note that the former converge to zero as \(\mu \to 1\). However, the high quality firm’s profits cannot be lower than its minmax, which is strictly positive as the firm can always make a profit margin of at least \((1 - \lambda) \varphi^H\) when selling the high quality product to the shopper high types.\footnote{This was not the case under product choices \((H, LH)\): in that case, the single-product firm’s minmax profits go down to zero as \(\mu \to 1\), as all shoppers would buy product \(H\) from the multiproduct firm if the single-product firm tried to raise its price above marginal cost.} Similarly, the profits of firm \(L\) are (weakly) above its minmax as firm \(H\) is pricing above its costs with probability one; the comparison is weak only when \(c^H \geq \theta^L q^H\): in this case, since firms are not competing for the low types, firm \(L\) makes monopoly profits in equilibrium, just as under its minmax.

### 6.2 Product Choices

We are now ready to analyze product line decisions given the continuation equilibria characterized above.

**Proposition 7** Assume \(\mu < 1\).

(i) The “overlapping” equilibrium \((LH, LH)\) constitutes a SPE for all \(\mu < 1\).

(ii) There exists \(\mu^*\) implicitly defined by

\[
(1 - \mu^*) (1 - \lambda) (\pi^H - \pi^L) = (1 + \mu^*) \lambda (\pi^L - \varphi^L)
\]

such that the “specialization” equilibrium \((L, H)\) constitutes a SPE if and only if \(\mu \geq \mu^*\). The critical threshold \(\mu^*\) is increasing in \(c^H\).

(iii) The “overlapping” equilibrium is unique if and only if \(\mu < \mu^*\).\footnote{In the proof of the Proposition we also show that there exists \(\mu^{**} \geq \mu^*\) such that if \(\mu \geq \mu^{**}\), there also exists a MSE such that firms randomize between \(L\) and \(H\). Since the condition for the existence of this equilibrium is more demanding that it is for the pure-strategy equilibrium \((L, H)\), it still holds true that the overlapping equilibrium is unique for \(\mu < \mu^*\).} A sufficient condition for uniqueness is \(c^H \geq \theta^L q^H\).

**Proof.** See the Appendix. \(\blacksquare\)

In Proposition 2 we showed that a SPE with overlapping product lines arises as soon as we add an arbitrarily small amount of non-shoppers. Proposition 7 now shows that this prediction remains valid for all values of \(\mu < 1\). The underlying logic remains the
same: the existence of the “overlapping” equilibrium hinges upon the incentives of firms to mimic those of a monopolist, and this holds true as long as the mass of non-shoppers \((1 - \mu)\) is positive, regardless of whether it is large or small.

Regarding the existence of the “specialization” equilibrium, Proposition 2 showed that it exists for \(\mu \to 1\) as long as \(c^H < \theta^L q^H\). Proposition 7 now shows that this prediction does not extend to all \(\mu < 1\). In particular, whereas \(c^H < \theta^L q^H\) is still needed, it is not sufficient: additionally, search costs have to be sufficiently low for the gains from softening competition to exceed the costs of giving up profitable opportunities to discriminate. To see this in more detail, consider the incentives to deviate from the “specialization” equilibrium by the firm carrying product \(L\). Adding product \(H\) would allow the firm to better discriminate the high types, thus making extra profits \((\pi^H - \pi^L)\) from selling product \(H\) to the non-shopper high types with probability \((1 - \mu) (1 - \lambda) / 2\). In contrast, adding product \(H\) would also intensify competition for product \(L\), forcing the firm to give up rents \((\pi^L - \varphi^L)\) on all the low types (excluding the non-shopper low types that visit the rival’s store) with probability \( (1 + \mu) \lambda / 2 \). The magnitude of the two effects coincides at \(\mu = \mu^*\), as implicitly defined in the statement of the Proposition. In turn, since in expectation firms only benefit from discriminating the non-shoppers, the softening of competition effect dominates the incentives to discriminate only when the mass of non-shoppers \((1 - \mu)\) is sufficiently small, i.e., when \(\mu \geq \mu^*\). Therefore, for \(\mu < \mu^*\) the “specialization” equilibrium breaks down, making the “overlapping” equilibrium the unique SPE of the game.

The fact that \(\mu^*\) is increasing in \(c^H\) means that, as the cost of high quality provision increases up to \(\theta^L q^H\), the set of \(\mu\) values for which the “overlapping” equilibrium is unique is enlarged; beyond that level, the “overlapping” equilibrium is the unique SPE for all \(\mu\). In case \(c^H \geq \theta^L q^H\), \(\mu^* = 1\). In words, the “specialization” equilibrium never exists in the presence of non-shoppers because if firm \(L\) adds product \(H\) it does not give up any rents on product \(L\).

We remain agnostic as to which equilibrium firms are more likely to play when there exist multiple equilibria \((c^H < \theta^L q^H \text{ and } \mu \geq \mu^*)\). Still, we want to stress that there are compelling theoretical reasons to believe that the “overlapping” equilibrium is relevant - beyond its empirical relevance, as discussed in Section 2 above. First, the Pareto criterion does not allow to rule out the “overlapping” equilibrium in general, despite the fact that it results in lower prices. In particular, the firm that carries product \(H\) at the “specialization” equilibrium is not necessarily better off than at the “overlapping” equilibrium as in the former it fails to capture the profits from serving the non-shopper low types. Furthermore, some authors have documented path dependency in equilibrium choices (Romero, 2015). Hence, the fact that the “overlapping” equilibrium exists for all
\( \mu < 1 \) while the “specialization” equilibrium only exists for \( \mu \geq \mu^* \) might be relevant if the game is played more than once. Consider for instance a simple repetition of our two-stage game and allow for the search costs to gradually fall. If the initial condition is high search costs, for which the “overlapping” equilibrium is unique, hysteresis would lead firms to keep on playing the same equilibrium even if the reduction in search costs implies that the “specialization” equilibrium eventually arises. The same would apply if the costs of quality are initially high and declining.\(^{41}\)

Last, combining the results in Proposition 1, Proposition 3, and Proposition 7, our last Lemma evaluates the impact of search costs on expected market prices and expected consumer surplus at SPE product choices.

**Lemma 3** (i) Expected prices decrease in \( \mu \in (0,1) \), with an upwards discontinuity at either \( \mu = 1 \) or at \( \mu = \mu^* \). Similarly, expected consumer surplus increases in \( \mu \in (0,1) \), with an downwards discontinuity at either \( \mu = 1 \) or at \( \mu = \mu^* \).\(^{42}\)

(ii) There exists \( \mu' \in (0,1) \) such that expected prices are higher and expected consumer surplus is lower at \( \mu = 1 \) (i.e., there are no search costs) than at \( \mu \in (\mu',1) \) (i.e., search costs are positive but not too high) whenever, in case of multiplicity, firms play the “overlapping” equilibrium.

**Proof.** See the Appendix.

The conventional wisdom that a reduction in search costs leads to lower prices applies in this model, but only when the reduction in search costs does not change equilibrium product lines. Indeed, when lower search costs induce firms to switch from the “overlapping” to the “specialization” equilibrium (either at \( \mu = 1 \) or at \( \mu = \mu^* \), depending on equilibrium selection), expected prices jump up as firms manage to mitigate competition through product choice. Hence, as long as search costs are not too high (i.e., for \( \mu > \mu' \)) and as long as firms play the “overlapping” equilibrium (i.e., which is unique for \( \mu < \mu^* \)), expected prices are lower in a market with search costs than in a frictionless market (\( \mu = 1 \)).

Similarly, as search costs go down, consumer surplus goes up with a discontinuity when

\(^{41}\)In contrast, if there are initially no search costs and the costs of quality provision are low, the market could remain at the “specialization” equilibrium as either search costs or quality costs go up, eventually giving rise to the “overlapping” equilibrium when the “specialization” equilibrium ceases to exist. However, given the overall current trend towards lower search costs, this possibility does not seem empirically relevant.

\(^{42}\)Whether the discontinuity occurs at \( \mu = 1 \) or at \( \mu = \mu^* \) depends on whether, in case of multiplicity, firms play either the “overlapping” or the “specialization” equilibrium, respectively.
firms switch from the “overlapping” to the “specialization” equilibrium. The discontinuity in consumer surplus is more pronounced than the discontinuity in expected prices because not only expected prices jump up, but also gross consumers’ surplus jumps down because of incomplete price discrimination at the “specialization” equilibrium (meaning that some high-types fail to buy their preferred good while some low-types fail to consume at all). Consumer surplus is the same at the “overlapping” equilibrium for $\mu'$ as compared to the “specialization” equilibrium for $\mu = 1$: in both cases, all consumers buy their preferred products and expected prices coincide. Hence, since consumer surplus at the “overlapping” equilibrium is increasing in $\mu$, it also follows that consumers are better off when search costs are positive but not too high (i.e., for $\mu > \mu'$) than when there are no search costs at all ($\mu = 1$). Figure 3 illustrates these results.

\footnote{Note that consumers never incur search costs since the shoppers have zero search costs and the non-shoppers do not search. Hence, search costs do not enter directly into the expression of consumer surplus; only indirectly to the extent that they affect prices and product choice.}
7 Extensions and Variations

In the preceding sections we have characterized product and price choices in a duopoly model with two possible quality levels and two consumer types, assuming no correlation between consumers’ search costs and quality preferences. In this section we discuss how one can relax each of these assumptions while preserving our main results. Our focus is on the existence of the “overlapping” equilibrium.

\( N \) products, \( N \) consumer types  Consider the case in which there are \( N \) potential products of qualities \( q^1 < \ldots < q^N \), and \( N \) consumer types with quality preferences \( \theta^1 < \ldots < \theta^N \).

As we did above, let us start by characterizing the monopoly solution and rephrasing our initial assumptions for the \( N \) products and \( N \) consumer types case. As it is well known, at the constrained monopoly solution, only the participation of the lowest quality product is binding, so that \( p^1 = \theta^1 q^1 \). For the remaining products, the constrained monopoly solution makes each consumer type indifferent between buying its preferred product or the one immediately below it in the quality ladder, i.e.,

\[
p^i = p^{i-1} + \theta^i (q^i - q^{i-1}).
\]

Thus, information rents accumulate across consumer types, and are therefore greater for higher types.

In the \( N \)-products case, assumption (A1) can be restated as requiring the monopolist to find it optimal to carry all the products, even if this requires leaving information rents to some types. Assumption (A2) can be restated by requiring that prices equal to marginal costs must induce full consumer separation. In other words, consumers of type \( i \) are willing to pay for the increase in quality from \( q^{i-1} \) to \( q^i \), while consumers of type \( i - 1 \) are not. Last, assumption (A3) can be restated by making firms prefer to sell product \( i \) to the \( i \)-types at its constrained monopoly price, rather than reducing it to \( \theta^{i-1} q^i \) so as to also serve lower type consumers.

Regarding the equilibrium for the case with no search costs, it is straightforward to show that the SPE must involve non-overlapping product choices (Proposition 1). The logic remains the same as in the 2-products case: namely, Bertrand competition drives prices down to marginal costs for overlapping products, so that dropping such products makes a firm better off by allowing it to save the fixed cost \( \epsilon \to 0 \). Since product choices do not overlap, prices are strictly above marginal costs.

Regarding the case with positive search costs, a similar logic as in the 2-products case also allows to conclude that both firms carrying all products constitutes a SPE for all \( \mu < 1 \) (Proposition 7). In the pricing stage, the upper bounds of the price supports in the mixed strategy equilibrium are the constrained monopoly prices given that at such
prices the firm only sells to the non-shoppers (since firms are symmetric, there cannot be a mass at the upper bound, so when one firm prices at the upper bound, the other firm has strictly lower prices with probability one). Thus, equilibrium profits are a fraction \((1 - \mu) / 2\) of the constrained monopoly profits just as in the 2-product case (Proposition 3).

Alternatively, if one firm deviates by dropping one or more products, the deviant reduces its profits. To see this, first note that the upper bounds in the support of the firm with fewer products are equal to the constrained monopoly prices (conditional on the products it carries), which it must play with positive probability. Such a probability mass is in place to equate the profits of the rival firm at the upper and lower bounds; without the mass, the firm that carries a larger product line would have incentives to keep on reducing prices below the lower bound, as its increased ability to price discriminate allows it to make higher profits from attracting the shoppers. Since it is not possible for the two firms to put a mass at the upper bound, it follows that when the firm with fewer products prices at the upper bound, the rival firm must be charging strictly lower prices with probability one. Again, this implies that in equilibrium, the firm carrying fewer product makes a fraction \((1 - \mu) / 2\) of the constrained monopoly profits conditional on the products it carries. The proof concludes by noting that \((A1)\) guarantees that such profits are lower than the profits from carrying the complete product line. Therefore, as we have argued, the exact same results as in the 2-product extend to the \(N\)-product case.

\(N\) products, \(N\) consumer types, \(N\) firm oligopoly We can extend the previous discussion by allowing for \(N\) firms. As already argued, \((i)\) Bertrand competition for overlapping products implies that the equilibrium when all consumers are shoppers has non-overlapping product lines; and \((ii)\) the presence of shoppers restores the monopolist’s incentives to carry the complete product line as each firm’s profits are proportional to monopoly profits (the factor being \((1 - \mu) / 2\) under duopoly or, more generally, \((1 - \mu) / N\) for the \(N\)-firms case). Since both forces are at play regardless of whether there are two or any arbitrary (finite) number of firms, the same result holds true.

Correlation between search costs and quality preferences Last, we have so far assumed that shoppers and non-shoppers are equally likely to be either high or low types. However, this may not hold in practice. For instance, if low types are lower income consumers with more time to search, then non-shoppers are more likely to be high types. Alternatively, if high types enjoy shopping for their preferred (high quality) product, then non-shoppers are more likely to be low types. Ultimately, this is an empirical question

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whose answer may vary depending on the type of product or context considered. However, as far as the predictions of the model are concerned, it is inconsequential whether the correlation between search costs and quality preferences is positive, negative or nonexistent.

To formalize this, one can introduce the parameters $\lambda_S$ and $\lambda^{NS}$, representing the fraction of low types among the shoppers and non-shoppers, respectively (if $\lambda_S > \lambda^{NS}$ there is positive correlation between search costs and quality types as the fraction of low types is higher among the shoppers than among the non-shoppers, or negative otherwise). The analysis of product and price choices without search costs remains intact since all consumers are shoppers by definition. As for the analysis with search costs, profits on good $H$ are proportional to $(1 - \lambda^{NS})$ and those on good $L$ are proportional to $\lambda^{NS}$, thus implying that the incentive structure remains unchanged. As such, the “overlapping” equilibrium always exists just as in the case with no correlation between search and quality preferences.

8 Conclusions

In this paper we have analyzed a simple model of product choice followed by price competition in markets with search costs. We have found that an arbitrarily small amount of search costs is enough to overturn the prediction that firms are always able to soften competition by carrying non-overlapping product lines, as in Champsaur and Rochet (1989). Our results extend to more general settings, including the case with more than two goods and consumer types, more than two firms, and the possibility that search costs and quality tastes are positively or negatively correlated.

Through product choice, search costs have important implications for market outcomes beyond their well studied price effects. Furthermore, we have shown that analyzing the price effects of search costs without endogenizing product choices can sometimes lead to overestimating the anticompetitive effects of search costs. In particular, a small amount of search costs can create head-to-head competition by inducing firms to carry overlapping products.

The multi-product nature of firms also adds important twists to the analysis of competition in the presence of search costs. An important departure from Varian (1980) is that goods within a store cannot be priced independently from each other. In particular, the incentives to separate both consumer types impose an upper (lower) bound on the highest price that can be charged for a high (low) quality good, given the price of the low (high) quality one. This holds true even for a single-product firm competing with a multi-product one, as price discrimination within the latter spreads to the former through
the effect of competition. In line with Varian (1980), we have also shown that search costs give rise to price dispersion when the two competing firms carry multiple products—a possibility not considered by Varian (1980). However, if one of the firms specializes in selling the high quality product, we have shown that a pure strategy equilibrium arises when search costs are high enough, i.e., the dispersion prediction might no longer hold. In particular, the market might be segmented between the non-shoppers who visit the single-product high quality store, and the remaining consumers who pay lower prices at the multi-product store.

Admittedly, there are several motives other than the ones studied in this paper that shape firms’ product choices. In particular, throughout the analysis we have assumed that firms incur an arbitrarily small fixed cost of carrying a product to highlight the strategic motives underlying product choice. However, higher fixed costs of carrying a product (which could arguably be higher for high quality products),\(^{44}\) could induce firms to offer fewer and possibly non-overlapping products. Our prediction is not that competitors should always carry overlapping product lines. Rather, our analysis suggests that if their product lines do not overlap in markets in which search costs matter, it must be for reasons other than firms’ attempts to soften competition through product choice—for instance, due to the presence of fixed costs.

To the extent that firms could collude to coordinate their product choices (as reported by Sullivan (2016) in the context of the super-premium ice cream market),\(^{45}\) competition authorities should remain vigilant if competitors’ product lines do not overlap—particularly so in markets in which fixed costs (at the product level) are not relevant but consumers find it costly to search.

\(^{44}\)In some cases, such costs can be substantial, e.g. firms have to advertise that they are carrying an additional product, or the transaction costs of dealing with an additional provider can sometimes be high. The marketing literature has analyzed several factors explaining the limited number of products sold per firm. For instance, Villas-Boas (2004) analyzes product line decisions when firms face costs of communicating about the different products they carry to their customers. They show that costly advertising can induce firms to carry fewer products as well as to charge lower prices for their high-quality goods.

\(^{45}\)See also the NY Times note quoted in the paper. Although it is difficult to divide “smooth” and “chunky” flavors in low and high-quality options, the logic of our result may apply as well. The ice-cream company that focuses on chunky flavors may need to reprice its existing offer downwards if it decides to also carry smooth flavors and compete head-to-head on these flavors with the rival company just carrying them. But this is profitable as long as exist some fraction of smooth non-shoppers.
Appendix: Proofs

Proof of Proposition 1 [SPE under $\mu = 1$] First, it is straightforward to show that the monopolization equilibrium $(LH, \emptyset)$ constitutes a SPE as if the inactive firm carried a product, prices for that product would go down to marginal cost and the firm would not recover the fixed cost $\epsilon \rightarrow 0$.

Second, we show that the specialization equilibrium $(L, H)$ constitutes a SPE. First, at subgames $(LH, LH)$, $(L, L)$ and $(H, H)$, both firms make zero profits. Second, at subgame $(L, LH)$ the low quality product is priced at marginal cost $c^L$ while the high quality product is sold at the highest price that satisfies the high types’ incentive compatibility constraint, i.e., $c^L + \theta^H \Delta q$. Firm $L$ makes zero profits while firm $LH$ gets a payoff of $(1 - \lambda)\varphi^H$, which equals its minimax. Third, at subgame $(H, L)$ the high quality product is priced at marginal cost $c^H$ while the low quality product is sold at the highest price that satisfies the low types’ incentive compatibility constraint and participation constraints, i.e., $\min\{c^H - \theta^L \Delta q, \theta^L q^L\}$. Firm $H$ makes zero profits while firm $LH$ makes profits $\lambda \pi^L$ if $c^H > \theta^L q^H$ or $\lambda \varphi^L$ otherwise, i.e., its minimax. Finally, at subgame $(L, H)$ the equilibrium is in mixed strategies. For the purposes of this proof, it suffices to put bounds on equilibrium profits. Minmax profits for each firm are computed by characterizing the firm’s best response to the rival pricing its good at marginal cost. Following our previous analysis, the minmax profits for the $H$ firm are $(1 - \lambda)\varphi^H > 0$, while the minmax profits for the $L$ firm are $\lambda \pi^L > 0$ if $c^H > \theta^L q^H$ or $\lambda \varphi^L > 0$ otherwise. Since at the mixed strategy equilibrium firms always price above marginal costs (otherwise they would have zero profits, but this cannot be since their minmax profits are positive), equilibrium profits are strictly above the minmax whenever the participation constraint is not binding. The only case where above marginal cost pricing does not necessarily imply that firm $L$’s profits are strictly above its minmax is when $c^H > \theta^L q^H$, as in this case firm $L$’s best response is the the same regardless of whether firm $H$ prices at $c^H$ or above.\(^{46}\) Indeed, for the case $c^H > \theta^L q^H$, equilibrium profits are exactly equal to the minmax $\lambda \pi^L$. To see this, note that at the MSE the upper bounds of firms’ price supports are the constrained monopoly prices. Furthermore, firm $L$ has to play a proba-

\(^{46}\)It is straightforward to see that in a mixed strategy equilibrium we must have $p^H > c^H$ and $p^L > c^L$; otherwise, each firm’s profits would be zero, but this leads to a contradiction since profits cannot be below the minmax. Hence, firm $H$ would never like to price lower than $p^L + \theta^H \Delta q > c^L + \theta^H \Delta q > c^H$. Since at a price $p^L + \theta^H \Delta q$ firm $H$ would at least be serving the high types, its profits must be strictly greater than its minmax $(1 - \lambda)\varphi^H$. Similarly, if $p^H > \theta^L q^H$, firm $L$ would be a monopolist over the low-types, so it could always secure profits of at least $\lambda \pi^L$. If $p^H < \theta^L q^H$, firm $L$ would never like to charge prices lower than $p^H - \theta^L \Delta q > c^H - \theta^L \Delta q$. Since at a price $p^H - \theta^L \Delta q$ firm $L$ would at least be serving the low types, its profits must be strictly greater than its minmax $\lambda \varphi^L$. 35
bility mass at its upper bound. Otherwise, firm $H$ would make zero profits at its upper bound (as all consumers would strictly prefer to buy from firm $L$), but this cannot be the case since its minmax is strictly positive. Last, the two firms cannot put positive mass at their upper bounds as firm $L$ would be better off putting all its mass slightly below its upper bound (so as to attract all consumers whenever the rival plays the mass at the upper bound). It thus follows that when firm $L$ plays its upper bound, the rival is pricing below its upper bound with probability one. Hence, at the upper bound firm $L$ only serves the low types, thus making profits that exactly equal its minmax, $\lambda \pi^L$.

We are now ready to show that $(L, H)$ is a SPE. Starting at $(L, H)$, firm $H$ does not want to carry good $L$ as at $(L, LH)$ its profits are equal to the minmax, while they are strictly above that level at $(L, H)$. Similarly, firm $L$ does not want to carry good $H$ as at $(LH, H)$ its profits are equal to the minmax, while at $(L, H)$ its profits are (weakly) greater than its minmax. When the comparison is weak because firms’ profits are $\lambda \pi^L$ at both subgames (i.e., when $c^H \geq \theta^L q^H$), the fixed cost $\epsilon \to 0$ of carrying a product breaks the indifference in favour of $(L, H)$.

Last, we characterize the MSE. Suppose that the rival chooses $L$ with probability $\alpha$ and $H$ with probability $(1 - \alpha)$ . Equating the profits from choosing $L$ and $H$,

$$\alpha \Pi (L, L) + (1 - \alpha) \Pi (L, H) = \alpha \Pi (H, L) + (1 - \alpha) \Pi (H, H)$$

Since $\Pi (H, H) = \Pi (L, L) = 0$, solving for $\alpha$,

$$\alpha = \frac{\Pi (L, H)}{\Pi (L, H) + \Pi (H, L)}.$$ 

Thus implying that equilibrium profits at the MSE are

$$\frac{\Pi (L, H) \Pi (H, L)}{\Pi (L, H) + \Pi (H, L)}.$$ 

This equilibrium constitutes a SPE if and only if it is not dominated to choosing $LH$, i.e.,

$$\Pi (L, H) [\Pi (L, L) - \Pi (LH, L)] + \Pi (H, L) [\Pi (L, H) - \Pi (LH, H)] \geq 0$$

The first term is negative, while the sign of the second term depends on $c^H$ : (i) if $c^H \geq \theta^L q^H$, it is negative, implying that $LH$ dominates the candidate MSE, which therefore does not exist; on the contrary, (ii) if $c^H < \theta^L q^H$, the second term is positive, implying that a MSE cannot be ruled out, particularly so for low $c^H$, which is when the second term is higher (note that as $c^H \to \theta^L q^H$ the second term is close to zero, so the MSE is ruled out for some $c^H < \theta^L q^H$). Therefore, for those parameter values for which the above inequality holds, a MSE exists. \textit{Q.E.D.}
Proof of Proposition 2 [SPE under $\mu \to 1$] The results on existence and uniqueness of the “overlapping” equilibrium $(LH, LH)$ under the assumption $\mu \to 1$ are a particular case of the proof of Proposition 7. The proof of non-existence of the “specialization” equilibrium $(L, H)$ for $\mu \to 1$, for the case $c^H \geq \theta^L q^H$ is also contained in the proof of Proposition 7. Hence, it only remains to prove that $c^H < \theta^L q^H$ implies the existence of the “specialization” equilibrium. The proof of Proposition 1 above shows that $\Pi (L, H)$ and $\Pi (H, L)$ are strictly above the minmax, while the proofs of Proposition 5 and 4 show that $\Pi (LH, H)$ and $\Pi (LH, L)$ are equal to their minmax as $\mu \to 1$. It follows that no firm wants to deviate from $(L, H)$. Q.E.D.

Proof of Lemma 1 Argue by contradiction and suppose that the firm chooses $\Delta p > \theta^H \Delta q$. Hence, all buyers visiting the store buy product $L$, and the firm makes a profit margin equal to $(p^L - c^L)$. If the firm reduced $p^H$ so that $\Delta p = \theta^H \Delta q$, it would still sell product $L$ to the low types at the same price, but would now sell product $H$ to the high types with a higher profit margin $p^H - c^H = p^L + \theta^H \Delta q - c^H > p^L - c^L$, where the inequality follows from $(A2)$. A similar reasoning applies to rule out $\Delta p < \theta^L \Delta q$. Q.E.D.

Proof of Proposition 3 [pricing at subgame $(LH, LH)$] The non-existence of a pure strategy equilibrium follows from standard arguments. Firms cannot tie in prices as a slight reduction in the price would allow a firm to attract all the shoppers. Firms cannot charge different prices either as the high-priced firm would only serve the non-shoppers and would thus be better off by either undercutting the rival’s price or by charging the (constrained) monopoly prices to maximize profits out of the non-shoppers; in turn, if the high-priced firm acts as the (constrained) monopolist, the other firm would find it profitable to slightly price below that level, thus not making it profitable any more for the rival to charge the (constrained) monopoly prices. Thus, the equilibrium must be in mixed-strategies. Since firms are symmetric, we focus on characterizing the symmetric mixed strategy equilibria. Standard arguments imply that there are no holes in the support and that firms play no mass point at any price of the support, including the upper bound (see, for instance, Narasimhan, 1998). (i) At the upper bound, firms serve the non-shoppers only. Since profits are increasing in prices subject to $(IC^H)$, the optimal prices at the upper bounds are $\bar{p}^H = \theta^H q^H - q^L \Delta \theta$ and $\bar{p}^L = \theta^L q^L$, so that $\Delta \bar{p} = \theta^H \Delta q$.

We now demonstrate $(ii)$, i.e., that at the lower bound $\Delta \bar{p} < \theta^H \Delta q$. Suppose otherwise that the price gap $p^H - p^L$ is constant and equal to $\theta^H \Delta q$ at and in the neighborhood of the lower bound (or throughout the entire price support for that matter). When a
firm plays \( \mathbf{p} = (p^H, p^L) \) it obtains

\[
\Pi(LH, LH; \mathbf{p}) = \left( \mu + \frac{1 - \mu}{2} \right) (1 - \lambda)(p^H - c^H) + \left( \mu + \frac{1 - \mu}{2} \right) \lambda(p^L - c^L).
\]

Using \( \Pi(\cdot; \mathbf{p}) = \bar{\pi} \equiv (1 - \mu) \left[ \pi^L + (1 - \lambda) \left( \theta^H \Delta q - \Delta c \right) \right] / 2 \), the payoff at the upper bound, and the assumption that \( p^H - p^L = \theta^H \Delta q \) we obtain

\[
p^H - c^H = \frac{1 - \mu}{1 + \mu} (p^H - c^H) + \lambda \frac{2\mu}{1 + \mu} \varphi^H
\]
and

\[
p^L - c^L = \frac{1 - \mu}{1 + \mu} (p^L - c^L) - (1 - \lambda) \frac{2\mu}{1 + \mu} \varphi^H.
\]

We now compute the cdf \( F(p^H) \) firms use in equilibrium to randomize prices. First, notice that if one firm plays something in the support, the other firm never wants to deviate and serve just the high type with a price \( \theta^H q^H \), because according to (A1) the payoff of doing so would be strictly lower. Thus, to obtain the cdf \( F(p^H) \) around the lower bound, notice that playing any pair \( p^H \) and \( p^L = p^H - \theta^H \Delta q \) around the lower bound yields an expected payoff of

\[
\Pi(\cdot; p^H, p^L) = (1 - \lambda)(p^H - c^H) \left[ \frac{1 - \mu}{2} + \mu(1 - F(p^H)) \right] + \lambda(p^L - c^L) \left[ \frac{1 - \mu}{2} + \mu(1 - F(p^H)) \right]
\]

where \( 1 - F(p^H) \) is the probability to attract both high and low type shoppers. Rearranging terms and using \( \Pi(p^H, p^L = p^H - \theta^H \Delta q) = \bar{\pi} \) leads to

\[
(p^H - p^H) \frac{1 - \mu}{2} = \left[ 1 - F(p^H) \right] \left[ \mu(p^H - c^H) - \lambda \mu \varphi^H \right].
\]

From this expression we obtain

\[
f(p^H) = \frac{1 - \mu}{2 \left[ \mu(p^H - c^H) - \lambda \mu \varphi^H \right]} > 0
\]
and

\[
f(p^L) = \frac{1 + \mu}{2 \left[ \mu(p^L - c^L) + (1 - \lambda) \mu \varphi^H \right]} > 0.
\]

Since \( p^H - c^H = p^L - c^L + \varphi^H \), from (9) we also obtain that

\[
f(p^L) = \frac{1 + \mu}{2 \left[ \mu(p^L - c^L) + (1 - \lambda) \mu \varphi^H \right]} > 0.
\]

If the lower bound \( p \) is indeed part of the equilibrium support, firms would not want to deviate from it. There are four possible (local) deviations to consider: (i) \( p^L \) and
\( p^H = p^H - \varepsilon \), (ii) \( p^L \) and \( p^H = p^H + \varepsilon \), (iii) \( p^H \) and \( p^L = p^L - \varepsilon \), and (iv) \( p^H \) and \( p^L = p^L + \varepsilon \), where \( \varepsilon \to 0 \). The first deviation is clearly not profitable. If the firm plays \( p^H = p^H - \varepsilon \), it sells the same amount but at a lower price. Playing (ii) \( p^H = p^H + \varepsilon \) is also unprofitable. It violates the IC for high type consumers; so the firm would end up selling only low quality products to both, all shoppers and the non-shoppers coming to the store. Deviation (iii) is also unprofitable for the same reason (ii) is. We are left with deviation (iv). Notice first that playing \( p^H \) and \( p^L = p^L + \varepsilon \) only affects profits from low type consumers. The change in profit is

\[
\Delta \Pi = \lambda \left(1 - \frac{\mu}{2}\right) \varepsilon + \lambda \mu \left[ (1 - F(p^L + \varepsilon))(p^L + \varepsilon - c^L) - (p^L - c^L) \right].
\]

The first term captures the gain from non-shoppers and the term in brackets captures the trade-off of losing the shoppers and charging them a bit more. We now take the derivative of \( \Delta \Pi \) with respect to \( \varepsilon \) and evaluate it at \( \varepsilon = 0 \) to obtain

\[
\frac{\partial \Delta \Pi}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \frac{\lambda(1 - \mu)}{2} + \lambda \mu [1 - f(p^L)(p^L - c^L)].
\] (11)

Replacing \( f(p^L) \) that follows from (10) into (11) we obtain \( \partial \Delta \Pi / \partial \varepsilon |_{\varepsilon=0} > 0 \), which contradicts that playing \( p \) was an equilibrium. Q.E.D.

**Proof of Lemma 2 [pricing at subgame \((LH, LH)\)]** We want to show that the equilibrium in the statement of the Proposition is indeed an equilibrium. First, firms could deviate by playing the price pairs in the support with different probabilities, while still choosing price pairs that satisfy incentive compatibility. However, this is unprofitable given that all price-pairs in the support give equal expected profits. Indeed, the equilibrium has been constructed so that

\[
(p^L - c^L) \left[ \frac{1 - \mu}{2} + \mu(1 - F^L(p^L)) \right] = \frac{1 - \mu}{2} (p^L - c^L) = \frac{1 + \mu}{2} (p^L - c^L)
\]

and

\[
(p^H - c^H) \left[ \frac{1 - \mu}{2} + \mu(1 - F^H(p^H)) \right] = \frac{1 - \mu}{2} (p^H - c^H) = \frac{1 + \mu}{2} (p^H - c^H)
\]

with the ratio (5) derived in order for the price pair \((p^H, p^L)\) to satisfy \( F^H(p^H) = F^L(p^L) \), i.e., the choice of \( p^L \) results in a choice of \( p^H \), so that the prices satisfying that ratio are played with equal probability. Therefore, expected profits at the proposed equilibrium are as in (3).

Second, firms could deviate by choosing \( p^L \) and \( p^H \) not satisfying equation (5) while still satisfying incentive compatibility. Again, these deviations are not profitable since
all the prices in the support give equal profits. Deviating to prices that do not satisfy incentive compatibility is unprofitable because of Lemma 1.

Last, firms could deviate by playing price pairs outside the support. Choosing any prices above \((\bar{p}^L, \bar{p}^H)\) as defined above is unprofitable, as at these prices the firm is only selling to the non-shoppers and \((\bar{p}^L, \bar{p}^H)\) are the optimal monopoly prices. Choosing any prices below \((\underline{p}^L, \underline{p}^H)\) as defined above is unprofitable, as at these prices the firm is inelastically selling to all consumers with probability one and would thus gain by raising the price up to \((\bar{p}^L, \bar{p}^H)\).

**Proof of Proposition 4 [pricing at subgame \((L, LH)\)]** It is easy to see that the equilibrium must be in mixed strategies. Since both firms are competing to attract the shoppers, any PSE candidate can be ruled out by either firm’s incentives to slightly undercut its rival. Note that by (A1) it does not pay firm LH to only serve the high types. Both firms choose \(p^L\) in \([\underline{p}^L, \theta^L q^L]\) and incentive compatibility restricts firm LH to price the high quality product at \(p^H = p^L + \theta^H \Delta q\).

We now show, by contradiction, that one of the two firms must be placing an atom at the upper bound. Suppose not, in which cases upper-bound payoffs are given by\(^{47}\)

\[
\Pi(L, LH; \bar{p}) = \frac{1 - \mu}{2} \pi^L \tag{12}
\]

\[
\Pi(LH, L; \bar{p}) = \frac{1 - \mu}{2} \left[ \pi^L + (1 - \lambda) \varphi^H \right] \tag{13}
\]

In the absence of atoms, when firm \(L\) prices at the upper bound, the shoppers low and all high types buy from the rival because with probability one firm LH would be pricing both goods at a lower price (after controlling for quality), whereas when LH prices at the upper bound, both shoppers’ types prefer to buy the rival’s low quality product. With these hypothetical “upper-bound” payoffs we can now find the lower bound of the price support \(\underline{p}^L\), which must be atomless to rule out any deviation. If we equalize (12) to what firm \(L\) would get by pricing at the lower bound and attracting all shoppers and half of the non-shopper low and high types, i.e.,

\[
\Pi(L, LH; \underline{p}^L) = \frac{1 + \mu}{2} (p^L - c^L).
\]

We obtain

\[
\underline{p}^L - c^L = \frac{1 - \mu}{1 + \mu} \pi^L \tag{14}
\]

where the subindex \(l\) indicates that \(\underline{p}^L\) is obtained using \(L\)’s indifference condition. Similarly, if we equalize (13) to what firm LH would get by pricing at the lower bound and

\(^{47}\text{Note that firm } L \text{ serves the non-shoppers, both the low and the high types at } \theta^L q^L.\)
attracting all shoppers as well as non-shopper high types, i.e.,

$$\Pi (LH, L; p) = \frac{1 + \mu}{2} \left( p^L - c^L + (1 - \lambda) \varphi^H \right).$$

We obtain

$$p^L_{lh} - c^L = \frac{1 - \mu}{1 + \mu} \pi^L - \frac{2\mu}{1 + \mu} (1 - \lambda) \varphi^H$$

(15)

where the subindex $lh$ indicates that $p^L_{lh}$ is obtained using $LH$’s indifference condition. Importantly, note that $p^L_{lh} - c^L$ is decreasing in $\varphi^H$. Simple inspection of (14) and (15) shows that $p^L_{lh} \neq p^L_{lh}$, except for $\mu = 0$, which contradicts the initial assumption that no firm places an atom at the upper bound for any $\mu > 0$. Furthermore, we have $p^L_{lh} > p^L_{lh}$ for all $\mu > 0$. This implies that $p^L_{lh}$ is the actual lower bound and that firm $L$ must be playing the upper bound with positive probability mass, while firm $LH$ plays no mass at any price. The size of that mass $\omega^L$, can be obtained by invoking $LH$’s indifference condition

$$\frac{1 + \mu}{2} \left( p^L - c^L \right) + (1 - \lambda) \varphi^H = \left( \frac{1 - \mu}{2} + \mu \omega^L \right) \left( \pi^L + (1 - \lambda) \varphi^H \right)$$

and after replacing $p^L_{lh} - c^L$ from (14),

$$\omega^L = \frac{1 + \mu}{2\mu} \frac{1 - \mu}{(1 + \mu) \pi^L + (1 - \lambda) \varphi^H} - \frac{1 - \mu}{2\mu}.$$

Equilibrium profits at this MSE are

$$\Pi (L, LH) = \frac{1 - \mu}{2} \pi^L$$

$$\Pi (LH, L) = \frac{1 + \mu}{2} \left( \frac{1 - \mu}{1 + \mu} \pi^L + (1 - \lambda) \varphi^H \right).$$

The characterization of the MSE ends with the probability distribution $F(p^L)$ used by the firms to set prices in the range $[p^L, \theta^L q^L)$. Since both firms play according to the same distribution, one can derive such distribution from either firm’s indifference condition. Q.E.D.

**Proof of Proposition 5 [pricing at subgame $(H, LH)$]** At the candidate PSE firm $H$ charges $p^H = \theta^H q^H$ while firm $LH$ charges $p^L = \theta^L q^L$ and $p^H = \theta^H q^H - \Delta \theta q^L$. At

\footnote{Notice that when firm $L$ plays the atom $\theta^L q^L$, which happens with probability $\omega^L$, firm $LH$’s upper bound for $p^L$ is not $\theta^L q^L$ but $\theta^L q^L - \varepsilon$ with $\varepsilon \to 0$. The reason is that in equilibrium there cannot be a tie on a price with positive probability.}

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these prices firm $H$ only sells to the non-shopper high types and firm $LH$ to all the rest.\footnote{Recall from (A2) that a firm carrying product $H$ always prefers to serve just the high types rather than having to reduce prices to also serve the low types. Allowing for this possibility would duplicate the number of cases to consider without adding any qualitatively different results.} Thus, firms’ profits at this PSE are

\[
\Pi(H, LH) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H
\]

(16)

\[
\Pi(LH, H) = \frac{1 + \mu}{2} [\lambda \pi^L + (1 - \lambda)(\pi^H - \Delta \theta q^L)].
\]

(17)

For this to be an equilibrium, neither firm would want to deviate from it. While this is evident for firm $LH$, firm $H$ could charge slightly less than $\theta^H q^H - \Delta \theta q^L$ to also attract the high type shoppers (note that at this price there are still unserved low types).\footnote{Note also that this deviation implicitly uses (A2), in particular, that serving the high types at price $\theta^H q^H - \Delta \theta q^L$ is more profitable for firm $H$ than serving all consumers at price $\theta L q^H$.} If it does, it would sell to all the high types except to the non-shoppers who buy from firm $LH$. It would thus make profits

\[
\Pi'(H, LH) = \frac{1 + \mu}{2} (1 - \lambda) (\pi^H - \Delta \theta q^L).
\]

(18)

Comparing the profit expressions for firm $H$, this deviation is unprofitable if

\[
\mu \leq \hat{\mu} \equiv \frac{q^L \Delta \theta}{\pi^H + (\pi^H - \Delta \theta q^L)} < 1.
\]

(19)

Hence, the PSE exists if and only if $\mu \leq \hat{\mu}$. Otherwise the equilibrium must be in mixed strategies.

We now characterize the MSE. The upper bounds for firm $LH$ must be $\bar{p}^L = \theta^L q^L$ and $\bar{p}^H = \theta^H q^H - \Delta \theta q^L$. For firm $H$, the upper bound could be either be the unconstrained monopoly price $\theta^H q^H$ or the constrained monopoly price $\theta^H q^H - q^L \Delta \theta$, but never something in between. We introduce $\omega^h$ and $\omega^{lh}$ to denote the masses that firms $H$ and $LH$ place at the upper bounds, respectively. As for the lower bounds, we also need to consider two possibilities, either $p^H \geq \theta L q^H$ or $p^H < \theta L q^H$. We go over these four possible cases next.

\textbf{Cases 1 and 2:} $p^H \geq \theta L q^H$ In these two cases, the low types never want to buy good $H$, so the $(IC^L)$ is never binding. Thus, firm $H$ never serves the low types and firm $LH$ can always price good $L$ at the monopoly price $\theta L q^L$. 
Case 1: $p^H \geq \theta L q^H$ and the upper bound for firm $H$ is $\theta L q^H$  

Firm $H$’s profits at the lower and upper bounds are, respectively

$$
\Pi(H, LH; \bar{p}) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H \\
\Pi(H, LH; \underline{p}) = \frac{1 + \mu}{2} (1 - \lambda) (p^H - c^H).
$$

Since these two have to be equal, it follows that

$$
p^H = c^H + \frac{1 - \mu}{1 + \mu} \pi^H. \quad (20)
$$

Firm $H$ must also be indifferent between playing $\theta L q^H$ (and serving only the non-shopper high types) and $\theta L q^H - \Delta \theta q^L - \epsilon$, with $\epsilon \to 0$ (and also serving shopper high types with some probability). If $\omega^{lh}$ is the probability mass that firm $LH$ places at the upper bound, then

$$
\frac{1 - \mu}{2} (1 - \lambda) \pi^H = \left(\frac{1 - \mu}{2} + \mu \omega^{lh}\right) (1 - \lambda)(\theta L q^H - \Delta \theta q^L - c^H)
$$

which yields

$$
\omega^{lh} = \frac{1 - \mu}{2 \mu} \frac{\Delta \theta q^L}{\pi^H - \Delta \theta q^L}. \quad (21)
$$

Note that the mass is decreasing in $\mu$: as $\mu \to 1$, $\omega^{lh} \to 0$, and as $\mu \to \hat{\mu}$, $\omega^{lh} \to 1$; hence, there is continuity between the PSE and the MSE.

On the other hand, whenever firm $LH$ prices the two goods at the upper bounds, $\bar{p}^L = \theta L q^L$ and $\bar{p}^H = \theta H q^H - \Delta \theta q^L$, it obtains

$$
\Pi(LH, H; \bar{p}) = \frac{1 + \mu}{2} \lambda \pi^L + (1 - \lambda) \left(\frac{1 - \mu}{2} + \mu \omega^{lh}\right) (\pi^H - \Delta \theta q^L)
$$

which must be equal to what it gets by playing at $\bar{p}^L = \theta L q^L$ and $\bar{p}^H$, as defined in (20),

$$
\Pi(LH, H; \bar{p}) = \left(\frac{1 - \mu}{2} + \mu\right) \lambda \pi^L + \left(\frac{1 - \mu}{2} + \mu\right) (1 - \lambda) (\bar{p}^H - c^H)
$$

Equalizing these two latter expressions implies that $\omega^h = \omega^{lh}$, as defined in (21). Thus, the two firms play the same mass at their respective upper bounds.

We still need to check, as we had initially assumed, that $\bar{p}^H \geq \theta L q^H$. This requires

$$
\mu < \mu^+ \equiv \frac{\pi^H - (\theta L q^H - c^H)}{\pi^H + (\theta L q^H - c^H)}. \quad (22)
$$

If $c^H \geq \theta L q^H$ then $\mu^+ \geq 1$ so this condition is always satisfied. In contrast, if $c^H < \theta L q^H$, then $\mu^+ < 1$. Hence, the equilibrium characterization above is only valid for $\mu \in (\hat{\mu}, \mu^+)$. 

We also need to make sure that firm $H$ does not want to deviate outside of the support to serve both high as well as low types. The condition to guarantee this, which only applies for the case $c^H < \theta^L q^H$, is
\[
\frac{1 + \mu}{2} (1 - \lambda) (p^H - c^H) > \frac{1 + \mu}{2} (\theta^L q^H - c^H)
\]
given that the low types never buy good $H$ unless it gives them positive utility. Rearranging this condition yields
\[
\mu < \mu^- \equiv \frac{(1 - \lambda) \pi^H - (\theta^L q^H - c^H)}{(1 - \lambda) \pi^H + (\theta^L q^H - c^H)}.
\]
It is easy to see that $\mu^- < \mu^+$ and from (A3) that $\mu^- > \hat{\mu}$. Hence, for $\mu \in [\hat{\mu}, \mu^-)$ this equilibrium is guaranteed to exist (later on we will see that for $\mu > \mu^-$ another equilibrium is also guaranteed to exist, so equilibrium existence is not at stake).

**Case 2: $p^H \geq \theta^L q^H$ and the upper bound for firm $H$ is $\theta^H q^H - q^L \Delta \theta$** Firm $H$ cannot play a mass point at the upper bound $\theta^H q^H - q^L \Delta \theta$. If it did, firm $LH$ could make more profits at prices slightly below the monopoly price than at the monopoly price. Since the monopoly price must be in the support of firm $LH$, it follows that $\omega^h = 0$. Thus, firm $LH$’s profits at the upper and bounds are
\[
\Pi (LH, H; \bar{p}) = \frac{1 + \mu}{2} \lambda \pi^L + \frac{1 - \mu}{2} (1 - \lambda)(\pi^H - \Delta \theta q^L)
\]
\[
\Pi (LH, H; p) = \frac{1 + \mu}{2} \lambda \pi^L + \frac{1 + \mu}{2} (1 - \lambda)(p^H - c^H).
\]
Equalizing the two we obtain
\[
p^H = c^H + \frac{1 - \mu}{1 + \mu} (\pi^H - \Delta \theta q^L).
\]
Since firm $H$ must also be indifferent between playing $p^H$ and $\theta^H q^H - \Delta \theta q^L - \epsilon$ (with $\epsilon \to 0$),
\[
\left( \frac{1 - \mu}{2} + \mu \omega^{lh} \right) (1 - \lambda)(\pi^H - \Delta \theta q^L) = \frac{1 - \mu}{2} (1 - \lambda)(\pi^H - \Delta \theta q^L)
\]
from which it follows that we should also have $\omega^{lh} = 0$. However, firm $H$ would then rather deviate to charging the unconstrained monopoly price to obtain profits $\frac{1 - \mu}{2} (1 - \lambda) \pi^H$. It follows that this case can never be an equilibrium: if $p^H > \theta^L q^H$, firm $H$ must have the upper bound at the unconstrained monopoly price $\theta^H q^H$. 

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Cases 3 and 4: $p^H < \theta^L q^H$  This case only arises when $c^H < \theta^L q^H$ as otherwise the firm would never price below $\theta^L q^H$. When this is the case, we have to consider the possibility that $(IC^L)$ is binding. In other words, firm $LH$ cannot always price good $L$ at $\theta^L q^L$ given that for $p^H < \theta^L q^H$ the low types would rather buy good $H$. Hence, firm $LH$ plays $p^L = \min\{\theta^L q^L, p^H - \theta^L \Delta q\}$ and both firms choose $p^H$ randomly. Note that for $p^H \geq \theta^L q^H$ this results in $p^L = \theta^L q^L$ with per-unit profits on good $L$ equal to $\pi^L$; whereas for $p^H < \theta^L q^H$ this results in $p^L = p^H - \theta^L \Delta q$ with per-unit profits on good $L$ equal to $p^H - c^H + \varphi^L$.

We also first note that when playing $p^H$, firm $H$ attracts all the shoppers, both high and low types, with probability one. The reason is simple. First, whenever firm $LH$ prices good $H$ above $\theta^L q^H$, its price for $L$ is $\theta^L q^L$ so the low types’ utility from choosing $L$ is zero and hence, they would rather buy $H$ at $p^H < \theta^L q^H$. And second, whenever firm $LH$ prices good $H$ below $\theta^L q^H$ but above $p^H$, the low types are indifferent between choosing $H$ or $L$ from firm $LH$. Hence, when firm $H$ prices at $p^H$, its price for good $H$ is lower than that of firm $LH$. It follows that the shopper low types prefer to buy good $H$ from firm $H$ rather than good $L$ from firm $LH$.

We again need to consider two cases for firm $H$’s upper bound.

Case 3: $p^H < \theta^L q^H$ and the upper bound for firm $H$ is $\theta^H q^H$  Firm $H$ has to be indifferent between playing $\theta^H q^H$, which it plays with some positive probability $\omega^h$, and the lower bound $p^H$. So, it must hold that

$$\frac{1 - \mu}{2} (1 - \lambda) \pi^H = \left( \frac{1 - \mu}{2} + \mu \right) (p^H - c^H)$$

leading to

$$p^H = c^H + \frac{1 - \mu}{1 + \mu} (1 - \lambda) \pi^H.$$  

We also need to show $p^H < \theta^L q^H$ as assumed, which requires $\mu > \mu^-$, as defined above in equation (23).

On the other hand, if firm $LH$ plays the upper bounds, $p^L = \theta^L q^L$ and $p^H = \theta^H q^H - \Delta \theta q^L$, it obtains

$$\Pi (LH, H; \bar{p}) = \left[ \frac{1 - \mu}{2} + \mu \left( 1 - F^h(\theta^L q^H) \right) \right] \lambda \pi^L + \left[ \frac{1 - \mu}{2} + \mu \omega^h \right] (1 - \lambda) (\pi^H - \Delta \theta q^L)$$

where $F^h(\theta^L q^H)$ is the probability that firm $H$ prices good $H$ below $\theta^L q^H$. The profit from pricing at the upper bound must be equal to pricing good $H$ at $\theta^L q^H$ and good $L$.
at $\theta^Lq^L$, giving firm LH a profit of
\[
\Pi (LH, H; \theta^Lq^L, \theta^Lq^H) = \left[ \frac{1 - \mu}{2} + \mu(1 - F^h(\theta^Lq^H)) \right] \lambda \pi^L + \left[ \frac{1 - \mu}{2} + \mu(1 - F^h(\theta^Lq^H)) \right] (1 - \lambda)(\theta^Lq^H - c^H). \tag{26}
\]

Note that the probability that shoppers buy from firm LH, regardless of their type, is $1 - F^h(\theta^Lq^H)$. If so, high types buy the high quality product, their preferred choice, while low types buy the low quality product, also their preferred choice. Since $\theta^Lq^H - c^H = \pi^L - \varphi^L > 0$, expression (26) can be conveniently re-written as
\[
\Pi (LH, H; \theta^Lq^L, \theta^Lq^H) = \frac{1 - \mu}{2} (1 - \lambda)\pi^L + \frac{1 + \mu}{2} (1 - \lambda)(\theta^Lq^H - c^H). \tag{27}
\]

Either (25) or (27) must also be equal to what firm LH gets by playing the lower bounds $p^H$ and $p^L = p^H - \theta^L\Delta q$, which is (note that the $(IC^L)$ is now binding at the lower bound)
\[
\Pi (LH, H; p) = \frac{1 + \mu}{2} \lambda (p^H - c^H + \varphi^L) + \frac{1 + \mu}{2} (1 - \lambda)(p^H - c^H). \tag{28}
\]

Rearranging this latter expression and using (24) (note that at the lower bound firm LH makes more profits than firm H since it can discriminate), equation (28) reduces to
\[
\Pi (LH, H; p) = \frac{1 - \mu}{2} (1 - \lambda)\pi^H + \frac{1 + \mu}{2} \lambda \varphi^L. \tag{29}
\]

Equalizing the above expression to (27) gives an implicit expression for $F^h(\theta^Lq^H)$,
\[
\frac{1 - \mu}{2} + \mu(1 - F^h(\theta^Lq^H)) = \frac{1}{2} \frac{(1 - \mu)(1 - \lambda)\pi^H + (1 + \mu) \lambda \varphi^L}{\pi^L - (1 - \lambda)\varphi^L} \equiv \Gamma. \tag{30}
\]

Notice that $\lim_{\mu \to 1} \mu(1 - F^h(\theta^Lq^H)) = \lambda \varphi^L / (\pi^L - (1 - \lambda)\varphi^L) < 1$.

To conclude the characterization of the equilibrium, we need to find an expression for $\omega^h$, the mass that firm H places at the upper bound, and make sure it is in the unit interval for all $\mu \in (\mu^-, 1)$. Using (30) while equalizing (25) and (29) yields
\[
\mu \omega^h (1 - \lambda)(\pi^H - \Delta q^L) = \frac{1 + \mu}{2} \lambda \varphi^L + \frac{1 - \mu}{2} (1 - \lambda)\Delta q^L - \Gamma \lambda \pi^L.
\]

Using $\pi^L - \varphi^L > 0$, it is easy to show that
\[
\lim_{\mu \to 1} \omega^h = \frac{\lambda \varphi^L(\pi^L - \varphi^L)}{(\pi^L + \varphi^H)(\pi^L - (1 - \lambda)\varphi^L)} \in (0, 1).
\]

It is also possible to show that $\omega^h < 1$ for $\mu = \mu^-$. At this point firm H’s lower bound approaches $\theta^Lq^H$, which is the deviation used in Case 1 to establish $\mu^-$. Hence, there is continuity in the mixed strategy equilibria on either side of $\mu^-$. Finally, note we do not need to check firms’ incentives to deviate outside of the support since at the lower bound firms are already serving all shoppers.
Case 4: $p^H < \theta^L q^H$ and the upper bound for firm $H$ is $\theta^H q^H - q^L \Delta \theta$ Firm $H$ cannot play a mass point at the upper bound. If it did, firm $LH$ could make more profits with a slight undercut of the monopoly price. Since the monopoly price must be in the support of firm $LH$, it follows that $\omega^h = 0$. Thus, equilibrium profits for the $LH$ firm at the upper bound are

$$\Pi(LH, H; \bar{p}) = \left(1 - \frac{\mu}{2} + \mu (1 - F^h(\theta^L q^H))\right) \lambda \pi^L + \frac{1 - \mu}{2} (1 - \lambda)(\pi^H - \Delta \theta q^L), \quad (31)$$

where $F^h(\theta^L q^H)$ is again the probability that firm $H$ prices $H$ good below $\theta^L q^H$. In equilibrium, these profits must be equal to those from pricing at the lower bound

$$\Pi(LH, H; p) = 1 - \frac{\mu}{2} \lambda (\pi^H - \theta^L \Delta q - c^L) + \frac{1 - \mu}{2} (1 - \lambda)(\bar{p}^H - c^H).$$

In addition, we know that firm $H$ must be indifferent between playing $\theta^H q^H - \Delta \theta q^L - \epsilon$ (with $\epsilon \to 0$) and $p^H$, implying

$$\left(1 - \frac{\mu}{2} + \mu \omega^h\right) (1 - \lambda)(\pi^H - \Delta \theta q^L) = \left(1 - \frac{\mu}{2} + \mu\right)(\bar{p}^H - c^H).$$

This gives us two equations for three unknowns, namely, $F^h(\theta^L q^H)$, $\omega^h$, and $p^H$. A third equation is obtained from $LH$’s indifference between pricing good $H$ at $p^H = \theta^L q^H$ (and $L$ at $\theta^L q^L$) and at any higher price in the support (and $L$ still at $\theta^L q^L$). At the upper bound, this indifference reduces to

$$\left[1 - \frac{\mu}{2} + \mu (1 - F^h(\theta^L q^H))\right] (1 - \lambda)(\theta^L q^H - c^H) = \frac{1 - \mu}{2} (1 - \lambda)(\pi^H - \Delta \theta q^L),$$

from where we obtain

$$\mu (1 - F^h(\theta^L q^H)) = \frac{1 - \mu}{2} \frac{\Delta \theta \Delta q}{\theta^L q^H - c^H}.$$ 

Replacing it into (31) yields firm $LH$’s equilibrium profits,

$$\Pi(LH, H) = \frac{1 - \mu}{2} \left[\left(1 + \frac{\Delta \theta \Delta q}{\theta^L q^H - c^H}\right) \lambda \pi^L + (1 - \lambda)(\pi^H - \Delta \theta q^L)\right].$$

Since in equilibrium $1 - F^h(\theta^L q^H) \leq 1$, which implies that $\mu \geq \Delta \theta \Delta q/[2(\theta^L q^H - c^H) + \Delta \theta \Delta q]$, it is evident that the above characterization cannot be an equilibrium, since as $\mu \to 1$ firm $LH$ would be making less than its minmax profit of $\lambda \varphi^L$.

**Equilibrium profits under** $(H, LH)$ Wrapping up, equilibrium pricing and equilibrium profits are characterized as follows:
If $\mu \leq \hat{\mu}$: PSE with firm $H$ charging the unconstrained monopoly prices, and firm $LH$ charging the constrained ones. Equilibrium profits are those in equations (16) and (17).

If $\mu \in (\hat{\mu}, 1)$: MSE with firm $H$ charging the unconstrained monopoly price at the upper bound. Equilibrium profits are

$$\Pi (H, LH) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H.$$

For firm $LH$, equilibrium profits depend on $\mu$: If $c^H \geq \theta^L q^H$ and $\mu \in (\hat{\mu}, 1)$, or if $c^H < \theta^L q^H$ and $\mu \in (\hat{\mu}, \mu^-]$,

$$\Pi (LH, H) = \frac{1 + \mu}{2} \lambda \pi^L + \frac{1 - \mu}{2} (1 - \lambda) \pi^H.$$

Otherwise,

$$\Pi (LH, H) = \frac{1 + \mu}{2} \lambda \left( \Delta c - \theta^L \Delta q \right) + \frac{1 - \mu}{2} (1 - \lambda) \pi^H. \quad (32)$$

Q.E.D.

**Proof of Proposition 6 [pricing at subgame $(L, H)$]** We start by showing that for low values of $\mu$ there exists a PSE. The natural candidate is $p^H = \theta^H q^H$ for firm $H$ and $p^L = \theta^L q^L$ for firm $L$. Firm $H$ only sells to the non-shopper high types, while firm $L$ sells to all the rest (except for the non-shoppers that visit firm $H$). Thus, firms’ profits are

$$\Pi (H, L) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H \quad (33)$$

$$\Pi (L, H) = \frac{1 + \mu}{2} \pi^L.$$

For this to be an equilibrium, it must be the case that neither firm wants to deviate from it. In particular, firm $H$ could charge $\theta^H q^H - q^L \Delta \theta$ to also attract the shopper high types. It would thus sell to all the high types (except for the non-shoppers who visit firm $L$) and would make profits

$$\Pi' (H, L) = \frac{1 + \mu}{2} (1 - \lambda) (\pi^H - q^L \Delta \theta). \quad (34)$$

Comparing the profit expressions for firm $H$, the PSE equilibrium exists if and only if $\mu \leq \hat{\mu}$ as defined in (19). Otherwise, the equilibrium must be in mixed strategies. Note we do not need to check that firm does not want to charge $\theta^L q^H - c^H$ so as to serve all the non-shoppers, including the low types, given that (A3) guarantees that $(1 - \lambda) \pi^H > \theta^L q^H - c^H$.

In order to characterize the MSE, let us start by discussing the properties of the price supports. The upper bound for firm $L$ must be $\theta^L q^L$, as for the low quality product the
constrained and unconstrained monopoly prices coincide. Following the PSE discussion above, the upper bound for firm H could either be the unconstrained monopoly price \( \theta^H q^H \) or the constrained monopoly price \( \theta^H q^H - q^L \Delta \theta \). In either case, firm H never plays a price in between these two prices. Before considering these two possibilities, we first note that firm L has to play a mass at its upper bound \( \theta^L q^L \). Otherwise, when firm H charged \( \theta^H q^H - q^L \Delta \theta \), since firm L would be charging prices below \( \theta^L q^L \) with probability one, firm H would make strictly lower profits than if it charged \( \theta^H q^H \), as it would serve the same set of consumers at a lower price. Accordingly, firm L must play a mass point at \( p^L = \theta^L q^L \), which we denote by \( \omega^L > 0 \). Similarly, let \( \omega^H \) denote the mass point that firm H puts on its upper bound. As for the lower bounds, we only need to consider case \( p^H < \theta^L q^L \) because \( p^H < \theta^L q^L \) is ruled out by (A3).

**Case 1: \( p^H > \theta^L q^H \) and the upper bound for firm H is \( \theta^H q^H \)** For given \( p^L \), it never pays firm H to charge less than \( p^L + \theta^H \Delta q \) since at this price it attracts all the shopper high types. Hence, we must have \( p^H \geq p^L + \theta^H \Delta q \). Second, for given \( p^H \), it never pays firm L to charge less than \( p^L - \theta^H \Delta q \) since at this price it attracts all the shoppers. Hence, we must have \( p^L \geq p^H - \theta^H \Delta q \). Putting these two conditions together, it follows that we must have \( p^H = p^L + \theta^H \Delta q \). Therefore, when firm H charges \( p^H \), with probability one firm L is charging prices above \( p^H - \theta^H \Delta q \). Hence, all the high types buy from firm H (except for the non-shoppers that visit L) implying that firm H’s profits in equilibrium must satisfy

\[
\Pi (H, L; p^H) = \frac{1 + \mu}{2} (1 - \lambda) (p^H - c^H) \tag{35}
\]

\[
= \frac{1 + \mu}{2} (1 - \lambda) (p^L - c^L + \varphi^H).
\]

Similarly, when firm L charges \( p^L \), with probability one firm H is charging prices above \( p^L + \theta^H \Delta q \). Hence, the firm serves its non-shoppers plus all the shoppers, both high and low types, implying the firm L’s profits in equilibrium must satisfy

\[
\Pi (L, H) = \frac{1 + \mu}{2} (p^L - c^L) \tag{36}
\]

Since firm H’s upper bound is \( \theta^H q^H \) and firm L is pricing below \( \theta^L q^L \) with probability one, firm H’s profits are

\[
\Pi (H, L) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H. \tag{37}
\]

Since firm H must be indifferent between playing \( \theta^H q^H \) and \( \theta^H q^H - \Delta \theta q^L \), it follows that the mass that firm L puts on the monopoly price in case 1, \( \omega^L_1 \), must satisfy

\[
\left(\frac{1 - \mu}{2} + \mu \omega^L_1\right) (1 - \lambda) (\pi^H - q^L \Delta \theta) = \frac{1 - \mu}{2} (1 - \lambda) \pi^H
\]
which is satisfied at \( \omega_l^1 \) equal to expression (21). Just as we argued then, there is continuity between the pure strategy and this mixed-strategy equilibrium.

Firm \( H \) must also be indifferent between playing the upper bound \( \theta^H q^H \) and \( p^H \). Equating profits yields \( p^H \) as in equation (20). Since we must have \( p^H = p^L + \theta^H \Delta q \), then

\[
p^L = c^L + \frac{1 - \mu}{1 + \mu} \pi^H - \varphi^H.
\]

Using equation (36), equilibrium profits for firm \( L \) are thus

\[
\Pi (L, H) = \frac{1 + \mu}{2} \left( \frac{1 + \mu}{1 + \mu} \pi^H - \varphi^H \right).
\]

When firm \( L \) prices at the upper bound \( \theta^L q^L \), it makes profits

\[
\Pi (L, H; p^L) = \left[ \frac{1 + \mu}{2} \lambda + \left( \frac{1 - \mu}{2} + \mu \omega^h \right) (1 - \lambda) \right] \pi^L.
\]

(38)

Equalizing these two expressions and rearranging terms yields

\[
\omega^h = \frac{1 - \mu}{2} \left( \pi^H - \pi^L - \varphi^H \right) - \frac{\lambda \pi^L + \varphi^H}{(1 - \lambda) \pi^L}.
\]

(39)

Note that for \( \mu \to \tilde{\mu} \), \( \omega^h \to 1 \). Hence, there is continuity between the pure strategy equilibrium and this mixed-strategy equilibrium. For

\[
\mu = \tilde{\mu} \equiv \frac{\pi^H - (\pi^L + \varphi^H)}{\pi^H + (\pi^L + \varphi^H) - 2 (1 - \lambda) \pi^L}
\]

we have \( \omega^h = 0 \). Since \( \omega^h \) cannot become negative, this equilibrium cannot exist for \( \mu > \tilde{\mu} \) (below we elaborate further on this).

Last, we need to check, as we have assumed, that \( p^H > \theta^L q^H \). This requires \( \mu < \mu^+ \) as defined in (22). If \( c^H \geq \theta^L q^H \) then \( \mu^+ \geq 1 \) so this condition is always satisfied. In contrast, if \( c^H < \theta^L q^H \), then \( \mu^+ < 1 \). Hence, the equilibrium characterization above is only valid for \( \mu < \mu^+ \) and \( \mu < \tilde{\mu} \). Using the fact that \( \pi^L + \varphi^H > \theta^L q^H - c^H = \pi^L - \varphi^L \) and \( \pi^H > \theta^L q^H - c^H \), it can be established that \( \tilde{\mu} < \mu^+ \). Therefore, the equilibrium characterized here only applies for \( \mu \in (\tilde{\mu}, \tilde{\mu}] \).

We still need to check that firm \( H \) does not want to deviate outside of the support for this characterization to be valid. If firm \( H \) deviates to serve all the low type shoppers with probability one by charging \( p^H = p^L + \theta^L \Delta q \), it would get a profit of \( \frac{1 + \mu}{2} \left( p^L + \theta^L \Delta q - c^H \right) \). For this latter to be strictly lower than \( \frac{1 + \mu}{2} (1 - \lambda) \left( p^H - c^H \right) \) we need

\[
\varphi^L + \varphi^H > \lambda \frac{1 - \mu}{1 + \mu} \pi^H
\]

(41)

\(^{51}\) Note that as \( \mu \to 1 \) the firm would be making zero profits, which cannot be the case since that is below its minmax. Hence, at some point his equilibrium must cease to exist, as we will see below.
to hold for all $\mu \in (\bar{\mu}, \tilde{\mu}]$. So it suffices to show that (41) holds for $\tilde{\mu}$, which it strictly does because of (A3).

To continue with the equilibrium characterization we need to consider now the possibility of a smaller upper bound for firm $H$.

**Case 2: $p^H > \theta^L q^H$ and the upper bound for firm $H$ is $\theta^H q^H - q^L \Delta \theta$.** Firm $H$ cannot play a mass point at the upper bound $\theta^H q^H - q^L \Delta \theta$. If it did, firm $L$ could make more profits at prices slightly below the monopoly price than at the monopoly price. Since the monopoly price must be in the support of firm $L$, it follows that $\omega^l = 0$. Thus, equilibrium profits for the $L$ firm at the upper and lower bounds are

$$\Pi(L, H; \bar{p}^L) = \left(\frac{1-\mu}{2} + \mu \lambda\right) \pi^L$$

$$\Pi(L, H; p^L) = \frac{1+\mu}{2} (p^L - c^L).$$

Equalizing the two yields

$$p^L - c^L = \left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right) \pi^L.$$

So that

$$p^H - c^H = \left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right) \pi^L + (\theta^H \Delta q - \Delta c). \quad (42)$$

Last, we again need to check that $p^H > \theta^L q^H$ as initially assumed, and this requires

$$p^H = c^H + \left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right) \pi^L + \varphi^H > \theta^L q^H$$

$$\mu < \frac{(1-\lambda)\pi^L - \theta^L q^H + c^H + \varphi^H + \lambda \pi^L}{(1-\lambda)\pi^L + \theta^L q^H - c^H - \varphi^H - \lambda \pi^L} \equiv \tilde{\mu}. \quad (43)$$

Note that $\tilde{\mu} \geq 1$ iff

$$\pi^L - \varphi^L \leq \varphi^H + \lambda \pi^L \quad (44)$$

which always holds by assumption (A2). Hence, $p^H \geq \theta^L q^H$ is always satisfied. In turn, this also implies that we do not need to consider the cases with $p^H < \theta^L q^H$.

Using equation (35) to obtain firm $H$’s equilibrium profits,

$$\Pi(H, L) = \frac{1+\mu}{2} (1-\lambda) \left[\left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right) \pi^L + \varphi^H\right]. \quad (45)$$

Since firm $H$ must be indifferent between playing $\theta^H q^H - \Delta q^L$ and $p^H$, it follows that the mass that firm $L$ puts at the upper bound in case 2, $\omega^l_2$, must satisfy

$$\left(\frac{1-\mu}{2} + \mu \omega^l_2\right) (1-\lambda) (\pi^L + \varphi^H) = \frac{1+\mu}{2} (1-\lambda) \left(\frac{1-\mu + 2\lambda \mu}{1+\mu} \pi^L + \varphi^H\right).$$
which, after some algebra, yields

$$\omega_2^l = \frac{\varphi^H + \lambda \pi^L}{\varphi^H + \pi^L}.$$  \hspace{1cm} (46)

The expressions for $\omega_1^l$ and $\omega_2^l$ cross at a single value of $\mu$, call it $\mu^+$. Since $\omega_1^l$ is decreasing in $\mu$ and $\omega_2^l$ is flat, $\omega_1^l \geq \omega_2^l$ if and only if $\mu \leq \mu^+$. At $\mu^+$, since $\omega_1^l = \omega_2^l$, the profits made by firm $H$ in cases 1 and 2 must coincide, i.e., expressions (37) and (45) must be equal. At $\mu^+$, using expressions (37) and (45),

$$\frac{1 - \mu}{2}(1 - \lambda)\pi^H = \frac{1 + \mu}{2}(1 - \lambda) \left( \frac{1 - \mu + 2\lambda \mu}{1 + \mu} \pi^L + \varphi^H \right)$$

and rearranging,

$$\frac{1 + \mu}{2} \left( \frac{1 - \mu}{1 + \mu} \pi^H - \varphi^H \right) = \left( \frac{1 + \mu}{2} \lambda + \left( \frac{1 - \mu}{2} \right) (1 - \lambda) \right) \pi^L.$$  

This equation is satisfied at $\mu = \tilde{\mu}$ (from equation (39), recall how we defined $\tilde{\mu}$). Hence, we must have $\mu^+ = \tilde{\mu}$.

We still need to check for case 2 that firm $H$ does not want to deviate outside of the support. Following the analysis of case 1 and noticing that now $\frac{p^H - c^H}{2}$ is given by (42), the condition for firm $H$ not to find it profitable to deviate to serve all the shopper low types with probability one by charging $\frac{p^H}{2} + \theta^L \Delta q$ reduces to

$$\Delta c - \theta^L \Delta q + (1 - \lambda)\varphi^H > \lambda \left( \frac{1 - \mu}{1 + \mu} (1 - \lambda) \right) \pi^L$$

for all $\mu \geq \tilde{\mu}$. Since the term in parenthesis in the RHS is less than 1, we just need $\Delta c - \theta^L \Delta q + (1 - \lambda)\varphi^H > \lambda \pi^L$, which rearranged is simply (A3).

The above characterization has important implications for equilibrium existence and uniqueness. First, for $\mu < \tilde{\mu}$, $\omega_1^l > \omega_2^l$ implies that firm $H$ makes more profits at the unconstrained monopoly price than at the constrained monopoly price. Hence, the equilibrium characterized in case 2 cannot exist: the unique equilibrium is the one characterized in case 1. Second, for $\mu \geq \tilde{\mu}$, profits for firm $L$ in case 1 fall below its minmax, so that the unique equilibrium is the one characterized in case 2. Last note that the equilibrium is continuous in $\mu$. In particular, the masses with which firms play their upper bounds are continuous in $\mu$ at $\tilde{\mu}$ as we move from the PSE to the MSE in case 1, and at $\tilde{\mu}$ when we move to the MSE in cases 1 to 2.

**Equilibrium profits under $(H, L)$** Depending on the value of $\mu$, the equilibrium is characterized as follows:
For $\mu \in [0, \hat{\mu}]$:

\[
\Pi (H, L) = \frac{1-\mu}{2} (1-\lambda) \pi^H \\
\Pi (L, H) = \frac{1+\mu}{2} \pi^L.
\]

(47)

For $\mu \in [\hat{\mu}, \tilde{\mu}]$:

\[
\Pi (H, L) = \frac{1-\mu}{2} (1-\lambda) \pi^H \\
\Pi (L, H) = \frac{1+\mu}{2} \left( \frac{1-\mu}{1+\mu} \pi^H - \varphi^H \right).
\]

(48)

For $\mu \in [\tilde{\mu}, 1]$:

\[
\Pi (H, L) = \frac{1+\mu}{2} (1-\lambda) \left[ \left( \lambda + \frac{1-\mu}{1+\mu} (1-\lambda) \right) \pi^L + \varphi^H \right] \\
\Pi (L, H) = \left[ \frac{1+\mu}{2} \lambda + \frac{1-\mu}{2} (1-\lambda) \right] \pi^L.
\]

(49)

where $\hat{\mu}$ and $\tilde{\mu}$ are defined, respectively, by (19) and (40). Q.E.D.

**Proof of Proposition 7 [product choices]** Each firm has four potential choices: \{∅, L, H, LH\}. On the one hand, to prove that (LH, LH) is a SPE of the game for all $\mu < 1$, just note that all payoffs $\Pi (LH, LH)$, $\Pi (H, LH)$ and $\Pi (L, LH)$ are proportional to $(1-\mu)/2$ so that (A1) allows to conclude that $\Pi (LH, LH)$ is the greatest among these, just as in the monopoly case.

On the other hand, to find the conditions under which (L, H) is an equilibrium, we need to assess firm L’s deviation gains when also carrying good H (it is easy to check that this is the critical deviation; for instance, let $\mu \to 1$ and use (A2) to note that firm L’s deviation gains are greater than firm H’s, i.e., $\Pi (LH, H) - \Pi (L, H) > \Pi (LH, L) - \Pi (H, L)$). Since the relevant payoffs are (49) and (32), this gain is equal to

\[
\Pi (L, H) - \Pi (LH, H) = \frac{1+\mu}{2} \lambda \left( \pi^L - \varphi^L \right) - \frac{1-\mu}{2} (1-\lambda) \left( \pi^H - \pi^L \right).
\]

(50a)

Solving for $\mu$, the above profit difference is positive iff $\mu \geq \mu^*$, where

\[
\mu^* = \frac{(1-\lambda) \left( \pi^H - \pi^L \right) - \lambda \left( \pi^L - \varphi^L \right)}{(1-\lambda) \left( \pi^H - \pi^L \right) + \lambda \left( \pi^L - \varphi^L \right)}.
\]

Note that when $c^H \geq \theta^L q^H$, $\pi^L = \varphi^L$ and $\mu^* = 1$, making $c^H \geq \theta^L q^H$ a sufficient condition for the uniqueness of (LH, LH). Furthermore, taking the derivative of $\mu^*$ with respect to $c^H$ shows that

\[
\frac{\partial \mu^*}{\partial c^H} = -2\lambda (1-\lambda) \frac{(\pi^H - \pi^L) - (\pi^L - \varphi^L)}{(1-\lambda) \left( \pi^H - \pi^L \right) + \lambda \left( \pi^L - \varphi^L \right))^2}.
\]
So that
\[
\text{sign}\left\{ \frac{\partial \mu^*}{\partial c^H} \right\} = -\text{sign}\left\{ \left( \pi^H - \pi^L \right) - \left( \pi^L - \varphi^L \right) \right\} = -\text{sign}\left\{ c^L - \theta^L q^H \right\} > 0.
\]

Last, there might also exist a symmetric MSE such that firms choose \(L\) and \(H\) randomly, just as we showed in the proof of Proposition 2. This equilibrium constitutes a SPE if and only if it is not dominated to choosing \(LH\), i.e.,
\[
\Pi (L, H) \left[ \Pi (L, L) - \Pi (LH, L) \right] + \Pi (H, L) \left[ \Pi (L, H) - \Pi (LH, H) \right] \geq 0
\]
or equivalently, iff
\[
\Pi (L, H) - \Pi (LH, H) \geq \frac{\Pi (L, H)}{\Pi (H, L)} \left[ \Pi (L, L) - \Pi (L, L) \right] > 0.
\]
Hence, whereas the existence of the asymmetric PSE \((L, H)\) requires the profit difference (50a) to be positive, the existence of the MSE requires such a difference to be greater than a strictly positive number. If we denote with \(\mu^*\) the critical value for the existence of the MSE, we must then have \(\mu^* \geq \mu^*\). It thus follows that for \(\mu < \mu^*\), the unique equilibrium (either pure or mixed) is \((LH, LH)\). Q.E.D.

**Proof of Lemma 3 [prices and consumers surplus at the SPE]**

(i) It is straightforward to see that, conditional on firms playing \((LH, LH)\), expected prices are decreasing in \(\mu\). Since there is full discrimination, total consumption of each good remains fixed so that total surplus is given by \(\lambda \pi^L + (1 - \lambda) \pi^H\), irrespectively of \(\mu\). Since profits in equation (3) decrease in \(\mu\), consumer surplus must increase in \(\mu\). In turn, this implies that expected prices must be decreasing in \(\mu\). For given parameter values, competition is stronger at subgame \((LH, LH)\) than at \((L, H)\). Hence, expected prices at the former must be lower and consumer surplus must be higher. Thus, as \(\mu\) goes down, expected prices (consumer surplus) at \((LH, LH)\) decrease continuously until they jump up when firms start playing \((L, H)\), either at \(\mu \to 1\) or at \(\mu \to \mu^*\) depending on equilibrium selection. Similarly, as \(\mu\) goes down, consumer surplus at \((LH, LH)\) increases continuously until it jumps down when firms start playing \((L, H)\), either at \(\mu \to 1\) or at \(\mu \to \mu^*\) depending on equilibrium selection.

(ii) Last, expected prices at \((LH, LH)\) are equal to the (constrained) monopoly prices for \(\mu = 0\) and to marginal costs for \(\mu \to 1\). By Proposition 1, expected prices at \((L, H)\) for \(\mu = 1\) are strictly above marginal costs. It follows that there must exist \(\mu' \in (0, 1)\) such that expected prices at \((LH, LH)\) for \(\mu \in (\mu', 1)\) are lower than at \((L, H)\) for \(\mu = 1\). If \(\mu' < \mu^*\), it follows that regardless of equilibrium selection, expected prices are lower at
\( \mu \in (\mu', \mu^*) \) than at \( \mu = 1 \). Similarly, for \( \mu = 0 \), expected consumer surplus at \((LH, LH)\) is equal to the expected information rents of the high types, \( \Delta \theta q^L \), for \( \mu = 0 \), and it is equal to total welfare for \( \mu \to 1 \) as in this case profits are zero. Total welfare is the same at \((LH, LH)\) for all \( \mu \) than at \((L, H)\) at \( \mu = 1 \) since in both cases discrimination is complete. The proof is completed by noting that, by (i) above, consumer surplus at \((LH, LH)\) is increasing in \( \mu \). Q.E.D.
References


