Duration Dependence: A Structural Approach*

Ioannis Kospentaris†

September 14, 2017

JOB MARKET PAPER

Abstract

This paper builds a directed search model of the labor market to evaluate the importance of three key reasons why the job-finding rate strongly falls with unemployment duration. The key reasons are: (i) unobserved worker heterogeneity, (ii) skill loss, and (iii) job-search effort decline. The model includes all of these channels and each mechanism is disciplined by a distinct source of micro data. The model predicts a job-finding profile over the unemployment spell very close to the one observed in US data. To evaluate the importance of each factor for the decline in job-finding, I use the model to compute counterfactual job-finding profiles, shutting down one channel at a time. Two novel results emerge: (i) the bulk of the effect of unobserved worker heterogeneity is concentrated in the first two months of the unemployment spell; the drop in job-finding rate found in longer spells is a result of skill loss and lower job-search effort. (ii) Job-search effort decline has a symmetric impact over the whole unemployment spell, while the effects of skill loss are concentrated in the last months of the spell. The results have clear implications for active labor market programs aimed at reducing long-term unemployment.

*I am deeply grateful to Gary Hansen, Lee Ohanian, Till von Wachter and Pierre-Olivier Weill for invaluable advice and support. I am also indebted to Andy Atkeson, Richard Domurat, Pablo Fajgelbaum, Andreas Gulyas, Manolis Hatzikostantinou, Adriana Lleras Muney, Christos Makridis, Joe Ostroy, Manisha Padi, Liyan Shi, and most importantly Ben Smith, as well as seminar participants in the Macro Proseminar at UCLA for many fruitful comments and discussions. I am solely responsible for any errors or omissions.

†University of California, Los Angeles; email: ikospentaris@g.ucla.edu
1 Introduction

It is a well-documented fact that workers who have just entered unemployment have a much better chance of finding a job than workers who have been jobless for a long time.¹ This phenomenon is called *duration dependence in unemployment* and economists have offered two types of explanations for its occurrence. The first is referred to as *true* duration dependence, which shows that long-term unemployment may have a causal effect on individual job-finding prospects. The most prominent channels of true duration dependence are: (i) loss of skills during time away from work, (ii) decline of job-search effort, and (iii) employer discrimination against the long-term unemployed. Alternatively, duration dependence may be *spurious*—meaning, it could reflect dynamic sorting among the unemployed such that, at longer durations, the unemployment pool contains more “bad apples” with unobserved personal characteristics that lead to their having worse job-finding prospects.

Although there is empirical evidence that all these mechanisms are at work, it is not clear how large is the importance of each channel. First, even though there is a substantial number of studies trying to measure how much of the observed duration dependence is true, the results are “mixed and controversial” (Ljungqvist and Sargent (1998)), with estimates varying from a small to a sizable role.² Second, even if the “true-vs-spurious” debate was resolved, we do not have a clear sense of how large is the contribution of each true mechanism on the observed job-finding rate decline. That is, a measurement of the relative effects of skill loss and declining search effort is not available.³

This paper evaluates the importance of three key reasons why the job-finding rate falls with unemployment duration. The key reasons are: (i) unobserved worker heterogeneity, (ii) human capital depreciation, and (iii) search effort decline. To this aim, it develops an equilibrium search model of the labor market that incorporates all of these mechanisms and uses it to measure the impact of each channel on duration dependence. Each mechanism in the model is disciplined by a distinct source of micro data and it is shown that the model predicts a job-finding profile over the spell of unemployment very close to the one observed in US data. To evaluate the importance of each factor for duration dependence, I use the model to compute counterfactual job-finding profiles, shutting down one channel at a time. The contribution of each mechanism is measured as the difference between the job-finding profile predicted by the full model and the job-finding profile predicted by the version of the model in which this mechanism was absent.

The results from this exercise make two novel contributions to the debate surrounding duration dependence. First, according to the model, the bulk of the effect of dynamic sorting—resulting in spurious duration dependence— is concentrated in the first two months of the unemployment spell. As

¹This is true even after taking account the age, education, industry, and other relevant observable worker characteristics. See Kroft et al. (2016), Elsby and Hobijn (2010), and Krueger et al. (2014), among others.
²In the recent literature, Ahn and Hamilton (2016) and Alvarez et al. (2016) estimate a small role for true duration dependence, while Abraham et al. (2016) and Bentolilla et al. (2017) find it to be substantial.
³However, there are papers measuring the impact of employer discrimination on duration dependence. See below and Section 6.
the unemployment spell evolves, the channels causing true duration dependence become quantitatively more important and for very high durations account for most of total duration dependence. Second, focusing on the true channels, skill depreciation and declining search effort affect the decline in job-finding rate in different ways. Through the lens of the model, declining search effort has a symmetric impact on job-finding over the whole unemployment spell, while the effects of skill depreciation are concentrated in the last six months in unemployment. These results emphasize that the importance of each mechanism for the observed duration dependence strongly depends on the stage of the unemployment spell. That is, at different points in the course of the unemployment spell, the relative importance of each channel is different.

To make sense of these findings, consider the following two comparisons. First, in the US, a newly unemployed worker has a 30% greater chance to be employed next month, compared to an observationally similar worker who is jobless for three months. According to my model, almost all of this difference can be attributed to unobserved differences between the average newly unemployed and the average worker who is unemployed for three months. Second, when comparing a worker unemployed for eight months with a worker unemployed for a year or more, the former has a 12% greater chance of finding a job. The model attributes most of that difference to the combination of skill decay and lower search effort exceeded by workers who are unemployed for a year or more. Finally, the model implies that skill loss accounts for a larger part of the difference than the job-search effort decline.

Over all, these results have two important implications for labor market policy. First, the effects of an active labor market program are heterogeneous for workers at different stages of the unemployment spell. According to the predictions of the model, job-search assistance programs help short- and long-term unemployed workers in a similar way. However, skill intensive programs improve the job-finding prospects of the long-term unemployed more than those of short-term unemployed workers. Second, in quantitative terms, the model predicts that the impact of skill intensive programs on raising job-finding rates should be greater than the impact of job-search assistance. Both policies have significant positive effects, yet the model goes one step further. It implies that greater weight should be put on skill intensive programs than job-search assistance policies to reduce long-term unemployment.

This paper adds to a growing literature on duration dependence, including Fernández-Blanco and Preugschat (2016), Jarosch and Pilossoph (2015) and Doppelt (2014). All these authors build search models that feature both spurious and true duration dependence. The only mechanism of true duration dependence in these papers, though, is employer discrimination against hiring the long-term unemployed. These authors abstract from other sources of true duration dependence that could affect job-finding rates. I complement their work by building a model that features unobserved worker heterogeneity and two different empirically plausible channels of true duration dependence, disciplined by micro data. The main novelty, thus, is to provide estimates for the importance of skill depreciation and worker search effort for the job-finding rate decline. An estimate for the effect of dynamic sorting is also provided, which is in line with previous studies.

An attractive feature of the theoretical framework built in this paper is its tractability. Equilibrium
existence is proved and several sharp predictions are established. Specifically, I show analytically that
the job-finding rate, reemployment wage, and path of search effort are unambiguously decreasing over
the spell of unemployment. Turning to empirics, the structural model is informed by three sources of
data. First, high-quality measurements of the effect of unemployment duration on reemployment wages,
which are available in the literature, discipline the extent of skill depreciation. Second, the results from
the influential audit study of Kroft et al. (2013) are used to inform unobserved worker heterogeneity.
Finally, to discipline worker search effort, I use weekly data from the weekly survey of unemployed
workers, conducted by Krueger and Mueller (2011) in New Jersey. It is important to highlight that the
model predicts a realistic job-finding profile over the course of unemployment, though job-finding rates
are not used to pin down any model parameters. The final contribution of the paper is to introduce a
new computational strategy for solving directed search models of the labor market, which is based on
the proof of equilibrium existence.

It should be emphasized that it is very difficult to obtain results of the type obtained here without
(i) the use of a structural framework that (ii) includes all of these channels. First, to evaluate the
magnitude of these mechanisms with data alone, one would have to follow a large sample of workers in
different areas for a fair amount of time, taking multiple measurements of their job search applications
at very high frequencies. Given the unusually high data requirements, using a structural framework to
make progress seems a natural choice. Second, as will be clarified later, it is the interaction of dynamic
sorting with the channels of true duration dependence that produces the quantitative results of the
model. Intuitively, failing to find a job reveals a lot about the quality of the newly unemployed because
these workers are evaluated often by firms due to their high levels of human capital and search effort.
The long-term unemployed have lower skill levels and exceed lower search effort, thus they are rarely
evaluated by firms. An extra period of unemployment is not very informative about the unobserved
quality of the long-term unemployed and, as a result, the impact of dynamic sorting on job-finding
becomes less important at high durations.

The paper proceeds as follows. Section 2 describes the model environment, defines an equilibrium,
and analytically establishes equilibrium existence and characterization. In Section 3, I present the
empirical evidence that informs the model. Section 4 discusses the fixed-point structure of equilibrium,
identification strategy of the model and calibration procedure. In Section 5, I present the quantitative
results. Section 6 contains a discussion of the relevant literature and Section 7 concludes. Finally,
Appendix I contains all proofs.

2 Model

This section introduces a tractable equilibrium model of the labor market that contains three important
channels of duration dependence: (i) unobserved worker heterogeneity, (ii) human capital depreciation,
and (iii) search effort decline. The model builds on the directed search approach of Moen (1997),
Acemoglu and Shimer (1999a,b), and Gonzalez and Shi (2010). I begin with a simplified version of the model that incorporates only skill depreciation and a declining quality of the unemployment pool, without job separations. I prove existence of equilibrium and characterize its basic properties. In the next part I present a richer version of the model with endogenous participation decision and exogenous separation shocks. This richer version will be used for the quantitative analysis of Sections 4 and 5. All theoretical properties proved for the simple model go through in the full model, albeit with more cumbersome notation.

2.1 The Basic Environment

Time is discrete and runs forever. All agents are risk neutral and discount the future at a common factor \( \beta \). There is a unit measure of workers, divided between the states of employment and unemployment. The measure of participating firms will be endogenously determined by free entry. In this section there is no separation of workers from jobs: if a worker gets hired, she keeps this job forever. The only source of separation is an exit shock \( \nu \) that forces a worker (employed or unemployed) out of the market. Workers who have exited are replaced by a measure \( \nu \) of newly unemployed workers.

**Workers.** Workers are heterogeneous in two aspects. First, they are of either high (\( H \)) or low (\( L \)) ability. Ability captures worker’s suitability at any given firm: a type-\( i \) worker turns out to be suitable for a given job with probability \( a^i \) (with \( a^H > a^L \)). If a worker is not suitable for a given job, the match yields zero output. That is, high-ability workers have a higher probability of being productive in a given job than low-ability workers. This notion of suitability can be thought of as an extreme form of a match-specific shock, which depends on worker’s type. Notice that even high-ability workers will be unsuitable for some jobs. There is a mass \( \pi \in (0, 1) \) of high-ability workers.

Second, if a worker is suitable for a match, her productivity is a function of her unemployment duration. A job-seeker who is unemployed for \( \tau \in [1, T] \) periods and turns out to be suitable for a given job, will produce \( y_\tau \), with \( y_\tau > y_{\tau+1} \), up to the final period \( T \). All workers with unemployment duration greater or equal to \( T \) form a homogeneous group. The output of a worker-firm match depends solely on worker’s skill level at the time of the match. The deterioration of worker’s productivity over the unemployment spell captures skill loss. Notice that the effect of skill depreciation affects both high- and low-ability workers in the same way.

**Firms.** Firms are homogeneous. They post vacancies and wages aimed at workers of a specific unemployment duration at cost \( \kappa \). Meeting workers is subject to matching frictions. Moreover, firms have access to a simple testing technology: after meeting a worker, a firm observes a private, match-specific signal, which perfectly identifies unsuitable workers. Unsuitable candidates are disregarded (since their productivity for the job at hand is zero) and only suitable workers are hired. The testing expenses are included in the vacancy creation cost. Neither workers nor other firms learn the match-specific signals.
generated by the testing process; they only observe the hiring decision. A worker who fails to find a job does not know whether her application has been considered by a firm and found unsuitable or it was not considered at all due to matching frictions.\footnote{As it will be seen later, the matching function will reflect that feature of the model: it will determine the number of productive matches rather than the number of meetings. The process of receiving applications and the process of evaluating applicants are combined in this model.}

**Labor Market.** The labor market consists of many different submarkets, indexed by the unemployment duration of workers who search for jobs in the submarket. Firms are free to enter any submarket and post any wage they want to attract workers of a specific unemployment duration. Search is directed in the sense that workers of different durations search in different submarkets. Hence, when firms post wages and vacancies in a submarket, they calculate the expected profit with workers of only one unemployment duration in mind. Assuming this market structure is without any loss of generality: it is a standard result in directed search models with heterogeneous workers, homogeneous firms and bilateral meetings that labor market participants endogenously choose to search in different submarkets (see Moen (1997), Acemoglu and Shimer (1999a), Mortensen and Wright (2002), Menzio and Shi (2010), Gonzalez and Shi (2010), Guerrieri et al. (2010)).\footnote{Several papers postulate that firms commit to hire workers of only one type in each submarket, as this paper does. See Doppelt (2014) and Flemming (2016), among others.} In other words, even if it was assumed that firms are free to post wages for any worker types they want, they would endogenously choose to post a wage directed to workers with a specific unemployment duration.

**Information Structure.** A worker’s ability is unobservable to both the worker herself and potential employers: there is symmetric incomplete information in the model, as in Gonzalez and Shi (2010), Fernández-Blanco and Preugschat (2016) and Doppelt (2014). On the other hand, worker’s unemployment duration, and thus her productivity in suitable matches, is public information. In other words, all firms know the output of a successful match with a worker of specific unemployment duration. Due to the fact that lack of information regarding a worker’s type is symmetric, the worker and the “labor market” (i.e. all firms and other workers) share the same belief about the probability a worker of a given duration be suitable for a job. Hence, workers of the same unemployment duration are observationally equivalent and a worker’s unemployment duration is a sufficient statistic for the probability the worker forming a successful match.

This information structure is based on Gonzalez and Shi (2010); it buys the model a lot of tractability for two reasons. First, it allows me to avoid the complexities arising in the case of adverse selection, analyzed in Guerrieri et al. (2010). Second, when this hiring protocol is combined with a constant returns to scale matching function, it implies that the ratio of suitable workers to vacancies is a summary statistic for all relevant information in a submarket. Hence, the only relevant state variable for workers and firms in a given submarket is the queue length of the submarket. As will be shown shortly, this is crucial for making the model block recursive.
Matching. In each submarket the number of matches is given by a Cobb-Douglas matching function. The inputs of this function are the number of vacancies, \( v_\tau \), posted in a submarket, as well as the total productive units of workers searching in this submarket: \( u^E_\tau = a^H u^H_\tau + a^L u^L_\tau \), where \( u^\tau_i \) denotes the measure of unemployed workers of type \( i \) searching in the submarket \( \tau \). The matching function for a specific submarket is:

\[
m_\tau = (u^E_\tau)^\alpha (v_\tau)^{1-\alpha}
\]  

(1)

When a firm is thinking in which submarket to post a wage, the only relevant piece of information is the vacancy filling probability in each submarket. Due to the constant returns to scale in the matching function, this probability depends only on the ratio of the effective units of search over the posted vacancies in each submarket, \( q \):

\[
\lambda_\tau = \frac{m_\tau}{v_\tau} = \lambda(q_\tau) = q_\tau^\alpha
\]

(2)

where \( q_\tau = \frac{u^E_\tau}{v_\tau} \) will be referred to as the queue length in submarket \( \tau \). Moreover, the only relevant pieces of information for a worker of unemployment duration \( \tau \) is the aggregate job-finding probability in this submarket, \( x_\tau \), as well as her belief about the probability of forming successful matches, \( \mu_\tau \). It is straightforward to repeat the calculation in (2) to show that:

\[
x_\tau = \frac{m_\tau}{u^E_\tau} = x(q_\tau) = q_\tau^{\alpha-1}
\]

(3)

In the next section I will show that a worker’s updated belief about her suitability is a function of parameters and the job-finding probability of the submarket she was looking for a job in the previous period.

To summarize, given the queue length in a submarket (which will be determined in equilibrium), an agent’s expected payoff is independent of the level and the composition of workers and firms in the submarket. Free entry of firms ensures that the wage in each submarket in a function of parameters and the submarket’s queue length only. This property of the model is known in the literature as block recursivity because it allows the calculation of the equilibrium queues and wages without keeping track of the distribution of worker types in different submarkets. The property of block recursivity crucially rests on the hiring protocol of Gonzalez and Shi (2010), the fact that search is directed, and the assumption of constant returns to scale in matching.

Learning from Unemployment Duration. While in unemployment a worker learns about her \( a \), the probability she will be productive in a randomly selected job.\(^6\) I define the worker’s expectation of \( a \) to be her belief and denote it as \( \mu \). For every worker who enters the labor market as newly unemployed,\(^6\)

\(^6\)It is important to stress again that every other participant in the labor market would have the same belief regarding a worker’s \( a \) as the worker herself. It will be shown shortly that the beliefs are functions of publicly observable information, hence the update is symmetric for every participant in the market.
her initial belief about her ability is \( \mu_1 = \pi a^H + (1 - \pi) a^L \).

The updating of beliefs depends on the queue length of the submarket into which the worker was searching in the last period. In her first period of unemployment the worker joins the submarket in which unemployed with duration \( \tau = 1 \) are searching. Let the queue length in this submarket be \( q_1 \) and the job-finding rate be \( x_1 = q_1^{\alpha - 1} \). If the worker gets a job, she moves to the employment state and she no longer needs to keep track of her beliefs. If she fails to find a job, she will update her beliefs accordingly. Let \( P^H \) denote the probability that \( a = a^H \). In her first period of unemployment, \( P^H \) is just equal to \( \pi \); if she fails to find a job in this period though, Bayes rule implies:

\[
P^H = P(a = a^H | x_1, no\ match) = \frac{\pi (1 - a^H x_1)}{\pi (1 - a^H x_1) + (1 - \pi) (1 - a^L x_1)} = \frac{\pi (1 - x_1 a^H)}{1 - x_1 \mu_1}
\]

For an arbitrary period of unemployment in some submarket with job-finding rate equal to \( x_\tau \), which did not lead to a successful match, this updating would read:

\[
P^H_{\tau + 1} = \frac{P^H_{\tau} (1 - x_\tau a^H)}{1 - x_\tau \mu_{\tau}}
\]

where \( \mu_{\tau} = P^H_{\tau} a^H + (1 - P^H_{\tau}) a^L \).

Substituting (4) into the definition of \( \mu_{\tau + 1} \) and doing some algebra yields:

\[
\mu_{\tau + 1} \equiv H(\mu_{\tau}, x_\tau) = a^H - \frac{(a^H - \mu_{\tau})(1 - x_\tau a^L)}{1 - x_\tau \mu_{\tau}}
\]

Notice that \( H(x_\tau, \mu_{\tau}) \) is decreasing in \( x_\tau \): the higher the job-finding rate in a submarket, the stronger the signal that the worker did not get match because of her low ability. It is important to stress again that workers of the same duration have the same beliefs about their suitability for a random job and update them in the same way, as the whole labor market does.

**Timing.** Each period of the model consists of four stages:

1. Exit of workers and entry of newly unemployed
2. Wage-posting
3. Matching
4. Production
Value Functions. To determine the optimal wage-posting policies by firms, I follow Acemoglu and Shimer (1999a,b) and rely on Bellman’s Principle of Optimality to compute the value of one-period deviations. A firm posts wage \( w \) this period and behaves optimally forever after; a worker applies to a vacancy paying wage \( w \) this period and behaves optimally forever after. Since agents behave optimally after the current period, the continuation values should include the maximized values of a vacancy or a worker, respectively. That is, the continuation values are the maximum values which a vacancy or an unemployed worker can get in the market. Of course, these are endogenous objects to be determined in equilibrium.

Consider a firm evaluating the prospect of posting wage \( w' \) aimed at workers of duration \( \tau \).\(^7\) In directed search models, workers adjust their behavior in response to different wages posted by firms. In this sense, when a firm posts wage \( w' \) in submarket \( \tau \) it anticipates a queue length \( q' \), which is a function of the posted wage: \( q' = Q_\tau(w') \). The function \( Q_\tau(\cdot) \) represents the firms’ rational expectations about the equilibrium relationship between posted wages to queue length; it is defined for any wage \( w \), not only for the wage that will be posted in equilibrium. It is an endogenous object to be determined in equilibrium under a rational expectations condition, which will be articulated in the next section.

The value of posting a vacancy with wage \( w \) for workers of unemployment duration \( \tau \) is given by:

\[
V_\tau(w) = -\kappa + \left[ \lambda(Q_\tau(w))J_\tau(w) + (1 - \lambda(Q_\tau(w)))V^* \right] \tag{6}
\]

where \( V^*_\tau = \max_w V_\tau(w) \). This expression captures the fact that the firm receives the maximum value of posting a vacancy in submarket \( \tau \) after the current period. The firm pays a cost \( \kappa \) to post the vacancy, which is the same for all submarkets. Of course, the probability the vacancy is filled is a function of the expected queue length: \( \lambda(w) = \lambda(Q_\tau(w)) \) and \( Q_\tau(w) \) will be determined in equilibrium. It denotes the queue length a firm anticipates when posts a vacancy paying wage \( w \) in submarket \( \tau \).

Following the same argument, the value of filling a vacancy with an unemployed of duration \( \tau \) is given by:

\[
J_\tau(w) = y_\tau - w + \beta(1 - \nu)J_\tau(w) \tag{7}
\]

The worker produces \( y_\tau \) units of produce and is paid the posted wage \( w \); when the exit shock hits, the vacancy is destroyed.

Turning to workers, the value of being unemployed for \( \tau \) periods and applying to a vacancy paying \( w \) with queue length \( q \) is:

\[
U_\tau(w, q) = b + \beta(1 - \nu)\left[ \mu_\tau x(q)(E_\tau(w) - U^*_\tau+1) + U^*_\tau+1 \right] \tag{8}
\]

where \( x(q) = q^{\alpha-1} \), \( \mu_\tau+1 = H(x(q), \mu_\tau) \) and \( U^*_\tau+1 = \max_{w,q} U_{\tau+1}(w, q) \). A worker receives \( b \) while unemployed, with \( b < y_\tau \). The job-finding probability is the product of the aggregate job-finding

\(^7\)An equivalent way to express that is to say that the firm creates a new submarket for workers of unemployment duration \( \tau \), posting a vacancy at wage \( w' \).
probability, \(x(q)\), as well as the probability the worker being suitable for the job. The worker does not know that probability, so she uses her beliefs \(\mu_\tau\) to calculate the value of unemployment; if she fails to find a job, she updates her beliefs following Bayes rule in equation (5). Finally, the workers get their maximum value of unemployment after the current period.

Similarly, the value of employment can be computed as:

\[
E_\tau(w) = w + \beta(1 - \nu)E_\tau(w)
\]

As long as the worker is employed, she receives the wage posted in the submarket she was searching when hired. When the exit shock hits, the worker exits the labor market.

### 2.2 Equilibrium

**Equilibrium Queue Lengths.** Recall that the queue length function \(Q_\tau(w)\) represents a firm’s rational expectations about the queue of workers it would face if it posted the wage \(w\) directed to unemployed workers of duration \(\tau\). The idea is that in equilibrium these expectations should be pinned down by subgame perfection: \(Q_\tau(w)\) would be the queue length faced by the firm in the subgame where it posts \(w\) but all other firms post the equilibrium wage aimed at workers of duration \(\tau\).

Following a common practice in the directed search literature, I do not explicitly study the game-theoretic formulation of the model. Rather, I impose the following equilibrium condition on queue lengths to capture the spirit of subgame perfection, which needs to hold for all \(\tau\):

\[
Q_\tau(w) = \begin{cases} 
0, & U_\tau(w, 0) < U_\tau^* \\
\in [0, \infty), & U_\tau(w, Q_\tau(w)) = U_\tau^* \\
\infty, & U_\tau(w, \infty) > U_\tau^* 
\end{cases}
\]

When the firm posts wage \(w\) there are three possible outcomes. First, if the wage is very low (or \(U_\tau^*\) is very high), then the firm attracts no workers and \(Q_\tau(w) = 0\). Moreover, workers must find it strictly suboptimal to apply to his job (since the wage is too low) even there are no other workers competing for that vacancy and, as a result, \(U_\tau(w, 0) < U_\tau^*\). Second, if the wage is very high (or \(U_\tau^*\) is very low), then the firm attracts all workers and \(Q_\tau(w) = \infty\). A worker must find it strictly optimal to come to apply to this firm, even when she has to compete with all other workers for the vacancy. Third, if the wage is in an intermediate range, then workers will apply to this vacancy until they are indifferent between applying to this job (receiving the value \(U_\tau(w, Q_\tau(w))\)) or to any other vacancy (receiving the value \(U_\tau^*\)). That is, the queue length \(Q_\tau(w)\) should solve the equation \(U_\tau(w, Q_\tau(w)) = U_\tau^*\).

Notice that, as argued in Shi (2002); Shi et al. (2006), the third case is impossible to take place: if the queue length is infinite, the probability a worker gets a job is zero, hence her expected utility from

---

8The game-theoretic foundations of the equilibrium queue lengths condition (10) are masterly analyzed in Burdett et al. (2001) and Galenianos and Kircher (2012).
searching in this submarket is zero, which is less than $U_\tau^*$, a contradiction of the requirement. Hence, the equilibrium queue length condition can be simplified as:

$$Q_\tau(w) = \begin{cases} 0, & U_\tau(w, 0) < U_\tau^* \\ \in [0, \infty), & U_\tau(w, Q_\tau(w)) = U_\tau^* \end{cases} \quad (11)$$

Finally, I show in Lemma 3 that the first case will never be observed in equilibrium. However, condition (11) is important because it pins down the out-of-equilibrium firms’ beliefs about workers’ responses to wage offers that are not observed in equilibrium.

**Definition of Equilibrium.** A competitive search equilibrium is a set of wages offered by firms $W^* \subset \mathbb{R}_+$, a set of queue length functions $\{Q^*_\tau\}$ with each $Q^*_\tau : \mathbb{R}_+ \mapsto \mathbb{R}_+$, a vector of workers’ utility levels $U^* \in \mathbb{R}_+^T$, a belief function $\mu_\tau : \mathbb{R} \mapsto [0, 1]$ and a set of value functions $\{J_\tau : \mathbb{R}_+ \mapsto \mathbb{R}_+, V_\tau : \mathbb{R}_+ \mapsto \mathbb{R}_+, E_\tau : \mathbb{R}_+ \mapsto \mathbb{R}_+, U_\tau : \mathbb{R}_+^2 \times T \mapsto \mathbb{R}_+\}$ defined in (6)-(9) above, with the following properties:

1. **Optimal Application.** $U^*_\tau = \sup_{w_\tau \in W^*} U_\tau(w_\tau, Q^*_\tau(w_\tau))$, for all $\tau$

2. **Profit Maximization and Free Entry.** $V^*_\tau(w_\tau^*) = 0 \geq V_\tau(w_\tau)$, for any $w_\tau$, for all $w_\tau^* \in W^*$ and for all $\tau$

3. **Consistency/Rational Expectations/Subgame Perfection.** $Q^*_\tau(w_\tau)$ satisfies the equilibrium queue lengths condition (11), for all $\tau$ and for all $w_\tau^* \in W^*$

4. **Beliefs Updating.** A worker with beliefs $\mu_\tau$ in submarket $\tau$ uses Bayes rule to update her beliefs: $\mu_{\tau+1} = H(x(Q^*_\tau(w_\tau)), \mu_\tau)$, if she fails to find a job in this submarket.

**Equilibrium as a Solution to an Auxiliary Problem.** An important result, due to Moen (1997) and Acemoglu and Shimer (1999a,b), is that the equilibrium can be characterized as the solution to an auxiliary constrained maximization problem. The objective function of the auxiliary problem is the value function of the agents on one side of the market; the constraint is that the agents on the other side of the market receive their optimal values. I extend this equivalence result to a framework with skill depreciation and a declining expected quality of the unemployment pool.

Consider the following constrained maximization problem:

$$V^*_\tau = \max_{w_\tau, q_\tau} -\kappa + \lambda(q_\tau) \frac{y_\tau - w_\tau}{1 - \beta(1 - \nu)}, \quad \forall \tau \leq T \quad (12)$$

subject to the constraints:

$$U^*_\tau \leq U_\tau(w_\tau, q_\tau) = b + \beta(1 - \nu) \left[ \mu_\tau x(q_\tau) \left( \frac{w_\tau}{1 - \beta(1 - \nu)} - U^*_\tau \right) + U^*_{\tau+1} \right] \quad (13)$$
and \( q_\tau \geq 0 \) with complementary slackness, where \( U_\tau^* = \sup U_\tau(w_\tau^*, q_\tau^*) \)

\[
V_\tau^* = 0, \quad \forall \tau \leq T
\]  

\[
\mu_{\tau+1} = H(x(q_\tau), \mu_\tau)
\]

In this auxiliary problem the firm takes the optimal values of workers as given. Solving this problem yields the optimal \( w_\tau^* \) and \( q_\tau^* \) as functions of \( U_\tau^* \), for all \( \tau \). The sequence of beliefs is constructed following equation (15) based on the sequence \( \{q_\tau^*\}_{\tau \leq T} \) The market values of workers are pinned down by solving equation (14) for all \( \tau \).

Suppose for now that this problem has a solution (not necessarily unique): \( \{w_\tau^*, q_\tau^*\}_{\tau \leq T} \) (I prove in the next section that the auxiliary problem does, indeed, have a solution). Then, the equivalence of the competitive search equilibrium with the solution to the auxiliary optimization problem is obtained through the following lemmas. All proofs can be found in Appendix I.

**Lemma 1 (Equilibrium \( \mapsto \) Auxiliary Problem).** Let \( w_\tau^* \in \mathbb{W}^* \) and \( q_\tau^* = Q_\tau^*(w_\tau^*) \), where \( \{\mathbb{W}^*, \{Q_\tau^*\}_{\tau \leq T}, U^*\} \) is an equilibrium allocation; then \( \{w_\tau^*, q_\tau^*\}_{\tau \leq T} \) solve problem (12) under constraints (13), (14) and (15), with \( U_\tau(w_\tau^*, q_\tau^*) = U_\tau^* \) if \( q_\tau^* > 0 \).

**Lemma 2 (Auxiliary Problem \( \mapsto \) Equilibrium).** If some \( \{w_\tau^*, q_\tau^*\}_{\tau \leq T} \) solve problem (12) under constraints (13), (14) and (15), then there exists an equilibrium \( \{\mathbb{W}^*, \{Q_\tau^*\}_{\tau \leq T}, U^*\} \) such that \( w_\tau^* \in \mathbb{W}^* \) and \( q_\tau^* = Q_\tau^*(w_\tau^*) \), \( \forall \tau \leq T \).

**Equilibrium Existence and Characterization.** The usefulness of Lemmas 1 and 2 is that they enable me to characterize equilibrium as the solution to the auxiliary profit maximization problem (12) under constraints (13), (14) and (15). A standard assumption, satisfied by my preferred Cobb-Douglas specification, is that \( \lambda(\cdot) \) is a strictly concave function. This guarantees the existence of an equilibrium in which workers of different unemployment durations search in different labor markets.

**Proposition 1.** There exists an equilibrium in which the labor market is segmented by unemployment duration.

The proof of existence constructs a fixed-point argument that uses Brouwer’s theorem. The strategy of the proof also suggests a computational strategy for how to compute the equilibrium, which is analyzed in Section 4.1. It is important to highlight that the algorithm fully exploits the block recursivity of the model: all endogenous variables are computed independently of the distribution of workers across states. Computing the masses of workers across different states becomes a matter of accounting.

An appealing feature of this model is its tractability. Indeed, one can analytically show that workers face declining job-finding probabilities and reemployment wages over a spell of unemployment. The tractability of the model is a result of the Gonzalez and Shi (2010) hiring protocol, as well as the
equivalence of competitive search equilibrium with the auxiliary problem. Extending the machinery of Moen (1997) and Acemoglu and Shimer (1999a, 1999b) to the current environment enables me to exploit the firms’ FOCs and analytically prove the following set of results.

**Proposition 2.** In any segmented markets equilibrium $q_\tau$ is increasing and $w_\tau$ is decreasing in $\tau$; also, the difference $y_\tau - w_\tau$ is decreasing in $\tau$. Hence, the value of a filled vacancy, $J(w_\tau)$, is decreasing in $\tau$.

Other papers in the recent macroeconomic literature on duration dependence feature some troubling implications. For example, for a given cohort of unemployed workers, the model of Gonzalez and Shi (2010) predicts increasing exit rates from unemployment for all workers. The model of Doppelt (2014) makes the same prediction but for a minority of workers. I prove that job-finding rates unambiguously decline for all workers following a specific cohort of unemployed. Finally, in the model of Fernández-Blanco and Preugschat (2016) reemployment wages may increase with unemployment duration. On the other hand, I prove that reemployment wages unambiguously fall over the spell of unemployment, as in the data.

It is worth underscoring that skill depreciation is the primary factor supporting these results, not the declining quality of unemployment pool. In other words, in a model with skill depreciation only, Proposition 2 would still hold. On the other hand, in a model which declining quality of unemployment pool is the only source of duration dependence, Proposition 2 would not be unambiguously true. In some simulations of this version of the model, I find wages to be increasing, queue lengths to be decreasing and the value of filling a vacancy to be increasing over the spell of unemployment. This echoes the counterfactual findings described above in papers that feature only unobserved worker quality. This shows that skill depreciation is necessary in order the model not to deliver counterfactual predictions. Unobserved heterogeneity and learning are necessary for the model to deliver convex job-finding rates, as explained in Section 4.2. Finally, these features of the model suggest a calibration strategy, since they demonstrate which mechanism accounts for each observable prediction of the model.

To close this section, I state two technical results, along with a more substantial one. It is common in directed search models that all submarkets open in equilibrium feature positive queue lengths (otherwise, firms would have profitable deviations). Moreover, as expected by the fact that high ability workers find jobs faster than their low ability counterpart, expected worker suitability declines over the spell of unemployment. The hiring protocol of Gonzalez and Shi (2010) captures the declining quality of unemployment pool in a straightforward and intuitive way.

**Lemma 3.** In any segmented markets equilibrium $q_\tau > 0$ for all $\tau$. Hence, the complementary slackness condition (13) holds with equality.

**Lemma 4.** Beliefs about worker suitability for a given job, $\mu_\tau$, are decreasing in $\tau$.

---

9This result is also important because it shows that negative duration dependence is not a trivial outcome when the duration of unemployment provides a signal of worker quality. Learning dynamics may lead workers to target jobs with lower queue lengths to increase the probability of getting hired. If this effect is strong enough, exit rates from unemployment will be increasing in unemployment duration.
Finally, since the employment prospects of workers deteriorate over time, the value of unemployment is strictly decreasing in unemployment duration. This result would also hold in a model with skill depreciation only. However, as mentioned above, the deterioration of employment prospects would not be fast enough to rationalize convex job-finding rates in this case.

**Proposition 3.** In any segmented markets equilibrium the value of unemployment, \( U^*_{\tau} \), is decreasing in \( \tau \).

### 2.3 Quantitative Extension

**Endogenous Search Effort.** The framework presented above can be easily extended to incorporate an extra force of duration dependence: declining search effort. I model search effort as a participation decision: a measure of unemployed workers will not be searching for jobs. In each period an unemployed worker is hit by an IID search cost shock \( \tilde{c} \). The support of \( \tilde{c} \) is a bounded interval in the real line, \( \text{supp}(\tilde{c}) = [-K_1, K_2] \), and its CDF is a continuous strictly increasing function \( F(\tilde{c}) \). The Bellman equation for an unemployed worker of duration \( \tau \) can be written as:

\[
U_\tau(w,q) = \beta \int_{-K_1}^{K_2} \max \left\{ -\tilde{c} + \beta (1 - \nu) \mu_\tau x(q) (E_\tau(w) - U^*_{\tau+1}), 0 \right\} dF(\tilde{c}) + \beta (1 - \nu) U^*_{\tau+1} \tag{16}
\]

The idea here is that if the search cost drawn at a period is low enough, then the worker participates in the labor market facing the job-finding prospects analyzed above. If the search cost is high though, the worker does not participate in the labor market and she enters next period as unemployed.

One can apply the standard quantile transformation to write equation (16) in a more concrete form. Define the function \( c'(z) \equiv F^{-1}(z) \), where \( z \) is a uniform random variable with [0, 1] support: \( z \sim U_{[0,1]} \).\(^{10}\) After the change of variables \( \tilde{c} \equiv c'(z) \), the value function of unemployment can be written as:

\[
U_\tau(w,q) = \beta \int_{0}^{1} \max \left\{ -c'(z) + \beta (1 - \nu) \mu_\tau x(q) (E_\tau(w) - U^*_{\tau+1}), 0 \right\} dz + \beta (1 - \nu) U^*_{\tau+1} \tag{17}
\]

Since \( F(\tilde{c}) \) is strictly increasing, its inverse is strictly increasing as well. Hence, the value function can be written as:

\[
U_\tau(w,q) = b + \max_{s \in [0,1]} \int_{0}^{s} -c'(z) + \beta (1 - \nu) \mu_\tau x(q) (E_\tau(w) - U^*_{\tau+1}) dz + \beta (1 - \nu) U^*_{\tau+1} \tag{18}
\]

Assuming that \( c'(z) \) is integrable, it has a well-defined antiderivative function \( c(z) \). If one assumes that \( c(0) = 0 \), one can write the value function in the familiar form:

\[
U_\tau(w,q) = \max_{s \in [0,1]} \left\{ b - c(s) + \beta (1 - \nu) s \mu_\tau x(q) (E_\tau(w) - U^*_{\tau+1}) + \beta (1 - \nu) U^*_{\tau+1} \right\} \tag{19}
\]

\(^{10}\)It is trivial to show that \( c'(z) \) has the same CDF as \( \tilde{c} \)
The interpretation of $s$ is different, though: instead of denoting the intensity of job search activity (intensive margin), here $s$ denotes the probability to participate in the labor market (extensive margin). This interpretation rests on the microfoundation presented above, in which the basic assumption is that search cost shocks are IID over time. Alternatively, one could think of this microfoundation as follows: only a measure $s$ of unemployed workers of duration $\tau$ participates in the labor market when applying to a job offering wage $w$ with a queue length $q$, while a measure $1 - s$ does not search for jobs. To summarize, equations (16) and (19) are equivalent and produce the same answer concerning worker job-search effort, supported by two different interpretations.

The FOCs for this problem are straightforward to interpret: the probability to participate in the labor market equalizes the marginal cost of participation with its marginal return. Evaluating the FOCs in equilibrium yields:

$$c'(s^*_\tau) = \beta(1 - \nu)\mu_\tau x(q_\tau) \left[ \frac{w_\tau}{1 - \beta(1 - \nu)} - U^*_{\tau+1} \right]$$

Assuming the standard power search cost function, $c(s) = \phi^{\eta s} / \eta$, and a Cobb-Douglas matching function, equation (20) becomes:\n
$$s^*_\tau = \left\{ \beta(1 - \nu)\phi^{-1} \mu_\tau q^{-1}_\tau x(q_\tau) \left[ \frac{w_\tau}{1 - \beta(1 - \nu)} - U^{\eta*}_{\tau+1} \right] \right\}^{\eta^{-1}}$$

Substituting back into (19) and a bit of algebra leads to:

$$q^{\alpha}_\tau \frac{w_\tau}{1 - \beta(1 - \nu)} = q^{\alpha}_\tau U^{\alpha*}_{\tau+1} + \frac{q_\tau}{\beta(1 - \nu)\mu_\tau} \left\{ \frac{U^{\alpha*}_{\tau+1} - b - \beta(1 - \nu)U^{\eta*}_{\tau+1}}{\phi^{\eta \tau - 1} \eta^{-1}} \right\}^{2^{-1}}$$

This is the enriched version of constraint (13) for the case with endogenous participation choice. One can substitute this constraint into firms’ profit and take FOCs with respect to $q_\tau$. This gives an expression for $q^{1-\alpha}_\tau$ as a function of $U^{*\alpha}_{\tau+1}$, $\mu_\tau$ and parameters only. Finally, one could substitute back into (22) to obtain an expression for equilibrium wage:

$$\frac{w_\tau}{1 - \beta(1 - \nu)} = \alpha \left( \frac{y_\tau}{1 - \beta(1 - \nu)} - U^{\alpha*}_{\tau+1} \right) + U^*_{\tau+1} = \alpha \frac{y_\tau}{1 - \beta(1 - \nu)} + (1 - \alpha)U^*_{\tau+1}$$

Based on (23) it is trivial to calculate the value of a filled vacancy for the firm:

$$J(w_\tau) = \frac{y_\tau - w_\tau}{1 - \beta(1 - \nu)} = (1 - \alpha) \left[ \frac{y_\tau}{1 - \beta(1 - \nu)} - U^*_{\tau+1} \right]$$

Proposition (2) ensures that $J(w_\tau)$ is decreasing in $\tau$; thus, the surplus of the match, $\frac{y_\tau}{1 - \beta(1 - \nu)} - U^{\alpha*}_{\tau+1}$, must also be decreasing in $\tau$. Moreover, simple substitution into the Free Entry condition and the

---

participation FOCs yields:

\[ q_\tau = \kappa^{\frac{1}{\alpha}} (1 - \alpha)^{-\frac{1}{\alpha}} \left[ \frac{y_\tau}{1 - \beta(1 - \nu)} - U^*_\tau + 1 \right]^{\frac{1}{\alpha}} \] (25)

\[ s^*_\tau = \left\{ \beta(1 - \nu) \phi^{-1} \mu_{\tau, 1}^{\frac{1 - \alpha}{\alpha}} (1 - \alpha)^{\frac{1 - \alpha}{\alpha}} \left[ \frac{y_\tau}{1 - \beta(1 - \nu)} - U^*_\tau + 1 \right] \right\}^{\frac{1}{\nu - 1}} \] (26)

which proves that under this specific parameterization the optimally chosen search effort is decreasing over unemployment duration.

**Lemma 5.** Under a power search cost function and a Cobb-Douglas matching function, workers’ participation probability is decreasing in \( \tau \).

**Exogenous Separations.** I also introduce exogenous separation shocks for the quantitative analysis: an employed worker loses her job with probability \( \delta \) each period. Since workers now move from employment to unemployment, I need to take a stance on how their unobserved ability evolves when they enter unemployment. I assume that every time a worker reenters unemployment her ability type is redrawn from the same distribution. That is, every time a worker reenters unemployment she will be of high ability with probability \( \pi \). This assumption says that there is a measure of \( \pi \) high ability workers in the population at every instant but the identities of those individuals change over time.

This assumption is not as restrictive as it may seem at a first glance. It is common in the literature to assume that inference on workers’ unobservable qualities reduces to workers’ current unemployment spell. Fernández-Blanco and Preugschat (2016) and Jarosch and Pilossof (2015) make the same assumption. In support of it, Eriksson and Rooth (2014) find that long unemployment spells in the past do not matter for employers’ hiring decisions, suggesting that subsequent work experience eliminates this negative signal. An equivalent way to implement that assumption would be to assume that workers “forget” what they have learned during their previous unemployment spells, as in Fernández-Blanco and Preugschat (2016) and Jarosch and Pilossof (2015). In my setting it seems more natural to assume that worker types are redrawn and I follow that route. An intuitive way to think about this idea is to have in mind that the model follows a specific cohort of unemployed over time.

### 3 Empirical Evidence

This section presents the empirical evidence that model the will speak to in Section 5 of the paper.

**Job-Finding Rates.** The main focus of this paper is to decompose duration dependence in unemployment into its key channels. The empirical evidence for duration dependence comes from the
empirical relationship between the observed job-finding probability and unemployment duration in the Current Population Survey (CPS). Specifically, I follow Kroft et al. (2016) and Jarosch and Pilossoph (2015) and estimate that relationship in two steps.

![Normalized Job Finding Probability](image)

Figure 1: Normalized Job-Finding Probabilities by Unemployment Duration. Data from CPS, 1994-2014, workers between 25 and 54 years old

First, I pool CPS data from 1994 to 2014, following the matching process outlined in Nekarda (2009), for workers between 25 and 54 years old. I regress the dummy for finding a job on unemployment duration and a standard set of demographic controls via weighted nonlinear least squares.\(^{12}\) Second, I estimate an exponential function for the average job-finding probability at duration \(\tau\) relative to the average job-finding probability of workers who have been unemployed one month or less:

\[
\frac{f(\tau)}{f(1)} = b_0 + (1 - b_0) \times \exp(-b_1 \times \tau) \tag{27}
\]

The empirical estimates are \(\hat{b}_0 = 0.480\) and \(\hat{b}_1 = 0.329\), very close to the estimates of Jarosch and Pilossoph (2015) and Kroft et al. (2016). Figure 1 plots the normalized job-finding probabilities (i.e. relative to the level in the first month) along with the fitted curve implied by specification (27). This fitted curve will serve as the main evaluation test of the model: the predicted job-finding probabilities

---

\(^{12}\)The controls include a gender dummy, a fifth degree polynomial in age, three race dummies (white/black/other), four education category dummies, and gender interactions with all these covariates.
will be compared to the estimates of specification (27), as a test for the success of the model to replicate the job-finding profile.

**Reemployment Wages.** In my model only workers suitable for a vacancy are hired. Unobserved heterogeneity implies that some workers are suitable for more jobs than other workers but this heterogeneity is not directly reflected in reemployment wages. Hence, the appropriate measure of wages for this paper should strip out the effects of unobserved heterogeneity in wages. Unfortunately, this cannot be done with CPS data. CPS follows workers for only eight months, hence there are very few workers that move from unemployment to employment twice. Thus, fixed effects cannot be used to strip out the effect of unobserved heterogeneity.

Fortunately, there are good measurements of this effect available in the literature. Schmieder et al. (2016) and Autor et al. (2015) provide causal estimates of non-employment duration on wages. They find that for each additional month in non-employment, wages decline by a bit less than one and two and a half percent, respectively. The sample in Autor et al. (2015), though, consists of SSDI applicants with low labor force attachment, hence this number may be too large for this paper. Moreover, Ortego-Marti (2017), controlling for unobserved heterogeneity with fixed effects in the PSID, finds that an extra month in non-employment lowers wage by one point two percent. Thus, I will use a monthly wage loss of one percent to discipline the decline of human capital in my model. Finally, note that both Schmieder et al. (2016) and Ortego-Marti (2017) report that this drop in wages is linear; that is, it is almost equal for each month in non-employment. I will exploit this feature of the data in my identification strategy, presented in Section 4.2.

**Search Effort.** It is notoriously difficult to obtain reliable measurements of job-search effort (see Hornstein and Kudlyak (2016)). Mukoyama et al. (2014), DeLoach and Kurt (2013), and Gomme and Lkhagvasuren (2015) use the minutes devoted to job-search as their measure of search effort. Using the American Time Use Survey (ATUS), along with CPS, allows them to obtain evidence of how this measure changes over the business cycle. These papers find mixed evidence regarding the cyclicity of time devoted to search.

A recent source of reliable evidence is the New Jersey survey of Krueger and Mueller (2011)— KM from now on. I choose to use that data for the following reasons: first, it is a panel survey. They followed the same unemployed workers over time; on the other hand, the ATUS is a cross-sectional survey. As a result, fixed effects cannot be incorporated directly. One needs to project time devoted to search by using the methods of job-search from CPS, as in Mukoyama et al. (2014). This method yields a monthly measure, yet it is plagued by the well-known reporting problems of CPS. Second, the KM survey was conducted in a weekly basis. Hence, the self-reported evidence on job-search are probably more accurate than those coming from CPS, which is a monthly survey. Finally, Krueger and Mueller (2011) oversampled long-term unemployed workers, which guarantees reliable reports of search effort for workers with high duration of unemployment.
I run two simple fixed effects regressions using the KM data.\textsuperscript{13} First, to determine the proper measure of search effort, I use the following specification:

$$Offer_{it} = a_i + \beta \times SE_{it-1} + \epsilon_{it}$$

(28)

where $Offer_{it}$ is a dummy of whether the individual was offered a job in week $t$ and $SE_{it-1}$ is a measure of search intensity in week $t - 1$. When $SE_{it-1}$ is the number of hours devoted to job-search (intensive margin), the estimate $\hat{\beta}$ is not significant, with a t-statistic equal to -0.49: one extra hour of job search has an insignificant effect on generating job-offers. This result was also obtained in a more sophisticated way by Krueger and Mueller (2011) and challenges the use of time devoted to search as the appropriate measure of search effort. On the other hand, when $SE_{it-1}$ is a dummy variable of whether the individual did anything to find a job in week $t - 1$ (extensive margin), the estimate of $\beta$ appears to be significant, with a t-statistic of 4.78. Hence, I choose to work with the extensive margin of participation as the proper measure of search effort in the KM data.

Second, I regress the dummy of search effort on unemployment duration and an individual fixed effect:

$$SE_{it} = a_i + \beta \times \tau_{it} + \gamma \times \tau_{it}^2 + u_{it}$$

(29)

where $\tau$ is the unemployment duration of the individual in week $t$ of the survey. The coefficient $\hat{\beta}$ is estimated to be equal to -0.024, with a t-statistic of -7.01, and $\hat{\gamma} = 0.0001$, with a t-statistic of 4.66. However, the KM survey was conducted from October 2009 to April 2010, a period of mass unemployment in New Jersey. Hence, the measured discouragement effects are likely higher than the effects in normal times. I choose to discipline the decline in search effort in my model assuming a weekly linear drop in participation of around two percent.

It is worth mentioning that this finding is consistent with the evidence reported in Faberman and Kudlyak (2014). Using data from a job website, they find that the weekly number of submitted applications declines as job-search continues, controlling for individual fixed effects. In their data, the drop seems to have a convex and not linear shape, though.

**Callback Rates.** To inform the distribution of unobserved heterogeneity in my model I use data from the audit study of Kroft et al. (2013), as reported in Kroft et al. (2016). This paper uses an audit study approach: they submitted carefully constructed fictitious job applications to posted job openings to investigate whether the duration of non-employment affects the likelihood to receive a callback when applying for a job. Kroft et al. (2013) report a steep decline of callbacks along duration, which can be seen in Figure 2.

Unfortunately, the external validity of this evidence is far from established. Jarosch and Pilossoph

\textsuperscript{13}Krueger and Mueller (2011) followed different groups of unemployed, starting from different initial unemployment durations; that is, they followed different “cohorts” of unemployed workers. All standard errors reported are clustered at this cohort level. Finally, the regressions use the weights provided in the survey.
Figure 2: Normalized Callback Probabilities by Unemployment Duration as approximated by equation (27) and reported in Kroft et al. (2016)

(2015) offer a thoughtful summary of the literature on audit studies and I summarize their main points here. Ghayad (2013) finds similar results as Kroft et al. (2013); Oberholzer-Gee (2008) finds declining callbacks only for very long unemployment spells; Eriksson and Rooth (2014) find large drops in callbacks for medium and low skilled jobs but not for high skilled jobs; most importantly, Farber et al. (2017) find no evidence of duration dependence in callbacks.

Farber et al. (2017) attribute most of the difference with Kroft et al. (2013) on the age composition of their samples: the former focus on older job applicants (mid-thirties to mid-fifties) while the latter on younger job-applicants (mid-twenties). There is an intuitive mechanism behind that difference: older applicants have longer employment histories that may outweigh any recent employment experience when resumes are evaluated by potential employers. Younger job-seekers, however, have short employment histories, hence recent unemployment experience may get higher weight in the evaluation of their applications. The fact that the applicants in Eriksson and Rooth (2014) and Ghayad (2013) are all in their twenties, with no more than five or six years of experience, supports that conclusion. Employers seem to employ unemployment duration as a signal of workers’ quality in cases where the information on workers’ CVs is not rich enough to allow for an informed decision (young workers, low-skilled applicants or applicants in slack labor markets; see Kroft et al. (2013)).

Jarosch and Pilossoph (2015), in the context of an equilibrium search model, find that interviews lost to statistical discrimination that would otherwise have led to jobs are very rare. Firms discriminate against long-term unemployed because they correctly anticipate being unable to form a viable match with them. Based on this result, this paper abstracts from stigma effects of unemployment duration—
meaning, that duration dependence in callbacks creates duration dependence in actual hiring. My interpretation of the results of audit studies is that discrimination in callbacks is an informed response to workers’ unobserved characteristics. In other words, employers’ beliefs, as captured by duration dependence in callbacks, are informative about unobservable worker quality among the population of job-seekers. I choose to discipline unobserved heterogeneity with the callback results from Kroft et al. (2013) because, given that their applicants were relatively young, the use of unemployment duration as a signal in this study has the best chance of being informative regarding the underlying worker characteristics, among the available audit studies.

4 Quantitative Analysis

4.1 The Fixed-Point Problem

The proof of equilibrium existence applies Brouwer’s fixed point theorem on the auxiliary optimization problem of maximizing (12) under the constraints (13), (14) and (15). Recall that the objective is the firm’s value of posting a vacancy, given that is supplies the worker with her market value, the Free Entry condition holds and beliefs about worker quality follow Bayes rule. The structure of this problem implies a straightforward algorithm for the computation of equilibrium.

The algorithm rests on the structure of this auxiliary problem and I conjecture that it could be used for all block recursive directed search models. It is similar in spirit to the famous Menzio and Shi (2011) method but it differs from their work in that it exploits the tractability of firm’s FOCs in the auxiliary problem, instead of the worker’s optimal submarket choice.

The equilibrium in my model is a fixed point in the space of workers’ market values, \( \{U^*_\tau\}_{\tau \leq T} \). One can see that by carefully inspecting the auxiliary optimization problem as analyzed in section 2.3. First, note that the market value constraint (22) defines a relationship between \( w_\tau, q_\tau, \mu_\tau \) and \( U^*_\tau, U^*_{\tau+1} \). This relationship is substituted into the objective function of the auxiliary problem to strip out wage from the value of a vacancy. Taking FOCs with respect to \( q_\tau \) yields an expression for queue lengths as a function of \( \mu_\tau, U^*_\tau \) and \( U^*_{\tau+1} \).

The next step is to substitute this expression back into the market value constraint (22). This will lead to an expression of equilibrium wages as a function of workers’ market values, as in equation (23). Notice that the beliefs do not appear in that equation. This is a result of the assumption made in Gonzalez-Shi hiring protocol that unsuitable workers are never hired, as well as of the assumption that workers redraw their types when enter unemployment. As a result, wages do not directly reflect the probability for a successful match.

Next, one can substitute the expression for wages in the Free Entry condition and express equilibrium queue lengths as a function of market values only, as in equation (25). Hence, a guess of \( \{U^*_\tau\}_{\tau \leq T} \) pins down the sequences \( \{w_\tau\}_{\tau \leq T} \) and \( \{q_\tau\}_{\tau \leq T} \), via equations (23) and (25).

Finally, given that \( \mu_1 \) is pinned down by the unobserved heterogeneity parameters, one can use (26)
to compute \( s_1 \). Then, \( \mu_2 \) is a function of \( q_1, s_1 \) and \( \mu_1 \). Using (26) again helps pin down \( s_2 \) and, by iteration, all subsequent \( \{ s_\tau \}_{\tau \leq T} \) and \( \{ \mu_\tau \}_{\tau \leq T} \). One then can use the definition of the value of unemployment to compute a new sequence \( \{ U^*_\tau \}_{\tau \leq T} \), based on the values of \( \{ w_\tau \}_{\tau \leq T}, \{ q_\tau \}_{\tau \leq T}, \{ s_\tau \}_{\tau \leq T} \) and \( \{ \mu_\tau \}_{\tau \leq T} \). If \( \{ U^*_\tau \}_{\tau \leq T} \) is close to the initial guess \( \{ U^*_\tau \}_{\tau \leq T} \), the fixed point is computed; if not, the algorithm should repeat the process.

In short, the algorithm works as follows:

1. Guess a sequence \( \{ U^*_\tau \}_{\tau \leq T} \).
2. Use the structure of the auxiliary problem to compute \( \{ w_\tau \}_{\tau \leq T} \) and \( \{ q_\tau \}_{\tau \leq T} \), via equations (23) and (25).
3. Use workers’ FOCs for optimal search effort and Bayes rule to compute \( \{ s_\tau \}_{\tau \leq T} \) and \( \{ \mu_\tau \}_{\tau \leq T} \).
4. Use the definition of the value of unemployment to compute the updated \( \{ U^*_\tau \}_{\tau \leq T} \).
5. If \( \{ U^*_\tau \}_{\tau \leq T} \) is close to \( \{ U^*_\tau \}_{\tau \leq T} \), stop; otherwise, go to step 1 and repeat.

4.2 Identification

The first step is to show that the structure of the model is sufficient to separately identify the effect of each force contributing to duration dependence. To put it differently, that there are some features of the data that the model would fail to capture if it did not incorporate all channels of duration dependence. In the model unemployed workers who participate in the labor market face duration dependence caused by skill depreciation and the declining quality of the unemployment pool. Hence, it should be shown that the effects of these two forces are not observationally equivalent through the lens of the model. Workers’ participation choice only amplifies these two channels but it does not constitute a separate mechanism of duration dependence that needs to be identified. The magnitude of this amplification will be disciplined directly by the Krueger and Mueller (2011) data on participation presented above.

Consider a version of the model in which the only force creating duration dependence is the declining quality of the unemployment pool. Workers’ productivity stays constant over the spell of unemployment. Without skill depreciation the monotonicity result of Proposition 2 ceases to hold. As a result, this version of the model produces a counterfactual prediction: reemployment wages are non-monotone in unemployment duration. In fact, they may even be increasing in unemployment duration for some parameterizations! Technically, this ambiguity in equilibrium wages is captured in the following expressions:

\[
\begin{align*}
  w_\tau &= \alpha y_\tau + (1 - \alpha) \left[ (1 - \beta(1 - \nu)(1 - \delta)) U^*_\tau + (1 - \beta(1 - \nu) \beta U^*_\tau \right] \\
  w_{\tau + 1} &= \alpha y_{\tau + 1} + (1 - \alpha) \left[ (1 - \beta(1 - \nu)(1 - \delta)) U^*_\tau + (1 - \beta(1 - \nu) \beta U^*_\tau \right]
\end{align*}
\]
In the version of the model without skill depreciation the productivity term is constant across $\tau$. The difference between (30) and (31) yields:

$$w_\tau - w_{\tau+1} = (1 - \alpha) \left[ (1 - \beta(1 - \nu)\delta)U^*_\tau_{\tau+1} - \beta(1 - \nu)\delta U^*_\tau_{\tau} - (1 - \beta(1 - \nu)(1 - \delta))U^*_\tau_{\tau+2} \right]$$  

(32)

Clearly, if the value of unemployment does not drop fast enough, then the negative terms in (32) may be greater than the positive term. This can be seen in Figure 7, in which I show the reemployment wages for versions of the model without skill depreciation. Under the calibrated parameters, reemployment wages are increasing in unemployment duration for the first three months in unemployment, a counterfactual prediction.

This feature of the model echoes the non-monotonicity result of Fernández-Blanco and Preugschat (2016). Their model does not feature skill loss and wages may increase with unemployment duration. This is a result of the directed search nature of these models; the intuition works as follows. Without human capital depreciation, and provided that the difference $a^H - a^L$ is small, the value of unemployment does not fall significantly with unemployment duration. In directed search models wages price waiting times. Hence, wages for longer durations would have to be larger to compensate the (equally productive) long-term unemployed for the lower matching probability they face. The model needs skill depreciation to guarantee that productivity falls sufficiently with unemployment duration to make equilibrium wages decline, an important feature if the data.

Turning to the version of the model without suitability considerations, assume that workers are homogeneous and productive for all jobs. Their productivity still declines with unemployment duration, capturing skill depreciation. Following the arguments of Proposition 2 one can show that this model predicts decreasing wages and job-finding rates. Thus, in principle, a version of the model with only human capital could rationalize both data series. This is not true though: the model cannot rationalize the small linear drop in wages and the large convex drop in job-finding rates at the same time.

To show that, let me derive the equilibrium expressions for job-finding rates and wages in a model with only human capital depreciation:

$$w_\tau = \alpha (y_\tau - \Delta_\tau) + \Delta_\tau$$  

(33)

$$q_\tau^{\alpha-1} \equiv f_\tau = \frac{1 - \alpha}{\kappa} \left( y_\tau - \Delta_\tau \right) \left( \frac{1 - \alpha}{\alpha} \right)$$  

(34)

where $\Delta_\tau \equiv (1 - \beta(1 - \nu)(1 - \delta))U^*_\tau_{\tau+1} - \beta(1 - \nu)\delta U^*_\tau_{\tau}$. The results in Schmieder et al. (2016) and Ortego-Marti (2017) show that the decline of reemployment wages over the unemployment spell is roughly linear. Hence, the path of human capital in the model should be calibrated such that the term $y_\tau - \Delta_\tau$ falls linearly. However, if this is the case, then the term $\left( y_\tau - \Delta_\tau \right) \left( \frac{1 - \alpha}{\alpha} \right)$ is a concave function. According to the Petrongolo and Pissarides (2001) survey, “A plausible range for the empirical elasticity on unemployment is 0.5 to 0.7...”, thus the range of the term $\left( \frac{1 - \alpha}{\alpha} \right)$ is from around 0.43 to 1, making it a concave function over $\tau$. However, it is clear from the data that the drop in job-finding rates has a
convex shape: it is large for short durations and small for high durations.

Intuitively, the productivity drop for workers of short durations is not enough to rationalize sharp declines in job-offer rates. Workers who are unemployed for few periods have almost the same productivity as workers unemployed for one period. Firms find it optimal to respond with slightly decreasing job-offer probabilities. Workers who have accumulated a lot of periods in unemployment have lost a large part of their productivity. As a result, they face sharp declines in job-finding rates, which also contradicts the data. The model needs the composition effect to produce job-finding rates that decline fast in low durations and slow in high durations. The intuition for this is, again, that when meeting rates are high and the worker fails to find a job, the probability to be suitable for a given job drops very fast. This force is needed on top of productivity drop to produce convex-shaped job-finding rates.

The argument analyzed above also suggests which data series informs each parameter of the model. The unobserved heterogeneity parameters are identified by the shape of callback rates coming from Kroft et al. (2013). The evidence on wages will determine the drop of productivity, \( y_\tau \); it will also pin down the vacancy creation cost \( \kappa \) through the Free Entry condition. Finally, the search cost parameters will be chosen so that participation in the model matches the weekly evidence on participation coming from the survey data of Krueger and Mueller (2011).

### 4.3 Calibration

I set a period in the model to be a week. I fix the maximum number of weeks in unemployment at \( T = 50 \). I normalize the productivity of newly unemployed workers to \( y_1 = 1 \) and a linear drop of worker productivity equal to \( d \% \) per month, motivated by the findings of Schmieder et al. (2016) and Ortego-Marti (2016, 2017). I impose a standard Cobb-Douglas matching function, as in equations (1)-(3).

Several parameters are set outside the model. The discount factor \( \beta \) is set to 0.999, consistent with a 5% annual interest rate. I target an average 40-year career for workers, implying \( \nu = 5 \times 10^{-4} \). The separation rate \( \delta \) is set to 0.009 to match the monthly separation rate of 3.4% from Shimer (2012). I set the value of leisure to \( b = 0.69 \), as in Fernández-Blanco and Preugschat (2016), which lies in the middle of the range of estimates provided by Chodorow-Reich and Karabarbounis (2015). Finally, I follow Shimer (2005b) and set the elasticity of the matching function with respect to unemployment to 0.72, which lies toward the upper end of the range of estimates reported in Petrongolo and Pissarides (2001).

There are seven parameters that are calibrated so the model matches the data reported in Section 3: \( \pi, a^H, a^L, d, \kappa, \eta \) and \( \phi \). Following the identification strategy outlined in the previous section, I choose the unobserved heterogeneity parameters \( (\pi, a^H \text{ and } a^L) \) to mimic the evidence in callback rates; \( d \) and \( \kappa \) to capture the empirical decline in wages; and the search cost parameters \( (\eta \text{ and } \phi) \) to replicate data on weekly participation. More specifically, following Fernández-Blanco and Preugschat (2016), I target the average, standard deviation and skewness of the callback rates from Kroft et al. (2013) to pin down \( \pi \),
### Table 1: Exogenously Set Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Discount Factor</td>
<td>0.999</td>
<td>Annual Interest Rate of 5 %</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Death Probability</td>
<td>$5 \times 10^{-4}$</td>
<td>40 year working life</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Separation Probability</td>
<td>0.009</td>
<td>Shimer (2012)</td>
</tr>
<tr>
<td>$b$</td>
<td>Value of Leisure</td>
<td>0.69</td>
<td>Fernández-Blanco and Preugschat (2016)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Matching Function Elasticity</td>
<td>0.72</td>
<td>Shimer (2005b)</td>
</tr>
</tbody>
</table>

$a^H$ and $a^L$. The targets for $d$ and $\kappa$ are the slope of reemployment wages and the average reemployment wage, based on Schmieder et al. (2017) and Ortego-Marti (2016, 2017). Finally, $\eta$ and $\phi$ are pinned down by the average and standard deviation of the participation profile over the unemployment spell from the Krueger and Mueller (2011) survey.

### Table 2: Jointly Calibrated Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>Share of high-ability workers</td>
<td>0.16</td>
<td>Average Callback Rate</td>
</tr>
<tr>
<td>$a^H$</td>
<td>Jobs suitable for high-ability workers</td>
<td>0.49</td>
<td>St. dev. of Callback Rates</td>
</tr>
<tr>
<td>$a^L$</td>
<td>Jobs suitable for low-ability workers</td>
<td>0.12</td>
<td>Skeweness of Callback Rates</td>
</tr>
<tr>
<td>$d$</td>
<td>Step of Productivity Decline</td>
<td>-0.7%</td>
<td>Slope of Reemployment Wages</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Vacancy Cost</td>
<td>2.49</td>
<td>Average Reemployment Wage</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Search Cost Elasticity</td>
<td>2.86</td>
<td>St. dev. of Participation Profile</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Disutility of Search</td>
<td>0.44</td>
<td>Average Participation Probability</td>
</tr>
</tbody>
</table>

It is important to notice that the evidence on job-finding rates over the unemployment spell is not included in the calibration targets. That is, none of the parameters is chosen such that the model replicates the job-finding data of Figure 1. On the contrary, the ability of the model to produce a duration dependence profile close to the observed one will be used as the main evaluation test for its validity.

As can be seen in Figure 3, the model does a good job matching the targeted features of the data. More specifically, it matches the participation profile very accurately and mildly underestimates the wage losses for long unemployment spells. It does a good job matching the average callback rate from Kroft et al. (2013) but it fails to match the skewness of the profile. As a result, it overestimates the drop in callbacks for low durations and underestimates it for high durations. The value of vacancy cost is at the upper end of estimates reported in the literature but, reassuringly, is roughly equal to 2.5 months production in the average match. This estimate is close to the values reported in other papers using directed search models, like Menzio and Shi (2011) or Flemming (2016). Finally, the model predicts a relatively low but realistic fraction of long-term unemployed equal to 12 %, though this was not included in calibration targets.
Table 3: Targeted Moments

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target Data Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi )</td>
<td>0.16</td>
<td>Average Callback Rate</td>
</tr>
<tr>
<td>( a^H )</td>
<td>0.49</td>
<td>St. dev. of Callback Rates</td>
</tr>
<tr>
<td>( a^L )</td>
<td>0.12</td>
<td>Skeweness of Callback Rates</td>
</tr>
<tr>
<td>( d )</td>
<td>-0.7%</td>
<td>Slope of Reemployment Wages</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>2.49</td>
<td>Average Reemployment Wage</td>
</tr>
<tr>
<td>( \eta )</td>
<td>2.86</td>
<td>St. dev. of Participation Profile</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0.44</td>
<td>Average Participation Probability</td>
</tr>
</tbody>
</table>

The unobservable heterogeneity parameters are broadly in line with the values reported in Fernández-Blanco and Preugschat (2016). The step of human capital decline is also close to the empirical estimates in the literature, including Ortego-Marti (2016, 2017), Schmieder et al. (2016) and Autor et al. (2015). Finally, there are very few estimates for the search cost parameters available in the literature. The most common approach is to normalize \( \phi = 1 \) and set \( \eta = 2 \). My estimates imply a higher elasticity, reflecting the large drop of participation in the KM survey. As mentioned earlier, though, this is likely an overestimate of the response of participation to the returns to job-search. This observation implies that the calibrated values used here are likely an upper bound for the true values of \( \phi \) and \( \eta \).
5 Results

Duration Dependence and Decomposition. As shown in the previous section, the parameter values in the model were chosen such that the model matches the available data on the channels creating duration dependence in unemployment. An important evaluation test for the model is whether it is able to predict a realistic duration dependence profile; that is, is the model-implied job-finding rate close to
As can be seen in Figure 4, the model predicts a duration dependence profile very close to the one observed in CPS, even though job-finding rates were not included in the calibration targets. More specifically, the model provides an excellent match for unemployment spells greater than four months and mildly overpredicts the initial steep decline in job-finding rates for shorter spells. This feature is implied by the failure of the model to match the skewness of the callback profiles accurately for short unemployment spells. It is reassuring that even though parameter values were chosen to make the model consistent with micro data on the sources of duration dependence, the model matches unemployment exit probabilities accurately. This fact demonstrates that the model is an appropriate framework to be used for evaluating the quantitative significance of the three mechanisms contributing to duration dependence in unemployment.

Let the forces behind duration dependence considered in this paper form the set:

$$I = \{\text{Unobserved Heterogeneity, Skill Depreciation, Search Effort}\}$$

To accurately evaluate the contribution of each factor $i \in I$ to duration dependence one should be able to answer the counterfactual question: “How large of a decline in job-finding probability would we observe had factor $i$ been absent?”. Moreover, the complete answer to this question should take into account the absence of the effect of factor $i$ on the other factors $j \in I$. In other words, for the appropriate decomposition exercise, one should be able to strip out not only the direct effect of $i$ on job-finding rates but also the interactions of $i$ with the rest of the factors contributing to duration dependence. The
model built in this paper will be used to perform this counterfactual exercise.

To be more specific, let the parameters capturing unobserved heterogeneity among workers be summarized by a vector $\xi = [\pi \ a^H \ a^L]$. The full model predicts the following equilibrium job-finding rate for each duration $\tau$:

$$ f_\tau = s_\tau(\mu_\tau, y_\tau) \times \mu_\tau(\xi, x_\tau) \times x_\tau(\mu_\tau, y_\tau) \quad (35) $$

where $y_\tau$ is the productivity level of workers with duration $\tau$. To evaluate the effects of skill loss, a version of the model without it will be used to compute the alternative job-finding profile $f^1$:

$$ f^1_\tau = s_\tau(\mu_\tau) \times \mu_\tau(\xi, x_\tau) \times x_\tau(\mu_\tau) \quad (36) $$

Finally, to evaluate the effects of search effort, a version of the model without it will be used to compute the following job-finding profile:

$$ f^2_\tau = \mu_\tau(\xi, x_\tau) \times x_\tau(\mu_\tau) \quad (37) $$

Notice that the profile $f^2$ includes only dynamic selection as a source of duration dependence. Hence, it captures the model’s prediction regarding the magnitude of spurious duration dependence.\(^{14}\)

The results of this exercise can be seen in Figure 5. Over all, and consistent with the findings of Alvarez et al. (2016) and Ahn and Hamilton (2016), unobserved worker heterogeneity accounts for the largest part of total duration dependence in unemployment. The role of true duration dependence, though, is quantitatively significant, especially for longer durations. The effect of skill loss plays a major role for spells greater than six months, while declining search effort affects the job-finding rate of all unemployed of duration greater than two months in the same way.

The model-implied significance of the forces causing true duration dependence can rationalize the findings of Abraham et al. (2016) and Bentolila et al. (2017). These authors find strong true duration dependence in the data. Importantly, my model can also shed light to the quantitative effect of each channel causing true duration dependence. The decline of search effort is more important for spells from two to six months, while skill depreciation becomes quantitatively dominant for spells longer than six months. Finally, the initial steep decline in job-finding probabilities can be attributed exclusively to dynamic selection, that is, to the fact that “good” workers find job faster than “bad” workers.

Another way to express the differential effects of various forces on the time-structure of duration dependence is to perform the following ratio analysis. First, to evaluate the effects of search effort, one could use the following counterfactual job-finding profile, which does not include search effort:\(^{15}\)

$$ f^3_\tau = \mu_\tau(\xi, x_\tau) \times x_\tau(\mu_\tau, y_\tau) \quad (38) $$

\(^{14}\)Actually, given the fact that the model cannot distinguish dynamic selection from unemployment stigma, this estimate should be interpreted as an upper bound for the extent of spurious duration dependence through the lens of the model. See, also, Section 6.

\(^{15}\)For robustness I perform the decomposition exercise outlined above by using $f^3$ instead of $f^1$. The magnitude of the effect attributed to skill depreciation and declining search effort changes slightly, due to the non-linearity of the model, but the message is roughly the same as that of Figure 5. This is also evident from Figure 6.
Next, define the following ratios:

\[ R_\tau = \frac{|f_\tau - f_\tau^2|}{f_\tau} \]  \hspace{1cm} (39)

\[ R_{HC}^\tau = \frac{|f_\tau^2 - f_\tau^3|}{f_\tau^2} \]  \hspace{1cm} (40)

\[ R_{SE}^\tau = \frac{|f_\tau^2 - f_\tau^1|}{f_\tau^2} \]  \hspace{1cm} (41)

The first ratio, \( R_\tau \), measures the contribution of true channels on the total amount of duration dependence predicted by the model, for different stages of unemployment duration. The next two ratios perform a similar measurement but focus on the overall true duration dependence the model predicts. \( R_{HC}^\tau \) captures the contribution of human capital depreciation to true duration dependence, while \( R_{SE}^\tau \) measures the impact of declining search effort on the model-implied true duration dependence. Notice that all ratio measures are indexed by the length of the unemployment spell, \( \tau \). This is done to highlight the fact that at different lengths of an unemployment spell, the quantitative importance of each mechanism for the observed duration dependence will be different.
Figure 5: Decomposition of Normalized Job-finding Probabilities by Unemployment Duration

Plotting the ratio measures in Figure 6 is illuminating. First, consider the classic question, “How much of the observed duration dependence is true?”. The model-implied response is, “It depends on the length of the unemployment spells of the workers at hand”. Over all, when comparing unemployed of most durations, dynamic selection accounts for the majority of the observed differences in job-finding. For the really long-term unemployed workers, though, the accumulated effect of true channels dominates and accounts for the biggest part of the observed differences in job-finding between workers.

To make a better sense of this finding, consider comparing the prospects of a newly unemployed with a worker looking for jobs for three months. In the data, the former has 30% greater probability to find
a job. According to my model, almost all of this difference can be attributed to unobserved differences between the average newly unemployed and the average worker who is unemployed for three months. On the other hand, when comparing a worker unemployed for eight months with a worker unemployed for a year or more, the former has a 12% greater chance of finding a job. The model attributes most of that difference to the combination of skill decay and lower search effort exceeded by the workers who are unemployed for a year or more.

Digging deeper into the structure of true duration dependence is, also, revealing. As expected from Figure 5, the effect of declining search effort rises fast in the second and third months of the unemployment spell and stays constant afterwards. On the contrary, the effect of skill loss needs time to accumulate and become significant for the true part of duration dependence. For spells up to nine months the importance of search effort is greater but for spells greater than nine months skill loss becomes the dominant cause of true duration dependence.

The quantitative results of the model highlight the importance of policies tight to the length of spell of different unemployed workers to improve their job-finding prospects. According to the model, there is little space for policymakers to improve the job-finding rates of the very short-term unemployed. The prospects of these workers decline very fast due to the declining unobserved quality of the unemployment pool. On the other hand, the model points to the appropriate short-term policy responses for improving the job-finding prospects of medium- and long-run unemployed workers. Specifically, the model implies that policymakers should invest in job-search assistance programs to help the medium-term unemployed and training programs to fight long-term unemployment.
Figure 6: Contribution of Various Channels to Duration Dependence by Unemployment Duration

It is worth emphasizing that the model-implied policy mix is fully in-line with the findings of Card et al. (2016) regarding the impact of active labor market policies on fighting long-term unemployment. In a meta-study of various active labor market programs around the world, Card et al. (2016) find that more than 70% of the training and job-search assistance programs in their sample had a significant positive
impact for the long-term unemployed. Importantly, though, they also find that “...job-search assistance programmes are not more effective for the long-term unemployed than for short-term UI recipients. At the same time, human capital intensive programmes – in particular training interventions, but also private sector employment programmes – are significantly more likely to bring about positive impacts for the long-term unemployed than for other participant groups.” (p. 19). The model developed in this paper illuminates how the interaction of dynamic selection with skill loss and declining search effort can generate these policy conclusions.

Finally, showing the predicted paths for reemployment wages is useful to understand the mechanics of the model. As mentioned earlier, in versions of the model without skill loss, reemployment wages are increasing for the first few months in unemployment, a counterfactual prediction. This highlights the importance of skill depreciation in order the model to match the data, as well as the fact that wages in the model are informed by the speed of skill depreciation. The fact that wages are higher in the version of the model with endogenous search effort is a result of directed search. Wages price waiting times in competitive search; waiting times (job-finding rates) are higher (lower) in the model with search effort, hence wages needs to be higher to compensate workers who do not differ in productivity over their unemployment spell.

**Intuition: How the Model works.** At this point it may be instructive to explain how the different channels of the model interact to make its predictions consistent with the data. First, consider the submarkets populated by job-seekers with short unemployment spells. Workers who are found suitable
in these submarkets have relatively high productivity, leading to high match surplus. As a result, firms post a lot of vacancies directed to newly unemployed workers, since they are very productive and have good chances of being suitable for a job’s tasks. Given the high probability of job applications be reviewed by firms, if a worker fails to find a job early on her spell, this says a lot about her quality: unemployment duration is a very informative signal for short durations, because of the large number of worker-firm meetings. Thus, \( \mu_\tau \) drops very fast in the first few periods of unemployment. Moreover, since the returns to job-search are high for newly unemployed, most workers engage in job search in the beginning of their unemployment spell.

As their unemployment spell evolves, workers become less productive, due to human capital depreciation. Hence, the match surplus declines and firms offer less job opportunities to workers with high unemployment durations: \( x_\tau \) declines, following the path of skill depreciation. However, because worker-firm meetings are scarce, the probability the CV of a worker be reviewed is low. Thus, failing to find a job is not very informative about workers’ quality: \( \mu_\tau \) drops slowly for high durations of unemployment. Finally, the returns to job search decrease, hence workers of high \( \tau \) exceed lower effort to find jobs: \( s_\tau \) drops due to discouragement.

**Expansions vs Contractions.** As can be seen in part (a) of Figure 7, the profile of duration dependence exhibits a small downward parallel shift in recessions.\(^{16}\) Following the exercise of Fernández-Blanco and Preugschat (2016), I compare the duration dependence profile of the calibrated full model with a version which features a 3% decrease in productivity for all unemployment spells. This 3% drop in the productivity of all workers intends to capture the effects of an aggregate productivity drop, mimicking a contraction in the model.

Notice that the model prediction for job-finding rates in a steady state with lower aggregate productivity is theoretically ambiguous. On the one hand, when firms post relatively less vacancies due to lower aggregate productivity, duration of unemployment is a weaker signal of workers’ quality. Hence, workers with long unemployment spells are more likely to be suitable for a job and, hence, \( \mu_\tau \) falls less steeply over duration. On the other hand, the returns to vacancy posting and search effort are lower when aggregate productivity is lower, making \( x_\tau \) and \( s_\tau \) decline faster than in a steady state with higher aggregate productivity.

Interestingly, in the calibrated model the latter effects dominate. As can be seen in part (b) of Figure 7, the model successfully predicts that the job-finding profile in a steady state with 3% lower aggregate productivity will be steeper than the profile in the standard calibration. Hence, the model has the potential to replicate the basic downward shift of the job-finding profile that takes place in recessions. A serious quantitative study of the effects of plausibly calibrated shocks in aggregate productivity on

---

\(^{16}\)Actually, one should account for the fact that the composition of the unemployment pool also changes in recessions, shifting to workers more attached to the labor force. I have done that, following the standard approach found in Elsby and Hobijn (2010), among others, and the results barely change. Accounting for the composition of the unemployment pool results in a slightly greater parallel drop of the duration profile but the difference is very small, still consistent with the model-implied drop.
duration dependence and the mechanisms underlying it is left for future research.

Figure 8: Normalized Job-Finding Probabilities in Expansions vs Contractions: Figure (a): CPS Data. Figure (b): Model Prediction
6 Discussion and Related Literature

This paper is related to several strands of literature in Macroeconomics and Labor Economics. This section describes how the paper fits into these strands and how its contributions advance the relevant lines of work.

Directed search. The model in this paper uses the machinery of directed search, developed by Moen (1997), Acemoglu and Shimer (1999a,b), Burdett et al. (2001), Mortensen and Wright (2002), Shi (2002); Shi et al. (2006), Shimer (2005a), and Inderst (2005). It generalizes the directed search framework to an environment in which interacting non-stationary forces cause duration dependence in unemployment. It establishes the equivalence between competitive search equilibrium and the solution of an auxiliary optimization problem, in the tradition of Acemoglu and Shimer (1999a,b), and characterizes the equilibrium analytically. It develops an algorithm to compute the equilibrium of directed search models that fully exploits the fixed-point structure of the auxiliary optimization problem. However, this paper remains silent regarding the efficiency properties of equilibrium, a theme analyzed very often in the directed search literature.

Models of Duration Dependence. This is the strand of the literature this paper is most related to. Early contributions include the random search models of Lockwood (1991), Pissarides (1992), Blanchard and Diamond (1994) and Acemoglu (1995). Gonzalez and Shi (2010) provide the first directed search framework that could speak to the question of duration dependence. They construct a model in which workers learn about their ability while searching for a job to explain the stylized fact that reemployment wages are decreasing in unemployment duration. This paper uses a variation of the matching process developed in Gonzalez and Shi (2010). Their paper is an exclusively theoretical contribution (provides no quantitative results) and, most importantly, has an important counterfactual implication: duration dependence in unemployment is positive. The hazard rates of individual workers out of unemployment are increasing over the unemployment spell. This paper fixes that problem by introducing human capital depreciation in the model. As I show above, my model is capable of successfully rationalizing both job-finding rates and reemployment wages data, as well as provide answers to relevant quantitative questions.

The most recent and closely related papers are Fernández-Blanco and Preugschat (2016), Jarosch and Pilossof (2015) and Doppelt (2014). Fernández-Blanco and Preugschat (2016) build a directed search model to rationalize the evidence presented in the audit study of Kroft et al. (2013). Jarosch and Pilossof (2015) evaluate the quantitative significance of firm discrimination against long-term unemployed on job-finding rates. Their results imply that the contribution of stigma to job-finding rates is weak, a finding this paper takes seriously and builds upon. Doppelt (2014) is a skillful quantitative extension of Gonzalez and Shi (2010) that features learning about a worker’s quality over her whole career.
A shared limitation of these studies is that the only source of true duration dependence is unemployment stigma—meaning, employer discrimination against long-term unemployed in hiring. As a result, they remain silent regarding the importance of skill depreciation and search effort decline for observed duration dependence.\textsuperscript{17} Moreover, their elegance and significance notwithstanding, these studies have some troubling implications. Reemployment wages in Fernández-Blanco and Preugschat (2016) may increase with unemployment duration; the firm-worker meeting rates in Jarosch and Pilossof (2015) are exogenous, so they cannot perform counterfactuals useful for policy analysis; finally, Doppelt (2014) shares the counterfactual result of Gonzalez and Shi (2010), with a minority of workers facing increasing job-finding rates over the unemployment spell. This paper predicts unambiguously decreasing job-finding rates and reemployment wages for all workers in the labor market, while the meeting rates are endogenous objects, determined in equilibrium.

It should be mentioned, however, that through the lens of the model analyzed here, dynamic sorting and employer stigma cannot be distinguished. The model does not incorporate a separate interview stage in the hiring process; hence, it is not capable of providing an estimate of the effect of employer discrimination on duration dependence. Since dynamic sorting is conflated with unemployment stigma, the results of my model should be interpreted as an upper bound for the magnitude of spurious and a lower bound for the magnitude of true duration dependence.

Other relevant contributions include Flemming (2016) and Potter (2017). Flemming (2016) rationalizes duration dependence in unemployment with learning-by-doing and home production. Her model predicts unambiguously decreasing job-finding rates and reemployment wages but the decline of the model-implied job-finding rates is linear. This result highlights the importance of a composition/learning mechanism to account for the convex shape of job-finding rates. Potter (2017) builds a partial equilibrium model to emphasize the effect of learning on workers’ search intensity, one of the mechanisms at work in this paper too. He uses the data from Krueger and Mueller (2011) survey but he works with the intensive margin of search effort, which is shown not to have a significant effect on job-finding probability. Finally, the consequences of human capital depreciation in random search models is studied by Ortego-Marti (2016, 2017) and Laureys (2014), in a line of work initiated by Pissarides (1992) and Ljungqvist and Sargent (1998).

**Empirical Work.** Turning to the empirical front, there is a large econometric literature trying to measure duration dependence by estimating duration models with observational data. This literature makes progress by imposing strong parametric identifying assumptions to identify true duration from dynamic selection. It is nicely summarized by Van den Berg (2001). More related to this paper is a series of recent contributions that estimate true and spurious duration either by using sophisticated econometric techniques or more reduced form methods. The former include Alvarez et al. (2016), Ahn and

\textsuperscript{17}To be fair, Doppelt (2014) provides a version of his model with skill depreciation. However, skill decay in Doppelt’s model attenuates the drop in job-finding rates! In other words, skill depreciation improves the prospects of workers in Doppelt’s model, which is at odds with the empirical findings of Card et al. (2016).
Hamilton (2016) and Bentolila et al. (2017), while the main paper in the latter is Abraham et al. (2016). Abraham et al. (2016) and Bentolila et al. (2017) find a strong role for true duration dependence. The results in Ahn and Hamilton (2016) and Alvarez et al. (2016) emphasize the importance of unobserved heterogeneity but they still find a small positive role for true duration dependence.

This paper analyzes duration dependence through the lens of an equilibrium search model. Hence, it can measure the magnitude of the effects of specific behavioral channels on observed duration dependence. The empirical approaches are primarily concerned with separating true from spurious dependence, thus they are not equipped to measure the magnitude different channels of true duration dependence. Other relevant empirical contributions include the papers measuring the effect of unemployment duration on reemployment wages (Schmieder et al. (2016), Autor et al. (2015), Nekoei and Weber (2017)), on search effort (Krueger and Mueller (2011), Faberman and Kudlyak (2014)), as well as a series of influential audit studies (Kroft et al. (2013), Eriksson and Rooth (2014), Ghayad (2013), Oberholzer-Gee (2008), Farber et al. (2017)). Finally, Card et al. (2016) is a meta-study of active labor market policies, the results of which are fully consistent with the results of this paper.

Methodology. This paper uses an equilibrium search model to assign magnitudes to forces causing spurious and true duration dependence. To measure the effect of each channel, it computes counterfactual job-finding profiles over the unemployment spell. For each counterfactual, a specific mechanism of duration dependence is shut down and the difference of the predicted job-finding profile with the profile of the full model is attributed to the missing channel. This methodology is employed by many recent studies in Macroeconomics: Burdett et al. (2016), Jarosch (2014), and Wolcott (2017) use rich search models to perform similar quantitative exercises. Moreover, Fernández-Blanco and Preugschat (2016), Jarosch and Pilossoph (2015) and Doppelt (2014) employ this methodology to evaluate the effects of employer discrimination against long-term unemployed in hiring. I am not aware of any paper, though, that uses a directed search model to evaluate the effects of dynamic selection, skill loss and search effort decline.

7 Conclusion

In short, this paper makes two contributions: (i) it introduces a directed search model of the labor market, featuring unobserved worker differences, skill loss in unemployment, and endogenous job-search effort; (ii) it uses the model to evaluate the significance of each factor for the observed duration dependence in unemployment. The results of interest include: (i) the model successfully replicates the job-finding profile in US data, even though the latter was not used to pin down any model parameters; (ii) in agreement with recent empirical literature, over all, the most important factor behind the total observed duration dependence is unobserved worker differences; (iii) the bulk of the effect of unobserved worker heterogeneity is concentrated in the first two months of the unemployment spell; the differences
among workers of longer spells should be attributed to skill loss and declining search effort; (iv) job-search effort is the dominant force behind drops in job-finding for spells between two and six months, while skill loss is the dominant force behind drops in job-finding for spells greater than six months. These results have sharp implications about how active labor market programs should be tailored to help short- and long-term unemployed workers find jobs.

To conclude, let me summarize some future research directions, based on this paper. First, an interesting direction would be to make the model stochastic to incorporate aggregate shocks. This would be a useful framework to study the effects of extensive unemployment benefits on duration dependence in recessions. Second, one could study the efficiency properties of the framework developed here and compare the results with Fernández-Blanco and Preugschat (2016). Third, the modeling of human capital could be extended to reflect skills as measured directly in the data, following Macaluso (2017). Finally, and more broadly, it would be of great interest the forces evaluated in this paper to be studied in a model of stock-flow matching (Ebrahimy and Shimer (2010)).
References


Flemming, J. (2016). Skill accumulation in the market and at home.


Wolcott, E. L. (2017). Employment inequality: Why do the low-skilled work less now?
## Appendix I: Proofs

**Lemma A.1 (Equilibrium $\mapsto$ Auxiliary Problem).** Let $w^*_\tau \in \mathbb{W}^*$ and $q^*_\tau = Q^*_\tau(w^*_\tau)$, where $\{\mathbb{W}^*, \{Q^*_\tau\}_{\tau \leq T}, U^*\}$ is an equilibrium allocation; then $\{w^*_\tau, q^*_\tau\}_{\tau \leq T}$ solve problem (12) under constraints (13), (14) and (15), with $U_\tau(w^*_\tau, q^*_\tau) = U^*_\tau$ if $q^*_\tau > 0$.

*Proof.* First, notice that the Beliefs Updating condition ensures that the constraint (15) is satisfied. Also, note that Optimal Application ensures that constraint (13) is satisfied.

Now, suppose that some $w^*_\tau$ and $q^*_\tau$ do not maximize (12). That is, there are $q'_\tau > 0$ and a $w'_\tau$ that achieve a strictly positive value for the firm, while satisfying constraints (13), (14) and (15). Formally:

$$-\kappa + \lambda(q'_\tau) \frac{y_\tau - w'_\tau}{1 - \beta(1 - \nu)} > 0$$

while $U_\tau(w'_\tau, q'_\tau) = U^*_\tau$. By the definition of competitive search equilibrium, it has to be the case that $U_\tau(w'_\tau, Q^*_\tau(w'_\tau)) \leq U^*_\tau$, due to Rational Expectations. Hence, $U_\tau(w'_\tau, q'_\tau) \geq U_\tau(w'_\tau, Q^*_\tau(w'_\tau))$. By definition:

$$U_\tau(w'_\tau, q'_\tau) = b + \beta(1 - \nu) \left[ \mu_\tau x(q'_\tau) \left( \frac{w'_\tau}{1 - \beta(1 - \nu)} - U^*_{\tau+1} \right) + U^*_{\tau+1} \right]$$

and

$$U_\tau(w'_\tau, Q^*_\tau(w'_\tau)) = b + \beta(1 - \nu) \left[ \mu_\tau x(Q^*_\tau(w'_\tau)) \left( \frac{w'_\tau}{1 - \beta(1 - \nu)} - U^*_{\tau+1} \right) + U^*_{\tau+1} \right]$$

thus, $x(Q^*_\tau(w'_\tau)) \leq x(q'_\tau) \iff Q^*_\tau(w'_\tau) \geq q'_\tau \iff \lambda(Q^*_\tau(w'_\tau)) \geq \lambda(q'_\tau)$. As a result:

$$-\kappa + \lambda(Q^*_\tau(w'_\tau)) \frac{y_\tau - w'_\tau}{1 - \beta(1 - \nu)} > 0$$

which contradicts the Profit Maximization and Free Entry conditions of equilibrium. \[\square\]

**Lemma A.2 (Auxiliary Problem $\mapsto$ Equilibrium).** If some $\{w^*_\tau, q^*_\tau\}_{\tau \leq T}$ solve problem (12) under constraints (13), (14) and (15) then there exists an equilibrium $\{\mathbb{W}^*, \{Q^*_\tau\}_{\tau \leq T}, U^*\}$ such that $w^*_\tau \in \mathbb{W}^*$ and $q^*_\tau = Q^*_\tau(w^*_\tau)$, $\forall \tau \leq T$.

*Proof.* Let me start with the constructive part of the claim. Define $\mathbb{W}^* = \{w^*_\tau\}_{\tau \leq T}$ and $Q^*_\tau(w^*_\tau) = q^*_\tau$, $\forall \tau \leq T$. Set the following recursively:

$$U^*_\tau = b + \beta(1 - \nu) \left[ \mu_\tau x(q^*_\tau) \left( \frac{w^*_\tau}{1 - \beta(1 - \nu)} - U(w^*_{\tau+1}, q^*_{\tau+1}) \right) + U(w^*_{\tau+1}, q^*_{\tau+1}) \right]$$

and

$$U^*_T = b + \beta(1 - \nu) \left[ \mu_T x(q^*_T) \left( \frac{w^*_T}{1 - \beta(1 - \nu)} - U(w^*_T, q^*_T) \right) + U(w^*_T, q^*_T) \right]$$

46
Now, define \( Q^*_\tau(w) \) to satisfy:

\[
U^*_\tau = b + \beta(1 - \nu) \left[ \mu_x(Q^*_\tau(w) \left( \frac{w_\tau}{1 - \beta(1 - \nu)} - U^*_\tau+1 \right) + U^*_\tau+1 \right]
\]

and

\[
U^*_T = b + \beta(1 - \nu) \left[ \mu_T(Q^*_\tau(w) \left( \frac{w_\tau}{1 - \beta(1 - \nu)} - U^*_T \right) + U^*_T \right]
\]

or \( Q^*_\tau(w) = 0 \) if there is no solution to any of these equations.

By construction, \( \{W^*, \{Q^*_\tau\}_{\tau \leq T}, U^*\} \) satisfy the Profit Maximization, Free Entry and Beliefs Updating conditions. It remains to be shown that it satisfies Optimal Application.\(^{18}\) Suppose to the contrary that there are equilibrium \( w'_\tau \) and \( Q^*_\tau(w'_\tau) > 0 \) that yield greater utility to the worker than \( U^*_\tau \):

\[
U^*_\tau < b + \beta(1 - \nu) \left[ \mu_x(Q^*_\tau(w'_\tau) \left( \frac{w'_\tau}{1 - \beta(1 - \nu)} - U^*_\tau+1 \right) + U^*_\tau+1 \right]
\]

But then there is a \( q'_\tau > Q^*_\tau(w'_\tau) > 0 \) such that:

\[
U^*_\tau = b + \beta(1 - \nu) \left[ \mu_x(q'_\tau) \left( \frac{w'_\tau}{1 - \beta(1 - \nu)} - U^*_\tau+1 \right) + U^*_\tau+1 \right]
\]

Then it is true that \( \lambda(q'_\tau) > \lambda(Q^*_\tau(w'_\tau)) \); that is, \( (w'_\tau, q'_\tau) \) yield strictly greater profit to the firm. That is, I have shown that \( (w'_\tau, q'_\tau) \) yield strictly greater profit than \( (w^*_\tau, q^*_\tau) \) while satisfying constraints (13), (14) and (15), a contradiction.

\[\blacksquare\]

**Proposition A.1.** There exists an equilibrium in which the labor market is segmented by unemployment duration.

**Proof.** First, consider the simple firms’ maximization problem (12) under constraint (13) only. The objective function is a continuous function. Also, every \( q_\tau \) is bounded below by zero and constraint (13) puts an upper bound on it for every duration \( \tau \). Therefore, Weierstrass Theorem ensures the existence of a solution to this simple maximization problem.

To proceed, let \( f : K \to K \), where \( K \equiv [a^L, a^H]^T \times [\frac{b}{1 - \beta(1 - \nu)}, \frac{y}{1 - \beta(1 - \nu)}]^T \) is a compact set. I define \( f \) to be the composite correspondence \( f \equiv \psi \circ g \), where \( \psi \) and \( g \) are defined as follows. First, let \( z \equiv (\{\mu_\tau\}_{\tau \leq T}, \{U_\tau\}_{\tau \leq T}) \) and \( g(z) \) be defined as the set of elements \( \{q_\tau, w_\tau, U^*_\tau\}_{\tau \leq T} \) that satisfy the zero-profit condition (14) and solve the firms’ profit maximization problem (12) under constraint (13). \( U^*_\tau \) is obtained by using the complementary slackness condition (13). Second, let \( \psi \) be defined as \( \psi(\{w_\tau\}_{\tau \leq T}, \{q_\tau\}_{\tau \leq T}, \{U^*_\tau\}_{\tau \leq T}) \equiv (\{\mu'_\tau\}_{\tau \leq T}, \{U'_\tau\}_{\tau \leq T}) \), where \( \{\mu'_\tau\}_{\tau \leq T} \) is uniquely determined by the

\(^{18}\) If Optimal Application is satisfied, then the Rational Expectations condition holds by the construction of \( Q^*_\tau(\cdot) \) and \( \{U^*_\tau\}_{\tau \leq T} \).
Bayesian updating equation (15) with $\mu_1 = \pi a^H + (1 - \pi)a^L$ and $U'_\tau = U^*_\tau$ for all $\tau$. Notice that the equilibrium can be identified as a fixed point of $f$.

I need to show that $f$ is a continuous function. First, $\psi$ is obviously continuous. It remains to be shown that $g(z)$ is singleton and continuous for every $z \in K$. After substituting constraint (13) in (12) firms’ problem becomes:

$$V^*_\tau = \max_{q_\tau} -\kappa + \lambda(q_\tau) \left( \frac{y_\tau}{1 - \beta(1 - \nu)} - U^*_{\tau+1} \right) + q_\tau \left( \frac{U^*_{\tau+1} - U^*_\tau - b}{\beta(1 - \nu)\mu_\tau} \right), \forall \tau \leq T$$

Having assumed that $\lambda(\cdot)$ is strictly concave ensures that this function is strictly concave in $q_\tau$, thus there is a unique optimum. Hence, $g$ is a function. Finally, the Maximum Theorem guarantees that $g$ is continuous at $z \in K$. Therefore, the composite function $f$ is also continuous. Hence, Brouwer’s Fixed Point Theorem ensures that $f$ has a fixed point in $K$, so there is an equilibrium with segmented labor markets.

**Proposition A.2.** In any segmented markets equilibrium $q_\tau$ is increasing and $w_\tau$ is decreasing in $\tau$; also, the difference $y_\tau - w_\tau$ is decreasing in $\tau$. Hence, the value of a filled vacancy, $J(w_\tau)$, is decreasing in $\tau$.

**Proof.** First, notice that by constraint (14) (Free Entry) the sequences $y_\tau - w_\tau$ and $q_\tau$ must move in opposite directions. That is, since $\lambda(q_\tau) \frac{y_\tau - w_\tau}{1 - \beta(1 - \nu)} = \kappa$, for all $\tau$, and $\lambda(\cdot)$ is strictly increasing, in order the Free Entry condition to hold, $y_\tau - w_\tau$ and $q_\tau$ have to be moving in opposite directions over different submarkets.

I will prove this statement by induction. To begin with, I will show that it is true for $\tau = T - 1$ and $\tau = T$. Following the algebra developed in section 4.1, one can compute equilibrium wages and the value of a filled job for submarkets $T - 1$ and $T$ as follows:

$$w_{T-1} = \alpha y_{T-1} + (1 - \alpha)(1 - \beta(1 - \nu))U^*_T$$
$$w_T = \alpha y_T + (1 - \alpha)(1 - \beta(1 - \nu))U^*_T$$
$$y_{T-1} - w_{T-1} = (1 - \alpha)(y_{T-1} - (1 - \beta(1 - \nu)U^*_T)$$
$$y_T - w_T = (1 - \alpha)(y_T - (1 - \beta(1 - \nu)U^*_T)$$

Subtracting the last two equalities yields:

$$(y_{T-1} - w_{T-1}) - (y_T - w_T) = (1 - \alpha)(y_{T-1} - y_T)$$

or just

$$w_{T-1} - w_T = \alpha(y_{T-1} - y_T) < y_{T-1} - y_T$$

The difference $w_{T-1} - w_T$ is positive and smaller than $y_{T-1} - y_T$. Also, $(y_{T-1} - w_{T-1}) > (y_T - w_T)$,
hence \( q_T \) has to be greater than \( q_{\tau-1} \).

To proceed with the induction, assume that \( w_\tau > w_{\tau+1}, (y_\tau - w_\tau) > (y_{\tau+1} - w_{\tau+1}) \) and \( q_\tau < q_{\tau+1} \); to complete the proof it needs to be shown that \( w_{\tau-1} > w_\tau, (y_{\tau-1} - w_{\tau-1}) > (y_\tau - w_\tau) \) and \( q_{\tau-1} < q_\tau \).

Subtracting wages yields:

\[
w_{\tau-1} - w_\tau = \alpha(y_{\tau-1} - y_\tau) + (1 - \alpha)(1 - \beta(1 - \nu))\left(U^*_\tau - U^*_\tau+1\right)
\]

The idea here is to use the information on \( w_\tau \) and \( w_{\tau+1} \) to gain information on the difference \( U^*_\tau+1 - U^*_\tau+2 \) which, in turn, will be useful for bounding the difference \( U^*_\tau - U^*_\tau+1 \) and the difference in wages.

Using the standard expression for equilibrium wages yields:

\[
U^*_\tau+1 = \frac{w_\tau - \alpha y_\tau}{(1 - \alpha)(1 - \beta(1 - \nu))}
\]

\[
U^*_\tau+2 = \frac{w_{\tau+1} - \alpha y_{\tau+1}}{(1 - \alpha)(1 - \beta(1 - \nu))}
\]

or just:

\[
U^*_\tau+1 - U^*_\tau+2 = \frac{w_\tau - w_{\tau+1} - \alpha(y_\tau - y_{\tau+1})}{(1 - \alpha)(1 - \beta(1 - \nu))}
\]

Now, consider the difference \( U^*_\tau - U^*_\tau+1 \):

\[
U^*_\tau - U^*_\tau+1 = b + \beta(1 - \nu) \left[ \mu_{\tau+1}(q_\tau) (\frac{w_\tau}{1 - \beta(1 - \nu)} - U^*_\tau+1) + U^*_\tau+1 \right] -
\]

\[
- b - \beta(1 - \nu) \left[ \mu_{\tau+1}(q_\tau) (\frac{w_{\tau+1}}{1 - \beta(1 - \nu)} - U^*_\tau+2) + U^*_\tau+2 \right] \geq
\]

\[
\geq \beta(1 - \nu) \left[ \mu_{\tau+1}(q_\tau) (\frac{w_\tau}{1 - \beta(1 - \nu)} - U^*_\tau+1 - \frac{w_{\tau+1}}{1 - \beta(1 - \nu)} + U^*_\tau+2) + U^*_\tau+1 - U^*_\tau+2 \right] =
\]

\[
= \beta(1 - \nu) \left[ \mu_{\tau+1}(q_\tau) (\frac{w_\tau}{1 - \beta(1 - \nu)} - \frac{w_{\tau+1}}{1 - \beta(1 - \nu)}) + (1 - \mu_{\tau+1}(q_\tau)) \left( U^*_\tau+1 - U^*_\tau+2 \right) \right] =
\]

\[
= \beta(1 - \nu) \left[ \mu_{\tau+1}(q_\tau) (\frac{w_\tau}{1 - \beta(1 - \nu)} - \frac{w_{\tau+1}}{1 - \beta(1 - \nu)}) + (1 - \mu_{\tau+1}(q_\tau)) \frac{w_\tau - w_{\tau+1} - \alpha(y_\tau - y_{\tau+1})}{(1 - \alpha)(1 - \beta(1 - \nu))} \right] =
\]

\[
= \beta(1 - \nu) \frac{(1 - \alpha \mu_{\tau+1}(q_\tau))(w_\tau - w_{\tau+1} - \alpha(y_\tau - y_{\tau+1})) (y_\tau - y_{\tau+1})}{(1 - \alpha)(1 - \beta(1 - \nu))}
\]

Hence, the difference \( w_{\tau-1} - w_\tau \) can be bounded as:

\[
w_{\tau-1} - w_\tau = \alpha(y_{\tau-1} - y_\tau) + (1 - \alpha)(1 - \beta(1 - \nu))\left(U^*_\tau - U^*_\tau+1\right) \geq
\]

\[
\geq \alpha(y_{\tau-1} - y_\tau) + \beta(1 - \nu)(1 - \alpha \mu_{\tau+1}(q_\tau))(w_\tau - w_{\tau+1}) - \beta(1 - \nu)\alpha(1 - \mu_{\tau+1}(q_\tau))(y_\tau - y_{\tau+1})
\]
Assuming a linear drop $D$ in workers’ productivity, as in the quantitative analysis of the paper, yields:

$$w_{\tau-1} - w_{\tau} = \alpha Dy_{\tau-1} - \beta(1 - \nu)x(1 - \mu_{\tau+1}x(q_{\tau+1}))(w_{\tau} - w_{\tau+1}) \geq$$

$$\geq \alpha Dy_{\tau}(1 - \beta(1 - \nu)(1 - \mu_{\tau+1}x(q_{\tau+1}))) + \beta(1 - \nu)(1 - \alpha \mu_{\tau+1}x(q_{\tau+1}))(w_{\tau} - w_{\tau+1}) \geq 0$$

Now, consider the following differences:

$$(y_{\tau-1} - w_{\tau-1}) - (y_{\tau} - w_{\tau}) = (1 - \alpha)(y_{\tau-1} - y_{\tau}) - (1 - \alpha)(1 - \beta(1 - \nu))(U^*_\tau - U^*_{\tau+1}) \geq$$

$$\geq (1 - \alpha)(y_{\tau-1} - y_{\tau}) - \beta(1 - \nu)(1 - \mu_{\tau+1}x(q_{\tau+1}))(w_{\tau} - w_{\tau+1}) + \beta(1 - \nu)(1 - \mu_{\tau+1}x(q_{\tau+1}))(y_{\tau} - y_{\tau+1}) \geq$$

$$\geq (1 - \alpha)(y_{\tau-1} - y_{\tau}) - \beta(1 - \nu)(1 - \mu_{\tau+1}x(q_{\tau+1}))(w_{\tau} - w_{\tau+1}) + \beta(1 - \nu)(1 - \mu_{\tau+1}x(q_{\tau+1}))(y_{\tau} - y_{\tau+1}) =$$

$$= (1 - \alpha)(y_{\tau-1} - y_{\tau}) + \beta(1 - \nu)(1 - \mu_{\tau+1}x(q_{\tau+1}))(y_{\tau} - w_{\tau}) - (y_{\tau+1} - w_{\tau+1}) \geq 0$$

Finally, since $q_{\tau}$ and $y_{\tau} - w_{\tau}$ move in opposite directions in equilibrium, the last step proves that $q_{\tau-1} \leq q_{\tau}$.

**Lemma A.3.** In any segmented markets equilibrium $q_{\tau} > 0$ for all $\tau$. Hence, the complementary slackness condition (12) holds with equality.

**Proof.** Suppose that there exists at least one duration group of workers such that its associated queue is 0. Let us denote by $\tau_0$ the first duration for which the queue length is 0. All queues associated with longer durations must also be 0, since $y_{\tau} < y_{\tau_0}$ for all $\tau > \tau_0$ and Proposition A.2 proved that the value of a filled vacancy is decreasing in $\tau$. Then, the unemployment value of workers with unemployment duration greater than or equal to $\tau_0$ must be $\frac{b}{1 - \beta(1 - \nu)}$, as they will remain unemployed forever.

Let $w_{\tau_0}$ be the profit maximizing wage for workers of duration $\tau_0$. Given that $y_{\tau_0} > b$, there exists an arbitrarily small, but positive $\epsilon$ such that $b + \epsilon < y_{\tau_0}$. Consider now the alternative wage $w'_{\tau_0} = b + \epsilon$. This wage offer will attract a positive queue of workers and delivers strictly higher profits than $w_{\tau_0}$, so $w_{\tau_0}$ and $q_{\tau_0} = 0$ cannot be profit-maximizing. Therefore, $q_{\tau} > 0$ for all $\tau$ in any equilibrium.

**Lemma A.4.** Beliefs about worker suitability for a given job, $\mu_{\tau}$, are decreasing in $\tau$.

**Proof.** By construction $a^L \leq \mu_{\tau} \leq a^H$ for all $\tau$. It is straightforward to notice that:

$$\mu_{\tau+1} \leq \mu_{\tau} \iff a^H = \frac{(a^H - \mu_{\tau})(1 - x_{\tau}a^L)}{1 - x_{\tau}\mu_{\tau}} \leq \mu_{\tau} \iff$$

$$(1 - x_{\tau}\mu_{\tau})a^H - (a^H - \mu_{\tau})(1 - x_{\tau}a^L) \leq (1 - x_{\tau}\mu_{\tau})\mu_{\tau} \iff$$

$$(1 - x_{\tau}\mu_{\tau})(a^H - \mu_{\tau}) \leq (1 - x_{\tau}a^L)(a^H - \mu_{\tau}) \iff a^L \leq \mu_{\tau}$$

which is always true, since in equilibrium $q_{\tau} > 0$.

50
Proposition A.3. In any segmented markets equilibrium the value of unemployment, $U^*_\tau$, is decreasing in $\tau$.

Proof. I prove this by induction. First, it is straightforward to notice that $U^*_{T-1} \geq U^*_T$ since $q_{T-1} < q_T$, $\mu_{T-1} \geq \mu_T$, $w_{T-1} \geq w_T$ and $x(\cdot)$ is strictly decreasing:

$$U^*_{T-1} = b + \beta(1 - \nu) \left[ \mu_{T-1} x(q_{T-1}) \left( \frac{w_{T-1}}{1 - \beta(1 - \nu)} - U^*_T \right) + U^*_T \right]$$

$$U^*_T = b + \beta(1 - \nu) \left[ \mu_T x(q_T) \left( \frac{w_T}{1 - \beta(1 - \nu)} - U^*_T \right) + U^*_T \right]$$

Now assume that $U^*_{\tau+1} \leq U^*_\tau$ to show that $U^*_\tau \leq U^*_\tau-1$:

$$U^*_{\tau-1} - U^*_\tau = b + \beta(1 - \nu) \left[ \mu_{\tau-1} x(q_{\tau-1}) \left( \frac{w_{\tau-1}}{1 - \beta(1 - \nu)} - U^*_\tau \right) + U^*_\tau \right] -
-b - \beta(1 - \nu) \left[ \mu_{\tau} x(q_\tau) \left( \frac{w_\tau}{1 - \beta(1 - \nu)} - U^*_{\tau+1} \right) + U^*_{\tau+1} \right] \\
\geq \beta(1 - \nu) \left[ \mu_{\tau-1} x(q_{\tau-1}) \left( \frac{w_{\tau-1}}{1 - \beta(1 - \nu)} - U^*_\tau - \frac{w_\tau}{1 - \beta(1 - \nu)} + U^*_{\tau+1} \right) + U^*_\tau - U^*_{\tau+1} \right] =
\beta(1 - \nu) \left[ \mu_{\tau-1} x(q_{\tau-1}) \left( \frac{w_{\tau-1}}{1 - \beta(1 - \nu)} - \frac{w_\tau}{1 - \beta(1 - \nu)} \right) + \left( 1 - \mu_{\tau-1} x(q_{\tau-1}) \right) \left( U^*_\tau - U^*_{\tau+1} \right) \right] \geq 0$$