Consumption volatility risk  
and the inversion of the yield curve*  

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Abstract

We propose a consumption-based model that allows for an inverted term structure of real and nominal risk-free rates. In our framework, the agent is subject to time-varying macroeconomic risk, and interest rates at all maturities depend on her risk perception which shape saving propensities over time. In bad times, when risk is perceived to be higher in the short- than the long-term, the agent would prefer to hedge against low realizations of consumption in the near future investing in long-term securities. This entails, in equilibrium, the inversion of the yield curve. Pricing time-varying consumption volatility risk is essential to obtain the inversion of the real curve and allows to price the average level and slope of the nominal one.

JEL classification: G12.

Keywords: real rates; uncertainty; habits; inverted yield curve; volatility risk.

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1. Introduction

The inversion of the term structures of interest rates, which happens when short-term yields are above long-term ones, is an occasional, yet not rare event. Postwar data on US Government bond yields show that while the term spread – i.e., the difference between long- and short-term yields – has been positive on average, several episodes of inversion have also been documented (Figure 1).

Nominal yields reflect a real as well as an inflation expectations component. While the latter is known to have a role in shaping the level and the slope of the yield curve, less evidence is available on the role of the real component. However, stylized facts point to the role of macroeconomic risk in affecting the term structure of interest rates: first, data on US TIPS (i.e., inflation-protected securities) from Gurkaynak et al. (2007, 2010) suggest that real yields fluctuates substantially over time and that the real term structure of interest rates, as well as the nominal one, has inverted during the last 10 years (see Figure 2); second, both the dispersion around survey expectations on consumption growth and option-implied volatilities in the stock market show strong correlations with the slope of the TIPS yield curve, and the TIPS and volatility term structures have experienced synchronized inversions (see Figure 3 and Figure 4).\(^1\)

This paper investigates the role of macroeconomic risk in shaping the term structure of interest rates. For this purpose, we propose a parsimonious consumption-based model of the term structure that allows the inversion of its real component. In our framework, the agent is subject to time-varying macroeconomic risk, and interest rates at all maturities depend on her risk perception which shape saving propensities over time. When risk is perceived to be higher in the long- than the short-run, the agent would be more willing to invest in short than long-term securities: in equilibrium, the yield curve is upward sloping. However, when risk is perceived to be higher in the short-term, the agent would prefer to hedge against low realizations of consumption in the near future buying long-term bonds, entailing the inversion of the equilibrium real yield curve.

Our model builds on the classic frameworks of Campbell and Cochrane (1999) and Wachter (2006). In these models, a representative agent has consumption preferences with respect to a habit level, and variations in the surplus over habit drive both the desire to smooth consumption over time and a precautionary motive that depends on changes in risk aversion. By inducing opposite consumption-saving desires, these two forces have opposite effects on the implied equilibrium risk-free rate, and, potentially, on the slope of the real

\(^1\)Three out of four episodes of inversions since 2008 are concurrent to inversions in the volatility term structure; the only inversion of the yield curve that is not matched by that of the volatility term structure is that of 2011-2012, which happened during the implementation of the Operation Twist by the Federal Reserve.
term structure. Assuming log-normal consumption growth, Campbell and Cochrane (1999) offset them to produce a constant risk free rate, while Wachter (2006) makes consumption smoothing motive always prevail such that reasonable estimates of consumption growth volatility do not allow the implied yield curve to invert.

With respect to those frameworks, our model features time-varying volatility of consumption growth and imperfect information. Consumption growth is a Markov switching process in which volatility varies between two regimes; agents do not observe the volatility state but infer it from the available draws of consumption. In equilibrium, real interest rates depend not only on the level of consumption with respect to habits, but also on expected volatility next period relatively to the long-run average. Indeed, a high perceived expected consumption growth volatility in the short-run induces the agent to shift her saving propensity to the long run: in times of low consumption relative to habits, this entails a prevailing desire for precautionary savings, while in times of high consumption, a prevailing desire of borrowing to consume more next period. In both cases, the equilibrium real yield curve is inverted.

Nominal yields are modeled by adding an exogenous inflation process to the aforementioned framework. With respect to the real term structure, the nominal one depends also on nominal factors (i.e., short-term inflation expectations and inflation volatility) and on the correlation between consumption growth and inflation. Assuming negative correlation, inflation volatility adds to consumption growth volatility as a second source of risk for the agent. The cumulated perceived risk matters for the slope of the implied equilibrium nominal yield curve in the same way as consumption volatility risk matters for the real curve.

Our model is mainly inspired by three studies. The key feature of consumption growth volatility being unobservable and time-varying is taken from Boguth and Kuehn (2013), who explore the connection between macroeconomic uncertainty and asset prices finding consumption growth volatility predicting returns for risk-exposed firms; the emphasis on long- vs. short-run risk is in the spirit of Bansal and Yaron (2004), that propose plausible solutions to asset pricing puzzles based on a persistent component in expected growth and on fluctuating uncertainty; the role of macroeconomic shocks on the slope of the yield curve is in line with Kurmann and Otrok (2013), who find that news about future total factor productivity (TFP) are the main factors behind the inversion of the curve and suggest the link of those shocks with consumption growth volatility. Intuitions over the connection between interest rates and macroeconomic risk are also in Breeden et al. (2015).

This paper is organized as follows. Section 2 describes the benchmark model and lays out some empirical findings on the relation between real rates and consumption. Section 3 presents the model of the real short rate with regime switches in the volatility of the surplus-consumption ratio and explains the mechanics of the inversion of the real and nominal term
2. Risk-free rates and consumption growth

Throughout this chapter, we explain the main arguments that motivate our research. First, we describe the features of the model proposed by Campbell and Cochrane (1999) (CC henceforth) that we take as a benchmark, focusing on the equilibrium risk-free rate; then, we make the point of the instability of the relationship between real short rates and consumption growth.

2.1. Benchmark model

Representative agents have preferences over consumption with respect to a slow-moving reference level $X_t$, that is an external habit level:

$$E_t \sum_{i=0}^{\infty} \beta^i (C_t - X_t)^{1-\gamma} - 1$$

(1)
Figure 2. Slope of the US TIPS yield curve: 5-year minus 2-year rates. Yields taken from Gurkaynak et al. (2010).

Figure 3. Slope of the real yield curve (3-year minus 2-year US TIPS yields) vs. standard deviation of one-year ahead expectations on US consumption growth. One-year ahead expectations constructed weighting current and next year private sector expectations taken from Consensus Economics. Monthly data are from January 2004 to March 2017. Yields are taken from Gurkaynak et al. (2010).
where $\beta$ is the subjective time discount factor and $\gamma$ the utility curvature. To capture the relation between $C_t$ and $X_t$, CC define the surplus-consumption ratio as the excess consumption relative to habits over the consumption level $C_t$:

$$S_t = \frac{C_t - X_t}{C_t}$$  \hspace{1cm} (2)

$S_t$ summarizes all the relevant information about the state of the economy and is the only state variable of the model. Note that $S_t \in [0, 1]$; throughout the paper, we will refer to bad states as states characterized by low $S_t$ and to good states as those with $S_t$ close to 1. Consumers’ relative risk aversion is time-varying and countercyclical:

$$\xi_t = \frac{\gamma}{S_t}$$  \hspace{1cm} (3)

For a constant $\gamma$, it falls during booms and increases during recessions.

In the model, consumption growth is exogenous and assumed to follow a random walk

$$\Delta c_{t+1} = g + v_{t+1}, \quad v_{t+1} \sim N(0, \sigma),$$  \hspace{1cm} (4)
In order to ensure that consumption growth remains always above habits, the surplus-consumption ratio $s_t$ is assumed to be an exogenous process calibrated in a way that ensures procyclicality: $s_t$ is mean reverting, autoregressive, correlated to shocks to consumption growth and heteroscedastic, with a positive time-varying coefficient $\lambda(s_t)$ loading on the innovation to consumption growth. The term $\lambda(s_t)$ is a sensitivity parameter defined as a square root function of past values of the $s_t$ process. $s_{t+1}$ follows

$$s_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda(s_t)(\Delta c_{t+1} - g)$$  

where $g$ is the average growth rate of consumption, $\phi$ the parameter regulating habit persistence,

$$\lambda(s_t) = \begin{cases} 
\frac{1}{\bar{s}} \sqrt{1 - 2(s_t - \bar{s})} - 1 & \text{if } s \leq s_{\max} \\
0 & \text{otherwise}
\end{cases}$$  

and

$$s_{\max} = \bar{s} + \frac{1}{2}(1 - \bar{S}^2)$$

$$\bar{S} = \sigma \sqrt{\frac{\gamma}{1 - \phi}}$$

As CC shows, the functional forms of $\lambda(s_t)$ and $\bar{s} = \ln \bar{S}$ are such that: (i) the risk-free rate is constant; (ii) habit is predetermined at the steady state $s_t = \bar{s}$; (iii) habit is predetermined near the steady state and moves nonnegatively with consumption everywhere\(^2\).

Wachter (2006) applies an alternative specification suggested by CC, that verifies requirements (ii) and (iii) but allows the short-term rate to be a linear function of the state. The functional form of $\lambda(s_t)$ is left unchanged, but $\bar{S}$ is now calibrated in the following way:

$$\bar{S} = \sigma_v \sqrt{\frac{\gamma}{1 - \phi - b/\gamma}}$$

From the Euler equation, the real one-period equilibrium risk-free rate is proportional to the deviations of $s_t$ from $\bar{s}$:

$$r_{t,t+1} = \bar{r} - b(s_t - \bar{s})$$  

where

$$\bar{r} = -\ln \delta + \gamma g - \frac{\gamma^2 \sigma^2}{2\bar{S}^2}$$

\(^2\)Habit is a non linear function of consumption. Around the steady state CC prove that $x_t = \ln(X_t)$ is such that

$$x_{t+1} \approx (1 - \phi)\bar{x} + \phi x_t + (1 - \phi)c_t$$
and

\[ b = \gamma(1 - \phi) - \frac{\gamma^2 \sigma^2}{\bar{S}^2} \]  

(12)

Importantly, being \( \{\delta, \gamma, g, \sigma, \phi\} \) all constant parameters, it follows that \( b \) is constant over time. The sign of the \( b \) term is key to get the relationship between real rates and surplus consumption; moreover, it has a clear economic interpretation.

Note that in an asset pricing framework, agents are not allowed to save to shift consumption bundles over time: in equilibrium, asset prices adjust to make her happy to consume the whole endowment in each period. Intertemporal consumption-saving preferences are governed by an intertemporal substitution and a precautionary saving motive. If \( b > 0 \), then the intertemporal substitution effect dominates: in good times, agents are more willing to consume than to save so the risk-free rate is driven down in equilibrium. On the contrary, if \( b < 0 \), then the precautionary motive dominates: in good times, the agent is less-risk averse so she would like to borrow to consume more today driving up the equilibrium interest rate.\(^3\)

In CC’s framework, \( b \) is set equal to 0 to completely offset these two effects. Instead, Wachter (2006) parameterizes \( b \) as a positive constant, so that the inter-temporal substitution effect always wins out, entailing a negative correlation between surplus consumption and equilibrium interest rates. Note that the term \( b \) determines not only the level, but also the slope of the equilibrium term structure of risk free rates: if \( b > 0 \), then the dominance of the intertemporal substitution motive is such that, in bad times, agents value consumption today more than consumption tomorrow and the equilibrium term structure is always upward sloping.

We now complete a preliminary analysis by having a closer look at the relationship between \( s_t \) and \( r_{t,t+1} \).

\( 2.2. \text{ Real rates and surplus consumption} \)

We have previously said that, in standard consumption-based models featuring habit, the equilibrium real risk-free rate is either constant or a negative function of the surplus-consumption ratio. Assuming Government bond yields in the United States as risk free, we investigate this issue empirically by comparing the historical dynamics of the real rate to that of the surplus-consumption ratio. Real rates – that cannot be proxied by TIPS in this analysis due to data availability – are estimated as the difference between the 3-month T-Bill rate and 3-month expected inflation, with the latter proxied by inflation forecasts made from an estimated autoregressive process (see Appendix A for details); the surplus-consumption ratio

\(^3\)In bad times, on the contrary, the intertemporal substitution propensity drives the equilibrium interest rate up, while the precautionary saving motive drives it down.
is instead constructed as the weighted average of past consumption growth with decreasing weights, as implied by the model and explained in Wachter (2006). Figure 5 displays the two series on a quarterly frequency from 1962 to 2014.

Figure 5. Real 3-month rate and surplus-consumption ratio. For the estimation method of real 3-month rates, see Appendix A. Surplus-consumption ratio is the weighted average of 40 quarters of past consumption growth with decreasing weights, as in Wachter (2006).

A quick graphical inspection suggests that the co-movement between the two is not stable over time: the correlation seems positive between late 60’s and late 70’s, then negative during the 80’s and 90’s, unclear on the rest of the sample. To analyze this relationship more formally, we estimate a time-varying $b$ by making rolling regressions on a 10-year window of the real 3-month rate on a constant and on our surplus-consumption proxy. The equation is

$$r_{t,t+1} = a_t + b_t \sum_{j=1}^{40} \phi^j \Delta c_{t-j} + \epsilon_{t+1} \quad (13)$$

The estimated $b_t$ coefficients are displayed in Figure ???. The coefficient exhibits large time variations, ranging from significantly negative to positive values. This entails that real rates depend positively on the surplus-consumption in some part of the sample and negatively in

4While surplus-consumption is theoretically influenced by all its own past values, we choose 40 quarters as the cut-off point.
some other parts, a feature of the data which is ruled out in CC’s and Wachter (2006)’s models.

![Coefficient b of regression](image)

**Figure 6.** Coefficient $b$ of regression $r_t = \text{constant} + bs_t + \text{error}$; real rates $r_t$ are estimated as the difference between the 3-month T-Bill rate and 3-month expected inflation; surplus-consumption ratio $s_t$ is constructed as the weighted average of past consumption growth with decreasing weights

3. **Model**

We propose a model which can accommodate for the time-varying correlation between surplus consumption ratio and risk free rates, and evaluate its implications for the slope of the term structure. First, we introduce a Markov switching process for the consumption growth and derive the pricing equation (Subsection 3.1); second, we discuss the behavior of the equilibrium risk-free rate and the equilibrium term structure (Subsection 3.2); third, we include inflation to explain the implication of the model for the nominal yield curve (Subsection 3.3).
3.1. Markov switching consumption growth and equilibrium risk-free rate

For the representative agent, we adopt the same set of preferences as CC and keep the same notation throughout the Section:

$$E_t \sum_{t=0}^{\infty} \beta^t (C_t - X_t)^{1-\gamma} - 1$$

(14)

We assume that, instead of being lognormal, consumption growth is a Markov switching process, in which volatility switches between two regimes.\(^5\) Denoting with \(g\) the constant drift, we assume that the process of log consumption growth \(\Delta c_{t+1}\) is

$$\Delta c_{t+1} = g + \sigma_{c_{t+1}} \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, 1)$$

(15)

with \(\sigma_{c}\) being either \(\sigma_h\) (high) or \(\sigma_l\) (low), with \(\sigma_h > \sigma_l\). Volatility is unobservable, depending on a latent variable \(\zeta_t\) indicating the state of the economy. Agents infer the state of the economy from observable consumption data. Denote by \(P\) the transition probability of being in state \(j = h, l\) coming from state \(i = h, l\)

$$P = \begin{bmatrix} p_{hh} & p_{hl} \\ p_{lh} & p_{ll} \end{bmatrix},$$

(16)

which is given and known to the agents at each point in time; new incoming information updates the likelihood of each state

$$\eta_t = \left[ \begin{array}{c} f(\Delta c_t | s_t = 1, X_{t-1}) \\ f(\Delta c_t | s_t = 2, X_{t-1}) \end{array} \right],$$

where \(X_{t-1}\) represents all information at time \(t-1\). Then, updated likelihoods and transition probabilities are used to form the posterior probability of being in each state based on the available data: call \(\xi_{t|t-1} \in \mathbb{R}^2\) the posterior belief vector at time \(t-1\), Bayes’ Law implies that

$$\xi_{t+1|t} = P' \frac{\xi_{t|t-1} \odot \eta_t}{1'(\xi_{t|t-1} \odot \eta_t)}$$

where \(\odot\) denotes element-by-element product and \(1\) is a 2-by-1 vector of ones.

\(^5\)Given that the trade-off between intertemporal substitution and precautionary saving motives does not depend on the drift of consumption growth, to keep the model as parsimonious as possible we do not impose latent states for it.
As consumption growth, the surplus consumption is also Markov switching:

\[ s_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda(s_t)\sigma_{\zeta+1} \epsilon_{t+1} \]  

(17)

where \( \phi \) is the AR coefficient (and the habit persistence coefficient). As in CC, habit is a slow moving average of consumption growth and is predetermined at and near the steady state. The steady state of surplus consumption is linked to the unconditional mean of the stochastic volatility process.

The stochastic discount factor (SDF) is a function of the surplus consumption:

\[ M_{t,t+1} = \delta \left( \frac{C_{t+1} S_{t+1}}{C_t S_t} \right)^{-\gamma} = \delta \exp \left\{ -\gamma \left[ g + (1 - \phi)(\bar{s} - s_t) + (\lambda(s_t) + 1)\sigma_{\zeta+1} \epsilon_{t+1} \right] \right\} \]  

(18)

Solving for the equilibrium risk-free rate involves the computation of the expectation of the SDF as a function of the two stochastic components of \( s_t \), i.e. \( \{\epsilon, \zeta\} \). After some algebra, we get

\[ r_{t,t+1} = \ln \frac{1}{E_t^c(\zeta) M_{t+1}} = -\ln \delta + \gamma g - \gamma(1 - \phi)(s_t - \bar{s}) - \ln E_t^{(c,\zeta)} \left( e^{-\gamma[\lambda(s_t)+1]\sigma_{\zeta+1} \epsilon_{t+1}} \right) \]  

(19)

where the last term on the right hand side is

\[ -\ln E_t^{(c,\zeta)} \left( e^{-\gamma[\lambda(s_t)+1]\sigma_{\zeta+1} \epsilon_{t+1}} \right) = -\ln \sum_{j \in \{h,l\}} \xi_{t+1|t}(j) E_t^{(c)} \left( e^{-\gamma[\lambda(s_t)+1]\sigma_j \epsilon_{t+1}} | \sigma_{\zeta+1} = \sigma_{j}, \xi_{t+1|t} \right) \]  

(20)

Equation 20 tells that, in a Markov switching world, agents have expectations about the future draws of consumption – that can be characterized by high or low volatility – and weight them by the posterior probability (i.e., the belief they have at time \( t \)) that such draws come from a high or a low volatility state. We interpret this factor as a precautionary saving effect, provided that Equation 19 differs from the risk free specification in CC model only for that\(^6\). In the extreme cases in which \( \xi_{t+1|t}(\sigma_h) = 0 \) or \( \xi_{t+1|t}(\sigma_h) = 1 \), the formula for the equilibrium risk free rate collapses to CC’s one.

The previous formula embeds one of the key results of our model: the intensity of the precautionary saving effect depends not only on the current state of the economy, summarized by \( s_t \), but also on agent’s beliefs and, precisely, on the posterior probability attached to the

\(^6\)In CC paper the close form solution of the risk free rate is

\[ r_{t,t+1} = \ln \frac{1}{E_t M_{t,t+1}} = -\ln \delta + \gamma g + \gamma(\phi - 1)(s_t - \bar{s}) - \frac{\gamma^2 \sigma^2}{2} (1 + \lambda(s_t))^2 \]  

(21)
two volatility states. Assume that $\sigma_l$ is low enough to let the intertemporal substitution effect dominate on the precautionary saving motive, and let $\sigma_h$ high enough to allow for the opposite. The steady state level of volatility $\sigma^*$ is such that $\sigma_l < \sigma^* < \sigma_h$, then our model is flexible enough to accommodate both times in which the intertemporal substitution motive prevails on the precautionary motive and times in which the opposite happens. In this model, negative rates and a downward-sloping yield curve appear as a deviation from a positive, upward-sloping steady state.

To summarize, the equilibrium one-period interest rate depends on the combination of the current state and on beliefs over next period consumption. Indeed, states in which $s_t$ is high might no longer be perceived as good states if $\sigma$ is also expected to be high in the short term: taken $s_t$ as given, when $\xi_{t+1|t}(\sigma_h)$ is higher than $\xi_{t+1|t}(\sigma_l)$, the equilibrium risk-free rate is driven up. Therefore, the combination of high $s_t$ and low $\xi_{t+1|t}(\sigma_h)$ defines good states, while bad states are those with low $s_t$ and high $\xi_{t+1|t}(\sigma_h)$. $\xi_{t+1|t}$ evolves based on the updated likelihood of the two states. Agents have imperfect information on the volatility, so a sequence of large shocks to consumption growth induce agents to weight more the high volatility state, while a sequence of small shocks slowly push them towards the low volatility state.

By introducing Markov switching consumption growth, we allow the trade-off between intertemporal substitution and precautionary saving motives to be endogenous. The flexibility of this specification allows to match the fact that the correlation between real short rates and surplus consumption is time-varying, and provides a rationale for the periods of positive correlations that appear from the empirical estimation of Equation 13.

### 3.2. The term structure of real risk-free rates

In the previous subsection, we have highlighted the key features underlying this model: time-varying posterior beliefs allow the inter-temporal and precautionary saving motives to dominate in different times, making the correlation of $r_t$ with $s_t$ also time-varying. Let’s now turn to the pricing of real risk-free bonds with maturities beyond one period to gain insights on the behavior of the entire term structure of interest rates.

The price at time $t$ of a real bond maturing after $n$ periods ($P_{n,t}$) is computed as the expectation of the future compounded SDFs until maturity. From the Euler equation:

$$P_{n,t} = E_t[M_{t+1}P_{n-1,t+1}]$$

$$= E_t[e^{\ln \delta - \gamma g + \gamma (1 - \phi) (s_t - \bar{s}) - \gamma [\lambda (s_t) + 1] \sigma_{t+1} \epsilon_{t+1}} P_{n-1,t+1}]$$

$$= \sum_{j \in \{h,l\}} \xi_{t+1|t}(j) E_t[e^{\ln \delta - \gamma g + \gamma (1 - \phi) (s_t - \bar{s}) - \gamma [\lambda (s_t) + 1] \sigma_{j,t+1} \epsilon_{t+1}} P_{n-1,t+1} | \sigma_{\xi_{t+1}} = \sigma_j, \xi_{t+1|t}]$$

(22)
with boundary condition \( P_{0,t} = 1 \); the yield-to-maturity is

\[
y_{n,t} = -\frac{1}{n} \ln P_{n,t}
\]  

(23)

As described in Equation 22, the real bond price is obtained by iterating forward one-period expectations of the bond price for \( n \) periods. While future states of the economy are not known at time \( t \), agents can only make expectations conditional on the available information at time \( t \). In order to account for all possible future states for both \( \epsilon \) and the posterior beliefs \( \xi \) for \( n \) periods, the bond price is solved numerically on a grid.

As explained in the previous Section, if we assume \( \sigma_h \) to be high enough to let the precautionary saving effect dominate, posterior beliefs biased towards \( \sigma_h \) are such that this scenario applies. In those cases, the precautionary saving motive implies agents’ willingness to save long-term, because they know that high volatility states have a limited duration and eventually go back to the low level: in this case, the “term structure of volatility beliefs” is downward sloping. Shifting saving propensity from the short to the long-run implies that the agent would be more willing to hold long- than short-term bonds, therefore in equilibrium short term securities will pay a premium with respect to long term one, entailing the inversion of the equilibrium yield curve.

3.3. Nominal yield curve

Denote by \( \pi_t = \ln \Pi_t \) the natural logarithm of the price level and introduce inflation \( \Delta \pi_t \) as a first order autoregressive, exogenous state process (AR(1)) following Cox et al. (1985) and Bekaert et al. (2004):

\[
\Delta \pi_{t+1} = \eta_0 + \psi_0 \Delta \pi_t + \sigma_{\Delta \pi} v_{t+1}
\]  

(24)

Denote also by \( \rho \) the linear correlation between \( v_{t+1} \) and \( \epsilon_{t+1} \) (i.e., the innovation in consumption growth). We can prove that the nominal bond price is equal to the expected discounted nominal payoff:\(^7\)

\[
P^S_{n,t} = E_t \left[ M^S_{t+1} P^S_{n-1,t+1} \right] = F^S_n(s_t) e^{A_n + B_n \Delta \pi_t}
\]  

(25)

\(^7\)Appendix B reports the proof of the nominal bond pricing formula
The SDF of the nominal security \( (M^S) \) is the ratio between the SDF of the real bond and the one-period gross inflation:

\[
M^S_{t+1} = e^{-\Delta \pi_{t+1}} M_{t+1} \quad (26)
\]

After some algebra, the nominal bond price becomes

\[
P^S_{n,t} = \text{const} \sum_{j \in \{h,l\}} \xi_{t+1|j} E_t^{(e)} \left[ M_{t+1} e^{\rho (B_{n-1} - 1) \sigma \Delta \pi_{t+1} F^S_{n-1,t+1}} | \sigma_{\xi_{t+1}} = \sigma_j, \xi_{t+1|j} \right] \quad (27)
\]

with

\[
\text{const} = e^{A_{n-1} + (B_{n-1} - 1) (\eta_0 + \psi_0 \Delta \pi_t) + 0.5 (B_{n-1} - 1)^2 \sigma^2_{\Delta \pi} (1 - \rho^2)}
\]

and

\[
M_{t+1} = e^{\ln \delta - \gamma g + \gamma (1 - \phi) (s_t - \bar{s}) - \gamma \lambda (s_t) + 1} \sigma_{\xi_{t+1}} \epsilon_{t+1}
\]

note that, assuming correlated innovations of the two state processes, the expected value in Equation 27 can be expressed as a function of \( \epsilon \) only. The yield-to-maturity of the nominal bond is

\[
y^S_{n,t} = -\frac{1}{n} \ln P^S_{n,t} \quad (28)
\]

The nominal bond price has two additional components with respect to the real bond price: a scale factor that depends on inflation volatility (in \( \text{const} \)) and an extra term in the expectation part of Equation 27, i.e. \( \exp \{ \rho (B_{n-1} - 1) \sigma_{\Delta \pi} \epsilon_{t+1} \} \). The extra term is key to get the intuition for the role of inflation. This term is a positive function of the product between \( \rho, \psi_0 \) (through \( B \)) and \( \sigma_{\Delta \pi} \). If \( \rho \) is negative, as reflecting the existing negative correlation between consumption growth and inflation, the extra term adds to the precautionary saving effect in its impact on the level and the slope of the term structure. Indeed, the agents’ desire to make precautionary saving/borrowing now depends not only on beliefs of the future consumption volatility states, but also on inflation volatility: the cumulated perceived risk matters for the slope of the nominal equilibrium yield curve in the same way as consumption volatility risk matters for the real one. If \( \sigma_{\Delta \pi} \) is sufficiently high, the nominal yield curve can invert even though posterior beliefs are such that the real one is upward sloping.

To complete the description of the model, we also compute the nominal risk premium up
to a constant term, which once again depends on surplus consumption and agents’ posterior probabilities:

\[ E_t\left(r^s_{n,t+1} - r^s_{1,t+1}\right) = \text{const} + E_t\left(\ln F^s_{n-1}(s_{t+1})\right) - \ln F^s_n(s_t) - \gamma(1 - \phi)(\bar{s} - s_t) + \ln \sum_{j \in \{h,l\}} \xi_{t+1|t}(j)e^{\frac{1}{2}\left(-\gamma(l\bar{s} + 1)\sigma_j - \rho\sigma_{\Delta s}\right)^2} \]  \hspace{1cm} (29)

Proof is in Appendix C.

4. **Empirical analysis**

This Section covers the application of the model described in Section 3 to US consumption and inflation data. The estimation of the parameters of the Markov switching process is carried out in Subsection 4.1. Then, we solve the model and discuss the behaviour of the slope of the term structure in Subsection 4.2. Finally we simulate from the model and report descriptive statistics in in Subsection 4.3.

4.1. **Parameter estimation**

We estimate the parameters of the Markov switching model by maximum likelihood. Real per capita consumption expenditures on nondurable goods and services are taken from the US Bureau of Economic Analysis. Following Yogo (2006), we restrict our sample to post 1952 data to avoid the exceptionally high consumption growth that followed World War II. Results are reported in Table 1, Panel A; sample data are from 1952Q1 to 2016Q3.

Average consumption growth is estimated at 0.49 per cent per quarter, while volatility equals 0.22 per cent in the low state and 0.56 per cent in the high state (i.e., the latter is 2.5 times bigger than the previous). The low volatility state is slightly less persistent: the probability that high consumption growth volatility will persist next period is 0.93, while for the low volatility state such probability is 0.88. Consumption growth and posterior probabilities are depicted in Figure 7.

Data on the monthly CPI index are taken from the Bureau of Labor Statistics database; inflation is constructed as quarter-on-quarter log returns, where quarterly CPI are values of the last month of the quarter. Estimates of the three parameters of the AR(1) process for inflation are reported in Table 1, Panel B. The long-term mean of the autoregressive process is 0.85 per cent, and inflation volatility is 0.82 per cent, higher than the volatility of consumption growth in high state. The correlation with consumption growth is estimated to be equal to -0.11.
Table 1: Parameter estimates of the consumption growth and inflation processes. Values are in percentage points. Non-annualized quarterly growth rates of consumption are computed using data on real consumption expenditures on nondurable goods and services taken from the US Bureau of Economic Analysis; inflation is constructed as quarter-on-quarter log returns, where quarterly CPIs are values of the last month of the quarter. CPI data are from the Bureau of Labor Statistics.

Figure 7. Output of the Markov switching estimate. Top panel: real per capita consumption growth. Bottom panel: expected volatility of consumption growth.
4.2. Model solution

We compute nominal and real bond prices numerically using the series method of Wachter (2005); for this purpose, a quadratic grid constructed as the combination of one grid for $s_t$ and one for $\xi_{t+1|t}$ is employed. Figure 8 shows equilibrium real yields in the extreme cases in which the agent perceives with certainty a low or a high volatility in the short term (left and right panel, respectively).

![Figure 8](image)

Figure 8. Continuously compounded yields on real bonds as a function of the surplus-consumption ratio implied by the posterior probabilities $P(\sigma = \sigma_h) = 0$ (left panels) and $P(\sigma = \sigma_h) = 1$ (right panels) and the parameters in Table 1 and Table 2.

In the low volatility case, short-term real yields are a decreasing function of the surplus-consumption ratio: real short rates are countercyclical (left panel, red line); moreover, the 5-year yields are always above the 3-month yields, i.e. the equilibrium real term structure is upward sloping for all values of the surplus consumption ratio. If the agent instead thinks that in the short-term the volatility of consumption growth will be high (high volatility case, right panel), the precautionary saving motive always prevails: short-term real yields are procyclical and the real term structure is inverted for all values of $S_t$. Note that the model can account for negative real rates.

Figure 9 shows how the short- and long-term real yields change as a function of the posterior probability to be in the low volatility state ($P(\sigma = \sigma_l)$) for a given $S_t$. Both short-
Figure 9. Continuously compounded yields on real bonds as a function of the posterior probability to be in the low volatility state in the short term implied by the parameters in Table 1 and Table 2. The solide black line represents the 5y yields; The dashed red line denotes 3m yields.
and long-term real yields are increasing with the probability of a low volatility state. The term structure is inverted when the high volatility state is perceived to be more likely (i.e., \( P(\sigma = \sigma_l) < 0.5 \)), while it is upward sloping in the opposite cases; the threshold value of the probability for which the term structure is inverted depends on the level of surplus consumption.

Figure 10. Continuously compounded short-term yields on real and nominal bonds as a function of the surplus-consumption ratio implied by the posterior probabilities \( P(\sigma = \sigma_h) = 0 \) (left panel) and \( P(\sigma = \sigma_h) = 1 \) (right panel) and the parameters in Table 1 and Table 2. The blue line represents the real yield; The green line denotes the nominal yield.

We now focus on the nominal curve, studying its sensitivity to the perceived consumption growth volatility across different calibrations of the long-term mean of the inflation process. Figure 10 depicts short-term nominal and real yields as a function of the surplus consumption when the agent expects low volatility state (left panel) or high volatility state (right panel). We can see that the results on the real risk free rate carry over to the nominal risk free rate. Note that nominal yields are always above real yields due to the effect of expected inflation. Figure 11 displays 3-month and 5-year nominal yields for different levels of expected inflation when the agent expects a low volatility state (lower panels) or high volatility states (upper panels). We consider expected inflation equal to its long-run mean (0.85 per cent, middle panels), and to plus and minus two unconditional standard deviations (right and left panels, respectively).
Figure 11. Nominal continuously compounded bond yields as a function of the surplus-consumption ratio implied by the posterior probabilities $P(\sigma = \sigma_h) = 1$ (upper panels) and $P(\sigma = \sigma_h) = 0$ (lower panels) and the parameters in Table 1 and Table 2, for different values of expected inflation: long-term expectation (middle panels), long-term expectation minus and plus two standard deviations (left and right panels). Blue lines represent 5y yields; green lines denote 3m yields.
The equilibrium nominal yield curve is very sensitive to changes in expected inflation. If the agent expects low volatility (lower panels), the higher the long-term inflation expectations, the smaller the difference between long and short term yields; for the case of high inflation expectations, we see that this difference is zero or negative for all the levels of surplus consumption. For the case of low volatility in the short term, provided that inflation expectations are mean reverting, variations in short-term yields are the main responsible for the inversion. This is coherent with the mechanics explained, in a different setup, by Kurmann and Otrok (2013).

If instead the agent expects high volatility states (top panels), the nominal yield curve is inverted for every values of the surplus consumption ratio, except in cases in which there are deflationary expectations and a high surplus consumption; moreover, the higher the long-term inflation expectations, the larger the gap between long- and short-term yields (top panels, from left to right). This suggests expected inflation is an important driver of the inversion of the nominal term structure.

4.3. Simulation

In order to match the slope of the term structure of nominal interest rates observed in the US market during the sample period, we simulate 100,000 observations of quarterly consumption growth and inflation. The model is calibrated using the parameters in Table 1 and Table 2. Mean and standard deviations of 3-month, 1-year, 3-year and 5-year zero yields are reported in Figure 12.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Utility Curvature $\gamma$</td>
<td>2.00</td>
</tr>
<tr>
<td>Habit persistence $\phi$</td>
<td>0.97</td>
</tr>
<tr>
<td>Discount rate $\delta$</td>
<td>0.98</td>
</tr>
<tr>
<td>Long-run mean of log surplus consumption $\bar{s}$</td>
<td>-3.25</td>
</tr>
<tr>
<td>Maximum value of log surplus consumption $s_{max}$</td>
<td>-2.75</td>
</tr>
</tbody>
</table>

Table 2: Assumptions on the parameters of the investors utility function

Model-implied values are very close, on average, to the observed ones. The mean of 3-month estimated nominal yields is 5.10 per cent, while the observed ones are on average 4.80
5. Conclusion

In this paper, we propose a consumption-based asset pricing model that allows not only the nominal, but also the real term structure of interest rates to invert. The main ingredients are time-varying volatility of consumption growth and imperfect information. Agents form posterior beliefs over future states of the economy. A high expected risk in the short term makes risk averse agent shifting her saving propensity from the short to the long-run, implying that she would be more willing to hold long- than short-term bonds and entailing the inversion of the equilibrium yield curve.

The proposed stochastic discount factor could, in principle, be used to price other type of assets. The impact of macroeconomic risk on equity pricing is investigated by Lettau et al. (2008) among others. The application on corporate bond pricing or derivative pricing can
be an avenue of future research. This model is designed for default-free economies: another interesting avenue of research could be that of investigating the evolution of a bond term structure containing a risk premium related to the default of the bond’s issuer. Equilibrium yield curves of different countries with different default risks could in this way be compared.
Appendix A. Market-implied real interest rates

Professional forecasters started to produce estimates of CPI inflation expectations at the beginning of the 80’s, so those can not be used to retrieve real rates (by subtracting inflation expectations from nominal rates) before that date. We instead follow the procedure proposed in Chapter 3 of the April 2014’s *World Economic Outlook* of the IMF: inflation expectations are computed as out-of-sample forecasts from a simulated autoregressive process of inflation. In this way we can estimate real rates for the whole sample (up to the 1960’s).

Denoting $P_t$ the monthly consumer price index at time $t$, an autoregressive model with 12 lags ($AR(12)$) is fitted on the variable $\gamma_t = \ln P_t - \ln P_{t-12}$; the estimation is carried out on a rolling window of 60 months in order to mitigate the effect of parameter instability. Model-based inflation expectations for horizon $j$ are computed using out-of-sample forecasts of $\gamma_t$. Real rates are then recovered as

$$r_{n,t} = r^s_{n,t} - (1 - g)(1 - g^n) \sum_{i=1}^{n} g^i E_t \pi_{t,t+i}$$

where $r_{n,t}$ and $r^s_{n,t}$ are the real and nominal rates at time $t$ on a bond of maturity $n$, $E_t \pi_{t,t+i}$ is the inflation expectation at time $t$ for period $t+i$ and $g = (1 + r^s)^{-i}$, with $r^s$ being the average nominal rate. The real rate is therefore equal to the nominal rate minus a weighted average of the inflation expectation over the entire life of the bond.

Appendix B. Pricing of real and nominal bonds

Let $P_{n,t}$ denote the price of a real bond maturing in $n$ periods, and $P^s_{n,t}$ the price of a nominal bond. Prices are computed as expectations of the future compounded SDFs until maturity.

The real price is determined recursively from the Euler equation (22) with boundary condition $P_{0,t} = 1$. Note that $P_{n,t}$ is a function of the posterior probability $\xi_{t+1|t}$. We solve for these functional equations numerically on a grid of values for the state variable $\xi_{t+1|t}$. Conditional on $\xi_{t+1|t}$, the price of the bond is a function of $s_t$ alone, so equation (22) can be
rewritten as

\[
P_{n,t} = E_t \left[ \delta \left( \frac{C_{t+1} S_{t+1}}{C_t S_t} \right)^{-\gamma} P_{n-1,t+1} \right]
\]

\[
= E_t [M_{t+1} P_{n-1,t+1}]
\]

\[
= \sum_{j \in \{h,l\}} \xi_{t+1|i}(j) E_t [M_{t+1} P_{n-1,t+1} | \sigma_{\xi_{t+1}} = \sigma_j, \xi_{t+1|i}]
\]

\[
= \sum_{j \in \{h,l\}} \xi_{t+1|i}(j) E_t [e^{\ln \delta - \gamma [g + (1 - \phi)(\bar{s} - s_t) + (\lambda(s_t) + 1)\sigma_j \epsilon + 1]} P_{n-1,t+1} | \sigma_{\xi_{t+1}} = \sigma_j, \xi_{t+1|i}]
\]

The last expectation can be solved using numerical integration on a grid of values for \(s_t\), conditional on being in state \(j\).

Analogously, the nominal bond price is equal to the expected discounted nominal payoff:

\[
P^s_{n,t} = E_t [M_{t+1} \prod_{t+1} P^s_{n-1,t+1}] \tag{30}
\]

In order to compute the nominal bond prices we introduce inflation as an additional state variable. Using the law of iterated expectations and conditioning on realizations of the shock to the level of the consumption growth, we can prove that

\[
P^s_{n,t} = F^s_{n,t} \exp \{ A_n + B_n \Delta \pi_t \} \tag{31}
\]

with

\[
F^s_{n,t} = E_t [M_{t+1} \exp \{ \rho(B_{n-1} - 1)\sigma_{\Delta \pi_{t+1}} \} F^s_{n-1,t+1}]
\]

\[
A_n = A_{n-1} + (B_{n-1} - 1) \eta_0 + 0.5(B_{n-1} - 1)^2 \sigma^2_{\Delta \pi}(1 - \rho^2)
\]

\[
B_n = (B_{n-1} - 1) \psi_0
\]

The boundary conditions are \(F^s_{0,t} = 1\), \(A_0 = 0\), and \(B_0 = 0\).

The proof is by induction. Suppose equation (31) is true for \(P^s_{n-1,t+1}\). Then, from the Euler equation it must be that

\[
P^s_{n,t} = E_t [M_{t+1} \prod_{t+1} \exp \{ A_{n-1} + B_{n-1} \Delta \pi_{n+1} \} F^s_{n-1,t+1}]
\]

\[
= E_t [M_{t+1} \exp \{-\eta_0 - \psi_0 \Delta \pi_t - \sigma_{\Delta \pi_{t+1}} + A_{n-1} + B_{n-1}(\eta_0 + \psi_0 \Delta \pi_t + \sigma_{\Delta \pi_{t+1}}) \} F^s_{n-1,t+1}]
\]

\[
= \exp \{ A_{n-1} + (B_{n-1} - 1)(\eta_0 + \psi_0 \Delta \pi_t) \} E_t [M_{t+1} F^s_{n-1,t+1} \exp \{(B_{n-1} - 1)\sigma_{\Delta \pi_{t+1}} \}]
\]
If we use the law of iterated expectations twice and condition on \( \xi_{t+1|t} \), that is the posterior probability at time \( t + 1 \), and then on \( \epsilon_{t+1} \), that is the error on the level of consumption growth we have

\[
P_{n,t}^S = \exp\{A_{n-1} + (B_{n-1} - 1) (\eta_0 + \psi_0 \Delta \pi_t) + \sum_{j \in \{h,l\}} \xi_{t+1|t}(j) E_t[M_{t+1} F_{n-1,t+1}^S \exp\{(B_{n-1} - 1) \sigma_{\Delta \pi} v_{t+1}\}] | \sigma_{\xi_{t+1}} = \sigma_j, \xi_{t+1|t}\]

given that

\[
(B_{n-1} - 1) \sigma_{\Delta \pi} v_{t+1} | \sigma_j \epsilon_{t+1} \sim N(\rho(B_{n-1} - 1) \sigma_{\Delta \pi} \epsilon_{t+1}, (B_{n-1} - 1)^2 \sigma_{\Delta \pi}^2 (1 - \rho^2))
\]

we have

\[
P_{n,t}^S = \exp\{A_{n-1} + (B_{n-1} - 1) (\eta_0 + \psi_0 \Delta \pi_t) + 0.5(B_{n-1} - 1)^2 \sigma_{\Delta \pi}^2 (1 - \rho^2) + \sum_{j \in \{h,l\}} \xi_{t+1|t}(j) E_t[M_{t+1} F_{n-1,t+1}^S \exp\{\rho(B_{n-1} - 1) \sigma_{\Delta \pi} \epsilon_{t+1}\}] | \sigma_{\xi_{t+1}} = \sigma_j, \xi_{t+1|t}\]

Therefore, equation (31) is satisfied with

\[
F_n^S(s_t) = E_t[M_{t+1} \exp\{\rho(B_{n-1} - 1) \sigma_{\Delta \pi} \epsilon_{t+1}\} F_{n-1,t+1}^S]
\]

\[
A_n = A_{n-1} + (B_{n-1} - 1) \eta_0 + 0.5(B_{n-1} - 1)^2 \sigma_{\Delta \pi}^2 (1 - \rho^2)
\]

\[
B_n = (B_{n-1} - 1) \psi_0
\]

**Appendix C. Nominal risk premium**

Let’s compute the nominal risk premium

\[
E_t\left(r^S_{n,t+1} - r^S_{1,t+1}\right)
\]

Using formula (25) we have that

\[
E_t\left(r^S_{n,t+1}\right) = E_t\left(\ln F^S_{n-1}(s_{t+1}) + A_{n-1} + B_{n-1}\Delta \pi_{t+1} - \ln F^S_n(s_t) + A_n + B_n\Delta \pi_t\right) =
\]

\[
= \text{cost} + E_t\left(\ln F^S_{n-1}(s_{t+1})\right) - \ln F^S_n(s_t) + B_{n-1}(\eta_0 + \psi_0\Delta \pi_t) - B_n\Delta \pi_t =
\]

\[
= \text{cost} + E_t\left(\ln F^S_{n-1}(s_{t+1})\right) - \ln F^S_n(s_t) + \psi_0\Delta \pi_t
\]
where the last equality comes from $B_n = (B_{n-1} - 1)\psi_0$. For the second term, we know that $r_{1,t+1}^s = 1/\ln(M_{t+1}^s)$ and

$$E_t\left(M_{t+1}^s\right) = E_t\left(e^{-\Delta \pi_{t+1}} M_{t+1}\right) = E_t[e^{-(\eta_0 + \psi_0 \Delta \pi_t + \sigma \Delta \pi_{t+1})} e^{\ln \delta - \gamma[g+(1-\phi)(\bar{s}-s_t)+(\lambda(s_t)+1)\sigma_{zt+1}t_{t+1}]}]$$

By using the same methodology that we applied for the formula of the nominal bonds, we have

$$E_t\left(M_{t+1}^s\right) = \exp(\ln \delta - \gamma (g + (1 - \phi)(\bar{s} - s_t)) - \eta_0 - \psi_0 \Delta \pi_t + 0.5 \sigma^2 \Delta \pi (1 - \rho^2))\sum_{j \in \{h,l\}} \xi_{t+1|t}(j) \exp(0.5(-\gamma(\lambda(s_t) + 1)\sigma_j - \rho \sigma \Delta \pi)^2)$$

so

$$r_{1,t+1}^s = 1/\ln(M_{t+1}^s) = \\
= -\ln \delta + \gamma (g + (1 - \phi)(\bar{s} - s_t)) + \eta_0 + \psi_0 \Delta \pi_t - 0.5 \sigma^2 \Delta \pi (1 - \rho^2) - \ln\left(\sum_{j \in \{h,l\}} \xi_{t+1|t}(j) \exp(0.5(-\gamma(\lambda(s_t) + 1)\sigma_j - \rho \sigma \Delta \pi)^2)\right)$$

Therefore the nominal risk premium is

$$E_t\left(r_{n,t+1}^s - r_{1,t+1}^s\right) = cost + E_t\left(\ln F_{n-1}^s(s_{t+1})\right) - \ln F_n^s(s_t) - \\
\gamma(1 - \phi)(\bar{s} - s_t) + \ln\left(\sum_{j \in \{h,l\}} \xi_{t+1|t}(j) \exp(0.5(-\gamma(\lambda(s_t) + 1)\sigma_j - \rho \sigma \Delta \pi)^2)\right)$$

(33)
References


