Abstract

We show that the popular approach to estimate the natural interest rate, as proposed by Laubach and Williams (2003), severely underestimates the volatility of the natural interest rate and may understate the actual decline of the natural rate over time. The key reason is that this procedure employs the Stock-Watson (1998) median unbiased estimator without taking into account the filter and parameter uncertainty surrounding the estimated latent factors. The resulting estimated variance ratios are downward biased, and the bias increases with the estimation uncertainty. We develop a modified version of the Laubach and Williams (2003) approach which, by explicitly incorporating uncertainty on factor estimates, removes the downward bias on the variance of the natural rate. We apply our procedure to U.S. data and provide evidence of a decline in the estimated natural interest rate which is more pronounced than what has been previously found in the literature.
1 Introduction

Since the onset of the Great Recession, economic growth in the U.S. and in many other advanced economies has been sluggish, nominal interest rates are steadily reaching their effective lower bounds, and the risk of deflation in the near future has not completely gone away. In this environment, slow potential growth may drive down the natural level of interest rate, which is the rate that equates savings and investment and closes the output gap. Very low, or even worse, negative levels of the natural interest rate in a economy trapped at its zero lower bound make the stabilization tools of monetary policy ineffective, which in turn leads to persistent and negative levels of the output gap.

Being the natural interest rate a hypothetical concept, it has to be measured from the data. The recent study of Holston et al. (2016) tackles the estimation of this rate and shows that over the recent decades, the natural interest rate has dramatically fallen in the U.S. and in other advanced economies, mostly because of a concurrent slowdown in potential growth. Since the measures of the natural interest rate are in general imprecise, as already advocated by Laubach and Williams (2003), some commentators have raised the possibility that the natural interest rate did not actually fell, as its estimate for the more recent period is not statistically different from what it was several decades ago (see for a discussion Taylor and Wieland, 2016).

In this paper we show that the popular approach to estimate the natural interest rate, as proposed by Laubach and Williams (2003), severely underestimates the volatility of the natural interest rate and consequently understates the actual decline of the natural rate over time. The key reason is that this procedure employs the Stock-Watson (1998) median unbiased estimator without taking into account the uncertainty surrounding the estimated latent factors, namely the potential output, the output gap, and the trend growth of the economy. The resulting estimated variance ratios are downward biased, and the bias increases with the uncertainty on the estimated factors. We develop a modified version of the Laubach and Williams (2003) approach which, by explicitly incorporating uncertainty on factor estimates, removes the downward bias on the variance of the natural rate. We apply our procedure to U.S. data and provide evidence of a decline in the estimated
natural interest rate, which is more pronounced than what has been found in Holston et al. (2016).

The Laubach-Williams framework posits that the natural interest rate is the sum of two nonstationary unobserved processes: the underlying trend growth of the economy and another unrelated factor (the $z$ component). Due to the so-called pile-up problem, according to which the variances of the innovations to the non-stationary processes are estimated to be zero, Laubach and Williams (2003) propose an estimation method which proceeds in sequential steps. The first step consists in estimating a simpler model as in Kuttner (1994) to recover an estimated measure of potential output. The estimated potential output is then used to obtain a Stock-Watson (1998) median unbiased estimator of the ratio of the volatility of the innovation to the trend growth over the volatility of the innovation to the level of potential output. This step treats the estimated potential output as true data, thus ignoring two sources of uncertainty: (i) the fact that the underlying potential output is unobserved and must be inferred from the model and the data (filter uncertainty); (ii) the fact that model’s true parameters are unknown and need to be estimated (parameter uncertainty). We show that by neglecting this estimation uncertainty, the estimated variance ratio is downward biased, and thus the variance of the innovation to the trend growth is underestimated. This pitfall also arises in the second step of the procedure, as we show that the ratio of the variance of the innovation to the $z$ component to the variance of the innovation to the output gap is also downward biased. Hence the Laubach-Williams approach leads to downward biases on the variances of the trend growth and the $z$ component. As a result, the estimated variance of the natural interest rate is also downward biased.

To what extent is this bias important? We show that the bias increases with the uncertainty about the the estimates of the latent states and model’s parameters. In turn, Laubach and Williams (2003) and Holston et al. (2016) have shown that estimation uncertainty is dramatic. For this reason, we expect the downward bias on the variance of the natural interest rate to be sizable. We develop a modified version of the Laubach-Williams procedure which deals with this bias, and compare its results with the standard procedure when applying it to real U.S. data.
A second question that we plan to address relates to whether the bias is stronger when estimating $\lambda_g$ or $\lambda_z$. While the Laubach-Williams procedure estimates $\lambda_g$ by assuming that just potential output is perfectly observed, the estimation of $\lambda_z$ posits this assumption for more latent factors (namely the output gap and the trend growth) as well as for some model’s parameters. For this reason, we expect the bias on $\lambda_z$ to be stronger. As a result, the standard Laubach-Williams procedure, by underestimating the role of the $z$ component, would attribute most of the variation in the real interest rate to the trend growth. This conjecture is in line with results provided in Holston et al. (2016), which show that most of the fall in the natural interest rate is attributable to a concurrent slowdown in potential growth.

The structure of the paper is as follows. Section 2 illustrates the issue when employing the Laubach-Williams approach. Section 3 proposes a modified version of the Laubach-Williams approach which deals with the downward bias in the variance ratios. Section 4 compares results of the two procedures in an empirical application to the U.S. data.

## 2 The issue in the Laubach-Williams framework

We consider a simplified version of the Laubach and Williams (2003) model, given by

\begin{align*}
\tilde{y}_t &= y_t - \tilde{y}_t^* \\
y_t^* &= y_{t-1}^* + g_{t-1} + \varepsilon_t^y \\
g_t &= g_{t-1} + \varepsilon_t^g \\
\tilde{y}_t &= \alpha \tilde{y}_{t-1} - \gamma (r_{t-1} - r_{t-1}^*) + \varepsilon_t^{\tilde{y}} \\
\pi_t &= \kappa \tilde{y}_{t-1} + \varepsilon_t^\pi \\
r_t^* &= g_t + z_t \\
z_t &= z_{t-1} + \varepsilon_t^z
\end{align*}

where equation (1) defines output gap $\tilde{y}_t$ as the deviation from the observed (log-) real GDP, $y_t$, and the unobserved level of potential output, $y_t^*$. Equation (2) defines the law
of motion of potential output, which depends on its past lag and on lagged trend growth of the economy \((g)\). Equation (3) is a simple IS relation which links the output gap to the real interest rate gap, which is given by the observed real interest rate, \(r\), minus the unobserved natural interest rate, \(r^*\). In turn, equation (5) posits a simple Phillips curve according to which the observed inflation rate, \(\pi_t\), depends on lagged output gap. Equation (6) defines the natural interest rate as the sum of two components: the trend growth of the economy and other factors, \(z_t\). Finally, equation (7) defines the law of motion of the \(z\) component, which follows a random walk. There are five serially and mutually uncorrelated zero mean disturbances in the model, \(\varepsilon_y^*, \varepsilon_g, \varepsilon_y, \varepsilon_t\), and \(\varepsilon^*_t\), with their respective variances given by \(\sigma^2_y, \sigma^2_g, \sigma^2_y, \sigma^2_\pi, \) and \(\sigma^2_z\).

Given the specification of the model, the level of real GDP and potential output are I(2) processes, while the natural rate of interest is I(1). When estimating the model by maximum likelihood and by applying the Kalman filter, researchers have encountered the so-called pile-up problem, according to which the estimated variances of the innovations to the non-stationary processes \(z\) and \(g\) are zero. To avoid this problem, Laubach and Williams (2003) propose an estimation method which proceeds in sequential steps. The first step consists in estimating a simpler model to recover a measure of potential output, by omitting the real rate gap term from equation (4) and by assuming that the trend growth rate is constant. The Stock and Watson’s (1998) median unbiased estimator of \(\lambda_g = \frac{\sigma_g}{\sigma^*_y}\) is obtained by testing for a structural break with unknown break date on equation (3). The second step consists of imposing the estimated value of \(\lambda_g\) from the first step and include the real interest rate gap in the output gap equation under the assumption that \(z\) is constant. Again, the Stock and Watson’s (1998) median unbiased estimator of \(\lambda_z = \frac{\gamma \sigma_z}{\sigma_y}\) is obtained by testing for a structural break with unknown break date on the IS curve of equation (4). The third and final step consists of imposing the estimated value of \(\lambda_g\) from the first step and \(\lambda_z\) from the second step and estimate the full model by maximum likelihood.

Does this procedure recover the true values \(\lambda_g\) and \(\lambda_z\)? We tackle this question by first considering the first step of the estimation procedure, which involves recovering \(\lambda_g\).
In this context, a measure of $\lambda_g = \frac{\sigma_g}{\sigma_{y^*}}$ is obtained by testing for a structural break with unknown break date on equation (3), which can be rewritten as

$$\Delta y_t^* = g_{t-1} + \varepsilon_t^{y^*}$$

where $\Delta y_t^*$ is the first difference of potential output. The Laubach-Williams procedure replaces the unobserved potential output by its estimate obtained in the first stage. This estimate, denoted by $\Delta \hat{y}_t^*$, is such that

$$\Delta \hat{y}_t^* = \Delta y_t^* + m_t$$

where $m_t$ is a measurement error which reflects two sources of uncertainty: (i) the fact that the underlying potential output is unobserved and must be inferred from the model and the data (filter uncertainty); (ii) the fact that model’s true parameters are unknown and need to be estimated (parameter uncertainty). For simplicity here we assume that $m_t$ is a zero mean serially uncorrelated process with variance $\sigma_m^2$. Hence equations (8) and (9) imply that

$$\Delta \hat{y}_t^* = g_{t-1} + m_t + \varepsilon_t^{y^*}$$

which states that the change in estimated potential output is the sum of two unobserved components: a I(1) process given by the trend growth, and a stationary process given by the sum of the measurement error, $m_t$, and the innovation to potential output, $\varepsilon_t^{y^*}$. By applying the Stock-Watson procedure on equation (10), we obtain that

$$\tilde{\lambda}_g = \sqrt{\frac{\sigma_g^2}{\sigma_{y^*}^2 + \sigma_m^2}} = \frac{\sigma_g}{\sigma_{y^*}} \sqrt{\frac{\sigma_{y^*}^2}{\sigma_{y^*}^2 + \sigma_m^2}} = \lambda_g \sqrt{\frac{\sigma_{y^*}^2}{\sigma_{y^*}^2 + \sigma_m^2}}$$

where $\tilde{\lambda}_g$ denotes the variance ratio obtained when the unobserved potential output is replaced by its estimate. The first term in the right-hand side is the true variance ratio, while the second term defines the bias induced by the fact that the Stock and Watson’s (1998) procedure is applied to an estimated series for the unobserved potential output. Since the term is in between 0 and 1, this bias is negative. The bias is maximum when the variance of the measurement error tends to infinity (the term goes to zero), and there is no bias when the variance of the measurement error is zero (that is the case in which
potential output is observable without error). The implication of this bias is that, for any level of variance of the innovation to potential output, $\sigma^2_y$, the Laubach-Williams procedure yields an estimated variance of the innovation to the trend growth, $\sigma^2_g$ which is downward biased.

Regarding the second step of the procedure, a measure of $\lambda_z = \frac{\gamma \sigma_z}{\sigma_y}$ is obtained by testing for a structural break with unknown break date on equations (4), (6), (7), which can be rewritten as

$$\frac{1}{\gamma} \ddot{y}_t - \frac{\alpha}{\gamma} \ddot{y}_{t-1} + r_{t-1} - g_{t-1} = z_{t-1} + \frac{1}{\gamma} \ddot{e}^y_t$$

(12)

The term in the left-hand side is unobserved, the procedure replaces it by an estimate obtained in the second stage. Denote the left-hand side term by $w_t$, then its estimate $\hat{w}_t$, is such that

$$\hat{w}_t = w_t + n_t$$

(13)

where $n_t$ is a measurement error which reflects the filter and parameter uncertainty. Again, for simplicity we assume that $n_t$ is a zero mean serially uncorrelated process with variance $\sigma^2_n$. Hence equations (12) and (13) now imply that

$$\hat{w}_t = z_{t-1} + n_t + \frac{1}{\gamma} \ddot{e}^y_t$$

(14)

which says that the estimated process $\hat{w}_t$ is the sum of two unobserved components: a I(1) process given by the $z$ component, and a stationary process given by the sum of the measurement error, $n_t$, and the innovation to the output gap, $\ddot{e}^y_t$. The Stock-Watson procedure applied on equation (14) yields

$$\tilde{\lambda}_z = \sqrt{\frac{\frac{\gamma^2 \sigma^2_z}{\sigma^2_y + \sigma^2_n}}{\frac{\sigma^2_y}{\sigma^2_y + \sigma^2_n}}} = \lambda_z \sqrt{\frac{\sigma^2_y}{\sigma^2_y + \sigma^2_n}}$$

(15)

where $\tilde{\lambda}_z$ denotes the variance ratio obtained when replacing the true value of $w_t$ by its estimate. The first term in the right-hand side is the variance ratio when $w_t$ is observed, while the second term defines the bias induced by the fact that the Stock and Watson’s (1998) procedure is applied to an estimated series for $w_t$. Again, since the term is in between 0 and 1, this bias is negative. The bias is maximum when the variance of the measurement error tends to infinity (the term goes to zero), and there is no bias when
the variance of the measurement error is zero (the term equals one). The implication of this bias is that, for any level of variance of the innovation to output gap, $\sigma^2_{\tilde{y}}$, the Laubach-Williams procedure yields an estimated variance of the innovation to the trend growth, $\sigma^2_{z}$ which is downward biased.

Which are the implications of these biases for the volatility of the estimated natural interest rate? Since from equation (6) the natural rate of interest is the sum of trend growth and the $z$ component, we also know that

$$\sigma^2_{r^*} = \sigma^2_g + \sigma^2_z$$ (16)

and by applying the restrictions imposed by $\lambda_g$ and $\lambda_z$, we obtain

$$\sigma^2_{r^*} = \lambda_g^2 \sigma^2_{\tilde{y}^*} + \frac{\lambda_z^2 \sigma^2_{\tilde{y}}}{\gamma^2}$$ (17)

so that the variance of the natural rate of interest positively depends on $\lambda_g$ and $\lambda_z$. Replace in equation (17) the true variance ratios by those obtained when neglecting estimation uncertainty, $\tilde{\lambda}_g$ and $\tilde{\lambda}_z$, to obtain:

$$\tilde{\sigma}^2_{r^*} = \lambda_g^2 \tilde{\sigma}^2_{\tilde{y}^*} + \frac{\lambda_z^2 \tilde{\sigma}^2_{\tilde{y}}}{\gamma^2} < \sigma^2_{r^*}$$ (18)

The Laubach-Williams approach leads to downward biases on the estimated variances of the trend growth and the $z$ component. As a result, the estimated variance of the natural interest rate is also downward biased.

There are two important questions to address. First, to what extent is this bias important? We have shown that the bias increases with the uncertainty about the the estimates of the latent states and model’s parameters. In turn, Laubach and Williams (2003) and Holston et al. (2016) have shown that estimation uncertainty is dramatic. For this reason, we expect the downward bias on the variance of the natural interest rate to be sizable. We develop a modified version of the Laubach-Williams procedure which deals with this bias, and compare its results with the standard procedure when applying it to real U.S. data.

A second question that we plan to address relates to whether the bias is stronger when estimating $\lambda_g$ or $\lambda_z$. While the Laubach-Williams procedure estimates $\lambda_g$ by assuming
that just potential output is perfectly observed, the estimation of $\lambda_z$ posits this assumption for more latent factors (namely the output gap and the trend growth) as well as for some model’s parameters. For this reason, we expect that the bias will be stronger for $\lambda_z$. As a result, the standard Laubach-Williams procedure, by underestimating the role of the $z$ component, would attribute most of the variation in the real interest rate to the trend growth. This conjecture is in line with results provided in Holston et al. (2016), which show that most of the fall in the natural interest rate is attributable to a concurrent slowdown in potential growth.

3 Solving the downward bias

— TO BE COMPLETED —

4 Empirical application to U.S. data

— TO BE COMPLETED —
References


