Abstract

The paper focuses on cointegration relationships that are affected by the presence of parameter instabilities that can appear in the whole time period. Usually, the estimation of structural breaks for cointegrated relationships requires excluding some observations at the extremes of the time period, avoiding the possibility of capturing parameter instabilities at the end-of-sample (EOS). In this paper we propose a procedure to estimate both structural breaks at the non-end-of-sample (NEOS) and parameter instabilities at the EOS. The new procedure is designed to improve the performance of the existing EOS instabilities tests that are proposed by Andrews (2003), Andrews and Kim (2006) and Kim (2010).

JEL classification: C22, C32

Keywords: cointegration, structural breaks, parameter instability

1 Introduction

Economies are in constant evolution and are affected by shocks that have different nature attending to their persistence characteristics –i.e., transitory and permanent shocks. These shocks might lead to changes in the relationships among macroeconomic variables of the economies. In empirical economics this issue has been addressed in most situations through the analysis of parameter instabilities of specified models.

Econometrics has designed different proposals to detect the presence of parameter instabilities that can be classified according to the model strategy – single-equation-based approaches such as the one proposed in Bai and Perron (1998, 2003) and system-based approaches like the ones in Bai, Lumsdaine and Stock (1998) and Qu and Perron (2007) – and the stochastic properties of time series – econometric techniques for I(0) and I(1) stochastic processes such as the ones in Bai and Perron (1998, 2003) and Kejriwal and Perron (2010). Another interesting feature that can be used to classify the different proposals of parameter instabilities analysis is the assumption about the subperiod in which those instabilities can take place. In this regard, it is quite common to discard some observations at the extremes of the period analyzed to define a region into which potential structural breaks can occur. This assumption deliberatively prevents the possibility of modeling parameter instabilities at the extremes of the time period. Fortunately, there are some contributions in the literature that explore this issue and propose statistics to test the presence of parameter instabilities at the end of the sample. Andrews (2003), Andrews and Kim (2006) and Kim (2010) propose statistics that are designed to detect parameter instabilities for I(0) processes and cointegrated models.

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This paper aims at analyzing the effects of unattended structural breaks that take place at the end of sample on the estimation of structural breaks on the non-end of sample period. We focus on both the case of I(0) stationary processes and the cointegration framework. The paper shows that unattended structural breaks at the end of sample can bias the analysis that is usually conducted in empirical applications, leading to conclude that there might be more structural breaks than exist. Further, the estimation of the break dates can also be affected.

The paper proceeds as follows. In Section 2 defines the model for which parameter instabilities are investigated, covering both I(0) stationary models and cointegration models. Section 3 studies the effects of unattended structural breaks on the estimation of the break dates through the minimization of the sum of squared residuals. Section 5 conducts an extensive Monte Carlo simulation experiment to assess the performance of popular parameter instabilities tests that are used in empirical applications. Section 4 proposes a unified strategy to model the presence of parameter instabilities throughout the analyzed period. Finally, Section 6 concludes. The proofs are collected in the appendix.

2 The model

Let us consider the stochastic process $y_t$ with the data generating process (DGP) given by:

$$ y_t = z'_t \gamma_j + u_t \quad t = T_j^0 + 1, \ldots, T_{j+1}^0, \quad (1) $$

$j = 0, 1, \ldots, J$, where $J$ denotes the number of structural breaks creating $J + 1$ regimes in the model and $T_j^0$ the break dates with the convention that $T_0^0 = 0$ and $T_{J+1}^0 = T$. It is assumed that there exists a fixed $\lambda_j^0$ for each $j$ such that $T_j^0 = \lfloor \lambda_j^0 T \rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part, which implies that the length of each regime grows proportionally to the sample size $T$.

The vector of regressors $z_t$ in (1) is defined by the combination of up to two different sets of regressors, i.e., $z_t = (z_{1,t}', z_{2,t}')'$, where $z_{1,t}$ contains the regressors that define the deterministic component – i.e., constant or linear time trend – and $I(1)$ non-stationary stochastic regressors and $z_{2,t}$ contains $I(0)$ stationary stochastic regressors. The model specification assumes that $u_t \sim I(0)$, so that cointegration is imposed in case that there are $I(1)$ stochastic regressors in (1) – note that this framework is defined in a way that regression analysis among a vector $Y_t = (y_t, z_{2,t}')'$ of $I(0)$ stochastic processes is allowed.

The specification of the model embeds a variety of structural break models that are proposed in the econometrics literature by altering the nature of the $p \times 1$ vector of regressors $z_t$ in (1). For example, Bai and Perron (1998) consider $I(0)$ stochastic regressors. Kejriwal and Perron (2010) consider $I(1)$ non-stationary stochastic regressors assuming that $u_t \sim I(0)$, so that cointegration is imposed. It is also possible to have both $I(0)$ and $I(1)$ regressors at the same time. Finally, Vogelsang (1997) specifies $z_t$ to be deterministic time trends.

All the structural break models mentioned above rely on the definition of a trimming parameter ($\varepsilon$) that prevents the presence of parameter instabilities at the extremes of the sample period – usually, $\varepsilon = 0.15$ and that, for convenience, is also used to define the minimum distance between two consecutive structural breaks for the multiple structural breaks case. Let $T_j$ be a generic break date and $\lambda_j = T_j/T$ be the corresponding generic break fraction. Then, the set of admissible break fractions is defined to be

$$ \Lambda = \{ (\lambda_1, \ldots, \lambda_J) | \varepsilon < \lambda_1 < \cdots < \lambda_J < 1 - \varepsilon \text{ and } |\lambda_j - \lambda_{j+1}| > \varepsilon \}. $$

This assumption implies that there cannot be additional structural breaks in the last $\varepsilon$ fraction of the sample regardless of the existence of breaks elsewhere in the sample. However, we could think of the possibility that parameter instabilities at the end of sample (EOS) period appear due to a recent change in the DGP. This situation is of great interest for empirical analyses – e.g., great recession effects on developed economies or the effects of Brexit to the
European Union – and might also affect the structural break analysis at the non-end of sample (NEOS) period.

To be specific and following Andrews (2003), we define the EOS to be the last \( m \) observations, so that the DGP at the EOS is given by:

\[
y_t = z_t \gamma_{eos} + u_t \quad t = T_{eos} + 1, \ldots, T,
\]

where \( T - T_{eos} = m \). With this set-up, the \((J + 1)\)-th regime \((t = T_J + 1, \ldots, T)\) is split in two:

\[
\begin{align*}
y_t &= z_t \gamma_{J} + u_t \quad t = T_J + 1, \ldots, T_{eos} \\
y_t &= z_t \gamma_{eos} + u_t \quad t = T_{eos} + 1, \ldots, T,
\end{align*}
\]

so that we allow for parameter instabilities at the EOS. Note this parameter instability analysis admits an additional interpretation in terms of testing the adequacy of the (arbitrary) specification of \( \varepsilon \) for the structural break analysis in the NEOS period.

3 Mutual interference between EOS and NEOS structural breaks

This section investigates how the existence of an EOS structural break affects the inference on the NEOS structural break and vice versa. The analysis distinguishes two different situations. First, we focus on univariate analysis of I(0) stochastic processes and, second, we cover the cointegration case.

3.1 The univariate I(0) stochastic process case

For the expositional simplicity, we consider a prototypical model specified by the following assumptions.

**Assumption 1** \( J = 1 \), and \( m/T \to \varepsilon_{eos} < \varepsilon \).

**Assumption 2** \( E(u_t) = 0 \) and \( Var(u_t) = \sigma^2 < \infty \). Let \( v_t = u_t z_t \), \( E(v_t) = 0 \), \( E(v_t v'_t) = \Sigma \) and \( E(v_t v'_{t-j}) = 0 \) for all \( j \neq 0 \). In addition, the partial sums of \( v_t \) satisfy the functional central limit theorem:

\[
T^{-1/2} \sum_{t=1}^{[Tr]} v_t \Rightarrow \Sigma_{1/2} W_p(r),
\]

where \( W_p(r) \) is the \( p \) dimensional standard Wiener process defined on the unit interval.

**Assumption 3** The regressors are such that

\[
p \lim_{T \to \infty} T^{-1} \sum_{t=1}^{[Tr]} z_t z'_t = rQ,
\]

uniformly in \( r \in [0, 1] \), where \( Q \) is some non-singular fixed matrix with \( ||Q|| \leq c < \infty \).

Assumption 1 allows for one NEOS structural break. Assumption 2 specifies that \( v_t = u_t z_t \) is serially uncorrelated and follows the functional central limit theorem. Assumption 3 rules out heterogenous distributions of the regressors across regimes. These assumptions are stronger than what can be found in the literature, but they are made only for expositional simplicity. Results to be presented below are well expected to hold under a more general set of assumptions. Assumption 1 also specifies that the length of the EOS is asymptotically an \( \varepsilon_{eos} \) fraction of the sample size, but is small enough to be outside the admissible set of NEOS break dates.
Let us first investigate the effects of an EOS structural break on the NEOS structural break analysis, by considering the case when there is no NEOS structural break but there is an EOS structural break. Typically, the NEOS structural break dates are estimated by minimizing the sum of squared residuals (SSR) over $\Lambda$. With $J = 1$ by Assumption 1,

$$\hat{\lambda}_1 = \arg \min_{\lambda_1 \in \Lambda} SSR(\lambda_1),$$

where

$$SSR(\lambda_1) = y'M(\lambda_1)y,$$

with $y = (y_1, \ldots, y_T)'$, $M(\lambda_1) = I - P(\lambda_1)$, $P(\lambda_1) = X(\lambda_1) (X(\lambda_1)'X(\lambda_1))^{-1} X(\lambda_1)'$, $X(\lambda_1) = [Z, Z(\lambda_1)]$, $Z = [z_1, \ldots, z_T]'$ and $Z(\lambda_1) = [0, \ldots, 0, z_{T_1+1}, \ldots, z_T]'$.

The null and alternative hypotheses for the NEOS structural break are given by:

$$\begin{cases} 
  H_0^{NEOS}: & \gamma_0 = \gamma_1 = \gamma \\
  H_1^{NEOS}: & \gamma_0 \neq \gamma_1 
\end{cases}.$$

Under Assumption 2, these hypotheses can be tested with the following simple statistic:

$$\sup_{\lambda_1 \in \Lambda} Wald(\lambda_1) = T \cdot \frac{SSR_{p} - SSR(\hat{\lambda}_1)}{SSR(\lambda_1)},$$

where $SSR_p = y'M_p y$, $M_p = I - P_Z$ and $P_Z = Z(Z'Z)^{-1}Z'$.

The following Theorem summarizes the effects of the EOS structural breaks on the NEOS structural breaks analysis.

**Theorem 1** Let $y_t$ be the stochastic process given by (1) and (2). Suppose that Assumptions 1 $\sim$ 3 hold and $\gamma \Delta \equiv \gamma_{cos} - \gamma \neq 0$. As $T \to \infty$, we have the following results under $H_0^{NEOS}$:

(i) $\hat{\lambda}_1 = 1 - \varepsilon$.

(ii) $\plim \frac{1}{T} \sup_{\lambda_1 \in \Lambda} Wald(\lambda_1) = \frac{p(\varepsilon_{cos}) - q(\varepsilon, \varepsilon_{cos})}{q(\varepsilon, \varepsilon_{cos}) + a}$, where $p(\varepsilon_{cos}) = (1 - \varepsilon_{cos})\varepsilon_{cos}$, $q(\varepsilon, \varepsilon_{cos}) = (\varepsilon - \varepsilon_{cos})\varepsilon_{cos}/\varepsilon$ and $a = \sigma^2/\gamma_\Delta Q\gamma_\Delta$.

The proof is given in the appendix. The result in (i) implies that $\hat{T}_1$ will collapse on the upper boundary of admissible values for the break dates. The one in (ii) shows the probability limit of the $sup_{\lambda_1 \in \Lambda} Wald(\lambda_1)$ test statistic, which can be viewed as the approximate Bahadur slope of the test statistic – see Kim and Perron (2009) for example. If the Wald statistic were "good" efficiency. However, as $\varepsilon$ gets large, the relative efficiency drops quickly and is only around 0.3 when $\varepsilon$ is at the popular choice of 0.15. The asymptotic relative approximate Bahadur efficiency of two tests is simply the limit of the ratio between the two approximate Bahadur slopes. Figure 1 displays the efficiency of $sup_{\lambda_1 \in \Lambda} Wald(\lambda_1)$ relative to $Wald(1 - \varepsilon_{cos})$ for $0.05 \leq \varepsilon \leq 0.15$ with $\varepsilon_{cos} = 0.05$ and $a = 1$. When $\varepsilon = 0.05$, the admissible set $\Lambda$ includes the true break fraction and there is no loss of efficiency. However, as $\varepsilon$ gets large, the relative efficiency drops quickly and is only around 0.3 when $\varepsilon$ is at the popular choice of 0.15.

Having specified the EOS as non-negligible fraction of the sample, one might wonder why $\Lambda$ is not enlarged to include $1 - \varepsilon_{cos}$ to resolve the issues raised above. However, our point here is that given data the behavior of $\hat{T}_1$ is exactly the same as what one would obtain when there is absolutely no break in the entire sample. Furthermore, the test statistic, while still consistent, suffers from great power loss, which is very troublesome. Since there is no NEOS structural break, one might regard the null rejection probability as size. Then, the size is not controlled
at all due to the consistency of the statistic. On the other hand, one might regard it as power, since there is an EOS break after all. Then, the test is not the best we can find as evidenced by the result in (ii).

Let us now investigate the effects of a NEOS structural break on the EOS structural break analysis. The null hypothesis for the EOS structural break is given by:

\[ H_0^{EOS} : \{ \gamma_{eos} = \gamma_J \text{ for all } t = T_{eos} + 1, \ldots, T \} \]

and the alternative hypothesis by:

\[ H_1^{EOS} : \{ \gamma_{eos} \neq \gamma_J \text{ for some } t = T_{eos} + 1, \ldots, T \} \]

and/or the distribution of \( \{ u_t \}_{t=1}^{T_{eos}+1} \) differs from the distribution of \( \{ u_t \}_{t=1}^{T_{eos}} \).

In consideration of the finite sample performance, Andrews (2003) and other authors propose testing procedures for the EOS structural break that involve complex steps. For simplicity, let us consider the following procedure that is somewhat simplified from the original version. Let us define the parameter estimators

\[
\hat{\gamma} = (Z'Z)^{-1}Z'y \\
\hat{\gamma}_t = (Z'_t Z(t))^{-1}Z'_t y(t),
\]

where \( y \) and \( Z \) are as defined right after (4), \( Z_t \) and \( y(t) \) delete observations indexed from \( t \) to \( t + m/2 - 1 \) from \( Z \) and \( y \) respectively. The test statistic \( P \) is given by

\[
P = (Y_{T_{eos}+1,T} - Z_{T_{eos}+1,T} \hat{\gamma})' (Y_{T_{eos}+1,T} - Z_{T_{eos}+1,T} \hat{\gamma})
\]

and the NEOS counterparts of \( P \) is given by

\[
P_t = (Y_{t,t+m-1} - Z_{t,t+m-1} \hat{\gamma}_t)' (Y_{t,t+m-1} - Z_{t,t+m-1} \hat{\gamma}_t),
\]

for \( t = 1, \ldots, T - 2m + 1 \) where \( Y_{a,b} \) and \( Z_{a,b} \) vertically stack \( y_{a,s} \) and \( z'_{a,s} \) for \( t = a, \ldots, b \). The empirical cumulative distribution function (cdf) of \( \{ P_t, t = 1, \ldots, T - 2m + 1 \} \) is then

\[
\hat{F}(x) = \frac{1}{T - 2m + 1} \sum_{t=1}^{T-2m+1} 1(P_t \leq x),
\]

and the critical value for the asymptotic size \( \alpha \) test is obtained as the \( 1 - \alpha \) percentile of \( \hat{F}(x) \).

The working mechanism of this test procedure is as follows. Given the consistency of \( \hat{\gamma} \) and \( \hat{\gamma}_t \), the distribution of \( P \) approaches that of \( U_{T_{eos}+1,T} U_{T_{eos}+1,T} \) and the distribution of \( P_t \) approaches that of \( U_{t,t+m-1} U_{t,t+m-1} \) where \( U_{a,b} \) stacks \( u_t \) for \( t = a, \ldots, b \). Under the null of \( H_0^{EOS} \), it is warranted that \( u_t \) is strictly stationary and thus the \( 1 - \alpha \) percentile of \( \hat{F}(x) \) becomes a correct critical value. However, when there is an unattended NEOS break, \( \hat{\gamma} \) and \( \hat{\gamma}_t \) are no longer consistent and the test loses its asymptotic validity. To make this point formally, we use the following assumptions.

**Assumption 4** \( J = 1 \) and \( m \) is fixed.

**Assumption 5** \( E(u_t) = 0, \text{Var}(u_t) = \sigma^2 < \infty \) and \( E(u_t z_t) = 0 \). In addition, \( E(z_t z'_t) > 0 \) and \( E(U_{1,m} U'_{1,m}) > 0 \) where \( U_{1,m} = (u_1, \ldots, u_m)' \).

**Assumption 6** The joint cdf of \( (z_t, u_t) \) is continuous.
Let \( u_t(\delta) = z_t^0 \delta + u_t \) for \( t = 1, \ldots, T \) and denote by \( F(x|\delta) \) the cdf of \( U(\delta)_{t+t+m-1} U(\delta)_{t+t+m-1} \) where \( U(\delta)_{a,b} \) stacks \( u_t(\delta)s \) for \( t = a, \ldots, b \). The following theorem summarizes the effects of unattended NEOS misspecification errors on the EOS instability test statistic.

**Theorem 2** Let \( y_t \) be the stochastic process given by (1) and (2). Suppose that Assumptions 4 \( \sim 6 \) hold and \( \gamma_\Delta \equiv \gamma_1 - \gamma_0 \neq 0 \). As \( T \to \infty \), we have under \( H_0^{EOS} \),

\[
\hat{F}(x) \to \lambda_1^1 \cdot F(x) - (1 - \lambda_0^1) \gamma_\Delta + (1 - \lambda_1^0) \cdot F(x|\lambda_1^0 \gamma_\Delta).
\]

As can be seen, the limit of the empirical cdf in which the EOS analysis bases on is affected when there is an unattended NEOS structural break. Consequently, the size of the \( P \) test statistic is not controlled except for a knife edge case and, furthermore, the unbiasedness of the statistic is not guaranteed.

### 3.2 The cointegration case

In this section we consider the model specification that accounts for the presence of one structural break that can affect the parameters of the I(1) stochastic regressors. Without loss of generality, we consider that there is no deterministic component in the model. We use the following assumptions instead of Assumptions 2 and 3.

**Assumption 7** The regressors are such that \( z_0 = \theta_p(1) \) and

\[
T^{-1/2} z_{[T\theta]} \Rightarrow B(r) = \Sigma_{2}^{1/2} W_p(r),
\]

where \( W_p(r) \) is the \( p \) dimensional standard Wiener process defined on the unit interval.

**Assumption 8** \( u_t \) is a white noise with \( E(u_t) = 0 \) and \( \text{Var}(u_t) = \sigma^2 < \infty \). In addition,

\[
\frac{1}{T} \sum_{t=1}^{[T\theta]} z_t u_t \Rightarrow \int_0^r B(r) dW(r),
\]

where \( B(r) \) is the same as in Assumption 7 and \( W(r) \) is the standard Wiener process independent of \( B(r) \).

Assumption 7 is a standard one. Assumption 8 holds when the regressors and the regression error are independent, which is again stronger than what is often used in the literature. These assumptions keep \( \sup_{\lambda_1 \in A} \text{Wald}(\lambda_1) \) in (5) as a valid test statistic.

The following theorem summarizes the effect of an EOS structural break for the cointegration case.

**Theorem 3** Let \( y_t \) be the stochastic process given by (1) and (2). Suppose that Assumptions 1, 7 and 8 hold and \( \gamma_{eos} \neq \gamma_1 \). As \( T \to \infty \), we have the following results under \( H_0^{EOS} \):

(i) \( \hat{\lambda}_1 \to D 1 - \varepsilon \).

(ii) \( \frac{1}{T} \sup_{\lambda_1 \in A} \text{Wald}(\lambda_1) \Rightarrow \eta(s|\gamma_\Delta) \equiv \frac{\gamma_\Delta^1 h(1 - \varepsilon) \gamma_\Delta}{\gamma_\Delta^0 h(1 - \varepsilon) \gamma_\Delta} \left( \int_{1 - \varepsilon}^{1} B(r) B(r') dr \right)^{-1} + \left( \int_{1 - \varepsilon_{eos}}^{1} B(r) B(r') dr \right)^{-1} \).

The proof is given in the appendix. As for the I(0) stationary case, \( \hat{T}_1 \) will collapse on the upper boundary of admissible values for the unknown break date. However, I(1) stochastic regressors make a difference since they lead to obtain a faster rate of divergence compared to the univariate I(0) stationary case – i.e., \( \theta_p(T^2) \) versus \( \theta_p(T) \). This distinctive feature is expected to intensify the pile-up effect shown by the estimated break date in finite samples – i.e., the estimated break date would be placed around the upper bound of the admissible set of break dates. The limiting expression for \( \sup_{\lambda_1 \in A} \text{Wald}(\lambda_1) \) shows that as \( \varepsilon \) gets bigger, the distribution of \( \eta(s) \) shifts toward smaller values, which implies the loss of power as in the case of I(0) regressors.
4 Unified parameter instability analysis

The analysis conducted so far calls for a testing strategy design that cover the situations in which the presence of structural instabilities can take place at the NEOS and EOS subperiods. The unified iterative estimation and testing strategy that we propose can be implemented in the following steps:

1. Define the value of the trimming parameter $\varepsilon$ and the maximum number of structural breaks ($J_{\text{max}}$) to take place at the NEOS period

2. Estimate the break dates at the NEOS period using a simple modification of the procedure suggested by Bai and Perron (1998, 2003) and Kejriwal and Perron (2010) – i.e., minimizing the SSR of the model given in (1) for all possible partitions defined by $J$.

   (a) The modification that we suggest considers one additional unknown structural break at the EOS period to reduce the risk of misplacing and/or misestimating the number and position of the structural breaks at the NEOS period.

   (b) As a result, we would estimate the break points considering that there are a maximum of $J_{\text{max}} + 1$ structural breaks, one of them placed at the EOS period – i.e., we will obtain up to $J$ estimated break points at the NEOS period plus one estimated break point at the EOS period

$$\left(\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_J, \hat{T}_{\text{eos}}\right) \in \Lambda \cup \left[\hat{T}_{\text{eos}} \in ((1 - \varepsilon + \varepsilon_{\text{eos}})T, (1 - \varepsilon_{\text{eos}})T)\right]; \quad 0 \leq J \leq J_{\text{max}}$$


   (a) If the EOS test rejects: we will have $J + 1$ structural breaks and define

   $$(\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_J, \hat{T}_{\text{eos}})$$

   (b) If the EOS test does not reject: we will have $J$ structural breaks and define

   $$(\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_J)$$

5 Monte Carlo simulation

5.1 EOS period is assumed to be stable

5.1.1 Effects of EOS break on NEOS break date estimation

As a complement of the derivations in Theorem 1, in this section we proceed to study the effects of unattended EOS structural breaks on NEOS structural break analysis in both finite and large sample sizes. Let us specify the DGP given by:

$$y_t = \mu + \theta DU_t + u_t, \quad (6)$$

with $\mu = 0$, $\theta = 1$, $DU_t = 1 (t > T_1)$, being $1 (\cdot)$ the indicator function, $\lambda_{10} = 0.95$ and $u_t \sim iidN (0, 1)$. The set of admissible break dates are defined as $\lambda_1 \in [\varepsilon, 1 - \varepsilon]$, with $\varepsilon = 0.15$ – also
used to denote the relative distance between two consecutive structural breaks. The number and position of structural breaks are estimated using the Bai and Perron (1998, 2003) sequential procedure with a maximum of $m = 5$ structural breaks with the nominal size set at the 5% level of significance. The sample sizes that are considered are $T = \{50, 100, 200, 300, 500, 1000\}$. Throughout this section, the simulation experiments are based on 1,000 replications and the computations are carried out using Matlab.

Panel A in Table 1 reports the frequency of structural breaks detection for the different sample sizes. The results indicate that the sequential test statistic in Bai and Perron (1998) tends to detect the presence of (at least) one structural break as $T$ gets large, although the structural break is placed outside the admissible set. On the one hand, although these results might be interpreted in terms of good performance of the sequential test statistic, it would be fair to recognize that the empirical power is not very high unless $T$ is large. On the other hand, if we were to interpret the results in terms of empirical size – since there is no structural break inside the admissible region – we would observe size distortions, which are more severe as $T$ increases. The second row of the table basically shows that in most of cases the procedure detects one structural break, with really few cases for which the estimated number of structural breaks is larger than one.

Figure 2 displays the histograms of the first estimated break fraction for the different sample sizes. It is worth mentioning that the break dates estimates are conditional on the outcome of the sequential test statistic in Bai and Perron (1998). Therefore, break date estimates are only recorded in those cases for which Bai and Perron (1998) sequential statistic detects the presence of at least one structural break. As can be seen, the probability mass shifts from the lower to the upper limit of $\Lambda$ as $T$ increases, corroborating the theoretical result obtained above. Figure 3 shows that the probability mass concentrates around the lower limit of $\Lambda$, and just in few occasions we observe a small mass of probability around the upper limit of $\Lambda$.

A similar analysis can be conducted for cointegrated relationships that accommodate the presence of parameter instabilities affecting both the deterministic and the stochastic regressors. The DGP used in this case is given by:

$$
\begin{align*}
y_t &= \mu + \theta DU_t + \beta_0 x_t + \beta_1 DU_t x_t + u_{1,t} \\
x_t &= x_{t-1} + u_{2,t},
\end{align*}
$$

with $\mu = 0$, $\theta = \beta_0 = \beta_1 = 1$ and $u_t = (u_{1,t}, u_{2,t})' \sim iid N(0, I_2)$. As above, $\lambda_1^0 = 0.95$ with the set of admissible break dates $\lambda_1 \in \Lambda = [\varepsilon, 1 - \varepsilon]$, $\varepsilon = 0.15$.

Panel B in Table 1 summarizes the estimated number structural breaks for the cointegration case using the sequential test statistic in Kejriwal and Perron (2010). As can be seen, the frequency of structural breaks detection increases with $T$, with values on the range 30% to 62%. In most cases the sequential procedure in Kejriwal and Perron (2010) detects only one structural break, which matches the DGP. Figure 4 indicates that the estimated structural break is mostly placed at the upper bound of $\Lambda$ – in the case where two structural breaks are detected, the second break also tends to be placed at the upper bound defined by $\Lambda$.

Although the picture that is obtained for both the I(0) and cointegration cases is qualitatively similar, it can be observed that the frequency of structural breaks detection is noticeable superior for the cointegration case – similar values are obtained for $T = 1000$. This feature can be explained by the different nature of the regressors that experience the effects of the structural break at EOS, which implies different rates of divergence of the estimator used to date the structural breaks – see the proofs of Theorems 1 and 3.

The main conclusion of this initial analysis is that the presence of structural breaks at the EOS can affect the performance of structural breaks test statistics that focus on the NEOS period. Further, we should consider that we can end up concluding that there is no parameter instabilities at the EOS using the statistics mentioned above if we pick up one structural break at the very end of the NEOS period.
5.1.2 Performance of the Bai-Perron and Kim test statistics. The I(0) case

In this section the finite sample performance of the UDmax and sequential test statistics in Bai and Perron (1998) and the $P(\hat{\alpha}, \Xi_T)$ and $P(\hat{\alpha}, I)$ tests in Kim (2010) are investigated. The DGP is given by (6) with $\mu = 0$, $\theta = 1$ and $u_t \sim iidN(0, 1)$. Four cases are studied depending on whether the structural break is placed at the NEOS and/or EOS subperiods – in the labels of the rows and columns, 1 indicates structural break, 0 otherwise:

\[
\begin{array}{c|cc}
\text{NEOS} & \lambda_{NEOS}^0, \lambda_{EOS}^0 & \text{EOS} \\
0 & (0, 0) & (0, 0.95) \\
1 & (0.5, 0) & (0.5, 0.95) \\
\end{array}
\]  

As above, the set of admissible break dates are defined as $\lambda_t \in \Lambda = [\varepsilon, 1 - \varepsilon]$, with $\varepsilon = 0.15$. The sample size is set at $T = \{50, 100, 200, 300, 500, 1000\}$ and 1,000 replications are carried out.

Table 2 reports the rejection frequency of the UDmax test statistic considering a maximum of $J = 5$ structural breaks. When there is no structural breaks at all ($\lambda_{NEOS} = \lambda_{EOS} = 0$), the empirical size of the UDmax approaches the nominal one as $T$ increases. As can be seen, size distortions appear for small $T$, although this is consequence of using the asymptotic critical values. The genuine empirical power of the UDmax statistic is investigated when one structural break is specified in the NEOS period – $(\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0)$. In this case, the rejection frequency is high even for small $T$.

The case $(\lambda_{NEOS}, \lambda_{EOS}) = (0, 0.95)$ defines a situation that violates the assumption in which the Bai-Perron framework builds upon – i.e., there are no further breaks at the extremes of the period – although we can observe that the rejection rate of the UDmax test increases as $T$ gets large. This is the consequence of the behavior described in Theorem 1, since the estimation of the EOS structural break on the NEOS subperiod tends to detect the presence of a structural break at the upper bound of $\Lambda$. Thus and although there is no structural breaks in the NEOS, the UDmax test does its best to pick up an out of range structural break. Finally, for $(\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0.95)$ the rejection rates of the UDmax statistic are similar to the $(\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0)$ ones, which is consistent with the existence of one NEOS structural break. In all, we can observe that the UDmax statistic shows good performance when detecting the presence of structural breaks, regardless of whether the structural break is placed at NEOS (proper detection) and/or EOS (misleading detection) subperiods.

Table 2 also reports the empirical performance of the Bai and Perron (1998) sequential test statistic and Kim (2010) $P(\hat{\alpha}, \Xi_T)$ and $P(\hat{\alpha}, I)$ test statistics, conditional on the inference drawn from the UDmax test. The conditional inference implies that the Bai and Perron (1998) sequential test statistic is used to estimate the number and position of the NEOS structural breaks for the cases in which the UDmax test rejects the null hypothesis of no structural breaks. Similarly, the computation of the $P(\hat{\alpha}, \Xi_T)$ and $P(\hat{\alpha}, I)$ test statistics depends on the outcome of the UDmax. On the one hand, if the UDmax does not reject the null hypothesis of no structural breaks, the $P(\hat{\alpha}, \Xi_T)$ and $P(\hat{\alpha}, I)$ tests are computed assuming parameter stability at NEOS. On the other hand, if the UDmax test rejects its null hypothesis, the sequential test statistic is used to estimate both the number and break dates positions, which are used to specify the model defined by (1). Then and based on this estimated model, the $P(\hat{\alpha}, \Xi_T)$ and $P(\hat{\alpha}, I)$ tests are computed to test the presence of EOS parameter instabilities.

Let us first focus on the performance of the Bai and Perron (1998) sequential statistic. For the no breaks case $(\lambda_{NEOS} = \lambda_{EOS} = 0)$ the rejection rates approaches the nominal size as $T$ increases, although mild size distortions are observed for small $T$. Note that when the null hypothesis of no structural breaks is rejected, the sequential procedure tends to detect one structural break in most of cases. When the structural break is placed at NEOS – $(\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0)$ – the empirical power of the sequential statistic rejects is high, with values that equals one for
Interestingly, for small sequential test shows a good performance in terms of detecting at least one structural break estimated in most of cases. When both NEOS and EOS structural breaks are specified, the sequential test shows a good performance in terms of detecting at least one structural break. Interestingly, for small \( T \) the sequential procedure tends to detect one structural break in the vast majority of cases, although for large \( T \) we observe that the probability of detecting two structural breaks significantly increases (41.7% for \( T = 1000 \)).

The empirical size of the \( P(\hat{\alpha}, \Xi_T) \) and \( P(\hat{\alpha}, I) \) tests proposed by Kim (2010) is close to the nominal size when there are no structural breaks. When there is just one NEOS structural break – i.e., \((\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0)\) – the \( P(\hat{\alpha}, \Xi_T) \) and \( P(\hat{\alpha}, I) \) tests become conservative as \( T \) increases – note that for small \( T \) the empirical size is close to the nominal one, but the rejection probability is virtually zero for large \( T \). The performance of these statistics when there is an EOS structural break depends on whether we also have a NEOS structural break. Thus, for the case in which \((\lambda_{NEOS}, \lambda_{EOS}) = (0, 0.95)\) the empirical power of the \( P(\hat{\alpha}, \Xi_T) \) and \( P(\hat{\alpha}, I) \) tests approaches one as \( T \) increases, although this is not the case for \((\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0.95)\) – rejection rates are marginally larger than the empirical size. Note that the performance of the Bai-Perron sequential statistic is also qualitatively different in both cases. For \((\lambda_{NEOS}, \lambda_{EOS}) = (0, 0.95)\) the sequential test tends to detect one structural break – as the \( P(\hat{\alpha}, \Xi_T) \) and \( P(\hat{\alpha}, I) \) tests do – whereas for \((\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0.95)\) the sequential test starts detecting two structural breaks in large samples (41.7% of cases for \( T = 1000 \)).

5.1.3 Performance of the Kejriwal-Perron and Kim test statistics. The cointegration case

In this section the finite sample performance of the UDmax and sequential test statistics in Kejriwal and Perron (2010) and the \( P(\hat{\alpha}, \Xi_T) \) test in Kim (2010) are investigated. The DGP is given by (7) with \( \mu = 0, \theta = \beta_0 = \beta_1 = 1 \) and \( u_t = (u_{1,t}, u_{2,t})' \sim iid N(0, I_2) \). As above, the four situations defined in (8) are considered.

The simulation results are reported in Table 3. The empirical size of the UDmax and the sequential test statistics is close to the 5% nominal size, although mild underrejection is found for small \( T \) – see the results for \((\lambda_{NEOS}, \lambda_{EOS}) = (0, 0)\). The \( P(\hat{\alpha}, \Xi_T) \) test overrejects, although this is a consequence of the conditional inference that is conducting – it is worth mentioning that the model that is estimated depends on the UDmax and sequential statistics inference. The UDmax and sequential statistics show good empirical power and ability for the estimation of the true number of structural breaks – see the results for \((\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0)\). In this case, the rejection probabilities of the \( P(\hat{\alpha}, \Xi_T) \) test are similar to the previous ones since there is no instabilities at the EOS.

For the case involving only an EOS structural break, the rejection probabilities of the UDmax and sequential statistics increase with \( T \), indicating that there is a structural break affecting the model, although it is located out of the range for these statistics. Note that when the presence of structural breaks is detected, the number of estimated breaks equals one in most of cases. In this situation, the \( P(\hat{\alpha}, \Xi_T) \) test features high empirical power values.

For the \((\lambda_{NEOS}, \lambda_{EOS}) = (0.5, 0.95)\) case the rejection probabilities for the UDmax and sequential statistics are quite high for all values of \( T \). Contrary to what is found for the \((\lambda_{NEOS}, \lambda_{EOS}) = (0, 0.95)\) case, the probability of detecting two structural breaks increases with \( T \). It seems that the sequential statistic points towards the correct number of structural breaks, although the break fraction of one of them cannot be consistently estimated. This rises the possibility that might be the sequential test can be used only to estimate the number of structural breaks, not their position – since one of them is out of the admissible set. Interestingly,
the rejection probabilities of the $P(\hat{\alpha}, \Xi_T)$ test are quite high, with values that approach one as $T$ increase. Compared to the results that have been obtained for the I(0) case, this is a distinctive feature that worth further analysis.

5.2 Unified strategy

5.2.1 The I(0) case

The DGP that is considered allow for up to two structural breaks:

$$y_t = \mu + \theta_1 DU_{1,t} + \theta_2 DU_{2,t} + u_t,$$

(6) with $\mu = 0$, $\theta_j = \{0, 1\}$, $j = 1, 2$, $DU_{1,t} = 1 (t > T_1)$, $DU_{2,t} = 1 (t > T_{cos})$, $u_t \sim iid N (0, 1)$.

As above, the four cases described in (8) are studied depending on whether the structural break is placed at the NEOS and/or EOS subperiods. The case of no structural breaks imposes $(\theta_1, \theta_2) = (0, 0)$. When there is only one structural break at NEOS period $(\theta_1, \theta_2) = (1, 0)$, whereas when there is one structural break at the EOS period $(\theta_1, \theta_2) = (1, 0)$. Finally, if we have two structural breaks, one at NEOS and another one at EOS period, $(\theta_1, \theta_2) = (1, 1)$.

The set of admissible break dates for the NEOS period are defined as

$$\Lambda = \{(\lambda_1, \ldots, \lambda_J)| \epsilon < \lambda_1 < \cdots < \lambda_J < 1 - \epsilon \text{ and } |\lambda_j - \lambda_{j+1}| > \epsilon\},$$

with $\epsilon = 0.15$ and $J = 1, \ldots, J_{max}$. The simulations consider considering a maximum of $J_{max} = 5$ structural breaks. For the EOS period we specify

$$\lambda_{eos} \in \Lambda_{eos} = [1 - \epsilon + \epsilon_{cos}, 1 - \epsilon_{cos}],$$

with $\epsilon_{cos} = 0.05$ and the sample size set at $T = \{100, 200, 300\}$.

Figure 6 depicts the densities of the estimated break dates for $J_{max} = 1$. When there are no structural breaks, the density of the break date estimated on the NEOS subperiod tends to be uniformly distributed across the admissible values of the break fraction defined by $\Lambda$. The density of $\hat{\lambda}_{eos}$ shows picks at the extremes of the $\Lambda_{eos} = [0.9, 0.95]$ set, although the probabilities of $\lambda_{eos}$ taking a given value in $\Lambda_{eos}$ are pretty similar – it should be noticed that the range of values defined by $\Lambda_{eos}$ is very narrow. When there is one NEOS structural break, the density of $\hat{\lambda}_1$ concentrates around the true $\lambda_1$ with a dispersion that reduces as $T$ increases. As expected, the density of $\hat{\lambda}_{eos}$ shows a similar behavior as for the no structural breaks case. The converse situation is found when there is an EOS break. Now, the density of $\hat{\lambda}_{eos}$ shows a spike on the true value of $\lambda_{eos}$, whereas $\hat{\lambda}_1$ is uniformly distributed across $\Lambda$.

5.2.2 The cointegration case

In this case the DGP accounts for up to two structural breaks:

$$y_t = \mu + \theta_1 DU_{1,t} + \theta_2 DU_{2,t} + \beta_0 x_t + \beta_1 DU_{1,t} x_t + \beta_2 DU_{2,t} x_t + u_t$$

$$x_t = x_{t-1} + u_{2,t},$$

with $\mu = 0$, $\beta_0 = 1$, $\{\theta_j, \beta_j\} = \{0, 1\}$, $j = 1, 2$, $DU_{1,t} = 1 (t > T_1)$, $DU_{2,t} = 1 (t > T_{cos})$, $u_t \sim iid N (0, 1)$. We consider the four cases described in (8) – i.e., (i) no structural breaks $(\theta_1, \theta_2, \beta_1, \beta_2) = (0, 0, 0, 0)$, (ii) one NEOS structural break $(\theta_1, \theta_2, \beta_1, \beta_2) = (1, 1, 0, 0)$, (iii) one EOS structural break $(\theta_1, \theta_2, \beta_1, \beta_2) = (0, 0, 1, 1)$, and (iv) one NEOS and one EOS structural breaks $(\theta_1, \theta_2, \beta_1, \beta_2) = (1, 1, 1, 1)$ – with $\epsilon = 0.15$ and $\epsilon_{cos} = 0.05$.

Figure 7 presents the density of $\hat{\lambda}_1$ and $\hat{\lambda}_{eos}$ for the four cases under investigation. Conclusions are similar to the ones reached for the I(0) case. When there are no structural breaks, the densities show that $\hat{\lambda}_1$ and $\hat{\lambda}_{eos}$ are uniformly distributed across their admissible set of potential values. The dynamic estimation algorithm places the break fraction around the true value
when there is a structural break, with a dispersion that decreases as $T$ increases. Consequently, the unified strategy that has been suggested seems to provide properly results when there are structural breaks at NEOS and/or EOS subperiods.

6 Conclusions

The paper has studied the effects of EOS structural breaks on the estimation of the number and position of structural breaks at the NEOS subperiod. It has been shown that the estimated break fractions for the NEOS subperiod can be affected by unattended EOS structural breaks. Besides, the computation of EOS parameter instabilities tests available in the literature are sensitive to proper modeling of the model at the NEOS subperiod. These features are found for both stationary and cointegrated models, which define a wide set of situations of interest for empirical analyses. This situation calls for the use of a unified framework that focus on potential parameter instabilities covering the whole time period.

References


A Appendix

A.1 Proof of Theorem 1

When Assumption 1 holds and $\gamma_{eos} \neq \gamma_1$, we can write the true DGP for $y$ as

$$y = Z\gamma + Z(1 - \varepsilon_{eos})\gamma_\Delta + u$$

where $Z = [z_1, \ldots, z_T]'$ and $Z(\lambda_1) = [0, \ldots, 0, z_{T_{eos}+1}, \ldots, z_T]'$. The SSR when the true break fraction is used is such that

$$\frac{1}{T}SSR(1 - \varepsilon_{eos}) = \frac{1}{T}y'M(1 - \varepsilon_{eos})y$$

$$= \frac{1}{T}u'M(1 - \varepsilon_{eos})u \overset{p}{\rightarrow} \sigma^2,$$

by Assumption 3. Similarly, the SSR for a generic break fraction is such that

$$\frac{1}{T}SSR(\lambda_1) = \frac{1}{T}y'M(\lambda_1)y$$

$$= \frac{1}{T}(d\gamma_\Delta + u)'M(\lambda_1)(d\gamma_\Delta + u)$$

$$= \frac{1}{T}\gamma_\Delta d'M(\lambda_1)d\gamma_\Delta + \frac{2}{T}u'M(\lambda_1)d\gamma_\Delta + \frac{1}{T}u'M(\lambda_1)u$$

$$\overset{p}{\rightarrow} (1 - \varepsilon_{eos} - \lambda_1)\varepsilon_{eos} (\gamma_\Delta ^TQ\gamma_\Delta) + \sigma^2,$$

where $d = Z(1 - \varepsilon_{eos}) - Z(\lambda_1)$. Note that we used the fact that $1 - \varepsilon_{eos} > \lambda_1$ in the above derivation. The SSR when no NEOS break is accounted for is such that

$$\frac{1}{T}SSR_r = \frac{1}{T}y'M_r y \overset{p}{\rightarrow} (1 - \varepsilon_{eos})\varepsilon_{eos} (\gamma_\Delta ^TQ\gamma_\Delta) + \sigma^2.$$  (A.3)

For the result in (i), we have using (A.2) and (A.1)

$$\hat{\lambda}_1 = \arg \min_{\lambda_1 \in A} SSR(\lambda_1)$$

$$= \arg \min_{\lambda_1 \in A} \left[ \frac{1}{T}SSR(\lambda_1) - \frac{1}{T}SSR(1 - \varepsilon_{eos}) \right]$$

$$= \arg \min_{\lambda_1 \in A} \frac{(1 - \varepsilon_{eos} - \lambda_1)\varepsilon_{eos}}{(1 - \lambda_1)} (\gamma_\Delta ^TQ\gamma_\Delta) + a_p(1) \overset{p}{\rightarrow} 1 - \varepsilon.$$

For the result in (ii), we have using (A.2), (A.3) and the fact that $\hat{\lambda}_1 \overset{p}{\rightarrow} 1 - \varepsilon$,

$$\operatorname{plim} \frac{1}{T} \sup_{\lambda_1 \in A} Wald(\lambda_1) = \operatorname{plim} \frac{SSR_r - SSR(\hat{\lambda}_1)}{SSR(\lambda_1)}$$

$$= \frac{(1 - \varepsilon_{eos})\varepsilon_{eos} - (\varepsilon - \varepsilon_{eos})\varepsilon_{eos}}{\varepsilon}$$

$$= \frac{p(\varepsilon_{eos}) - q(\varepsilon, \varepsilon_{eos})}{q(\varepsilon, \varepsilon_{eos}) + a}.$$
A.2 Proof of Theorem 2

Define

\[ \hat{F}_1(x) = \frac{1}{T_1} \sum_{t=1}^{T_1} 1(P_t \leq x) \]

\[ \hat{F}_2(x) = \frac{1}{T - T_1 - 2m + 1} \sum_{t=T_1+1}^{T-2m+1} 1(P_t \leq x). \]

Then, we can write that

\[ \hat{F}(x) = \frac{T_1}{T-2m+1} \hat{F}_1(x) + \frac{T - T_1 - 2m + 1}{T-2m+1} \hat{F}_2(x). \]

When there is an unattended NEOS break as stated in the theorem, the estimated residual in the first regime is

\[ \hat{u}_t = z_t' (\gamma_0 - \hat{\gamma}_t) + u_t \]

and the one in the second regime is

\[ \hat{u}_t = z_t' (\gamma_1 - \hat{\gamma}_t) + u_t. \]

It can also be shown that

\[ \hat{\gamma} \xrightarrow{p} \gamma_0 + (1 - \lambda_1^0) \gamma_\Delta, \]  

(A.4)

and

\[ \hat{\gamma}_t \xrightarrow{p} \gamma_0 + (1 - \lambda_1^0) \gamma_\Delta, \]  

(A.5)

uniformly in \( t \). It follows then \( \hat{F}_1(x) \xrightarrow{p} F(x\mid (1 - \lambda_1^0) \gamma_\Delta) \) and \( \hat{F}_2(x) \xrightarrow{p} F(x\mid \lambda_1^0 \gamma_\Delta) \).

A.3 Proof of Theorem 3

As a matter of notation, we simply write \( \int BB' \) for \( \int B(r)B(r)'dr \). First, note that we can write

\[ \frac{1}{T^2} SSR(\lambda_1) = \frac{1}{T^2} \gamma_\Delta' d'M(\lambda_1) d\gamma_\Delta + \frac{1}{T^2} u'M(\lambda_1) d\gamma_\Delta + \frac{1}{T^2} u'M(\lambda_1) u \]  

(A.6)

where \( d = Z(1 - \varepsilon^{\cos}) - Z(\lambda_1) \). The first term in (A.6) is such that

\[ \frac{1}{T^2} d'M(\lambda_1) d = \frac{1}{T^2} d'd - \frac{1}{T^2} d'X(\lambda_1) \left( \frac{1}{T^2} X(\lambda_1)'X(\lambda_1) \right)^{-1} \frac{1}{T^2} X(\lambda_1)'d \]

where

\[ \frac{1}{T^2} d'd \Rightarrow \int_{\lambda_1}^{1-\varepsilon^{\cos}} BB', \]

\[ \frac{1}{T^2} d'X(\lambda_1) \Rightarrow \left( \int_{\lambda_1}^{1-\varepsilon^{\cos}} BB' \left( I_p \quad I_p \right) , \right. \]

\[ \frac{1}{T^2} X(\lambda_1)'X(\lambda_1) \Rightarrow \left( \int_{\lambda_1}^{1-\varepsilon^{\cos}} BB' \int_{\lambda_1}^{1-\varepsilon^{\cos}} BB' \right) \equiv Q_{XX} \]
by Assumption 7. Hence,

$$\frac{1}{T^2} \gamma^1 dM(\lambda_1) d\gamma$$

(A.7)

\[
\Rightarrow \gamma^1 \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right) \left[ \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right)^{-1} - \left( I_p \ I_p \right) Q_{XX}^{-1} \left( I_p \ I_p \right) \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right) \gamma^1 \right] \\
\quad = \gamma^1 \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right) \left[ \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right)^{-1} - \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right)^{-1} \right] \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right) \gamma^1 \\
\quad = \gamma^1 \left[ \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right)^{-1} + \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right)^{-1} \right]^{-1} \gamma^1 > 0
\]

where the positive definiteness holds when $\varepsilon^{\text{cos}} > \lambda_1$. The first equality after the weak convergence follows from the inverse of a partitioned matrix applied to $Q_{XX}$ and the last equality holds because $M^{-1} - (M + N)^{-1} = M^{-1}(M^{-1} + N^{-1})^{-1}M^{-1}$ for some non-singular matrices $M$ and $N$ such that $M + N$ and $M^{-1} + N^{-1}$ are non-singular.

Using Assumption 8, it can be shown that both the second and third terms in (A.6) are $O_p(T^{-1})$. Therefore, the asymptotic behavior of $\lambda_1$ is determined as the minimizer of the first term in (A.6). The final limiting expression in (A.7) monotonically decreases as $\lambda_1$ increases toward $1 - \varepsilon^{\text{cos}}$ and thus $\lambda_1 \overset{p}{\to} 1 - \varepsilon < 1 - \varepsilon^{\text{cos}}$.

Similarly, it can be shown that

$$\frac{1}{T^2} SSR_r \Rightarrow \gamma^1 \left[ \left( \int_{0}^{1-\varepsilon^{\text{cos}}} BB' \right)^{-1} + \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right)^{-1} \right]^{-1} \gamma^1,$$

and thus

$$\frac{1}{T} \sup_{\lambda_1 \in A} Wald(\lambda_1) = \frac{SSR_r - SSR(\hat{\lambda}_1)}{SSR(\hat{\lambda}_1)} \Rightarrow \gamma^1 h(0) \gamma^1 - \gamma^1 h(1 - \varepsilon) \gamma^1,$$

where

$$h(s) = \left[ \left( \int_{s}^{1-\varepsilon^{\text{cos}}} BB' \right)^{-1} + \left( \int_{1-\varepsilon^{\text{cos}}}^{1} BB' \right)^{-1} \right]^{-1}.$$
Table 1: Detection of structural breaks using the sequential procedure in Bai and Perron (1998, 2003)

Panel A: The I(0) case

<table>
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<tr>
<th></th>
<th>T = 50</th>
<th>T = 100</th>
<th>T = 200</th>
<th>T = 300</th>
<th>T = 500</th>
<th>T = 1000</th>
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<tr>
<td>At least one break</td>
<td>0.128</td>
<td>0.138</td>
<td>0.129</td>
<td>0.201</td>
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<td>0.124</td>
<td>0.195</td>
<td>0.316</td>
<td>0.639</td>
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<td>J = 2</td>
<td>0.010</td>
<td>0.003</td>
<td>0.005</td>
<td>0.006</td>
<td>0.008</td>
<td>0.021</td>
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<tr>
<td>J &gt; 2</td>
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<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
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Panel B: The cointegration case

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<th>T = 300</th>
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<th>T = 1000</th>
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<tr>
<td>At least one break</td>
<td>0.303</td>
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<td>0.617</td>
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<td>J = 1</td>
<td>0.301</td>
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<td>0.558</td>
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<td>0.000</td>
<td>0.000</td>
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Table 2: Finite sample performance of the UDmax and sequential statistics in Bai and Perron (1998) and the P tests in Kim (2010)

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<th>(λ_{NEOS}, λ_{EOS})</th>
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<td></td>
<td>50</td>
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<td>(0, 0)</td>
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<td><strong>NEOS</strong></td>
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<tr>
<td>UDmax</td>
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<td>Seq (J &gt; 0)</td>
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</tr>
<tr>
<td>Seq (J = 1)</td>
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</tr>
<tr>
<td>Seq (J = 2)</td>
<td>0.007</td>
</tr>
<tr>
<td>Seq (J &gt; 2)</td>
<td>0.001</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td></td>
</tr>
<tr>
<td>P(\hat{\alpha}, \hat{\beta}_T)</td>
<td>0.029</td>
</tr>
<tr>
<td>P(\hat{\alpha}, I)</td>
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<td>(0.5, 0)</td>
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<td>Seq (J = 2)</td>
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<tr>
<td>Seq (J &gt; 2)</td>
<td>0.001</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td></td>
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<tr>
<td>P(\hat{\alpha}, \hat{\beta}_T)</td>
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</tr>
<tr>
<td>Seq (J = 1)</td>
<td>0.119</td>
</tr>
<tr>
<td>Seq (J = 2)</td>
<td>0.009</td>
</tr>
<tr>
<td>Seq (J &gt; 2)</td>
<td>0.000</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td></td>
</tr>
<tr>
<td>P(\hat{\alpha}, \hat{\beta}_T)</td>
<td>0.076</td>
</tr>
<tr>
<td>P(\hat{\alpha}, I)</td>
<td>0.132</td>
</tr>
<tr>
<td>(0.5, 0.95)</td>
<td></td>
</tr>
<tr>
<td><strong>NEOS</strong></td>
<td></td>
</tr>
<tr>
<td>UDmax</td>
<td>0.873</td>
</tr>
<tr>
<td>Seq (J &gt; 0)</td>
<td>0.849</td>
</tr>
<tr>
<td>Seq (J = 1)</td>
<td>0.783</td>
</tr>
<tr>
<td>Seq (J = 2)</td>
<td>0.065</td>
</tr>
<tr>
<td>Seq (J &gt; 2)</td>
<td>0.001</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td></td>
</tr>
<tr>
<td>P(\hat{\alpha}, \hat{\beta}_T)</td>
<td>0.037</td>
</tr>
<tr>
<td>P(\hat{\alpha}, I)</td>
<td>0.071</td>
</tr>
</tbody>
</table>
Table 3: Finite sample performance of the UDmax and sequential statistics in Kejriwal and Perron (2010) and the P test in Kim (2010)

<table>
<thead>
<tr>
<th>$(\lambda_{NEOS}, \lambda_{EOS})$</th>
<th>Sample size $(T)$</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NEOS</strong></td>
<td><strong>UDmax</strong></td>
<td>0.021</td>
<td>0.038</td>
<td>0.042</td>
<td>0.053</td>
<td>0.033</td>
<td>0.040</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 0)$</td>
<td>0.022</td>
<td>0.041</td>
<td>0.044</td>
<td>0.053</td>
<td>0.036</td>
<td>0.044</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 1)$</td>
<td>0.020</td>
<td>0.040</td>
<td>0.044</td>
<td>0.051</td>
<td>0.036</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 2)$</td>
<td>0.002</td>
<td>0.001</td>
<td>0.000</td>
<td>0.002</td>
<td>0.000</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 2)$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td>$P(\hat{\alpha}, \Xi_T)$</td>
<td>0.093</td>
<td>0.116</td>
<td>0.104</td>
<td>0.107</td>
<td>0.128</td>
<td>0.124</td>
</tr>
<tr>
<td>$(0.5, 0)$</td>
<td><strong>UDmax</strong></td>
<td>0.966</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 0)$</td>
<td>0.967</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 1)$</td>
<td>0.956</td>
<td>0.981</td>
<td>0.975</td>
<td>0.967</td>
<td>0.966</td>
<td>0.955</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 2)$</td>
<td>0.011</td>
<td>0.019</td>
<td>0.025</td>
<td>0.032</td>
<td>0.034</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 2)$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.003</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td>$P(\hat{\alpha}, \Xi_T)$</td>
<td>0.078</td>
<td>0.104</td>
<td>0.099</td>
<td>0.098</td>
<td>0.119</td>
<td>0.123</td>
</tr>
<tr>
<td>$(0, 0.95)$</td>
<td><strong>UDmax</strong></td>
<td>0.300</td>
<td>0.425</td>
<td>0.536</td>
<td>0.580</td>
<td>0.626</td>
<td>0.608</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 0)$</td>
<td>0.303</td>
<td>0.431</td>
<td>0.547</td>
<td>0.583</td>
<td>0.630</td>
<td>0.617</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 1)$</td>
<td>0.301</td>
<td>0.421</td>
<td>0.536</td>
<td>0.558</td>
<td>0.622</td>
<td>0.603</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 2)$</td>
<td>0.001</td>
<td>0.010</td>
<td>0.011</td>
<td>0.025</td>
<td>0.007</td>
<td>0.013</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 2)$</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td>$P(\hat{\alpha}, \Xi_T)$</td>
<td>0.744</td>
<td>0.902</td>
<td>0.955</td>
<td>0.983</td>
<td>0.993</td>
<td>0.996</td>
</tr>
<tr>
<td>$(0.5, 0.95)$</td>
<td><strong>UDmax</strong></td>
<td>0.701</td>
<td>0.636</td>
<td>0.659</td>
<td>0.667</td>
<td>0.649</td>
<td>0.672</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 0)$</td>
<td>0.707</td>
<td>0.638</td>
<td>0.662</td>
<td>0.667</td>
<td>0.652</td>
<td>0.675</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 1)$</td>
<td>0.626</td>
<td>0.287</td>
<td>0.224</td>
<td>0.181</td>
<td>0.138</td>
<td>0.165</td>
</tr>
<tr>
<td></td>
<td>$Seq (J = 2)$</td>
<td>0.080</td>
<td>0.340</td>
<td>0.410</td>
<td>0.469</td>
<td>0.486</td>
<td>0.482</td>
</tr>
<tr>
<td></td>
<td>$Seq (J &gt; 2)$</td>
<td>0.001</td>
<td>0.011</td>
<td>0.028</td>
<td>0.017</td>
<td>0.028</td>
<td>0.028</td>
</tr>
<tr>
<td><strong>EOS</strong></td>
<td>$P(\hat{\alpha}, \Xi_T)$</td>
<td>0.706</td>
<td>0.865</td>
<td>0.930</td>
<td>0.955</td>
<td>0.975</td>
<td>0.986</td>
</tr>
</tbody>
</table>

Figure 1: Asymptotic Relative Approximate Bahadur Efficiency
Figure 2: The I(0) case. First estimated break fraction

Figure 3: The I(0) case. Second estimated break fraction
Figure 4: The cointegration case. First estimated break fraction

Figure 5: The cointegration case. Second estimated break fraction
Figure 6: Densities of break fraction estimates. Univariate I(0) case for $T = 100$ (dotted line), $T = 200$ (dashed line) and $T = 300$ (solid line)
Figure 7: Densities of break fraction estimates. Cointegration case for $T = 100$ (dotted line), $T = 200$ (dashed line) and $T = 300$ (solid line)