Optimal Design and Quantitative Evaluation of the Minimum Wage *

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[PRELIMINARY AND INCOMPLETE]

Abstract

We analyze a labor market with random search and ex-ante firm heterogeneity featuring inefficient entry. Within this model, setting the minimum wage using a Vickery-Clarke-Groves mechanism delivers efficiency, by inducing the marginal firm to internalize the social cost of vacancy posting. In a dynamic stochastic model calibrated to the U.S. business cycle, a version of the mechanism with weak information requirements (the quasi-efficient mechanism) generates 18.9% gains in real income, which are 98.8% of the fully-efficient gains. The quasi-efficient mechanism is flexible over the business-cycle; by contrast, a constant minimum wage can generate up to 16.0% gains in real income.

Keywords: Minimum Wage Determination, Quantitative Evaluation, Vickrey-Clarke-Groves Auction

JEL Codes: J2, J3, J5

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1 Introduction

Few policies have more friends and foes than the minimum wage. A common view of minimum wage policy, dating at least back to Stigler (1946), is that it represents a trade-off between increased living standards of low paid workers and higher unemployment (see Flinn, 2006, and Lee and Saez, 2012, for recent investigation into the nature of this trade-off). By contrast, in this paper we analyse the minimum wage as a policy that prevents excessive entry from low productivity firms, and propose a market-based mechanism to find the efficient minimum wage. To this end, we build a version of the Diamond-Mortensen-Pissareides model (after Diamond, 1982; Pissarides, 1985; and Mortensen and Pissarides, 1994; henceforth referred as DMP) with firms that are ex-ante heterogeneous in productivity, and show that the efficient minimum wage can be set using a slightly modified version of the Vickrey-Clarke-Groves mechanism (after Vickrey, 1961; Clarke, 1971; and Groves, 1973; henceforth referred as VCG). The idea behind the VCG mechanism is to charge each bidder of an auction the cost they impose to their competitors, while ensuring that bidders truthfully report their privately-known valuations. Similarly, in the DMP environment, the VCG mechanism charges each firm (at least) the cost they impose on other firms — paid as a wage to the worker — while ensuring that firms truthfully report their privately-known productivities. The cost imposed on other firms is the congestion externality that yields excessive entry in the first place and, by turning it into the minimum wage, the mechanism ensures efficiency. A feature of the VCG mechanism is that it does not require the public sector to have much information, which we believe is valuable for the purpose of policy setting.

We next extend the framework to a dynamic environment with shocks but find that the mechanism relies too directly on individual reports for truth-telling to hold. Absent shocks though, the information requirement of the mechanism is similar to the static case. We then propose a quasi-efficient mechanism to set the minimum wage in the environment with shocks: implementing, period-by-period, the minimum wage that would deliver efficiency if the economy was in a steady state. We then calibrate the model to the U.S. economy and conduct several policy experiments to assess the value of setting the minimum wage using the quasi-efficient mechanism. We find that its implementation would generate 18.9% gains in real income. These gains already represent 98.8% of the gains that would be achieved in the fully-efficient allocation. A peculiar characteristic of the minimum wage in the data is its rigidity. By contrast, the quasi-efficient mechanism implies a minimum wage that responds to business cycle movements. We find that the largest gains that can be attained with a constant, fixed minimum wage are 16.0%, implying that the gains from the quasi-efficient mechanism being flexible and adjusting to the business cycle are 18.1%.
Our model of the labor market is a version of the DMP model where ex-ante heterogeneous firms post costly vacancies and are randomly matched with homogeneous workers. In this environment, vacancy creation generates a congestion externality: a vacancy of a low productivity firm decreases the probability of high productivity firm matching with a worker.\footnote{Julien and Mangin (2016) refer to this as the \textit{output externality}.} By not internalizing these costs, there are too many low productivity firms in the decentralized market.\footnote{See Albrecht et al. (2010), for a similar finding in the case of worker heterogeneity, and Julien and Mangin (2017) for a modified Hosios condition that makes this type of model be efficient.} A minimum wage prevents these firms from operating, as it forces firms to internalize the externality they generate when creating a vacancy. But not any minimum wage will do it; it must be set to equate the social marginal benefit and cost of vacancy creation. We show that a mechanism based on the VCG mechanisms delivers this efficient minimum wage.

The mechanism we propose works as follows: first, all firms may choose to create vacancies and unemployed workers may choose to search. A random matching process generates meetings between searching workers and vacancy-posting firms. For any meeting between a firm and a worker, the firm submits the maximum willingness to pay for that match to the Central Clearing House (CCH). With all the submitted reports, the CCH computes the externality that each firm represents to the overall economy as measured by the VCG price. This price is computed as the difference in the expected match value of all other firms, depending on whether or not a given firm posts a vacancy. After computing the VCG price, the CCH imposes this price as the minimum wage. Forecasting the outcome of this mechanism, no firm with productivity below the efficient threshold will choose to search and efficiency obtains.

In a static framework, the only objects needed to implement the mechanism are knowledge of the matching technology and the average of match values. We assume that the value of matches are private information of each firm; hence, the use of the VCG mechanism to induce truthful reporting. Implementation of the mechanism in a dynamic, stationary environment is not much more informationally demanding (only estimates of the discount factor and the destruction rate of jobs in the economy are needed). In a dynamic, stochastic environment the mechanism requires forming expectations about future unemployment and average firm productivity — objects that could be easily computed. However, the information requirement also includes forming expectations of what is the wage that the marginal firm in the current period will pay going forward. Forming these expectations requires two pieces: first, knowing the wage protocol, and second, using the self-reported productivity of the lowest-productivity firm. Unfortunately, this very last requirement implies that truthful
reporting is not necessarily holding anymore, as the lowest-productivity firm could make use of her own report to affect the minimum wage. Hence, we investigate an alternative to the minimum wage setting that does not have the truthful reporting problem. In particular, since the stationary version of the mechanism does not require forming expectations of the wage paid by the marginal firm going forward (it assumes it is constant, and equal to the minimum wage), we propose using the version of the mechanism of the stationary environment into the dynamic environment with shocks. In it, the CCH behaves myopically, acting as if the the current level of unemployment, aggregate shock and distribution of productivities were to be constant going forward. We show that using this quasi-efficient mechanism, even though it is ad-hoc, already delivers large real income gains.

In order to quantify the magnitude of the gains that our mechanism generates, we calibrate a dynamic stochastic version of our model to the U.S. business cycle. Given the resemblance between our model and Hagedorn and Manovskii (2008), we follow a similar calibration strategy, matching a group of $t$ features of the U.S. labor market. The main difference between our model and Hagedorn and Manovskii (2008) is that, in our case, there is heterogeneity in firm productivity, which implies that there is a distribution of wages every period. We calibrate the distribution of firm productivity to match the wage distribution in the U.S.

We compute the outcome of the economy with the current minimum wage and compare it to the outcome that the same economy would have if the minimum wage was set using the quasi-efficient mechanism. We find that real income attained from using the quasi-efficient mechanism is 18.9% larger. These gains represent 98.8% of the total gains that would be attained if the economy could implement the fully efficient allocation. These gains are driven in part by the minimum wage inducing higher efficiency firms being present, and in part by the flexibility the minimum wage has to adjust to the business cycle. In order to disentangle the two forces, we compute what is the largest real income that can be attained with a constant minimum wage, and find that it would increase real income by 16%, which implies that the gains from flexibility to the business cycle are 15%.

**Literature review**

Our results hinge on the assumption that there is excessive entry of firms in the market. The well-known Hosios condition (Hosios, 1990) establishes that the market outcome is efficient (that is, there is no excessive entry) if the share that workers capture from the bargaining, $\alpha$, is equal to one minus the elasticity of the matching function, $m(u, v)$, with respect to job postings, $v$ (with $u$ being the number of job seekers). If, instead, $\alpha < 1 - m_u(u, v) \times v/m(u, v)$, the labor market has excessively high firm entry. Unfortunately for us, it is common practice in the literature to pick parameters that satisfy the Hosios condition.
because calibrating $\alpha$ is typically not crucial.\textsuperscript{3} However, Hagedorn and Manovskii (2008), who develop a model similar to ours, calibrate their model and find that the empirically relevant case is when there is excessive firm entry.\textsuperscript{4} Another example is Lagos (2006), who, even though in the benchmark calibration of the model imposes the Hosios condition to hold, performs an alternative calibration targeting $\alpha$ and also finds that excessive entry is the empirically relevant case.\textsuperscript{5}

This is not the first paper to think of the minimum wage as an efficiency enhancing policy. Swinnerton (1996), building on the model of Albrecht and Axell (1984), is the first to formalize the idea that a binding minimum wage can prevent the least productive firms from operating on the market. The difference between his work and ours, besides technical issues — we build on the DMP model and analyse the economy in a dynamic set-up with shocks — is that we propose a mechanism to find the optimal minimum wage and perform a quantitative exercise to show how large the real income gains of pursuing this policy can be. Other papers have explored the role that the minimum wage can play as a tool to solve information imperfection in the labor market: Drazen (1986) analyses the role that the minimum wage plays when there is adverse selection, and Lang (1987) analyses the role of the minimum wage when wages are used for signalling. van den Berg (2003) shows how using the minimum wage can enhance efficiency when wages are used as an equilibrium selection device. Galenianos et al. (2011) also study the role that the minimum wage has in a directed search model with perfect information and heterogeneous firms, and find that it can exacerbate, not improve, inefficiencies. Their result hinges on the assumption that there is a finite number of firms in the market that have market power, which imply inefficiently few firms in the market. Hence, forcing some firms out worsens the situation. Instead, they show that moderate unemployment benefits increase welfare in their set-up. By comparison, in our model there is an excess of firms, and the minimum wage gets rid of the less productive

\textsuperscript{3}See, for example, Shimer (2005), Pries and Rogerson (2005), and Pissarides (2009) as examples where the Hosios condition is assumed, but the list is far from exhaustive.

\textsuperscript{4}Hagedorn and Manovskii (2008) choose the matching function to be $m(u, v) = uv/(u^l + v^l)^{1/l}$, and calibrate the bargaining parameter as well as the parameter governing the matching function. The matching function implies that $1 - m_v(u, v) \times v/m(u, v) = (1 + \theta^l)^{-1}$, where $\theta = v/u$ is the market tightness. The value for market tightness that they use is 0.634, which they find by combining estimates of Shimer (2005) on the monthly job finding rate, and of den Haan et al. (2000) on the job filling rate. The calibrated parameter of the matching function is $l = 0.407$, and the worker’s bargaining parameter is $\alpha = 0.052$. Hence, $(1 + \theta^l)^{-1} = 0.546$, a number larger than $\alpha$, which means that excessive entry of firms is the relevant case for their model. They also mention that Hall (2005) obtains a $\theta = 0.539$ from direct estimates of the Job Opening and Labor Turnover Survey. Using this number, $(1 + \theta^l)^{-1} = 0.563$, which again implies excessive entry.

\textsuperscript{5}Lagos (2006) sets $1 - m_v(u, v) \times v/m(u, v) = 0.72$ following Petrongolo and Pissarides (2001), and imposes the Hosios condition to hold ($\alpha = 0.72$) in the benchmark calibration. However, in a robustness exercise meant to get the income share of workers right while keeping the value of $1 - m_v(u, v) \times v/m(u, v) = 0.72$ (Appendix B of his paper), he finds that $\alpha = 0.1$, implying that there is excessive entry of firms as well.
ones. Moreover, we show how to introduce a mechanism to set the efficient minimum wage. We believe that using a mechanism to achieve efficiency in a model built in the spirit of theirs — which should come from setting unemployment benefits optimally — can be a fruitful avenue of future research.

Other papers have explored the role of the minimum wage as a tool that generates a trade-off between increased living standards of low paid workers and higher unemployment. For example, Flinn (2006) shows that a minimum wage increase can be welfare-improving to labor market participants in both the supply and demand sides of the labor market. Lee and Saez (2012) shows that a minimum wage, even though it leads to unemployment, is desirable if the government values redistribution, and that, accompanied by subsides for low-skilled workers, generates Pareto improving allocations. Other examples of this type of analysis include Boadway and Cuff (2001) and Gorostiaga and Rubio-Ramírez (2007).

The reminder of the paper is organized as follows: in section 2 we document the evolution of the minimum wage and how it relates to macroeconomic variables; in section 3, we parsimoniously build the static model and show that the VCG mechanism induces the efficient allocation; in section 4 we extend the model into a dynamic, stochastic set-up and propose the quasi-efficient mechanism; in section 5 we show how to calibrate the model to the U.S. labor market; in section 6 we develop our quantitative exercises; finally, section 7 concludes.

2 Minimum wage in the United States

In the United States, the minimum wage is set by a Federal law approved by Congress, which creates a lack of flexibility for the minimum wage to adjust to economic fluctuations. Since 2009, the minimum wage is $7.25, having been increased to $5.85 in 2007, and to $6.55 in 2008, within the same law change. Between 1982 and 2017, the Federal minimum wage law has changed two other times, first in 1990 (with gradual increments in 1990 and 1991), and then in 1996 (with gradual increments in 1996 and 1997). The blue line in Figure 1 shows the evolution of the average nominal minimum wage for the U.S. economy, and the red line shows the evolution of the federal nominal minimum wage. Clearly, the evolution of the nominal minimum wage is sluggish.

Moreover, in Figure 2 we show the evolution of the real minimum wage. The picture
shows that the real minimum wage only increases with each new legislation, and that it decreases during the other periods, by effect of inflation. By contrast, the minimum wage we propose in this paper adjusts weekly. The coefficient of variation of the real minimum wage in the last 30 years is XX, whereas using the coefficient of variation using the mechanism we propose in the paper is much larger, YY. As we show in the quantitative part of the paper, increasing flexibility of the minimum wage generates gains of 18.9% in real income vis-à-vis 16% that would be generated with the fixed minimum wage that generated the largest increase in real income.

The minimum wage is somewhat binding in the U.S. In Figure 3 we plot the fraction of workers receiving the minimum wage. After a rapid fall during the 1980’s, where no adjustment of the minimum wage took place, for the following decades, the fraction of people making exactly the minimum wage has adjusted upward every time the new minimum wage has been introduced, and it has decreased later on as the real minimum wage started to decrease. The average fraction of the workforce at the minimum wage between 1990 and 2015 is 4.82%. By comparison, with our mechanism, we find that the average fraction of workforce at the minimum wage is MM%. The reason for this difference is that the minimum wage in our framework is much more binding. In fact, the average minimum wage that our mechanism implies is NN$, compared to the MM$ effectively paid.
Figure 2:

US Minimum Wage in Real Terms

Figure 3:

Minimum Wage Earners in the US
3 The VCG mechanism in the labor market

The rational behind the Vickrey-Clarke-Groves (VCG) mechanism is to compare the value generated when a “bidder” enters an auction against when she does not. If the difference in value is large enough, the bidder should purchase the good; if it is not, the bidder should not purchase it. In this section we show how this logic can be applied to the labor market to set an optimal minimum wage.

We start building a simple model with three heterogeneous firms, two homogeneous workers and no frictions. Then, we show that adding search frictions in the model may generate an inefficient allocation, but that setting the minimum wage by means of the VCG mechanism restores efficiency. Then, we add elements to the model, namely outside options for workers, many firms, many workers and search costs, and show that the result of efficiency attained when using the VCG mechanisms remains. Last, we formally define the mechanism in the static framework, and show how it achieves efficiency.

3.1 Three heterogeneous firms, two workers and no frictions

Suppose there are three firms with productivities $p_1 > p_2 > p_3$ and two homogeneous workers. Furthermore, start assuming there is no friction in the market — that is, each worker gets matched with the best available firm. In this environment, an equilibrium consists of firms 1 and 2 getting matched each with a worker, and firm 3 not getting matched. Moreover, the equilibrium is efficient.

Now, consider the value that each firm generates in the market by participating, and compare that value with what would be generated if the firm chose not to participate. Start with firm 1. If she does not participate, firms 2 and 3 get matched each with one worker, and the resulting value is $p_2 + p_3$. If, instead, firm 1 participates, the value generated by the other firms is $p_2$ (again only firms 1 and 2 get matched with a worker). Comparing the two, we get that the value of firm 1’s participation to other firms is $W_1 = (p_2 + p_3) - p_2 = p_3$. Similarly, the value of firm 2’s participation on other firms is $W_2 = p_3$. For firm 3 the story is different. If she does not participate, the value generated by other firms is $p_1 + p_2$; if she participates, the value generated is also $p_1 + p_2$ (no worker gets matched with firm 3 if she participates). Hence, the value of firm 3’s participation on other firms is $W_3 = (p_1 + p_2) - (p_1 + p_2) = 0$.

Let $\hat{w} = \max\{W_i\}_{i=1,2,3} = p_3$ be the maximum value generated by the participation of a firm to other firms, and make each firm to pay $\hat{w}$ to participate in the market. Then, only firms 1 and 2 will participate, but not firm 3. Hence, setting $\hat{w}$ as the cost to participate — later on we will make this cost as the minimum wage — induces efficiency in the market. In this example, a policy that makes firms pay $\hat{w}$ is not necessary, because the equilibrium
without the policy is already efficient. As we shall see next, though, adding search frictions makes the equilibrium without the policy not efficient, but a policy that makes firms pay $\hat{w}$ induces efficiency.

Last, note that when determining each $W_i$, we do not use information of the productivity of firm $i$ ($p_i$ does not show up in the calculation). This feature is key to ensure that firms’ reporting of $p_i$ is truth-telling.

### 3.2 Adding search frictions: rationale for a minimum wage

Suppose we introduce a friction in the previous economy: matches happen with matching technology $M(m, n)$, where $m$ is the number of workers (always 2), and $n$ is the number of firms (2 if a firm does not participate, 3 if she does).

Following the logic from before, the value that is created if firm 1 does not participate in the market is $M(2, 2)/2 \times (p_2 + p_3)$, which is the probability that a match happens, multiplied by the value of all matches. Likewise, the value generated by other firms if firm 1 participates is $M(2, 3)/3 \times (p_2 + p_3)$. Note that the existence of a matching technology radically changes the expression for the value generated with and without participation. Before, participation of firm 1 removed the productivity of firm 3. Now, the productivity of firm 3 still appears even if firm 1 participates. Instead, participation of firm 1 changes the probability that a firm has to get matched with a worker.

We now focus our attention on firm 3, because in our previous example, efficiency obtained if firm 3 did not participate. If firm 3 does not participate, the value generated is $M(2, 2)/2 \times (p_1 + p_2)$; if she participates, the value generated by other firms is $M(2, 3)/3 \times (p_1 + p_2)$.

Hence, the value of firm 3 participation on other firms is

$$W_3 = (M(2, 2)/2 - M(2, 3)/3) \times (p_1 + p_2).$$

Let $w_3$ refer to the wage that firm 3 pays to the worker. Then, firm 3’s expected profits from participating are

$$E(\pi(p_3)) = M(2, 3)/3(p_3 - w_3) > 0.$$  

From an efficiency standpoint, firm 3 should only participate if the social value generated by her participation, $Z_3$, is large enough, i.e.

$$Z_3 = M(2, 3) \times \frac{p_1 + p_2 + p_3}{3} - M(2, 2) \times \frac{p_1 + p_2}{2} > 0.$$  

Note that even though firm 3 has the smallest productivity, in this case it is not necessarily true that she should not participate: under some parametrizations $Z_3$ may be positive.
(p_1 + p_2 + p_3)/3 < (p_1 + p_2)/2, but M(2, 3) > M(2, 3). Moreover, even if firm 3’s participation in the market is inefficient, she still decides to participate because her expected profits are positive. In order to restore efficiency in this market, we simply need to ensure that \( E(\pi(p_3)) = Z_3 \), that is, we need to align the incentives of firm 3 with social gains. Note that if we force \( w_3 \) to be

\[
w_3 = \frac{3}{M(2, 3)} \times \left( \frac{M(2, 2)}{2} - \frac{M(2, 3)}{3} \right) \times (p_1 + p_2) \equiv w_3^{VCG},
\]

then the equality is satisfied. Moreover, the expression in equation (1) exactly coincides with the VCG price in this environment. The expressions for the VCG price of firms 1 and 2 are very similar, with the only difference being the sum of other firms’ productivities at the right part of the expression. Since the sum of other firms’ productivities is highest for the case of firm 3, then we set the minimum wage to be \( \hat{w} = w_3^{VCG} \), and efficiency is achieved. In order to compute the VCG price, \( w_i^{VCG} \), we do not use information on \( p_i \), so no firm has the incentive to lie about their true productivity match.

As we shall see next, the logic behind this argument stays unchanged when we add outside options, search costs, and make the number of firms and workers be arbitrarily large.

### 3.3 Adding outside options

When we add workers having outside options, we need to take into account that if a worker is not matched, she still receives a flow value of non-market activity \( b \). One way of thinking about this element is that the outside option changes the problem from before by adding an “non-market firm” ready to hire the two workers in case that they do not get matched with a firm — if they do so, they create value \( p_{nm} = b \). Hence, an outside option includes the value created when vacancies are not filled. Returning to firm 3, in this environment the baseline value created if she does not participate is

\[
\frac{M(2, 2)}{2}(p_1 + p_2) + (2 - M(2, 2))b = M(2, 2) \left( \frac{p_1 + p_2}{2} - b \right) + 2b.
\]

Similarly, the value of other firms if firm 3 participates is given by

\[
M(2, 3) \left( \frac{p_1 + p_2}{3} - b \right) + 2b.
\]
Combining the two expressions from before, we get that the VCG price for firm 3 is given by
\[
    w_{3}^{\text{VCG}} = \frac{3}{M(2, 3)} \times \left( M(2, 2) \left( \frac{p_1 + p_2}{2} - b \right) - M(2, 3) \left( \frac{p_1 + p_2}{3} - b \right) \right),
\]
which, for the case of \( b = 0 \), gives exactly equations (1). The expression for the VCG price for firms 1 and 2 are the same as equation (2) up to appropriate changes.

As before, there is no reason why \( p_3 \) should be higher or lower than \( w_3^{\text{VCG}} \), and forcing firm 3 to pay at least \( w_3^{\text{VCG}} \) is a sufficient condition to guarantee efficiency in the model.

### 3.4 Adding many firms, many workers, and search costs

We now generalize the previous economy by having \( m \) workers, with a flow utility from non-market activities \( b \), and \( n \) firms, with productivities ordered according to \( p_1 > p_2 > ... > p_n > b \). When a firm posts a job offer, she incurs search cost \( s \). As before, there is a matching function, \( M(m, n) \), that determines how many matches happen between \( m \) firms posting and \( n \) workers searching.

The value generated by other firms if firm \( i \) does not participate is
\[
    \frac{M(m, n - 1)}{n - 1} \left( \sum_{j \neq i} p_j \right) + (m - M(m, n - 1))b,
\]
and the value generated by other firms if firm \( i \) decides to participate is
\[
    \frac{M(m, n)}{n} \left( \sum_{j \neq i} p_j \right) + (m - M(m, n))b.
\]

Again, combining both expressions we can compute the VCG price, which is
\[
    w_i^{\text{VCG}} = \frac{n}{M(m, n)} \left( M(m, n - 1) \left\{ \frac{1}{n - 1} \sum_{j \neq i} p_j - b \right\} - M(m, n) \left\{ \frac{1}{n} \sum_{j \neq i} p_j - b \right\} \right). \tag{3}
\]
Note that equation (3) is a direct generalization of equation (2) for the case of many firms and workers. Note that the job posting cost does not appear in the expression. As before, firm \( i \) cannot affect \( w_i^{\text{VCG}} \), implying, again, that setting the minimum wage according to \( \hat{w} = \max\{w_i^{\text{VCG}}\} \) induces truth-telling in this environment.
3.5 Efficiency with the minimum wage equal to the VCG price

Next, we show that setting minimum wage $\hat{w}$ is sufficient to achieve efficiency. We do so in two steps: first, we derive the equation that determines the marginal firm that maximizes the social planner’s problem; second, we show that, if the minimum wage is the one given in equation (3), the marginal firm from the market economy coincides with the one of chosen by the social planner.

The social planner wants to maximize net output. The function that determines the amount of output produced if the $x$-highest firms participate in the market is

$$\Omega_x = M(m, x) \left( \frac{1}{x} \sum_{j=1}^{j=x} p_j - b \right) - xs,$$

where the first term is the number of matches that occur multiplied by the average gain per match and the second term is the cost incurred by the $x$ firms searching. We assume that the matching function and the set of firm productivities are such that i) $\Omega_{x+1} - \Omega_x$ is initially positive and monotonically decreasing, and ii) the $(x^*)^{th}$ largest firm, with $x^* < n$, has productivity $p_{x^*}$ that satisfies $\Omega_{x^*} - \Omega_{x^*-1} = s$. With this information, we know that the social planner chooses to have the largest firms producing, and the last firm chosen to enter the market is exactly $x^*$.

In the market economy, firm $i$’s profits are equal to the probability of getting matched times the surplus made after paying a wage, net of search costs. Namely,

$$\pi(p_i) = \frac{M(m, x^{ZP})}{x^{ZP}} (p_i - w_i) - s,$$

where $x^{ZP}$ is the mass of firms operating in the economy, with $ZP$ denoting that the mass depends on what is the firm that makes zero profits, and $w_i$ is the salary that the firm pays to the worker. Consider the firm that makes zero profits when the minimum wage is determined by the VCG price, $\hat{w} = \max\{w_i^{VCG}\}$, and $w_i^{VCG}$ is determined by equation (3). Using the expression for profits, equation (5), and setting it equal to zero, we get that the
firm that makes zero profits has productivity $p_{ZP}$ that satisfies

$$\pi(p_{ZP}) = 0 \iff s = \frac{M(m, x_{ZP})}{x_{ZP}}(p_{ZP} - \hat{w}) \iff$$

$$s = -M(m, x_{ZP} - 1) \left\{ \frac{1}{x_{ZP} - 1} \sum_j p_j - b \right\} + M(m, x_{ZP}) \left\{ \frac{1}{x_{ZP}} \left( \sum_j p_j + p_{ZP} \right) - b \right\}$$

$$\iff s = \Omega_{ZP} - \Omega_{ZP-1}. \quad (6)$$

Since the marginal firm of the planner satisfies $\Omega_x - \Omega_{x-1} = s$, and given that $\Omega_x - \Omega_{x-1}$ is monotonically decreasing, firm in position $x^*$ is also firm in position $x_{ZP}$. Hence, using the minimum wage implied from the VCG mechanism, equation (3) implements the efficient allocation in this richer environment.

### 3.6 Continuum of firms and workers, and the mechanism

Assume now that the labor market has a mass $m < 1$ of workers, and a mass $n = 1$ of firms. Firms are heterogeneous in their productivities $p$, with $p \in [\underline{p}, \overline{p}] = \mathcal{P}$, and distributed according to $J(p)$. As before, $p > b$.

The mechanism works in two stages. In the first stage, all firms and workers who choose to may search. In the second, after search has occurred, all matched firms report the valuation of their vacancy to the Central Clearing House (CCH), which then sets a minimum wage based on the distribution of reports. Note that we assume firm productivities and search costs are unobservable to the CCH. We do however assume that the outside option $b$ and matching technology $M$ are publicly known.

Let $S(p) = (\delta(p), \gamma(p)) \in \mathcal{S}$ denote a strategy profile of firm with productivity $p$, which consists of whether to search ($\delta(p) = 1$) or not ($\delta(p) = 0$), and the valuation that the firm reports to the CCH, $\gamma(p)$. Note that the total mass of searching firms be $n = \int \delta(x)dJ(x)$.

Next, we show how the VCG price from equation (3) extends to the continuous case case. We do so in the following Lemma.

**Lemma 1** The VCG price when there is a continuum of bidders is given by

$$w^{VCG} = \int \delta(x)\gamma(x)dJ(x, S) - M_2(m, n) \frac{n}{M(m, n)} \left[ \int \delta(x)\gamma(x)dJ(x, S) - b \right] \quad (7)$$

**Proof.** See Appendix A.1. ■

The mechanism, $\Gamma = (S, t)$, amalgamates all the strategy profiles $S$, and establishes what is the payment function $t : R_+ \times \mathcal{S} \to R_+$ that each strategy profile has to follow. For any
two identical bids \( \gamma \), the payment is exactly the same. Hence, payments are independent of the (unobservable) productivity \( p \) of the firm making it. Moreover, the payment function is only defined over submitted bids. The payment function in our mechanism consists of making firms pay the maximum between the bargained wage that comes from productivity \( p \), and the VCG price that the firm should pay. Formally, the mechanism is defined as follows:

**Definition 2** The mechanism \( \Gamma \) is a strategy profile \( S \) and payment function, \( t : R_+ \times \mathcal{F} \rightarrow R_+ \), such that \( t(\gamma, S) = \max\{w^{VCG}, w(p)\} \), for all \( \gamma \), where \( w^{VCG} \) is given by equation (7), and \( w(p) \) denotes the wage outcome of bargaining with the worker.

The mechanism works by making the cost of participating in the labor market at least as large as the social cost of having the firm participate — which is a re-interpretation of the VCG intuition. Moreover, the mechanism generates truthful reports by firms because outcomes depend on averages rather than individual reports. Let

\[
\pi(p, S) = \delta(p) \left\{ \frac{M(m, n)}{n} [p - t(\gamma; S)] - s \right\}
\]

be the profits that firm with productivity \( p \) attains by choosing the truth-telling strategy profile \( S \). Then, consider any other strategy profile \( \tilde{S}_p = (\hat{\delta}(p), \hat{\gamma}(p)) \). An equilibrium induced by the mechanism \( \Gamma \) has to satisfy that the profits from firm \( p \) will be at least as large by choosing the truth-telling strategy as by choosing any other strategy. Formally, \( \pi(p, S) \geq \pi(p, \tilde{S}_p) \) for all \( \tilde{S}_p, p \in \mathcal{P} \).

The next step consists of establishing that the mechanism we propose induces the truth-telling equilibrium that coincides with the social planner solution. We do so in the following theorem.

**Theorem 3** The mechanism \( \Gamma \) truthfully implements the planners solution to the static search model.

**Proof.** See Appendix A.2. \( \blacksquare \)

We have now established that introducing a mechanism along the lines of the one we have defined in this section allows the labor market to operate efficiently. The next step in the analysis is to show how this mechanism operates in a dynamic framework. We do so in the next section.
4 Infinite Horizon

In this section we extend the model to a dynamic, infinite horizon environment, and show how to embed the mechanism we developed to set the optimal minimum wage in it. Since the goal is to make a quantitative analysis, the dynamic version of the model in this section is written in recursive form.

4.1 Decentralized economy

First we describe how firms and workers operate in the decentralized economy. The timing of the model is the following: upon each period beginning i) the aggregate productivity shock $z$ is revealed; ii) existing matches produce; iii) a fraction $\delta$ of existing matches are destroyed, which make workers enter unemployment; iv) $m$ workers and $n$ firms search; finally, v) $M(m,n)$ new matches are formed. All workers must wait one period to search after a job destruction episode, and time is discounted at rate $\beta$. As before, the model consists of workers searching, firms posting job offers, and matches happening at random. Upon matching, firm and worker engage in Nash-bargaining to determine wages. Last, the government also introduces a minimum wage, $\hat{w}$, that may be a function of the aggregate productivity shock, $z$, unemployment, $u$, and average firm productivity, $\hat{p}$. As we shall see in the workers’ problem, they have a reservation wage, $w_r$, below which they do not accept an offer. Then, the wage that determines a lower bound on the wage that firms may offer, $\bar{w}(z,u,\hat{p})$, is given by the maximum between the minimum wage, $\hat{w}$, and the reservation wage, $w_r$.

There is a mass 1 of workers in the economy. A worker decides whether to work or to remain unemployed. The value of being unemployed, $U$, is the same for all workers, while the value of working, $V(w)$, depends on the wage, $w$, that the firm pays. The decision on whether or not to work is given by $\hat{V}(w) = \max\{V(w), U\}$, which implies that the reservation wage solves the implicit equation $V(w_r) = U$.

Given a distribution of wages, $F$, and contact rate between firms and workers, $\lambda$, the value of unemployment, $U$, can be expressed as

$$U = b + \beta E \left[ \lambda \int_{w'} V'(x)dF(x) + (1 - \lambda)U' \right], \quad (8)$$

where $'$ denotes the variable is that of the following period. Similarly, the value of accepting a job with wage $w$ can be expressed as

$$V(w) = w + \beta E[\delta U' + (1 - \delta)V(w')]. \quad (9)$$
Every period there is a mass of heterogeneous firms, whose productivities $p \in [\underline{p}, \bar{p}]$ are distributed according to $J(p)$, that choose whether to pay search cost $s$ and get matched with a worker the following period, or not. The entry condition, which says that the value of the firm from entry has to cover search costs, $s$, is

$$\frac{M(m, n)}{n} \beta \Pi(z, w \geq \bar{w}(z, \hat{p}, u), p) \geq s.$$  \hspace{1cm} (10)$$

where $\Pi(z, w \geq \bar{w}(z, \hat{p}, u), p)$ depends on the firm’s output, which is a combination of the aggregate, time-varying shock, $z$, and the idiosyncratic, time-invariant productivity $p$, the wage paid, $w$, and the continuation value. Namely,

$$\Pi(z, w \geq \bar{w}(z, \hat{p}, u), p) = E \left[ z + p - w + \beta(1 - \delta)\Pi(z', w' \geq \bar{w}(z', \hat{p}', u'), p) \right].$$  \hspace{1cm} (11)$$

With the information on how firms and workers behave in the economy, we are ready to define an equilibrium in this economy. We do so next,

**Definition 4** Given a minimum wage function $\bar{w}(z, \hat{p}, u)$, an aggregate productivity shock $z$, the unemployment level $u$, and average productivity $\hat{p}$, a recursive equilibrium in this economy consists of a cut-off productivity of firms entering in the market, $\{p^*\}$, a reservation wage, $w^r$, mass of workers seeking a job, $n$, mass of firms posting job offers, $m$, and wages paid by firms, $\{w(p)\}_{p \in [\underline{p}, \bar{p}]}$, such that:

- i) workers choose whether to remain unemployed or working solving the problem $\hat{V}(w) = \max \{V(w), U\}$ where $U$ is characterized by equation (8) and $V(w)$ is characterized by equation (9);
- ii) reservation wage $w^r$ solves $V(w^r) = U$;
- iii) wages satisfy that $\min w(p) \geq \hat{w}(z, \hat{p}, u)$, and all unemployed workers search, $n = u$;
- iv) the cut-off firm, $p^*$, is characterized by equation (10) holding with equality, making

$$m = 1 - J(p^*) \equiv \bar{J}(p^*)$$

firms search;
- v) firms’ value function is characterized by equation (11);
- vi) next period productivity shock is characterized by

$$\log(z') = \mu \log(z) + \epsilon, \text{ where } \epsilon \sim N(0, \sigma);$$
• vii) next period unemployment is given by

\[ u' = u - M(u, \bar{J}(p^*)) + \delta(1 - u); \]

• viii) next period’s average productivity is given by

\[ \hat{p}' = \hat{p}(1 - \delta)(1 - u) + M(u, \bar{J}(p^*))E(p|p > p^*) \]

\[ \frac{1 - \delta}{(1 - \delta)(1 - u) + M(u, \bar{J}(p^*))}. \]

With the definition of equilibrium we can characterize the cut-off productivity firm. Note that conditional on making an offer with a wage large enough that it gets accepted, the profits of a firm, equation (11), can be conveniently re-written as

\[ \Pi(z, w \geq \hat{w}(z, \hat{p}, u), p) = \left( \frac{p^*}{1 - \beta(1 - \delta)} + E \left[ \sum_{s=0}^{\infty} (\beta(1 - \delta))^{s} (z^{(s)} - w^{(s)}) \right] \right), \tag{12} \]

where \( x^{(s)} \) denotes the value of variable \( x \), \( s \) periods ahead. Combining this expression with the equation for the marginal firm entering the market, equation (10), the latter holding with equality, we get that the cut-off productivity is characterized by

\[ \frac{M(u, \bar{J}(p^*))}{\bar{J}(p^*)} \beta \left( \frac{p^*}{1 - \beta(1 - \delta)} + E \left[ \sum_{s=0}^{\infty} (\beta(1 - \delta))^{s} (z^{(s)} - w^{(s)}) \right] \right) = s. \tag{13} \]

Equation (13) is the market equilibrium cut-off and it can be compared to the optimal optimal cut-off productivity — characterized by the social planner problem — to see that the market equilibrium in this economy is inefficient. More importantly, the market cut-off and the optimal cut-off expressions can be combined to determine the optimal minimum wage in this economy. We do it next.

### 4.2 Social Planner problem

Given that agents have linear utility, the social planner chooses to maximize the present value of net output. A given period’s net output consists of all the output produced by existing firms minus the search cost of all firms looking for a worker, plus the unemployment flow from non-market activities times the amount of unemployed workers. Then, the present value of net output is simply given by
\[ \Omega(u, \hat{p}, z) = \max_{p^*} \left\{ (z + \hat{p})(1 - u) - s\bar{J}(p^*) + ub + \beta\Omega(u', \hat{p}', z') \right\} \]

subject to: \[ u' = u - M(u, \bar{J}(p^*)) + \delta(1 - u), \]

\[ \hat{p}' = \frac{\hat{p}(1 - \delta)(1 - u) + M(u, \bar{J}(p^*))E(p|p > p^*)}{(1 - \delta)(1 - u) + M(u, \bar{J}(p^*))}, \]

\[ \log(z') = \mu \log(z) + \epsilon, \text{ where } \epsilon \sim N(0, \sigma), \]

where

\[ J(p^*) = \int_{p^*}^{\hat{p}} dJ(x), \]

and

\[ E(p|p > p^*) = \int_{p^*}^{\hat{p}} xdJ(x) / \int_{p^*}^{\hat{p}} dJ(x). \]

With this problem, we can characterize the level of optimal entry that the social planner chooses to have, given by cut-off \( p^* \). We do so in the following Lemma.

**Lemma 5** Optimal entry in the market is given by

\[ \beta M_2(u, \bar{J}(p^*)) \left( z' + \hat{p}' - b + \beta \frac{\Delta_p^{(2)}(1 - \delta) + M_1(u, \bar{J}(p^*)')\Phi_p^{(2)}}{1 - \hat{\beta}_M} \right) \]

\[ + \sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \hat{\beta}_{SP}^{(j)} \left( \frac{z^{(i)} + \hat{p}^{(i)} - b}{+\beta \frac{\Delta_p^{(i+1)}(1 - \delta) + M_1(u^{(i)}, \bar{J}(p^*)^{(i)})\Phi_p^{(i+1)}}{1 - \hat{\beta}_M}} \right) \]

\[ = \frac{\beta}{1 - \hat{\beta}_M} \left( M_2(u, \bar{J}(p^*))\Phi_p^{(0)} + \frac{M(u, \bar{J}(p^*))}{\bar{J}(p^*)}(E(p|p > p^*) - p^*) \right) + s \]

where \( \hat{\beta}_M = \beta(1 - \delta) \), \( \hat{\beta}_{SP}^{(j)} = \beta(1 - \delta - M_1(u^{(j)}, \bar{J}((p^*)^{(j)}))) \), \( \Delta_p^{(i+2)} = \hat{p}^{(i+1)} - \hat{p}^{(i+2)} \), \( \Phi_p^{(i+2)} = \hat{p}^{(i+2)} - E(\hat{p}^{(i+1)}) \).

**Proof.** See Appendix A.3. \( \blacksquare \)

The term \( \beta_M \) in equation (15) refers to the effective discount factor by the market, that is, the actual discount factor, \( \beta \), times the probability of survival, \((1 - \delta)\). Similarly, the term \( \beta_{SP} \) refers to the effective discount factor by the social planner, which is a slight modification from the market discount factor in that it also takes into account the change in the number of matches that arises from changing the unemployment pool, summarized by the term \( M_1(u, \bar{J}(p)) \). Terms \( \Delta_p \) and \( \Phi_p \) refer to changes in average firm productivity and difference between next period’s aggregate productivity and average productivity of new matches, respectively.

Next we explain equation (15), which is the determinant of optimal entry in the dynamic
model. The equation equalizes the gains from having one extra firm searching for a worker, to not having that extra firm searching. The left hand side of the equation summarizes the gains from having one extra firm. If the match occurs, it does not start producing until the following period; hence the $\beta$ term times the marginal increase in matches generated by the extra firm, $M_2(u, J(p))$. The gains from the extra firm operating are given by the productivity $z', \beta'$, minus the cost of having the worker not deriving non-market value, $b$, plus the same object in all the future periods — appropriately discounted at the social planner’s rate, $\beta_{SP}$. Second, there are the discounted gains (or costs) of changes in future average productivity ($\Delta_p$) induced by the new match, plus changes between average productivity and new firms’ productivity ($\Phi_p$) induced by a reduced unemployment pool ($M_1(u, J(p))$).

On the right hand side of the equation, we have the gains from not having one extra firm. The first term is the discounted value, starting next period (given by term $\beta/(1 - \beta_M)$), of not having the position filled by the marginal productivity firm, but having it filled by the average firm productivity instead ($E(p|p > p^*) - p^*$), adjusted by the probability of getting that match ($M(u, J(p^*)) / \bar{J}(p^*)$). On top of that, there is the difference between average productivity and new firm’s productivity $\Phi_p$ times the marginal increase in matches generated by the extra firm $M_2(u, J(p^*))$, discounted at the same rate as the previous term. The last term is simply the savings from not posting an offer, $s$.

The market equilibrium, unfortunately, does not deliver efficient entry, as highlighted by equation (15), unless the minimum wage takes special values. Next, we show how the minimum wage should be, for efficiency to be in effect.

### 4.3 Efficiency in the market

In our economy, as long as the productivity cut-off is the efficient one, then efficiency is ensured — the planner cannot modify anything else. Hence, the question of how the market allocation can achieve efficiency is as simple as to make the marginal firm in the market economy, described by equation (13), coincide with the marginal firm that the planner would choose, described by equation (15). As we show in the following theorem, there is a minimum wage that guarantees efficiency.
Theorem 6 A minimum wage equal to

\[ \hat{w} = \frac{1}{1 - \hat{\beta}_M} E(p|p > p^*) + \left( z + E \left[ \sum_{s=1}^{\infty} \tilde{\beta}_M(z^{(s)} - w^{(s)}) \right] \right) \]

\[ + \frac{\bar{J}(p^*)}{M(u, \bar{J}(p^*))} \times E \left[ \frac{1}{1 - \hat{\beta}_M} \left( M_2(u, \bar{J}(p^*)) \Phi_p^{(0)} \right) \right. \]

\[ - M_2(u, \bar{J}(p^*)) \left( z' + \hat{p}' - b + \beta \frac{\Delta p^{(2)}(1 - \delta) + M_1(u, \bar{J}(p^*)) \Phi_p^{(2)}}{1 - \hat{\beta}_M} \right) \]

\[ - M_2(u, \bar{J}(p^*)) \sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \tilde{\beta}_{SP}^{(j)} \left( \frac{\tilde{\beta}_M \Delta p^{(i+1)}(1 - \delta) + M_1(u^{(i)}, \bar{J}(p^*)^{(i)}) \Phi_p^{(i+1)}}{1 - \hat{\beta}_M} \right) \]

where \( \tilde{\beta}_{SP}^{(j)} = \beta(1 - \delta - M_1(u^{(j)}, \bar{J}(p^*)^{(j)})), \tilde{\beta}_M = \beta(1 - \delta), \Delta p^{(i+2)} = \hat{p}^{(i+2)} - \hat{p}^{(i+1)}, \Phi_p^{(i+2)} = \)

\[ \hat{p}^{(i+2)} - E(p^{(i+1)}), \] ensures that the market allocation is efficient.

Proof. Combine the expression for the cut-off firm in the market, equation (13), with the expression for optimal entry of the social planner, equation (15) and rearranging, we get equation (16).

EXPLAIN EQUATION FROM THEOREM

4.4 The Quasi-Efficient Mechanism

As we have discussed before, the CCH cannot compute the efficient minimum wage using equation (16), as if she did so, the purpose of the mechanism would be defeated. In this section we show that if the economy is in a steady state, these considerations do not take place. Then, we use this result to propose an ad-hoc mechanism to determine the minimum wage, which we label as the quasi-efficient mechanism: to use the version of the efficient minimum wage in the case of the steady state, and use it to establish the minimum wage in the case with shocks.

To this end, it is important to first establish how the efficient minimum wage looks like in the steady state. We do so in the next corollary.

Corollary 7 In the steady state, the efficient minimum wage is characterized by

\[ \hat{w} = z + E(p|p > p^*) - \frac{\bar{J}(p^*)}{M(u, \bar{J}(p^*))} \frac{1 - \tilde{\beta}_M}{1 - \tilde{\beta}_{SP}} M_2(u, \bar{J}(p^*)) (z + \hat{p} - b) \]

where \( \tilde{\beta}_{SP} = \beta(1 - \delta - M_1(u, \bar{J}(p^*))), \tilde{\beta}_M = \beta(1 - \delta), \) ensures that the market allocation is efficient.
Proof. It is a direct consequence of equation (16), with \( u = u', z = z', \hat{p} = \hat{p}', \Phi_p^{(i)} = 0, \Delta_p^{(i)} = 0 \). □

The expression for the minimum wage in the steady state coincides with the first two lines of equation (16). As we discussed when explaining that expression, the minimum wage in the steady state is, up to a change in the discount factors, the same as in the static model.

We propose to use the minimum wage that emerges from equation (17) into the dynamic model with shocks. Using this minimum wage is useful because, first, it is informative of how large is the bias vis-a-vis the optimal minimum wage — the objects in the expectations bracket of equation (16). Moreover, in that expression, the minimum wage is a function of the average productivity, unemployment rate, and aggregate technology shock. This implies that, if used in a world where there are aggregate shocks generating aggregate fluctuations, the minimum wage will adjust to the business cycle. Even though the adjustment is not the efficient one, as we shall see in the quantitative exercises that we do next, the minimum wage adjusted using our quasi-efficient mechanism is better than using a constant minimum wage.

5 Calibration

We solve the model at a weekly frequency, setting the rate of time preference \( \beta = 0.99^{\frac{1}{2}} \). The AR(1) process for aggregate productivity, \( z \), is set to match the same process in Hagedorn and Manovskii (2008, henceforth referred to as HM), discretized through a ten-state Markov chain, normalized to 1. The destruction rate of matches is exogenous and set at \( \delta = 0.0081 \) following HM and Shimer (2005).

The remaining parameters of the model are calibrated internally. In particular, we do not assume that the matching function is Cobb-Douglas with parameter equal to the worker bargaining power \( \alpha \) — i.e. we do not impose the Hosios condition. Instead, we set the functional form of the production function to be \( M(u, \nu) = u\nu(u^{\kappa} + \nu^{\kappa})^{-1/\kappa} \) (as in den Haan et al., 2000; and HM) which, due to the discrete-time formulation of our problem, is useful as it guarantees that matching probabilities of firms and workers lie between \((0, 1)\). Further, we assume the distribution of firm productivities is normal and target the mean \( \mu \) and variance \( \sigma \) against moments in the data.

The business cycle moments that we target in the data are directly from HM (which in turn relies on Shimer, 2005; and den Haan et al., 2000, for the computation of some values). We target a market tightness \( \nu/u = 0.634 \), and an elasticity of (detrended) wages with respect to productivity of 0.449. Posting costs as a fraction of average weekly output are targeted at 0.584 as estimated by HM and the job finding rate is targeted at 0.139. Finally,
Table 1: Calibration Parameters, Values and Moments

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Value</th>
<th>Target</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.052</td>
<td>v/u</td>
<td>.668</td>
<td>.634</td>
</tr>
<tr>
<td>$s$</td>
<td>0.583</td>
<td>$s/(\text{avg. } p)$</td>
<td>.601</td>
<td>.584</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.3133</td>
<td>Elast. wages to $p$</td>
<td>.467</td>
<td>.449</td>
</tr>
<tr>
<td>$b$</td>
<td>0.34</td>
<td>Finding Rate</td>
<td>.133</td>
<td>.139</td>
</tr>
<tr>
<td>$w$</td>
<td>0.5106</td>
<td>Frac of earners at $w$</td>
<td>.048</td>
<td>.048</td>
</tr>
<tr>
<td>$\mu$</td>
<td>-1.5</td>
<td>Avg. $z+p$</td>
<td>.973</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.73</td>
<td>SD of earners at $w$</td>
<td>.001</td>
<td>.080</td>
</tr>
</tbody>
</table>

we set the benchmark minimum wage to target an average fraction of 4.8% of earners with binding wages as we computed in section 2. The mean of firm productivity is set to target an average gross productivity ($z+p$) of 1 (alternatively, targeting a normalized value of average firm productivity at zero). The variance of firm productivity is set to target the standard deviation of the fraction of earners at the minimum wage over the business cycle, 0.001.

The model moments are computed by simulating out one hundred thousand periods of the aggregate shock process and then simulating the model over this sequence. The first thirty thousand periods are discarded leaving approximately 1,346 years of simulated data over which we compute the moments reported above. These parameters and the moments from the model and data are summarized below in Table 1.

The model successfully reproduces the business cycle moments targeted and the average fraction of earners at the minimum wage. The one moment the model has trouble hitting, however, is the cyclical moment of the minimum wage (the standard deviation of the fraction of minimum wage earners). This is a shortcoming directly related to our assumption that wages are renegotiated every period. While this assumption allows us to avoid having to carry around a distribution in the state space, it also means that in recessions a substantial number of workers are bargained down to the binding minimum wage. Empirically we know that wages are fairly downward sticky, and the difficulty the model has in matching this moment highlights the counter-factual nature of downward wage movement in recessions.

5.1 Calibrating the minimum wage

A number of influential papers point out the difficulty the standard DMP model has in jointly reproducing empirically plausible wage dispersion and business cycle volatility in the absence of worker heterogeneity or on the job search (see Hornstein et al., 2011; and Bils et al., 2011). In our calibration we sidestep this issue entirely by focusing on distributional measures instead of, for example, a level-to-level comparison as is the mean-min ratio anal-
ysed in Hornstein et al. (2011). We do this primarily because, while the aim of the paper is to analyze the design of efficient minimum wage policies, the role of the minimum wage (and indeed the mechanism developed above) is independent of any specific wage formation protocol. In fact, the mechanism requires only that wages be strictly monotonic in productivity. While the quantitative exercises assume wages are set through Nash bargaining between workers and firms, alternative protocols may be specified that imply greater or lesser earnings dispersion in equilibrium. Any change in these protocols however has no impact on the optimal minimum wage policy. Intuitively, what matters in this model is how binding the minimum wage is. That is, how many jobs are effected at the margin. Ultimately this is the most informative measure of how much the minimum wage matters for productivity, which is is the primary role analysed here. As such, we choose to directly target measures of “bindingness.”

6 Results

We run three quantitative experiments on the benchmark model. The first is the implementation of the quasi-efficient mechanism described in Section 4.4. The second experiment computes the fully efficient planner’s solution as a guideline for potential first-best gains. Finally, the third experiment takes the benchmark economy, assumes a fixed minimum wage, and quantitatively assesses the best constant minimum wage over simulated business cycles. Results from the three experiments and the benchmark are compared below Table 2.

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Best-Fixed MW</th>
<th>Quasi-Efficient</th>
<th>Fully Efficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg. Net Output</td>
<td>1</td>
<td>1.16</td>
<td>1.189</td>
<td>1.193</td>
</tr>
<tr>
<td>CoV Net Output</td>
<td>0.023</td>
<td>0.021</td>
<td>0.019</td>
<td>0.022</td>
</tr>
<tr>
<td>CoV u</td>
<td>0.031</td>
<td>0.057</td>
<td>0.013</td>
<td>0.012</td>
</tr>
<tr>
<td>Corr(Y, z)</td>
<td>0.977</td>
<td>0.983</td>
<td>0.988</td>
<td>0.999</td>
</tr>
<tr>
<td>Corr(u, z)</td>
<td>-0.971</td>
<td>-0.947</td>
<td>-0.628</td>
<td>-0.854</td>
</tr>
<tr>
<td>Avg. MW</td>
<td>0.511</td>
<td>0.93</td>
<td>0.945</td>
<td>–</td>
</tr>
<tr>
<td>CoV MW</td>
<td>0</td>
<td>0</td>
<td>0.021</td>
<td>–</td>
</tr>
</tbody>
</table>

CoV is Coefficient of Variation. Average net output is normalized to unity in the benchmark. Correlations are computed using HP filtered data detrended with a smoothing parameter of 129,600.

8In Mahone and Pujolas (2017) we show explicitly how the introduction of ex-ante, as opposed to ex-post, heterogeneity circumvents the problems discussed in the aforementioned papers by breaking the equivalence between the average firm and the marginal firm at entry.
Comparing the benchmark to the best-fixed minimum wage economy, we see that simply raising the constant minimum wage can yield welfare gains of 16%. This leads however to a notably larger variation in unemployment over the cycle. The quasi efficient (and full efficient) counterparts yield somewhat larger gains in average output but also feature substantially reduced correlations between unemployment and aggregate shocks as compared to the fixed minimum wage economies. This comes from the minimum wage setting mechanism adjusting the minimum wage over the cycle.

Figure 4 displays the variation of unemployment rates in the best-fixed and quasi-efficient economies (normalized to one over the cycle), and illustrates clearly the extent to which the mechanism reduces the cyclicality of unemployment and, consequently, finding rates.

Figure 4:

![Unemployment Variation over the Business Cycle](image)

6.1 Flexible Minimum Wage Policy

One of the clear implications of the proposed policy mechanism is that minimum wages will respond to aggregate movements in productivity. Intuitively, when times are bad the minimum wage can be reduced so it is less “binding” than in good times. This would suggest that the gains to flexibility are largest precisely when having a binding minimum wage matters most. Figure 5 displays the difference between output in the quasi efficient and best-fixed worlds \((y_Q - y_B)\) against the aggregate shock process \(z\). Both series have been normalized to have a mean zero and standard deviation of one for ease of comparison.

The measured gains to flexibility and shock process \(z\) have a negative correlation of -0.25,
clearly visible in Figure 5, suggesting that the loosening of the constraint imposed by the minimum wage is particularly valuable (in output terms) when times are bad. This is a very policy-relevant result given the current unresponsiveness of minimum wages to the business cycle in current policy.

It is also worth remembering that the economies studied here are populated entirely by risk neutral agents, so the welfare gains obtained through a flexible policy over the business cycle are entirely through output. Welfare gains with risk averse workers would likely be substantially larger given the mechanism’s active smoothing of unemployment.

6.2 Robustness

We perform two robustness checks on our result. The first is solving the model at an annual frequency. The "myopic" nature of the solution is probably less important if we reset the min wage every week. The second is solving the model for a subset of occupations for which the minimum wage is likely more relevant. This deals with potential complaints regarding directed search and the assumed homogeneity of workers.
7 Conclusion

TBD
References


A Appendix

A.1 Proof of Lemma 1

The intuition from the VCG mechanism is to charge the boundary firm so that her gain is exactly equal to the social gain from her entry. The welfare function from having all firms with productivity \( p \geq p^* \) operating, in the case of a continuum of firms is

\[
\Omega(p^*) = M(m, \bar{J}(p^*)) \left[ \frac{1}{\bar{J}(p^*)} \int_{p^*}^p x dJ(x) - b \right] - \bar{J}(p^*) s, \tag{18}
\]

where \( \bar{J}(p^*) = 1 - J(p^*) \). The first term is the number of matches multiplied by the average gain over unemployment, and the second term is the search costs incurred. Notice that with only a fraction \( (1 - J(p^*)) \) firms participating, the truncated CDF is \( (J(x) - J(p^*))/J(p^*) \), yielding a PDF of \( J'(x)/\bar{J}(p^*) \).

To derive the minimum wage charged by the mechanism to the boundary firm in the continuous case, we compute the per worker wage bill of a firm that controls a positive mass of productivities next the boundary — using the discrete version of the mechanism — and make the mass of firms controlled limit to zero.

Consider a firm that controls a positive mass \( (p^*, p^* + \epsilon) \) of productivities, where \( p^* \) is the boundary productivity. Then, dividing the total wage bill that this firm pays by the mass of productivities that this firm controls, \( \bar{J}(p^*) - \bar{J}(p^* + \epsilon) \), we can obtain the per-firm wage bill. Taking the limit of \( \epsilon \) to zero allows us to obtain the wage charged for firms with the boundary productivity.

The total wage bill charged to the boundary firm in the discrete example, equation (3), is

\[
\hat{w} = \frac{\bar{J}(p^*)}{M(u, \bar{J}(p^*))} \left[ M(u, \bar{J}(p^* + \epsilon)) \left\{ \frac{1}{\bar{J}(p^* + \epsilon)} \int_{p^* + \epsilon}^{p^*} x dJ(x) - b \right\} - M(u, \bar{J}(p^*)) \left\{ \frac{1}{\bar{J}(p^*)} \int_{p^* + \epsilon}^{p^*} x dJ(x) - b \right\} \right], \tag{19}
\]

If we plug this into the expected profit function of the firm,

\[
\frac{M(u, \bar{J}(p^*))}{\bar{J}(p^*)} \left[ \int_{p^* + \epsilon}^{p^* + \epsilon} x dJ(x) - w \right] - \int_{p^*}^{p^* + \epsilon} s dJ(x), \tag{20}\]

we obtain exactly \( \Omega(p^*) - \Omega(p^* + \epsilon) \), so that this marginal “mass” firm earns (in total) the net social gain of her entry. The per-firm net social gain is \( (\Omega(p^*) - \Omega(p^* + \epsilon))/(\bar{J}(p^*) - \bar{J}(p^* + \epsilon)) \).

To find the associated per-worker wage being charged, \( \hat{w}_f(\epsilon) = \hat{w}/(\bar{J}(p^*) - \bar{J}(p^* + \epsilon)) \), we can write
\[ \dot{w}(\epsilon) = \frac{\bar{J}(p^*)}{M(u, \bar{J}(p^*))} \left[ M(u, \bar{J}(p^*) + \epsilon) \left\{ \frac{1}{J(p^* + \epsilon)} \int_{p^* + \epsilon}^{p} xdJ(x) - b \right\} \right. \\
- \left. M(u, \bar{J}(p^*)) \left\{ \frac{1}{J(p^*)} \int_{p^* + \epsilon}^{p} xdJ(x) - b \right\} \left[ J(p^*) - J(p^* + \epsilon) \right]^{-1} \right] \]

Taking limits as \( \epsilon \to 0^+ \) using L'Hôpital's rule

\[ \lim_{\epsilon \to 0^+} \dot{w}(\epsilon) = \frac{\bar{J}(p^*)}{M(u, \bar{J}(p^*))} \left[ \lim_{\epsilon \to 0^+} \frac{M(u,J(p^* + \epsilon))}{J(p^* + \epsilon)} - \lim_{\epsilon \to 0^+} \frac{M(u,J(p^*))}{J(p^*)} - \lim_{\epsilon \to 0^+} \frac{M(u,\bar{J}(p^* + \epsilon)) - M(u,\bar{J}(p^*))}{\bar{J}(p^*) - J(p^* + \epsilon)} b \right] \]

we get that

\[ \dot{w}(\epsilon) = \frac{\bar{J}(p)}{M(u, \bar{J}(p))} \left[ \int_{p^* + \epsilon}^{p} xdJ(x) \frac{M(u, \bar{J}(p))}{J(p)} - M_2(u, \bar{J}(p)) \right] + M_2(u, \bar{J}(p)) b \]

Hence,

\[ \dot{w} = \frac{\bar{J}(p^*)}{M(u, \bar{J}(p^*))} \left[ \frac{M(u, \bar{J}(p^*))}{J(p^*)} \mu(p^*) - M_2(u, \bar{J}(p^*)) \{ \mu(p^*) - b \} \right] \tag{21} \]

where \( \mu(p^*) = \int_{p^*}^{p} xdJ(x)/J(p^*) \). Notice that this is exactly the minimum wage is the perform (divided by the point mass \( J'(p^*) \)) marginal social gain from entry adjusted for matching likelihood.

We can easily verify that this is also the wage that must be charged for the firms with the (efficient) boundary productivity are just willing to enter. Recall from the function \( \Omega(p^*) \) above that we can write

\[ \Omega'(p^*) = M_2(u, \bar{J}(p^*)) J'(p^*) [\mu(p^*) - b] + M(u, \bar{J}(p^*)) \mu'(p^*) - J'(p^*) s. \tag{22} \]

Since firms pay a search cost \( s \), we want the boundary productivity’s expected gains (absent search costs) to equal

\[ \frac{M_2(u, \bar{J}(p^*)) J'(p^*) [\mu(p^*) - b] + M(u, \bar{J}(p^*)) \mu'(p^*)}{J'(p^*)} = s \]

which simplifies to

\[ M_2(u, \bar{J}(p^*)) [\mu(p^*) - b] + \frac{M(u, \bar{J}(p^*))}{J'(p^*)} \mu'(p^*) = s. \tag{23} \]
Then, we equate pre-search costs gains of the marginal firm to the left hand side of equation 23, to get that the optimal minimum wage, $w^{optimal}$ has to satisfy

$$\frac{M(u, \bar{J}(p^*))}{\bar{J}(p^*)} [p^* - w^{optimal}] = M_2(u, \bar{J}(p^*))[\mu(p^*) - b] + \frac{M(u, \bar{J}(p^*))}{\bar{J}(p^*)} \mu'(p^*)$$

(24)

and noting that we can write $\mu'(p^*) = \frac{-\bar{J}'(p^*)}{\bar{J}(p^*)} \mu(p^*) + p \frac{\bar{J}'(p^*)}{\bar{J}(p^*)}$ we obtain that

$$w^{optimal} = \frac{\bar{J}(p^*)}{M(u, \bar{J}(p^*))} \left\{ \frac{M(u, \bar{J}(p^*))}{\bar{J}(p^*)} \mu(p^*) - M_2(u, \bar{J}(p^*))[\mu(p^*) - b] \right\}.$$  

(25)

Note that equations (21) and (25) are exactly the same, which implies that the minimum wage derived from the mechanism coincides with the minimum wage that induces efficiency in the market. Moreover, both expressions are equal to equation (7), which proves the result of the Lemma.

A.2 Proof of Thorem 3

First, consider all possible bidding firms given any particular $S \in \mathcal{S}$. Note that because $t_T(\gamma, S)$ depends only on the average reported valuation, no individual firm has an incentive to misreport (firms are atomistic). Hence for any $S$, valuations will be truthfully reported, so $\gamma(p) = p$.

Now consider the unique strategy profile $S^*$ such that the planners’ solution is efficiently implemented. In this case, $\delta^*(p) = 0 \ \forall p < p^*$ and $\delta^*(p) = 1, \ \forall p \geq p^*$ and $\gamma^*(p) = p \ \forall p$.

Moreover, under truthful reporting and efficient entry, the proposed minimum wage $\hat{w}$ implies that $\pi(p^*) = 0$, which is a direct consequence of Lemma 1.

Finally, given that the wage charges the marginal firm such that their profits are equal to the social value of their entry, notice that no firm with $p < p^*$ has an incentive to enter, since their expected profits will be negative.

A.3 Proof of Lemma 5

To prove the result, it is useful to first characterize some parts of the problem. In particular, the derivatives of the mass of operating firms with respect to the marginal firm is

$$\frac{d\bar{J}(p^*)}{dp^*} = -\frac{dJ(p^*)}{dp^*},$$

and the derivative of the expected entry productivity with respect to the marginal firm is

$$\frac{dE(p|p > p^*)}{dp^*} = -p^*\bar{J}(p^*) \frac{dJ(p^*)}{dp^*} + \int_{p^*}^\hat{p} x dJ(x) \frac{dJ(p^*)}{dp^*} \frac{1}{J(p^*)^2},$$

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Again, using consecutive periods we get

\[ 0 = -s \frac{d\bar{J}(p^*)}{dp^*} - \beta \Omega_1(u', \hat{p}', z') \left( M_2(u, J(p^*)) \frac{d\bar{J}(p^*)}{dp^*} \right) \]

\[ + \beta \Omega_2(u', \hat{p}', z') \left( M_2(u, \bar{J}(p^*)) \frac{(E|p > p^*| - \hat{p}')}{1 - u'} - \frac{M(u, \bar{J}(p^*)) (E|p > p^*| - p^*)}{J(p^*)} \right) \frac{d\bar{J}(p^*)}{dp^*} \]

since \( \frac{dJ(p^*)}{dp^*} \) appears everywhere, the problem is

\[ 0 = -s - \beta \Omega_1(u', \hat{p}', z') M_2(u, \bar{J}(p^*)) \]

\[ + \beta \Omega_2(u', \hat{p}', z') \left( M_2(u, \bar{J}(p^*)) \frac{(E|p > p^*| - \hat{p}')}{1 - u'} - \frac{M(u, \bar{J}(p^*)) (E|p > p^*| - p^*)}{J(p^*)} \right) \frac{d\bar{J}(p^*)}{dp^*} \]

The envelope condition for \( \hat{p} \) is

\[ \Omega_2(u, \hat{p}, z) = (1 - u) + \beta \frac{(1 - \delta)(1 - u)}{1 - u'} \Omega_2(u', \hat{p}', z'). \]

Again, using consecutive periods we get

\[ \Omega_2(u, \hat{p}, z) = (1 - u) \]

\[ + \beta \frac{(1 - \delta)(1 - u)}{1 - u'} \left( (1 - u') + \beta \frac{(1 - \delta)(1 - u')}{1 - u''} \Omega_2(u'', \hat{p}'', z'') \right) \]

\[ = (1 - u) + \beta (1 - \delta)(1 - u) \]

\[ + \beta^2 \frac{(1 - \delta)^2(1 - u)}{1 - u''} \Omega_2(u'', \hat{p}'', z'') \]

\[ = (1 - u) + \beta (1 - \delta)(1 - u) + (\beta (1 - \delta))^2 (1 - u) \]

\[ + \beta^3 \frac{(1 - \delta)^3(1 - u)}{1 - u''} \Omega_2(u'', \hat{p}'', z'') \]

\[ = (1 - u) \sum_{s=0}^{\infty} (\beta (1 - \delta))^s \]

\[ = \frac{1 - u}{1 - \beta (1 - \delta)}. \]

The envelope condition for \( u \) is

\[ \Omega_1(u, \hat{p}, z) = -(z + \hat{p}) + b + \beta (1 - \delta - M_1(u, \bar{J}(p^*)) \Omega_1(u', \hat{p}', z') \]

\[ + \beta \frac{(1 - u')}{1 - \beta (1 - \delta)} \frac{(\hat{p}' - \hat{p})(1 - \delta) + M_1(u, \bar{J}(p^*)) (E|p > p^*| - \hat{p}')}{1 - u'}. \]
Define it as

\[ \Omega_1(u, \hat{p}, z) = A + B\Omega_1(u', \hat{p}', z') \]
\[ \Omega_1(u, \hat{p}, z) = A + B(A' + B'\Omega(u'', \hat{p}'', z'')) \]
\[ \Omega_1(u, \hat{p}, z) = A + BA' + BB'(A'' + B''\Omega(u'', \hat{p}'', z'')) \]
\[ \Omega_1(u, \hat{p}, z) = A^{(0)} + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} B^{(j)} A^{(i)}, \]

where

\[ A^{(i)} = -z^{(i)} - \hat{p}^{(i)} + b + \beta \frac{(\hat{p}^{(i+1)} - \hat{p}^{(i)})(1 - \delta) + M_1(u^{(i)}, \bar{J}(p^*)(^{(i)}) (E(p|p > p^*)^{(i)}) - \hat{p}^{(i+1)})}{1 - \beta(1 - \delta)}, \]
\[ B^{(i)} = \beta(1 - \delta - M_1(u^{(i)}, \bar{J}((p^*)^{(i)}))), \]

and \(^{(i)}\) denotes the variable \(i\) periods ahead.

Combining the two envelope conditions into the first order condition, we get that

\[ 0 = -s - \beta M_2(u, \bar{J}(p^*)) \left( -z' - \hat{p}' + b + \beta \frac{(\hat{p}' - \hat{p}')(1 - \delta) + M_1(u, \bar{J}(p^*)'(E(p|p > p^*)' - \hat{p}''))}{1 - \beta(1 - \delta)} \right) \]
\[ \frac{-z^{(i)} - \hat{p}^{(i)} + b + \beta \frac{(\hat{p}^{(i+1)} - \hat{p}^{(i)})(1 - \delta)}{1 - \beta(1 - \delta)} + M_1(u^{(i)}, \bar{J}(p^*)^{(i)}) (E(p|p > p^*)^{(i)}) - \hat{p}^{(i+1)})}{1 - \beta(1 - \delta)} \]
\[ - \beta M_2(u, \bar{J}(p^*)) \sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \beta(1 - \delta - M_1(u^{(j)}, \bar{J}((p^*)^{(j)}))) \]
\[ + \frac{\beta}{1 - \beta(1 - \delta)} \left( M_2(u, \bar{J}(p^*)) (E(p|p > p^*) - \hat{p}') - \frac{M(u, \bar{J}(p^*))}{\bar{J}(p^*)} (E(p|p > p^*) - p^*) \right) \]

Rearranging we get the desired result. \(\blacksquare\)