A dOOR to Ordinality in Assignment Problems: Outside Option Robustness

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Abstract

We argue that any assignment rule in school choice and other assignment problems with unequal outside options (e.g., access to private schools) has to be Outside Option Robust: no agent shall envy the probabilities to public schools preferred to her own outside option assigned to any other agent. When analyzing assignment rules, OOR turns out to be a refinement of ex-ante envy-freeness. Very importantly, in preference-dense economies, OOR implies that all information on preferences among public schools beyond their ordinal component can be ignored. Moreover, agents with same ordinal preferences among non-outside-option objects obtain the same probabilities for objects they prefer to the outside option. Connecting to a recent paper by Liu and Pycia, this implies that there is a unique anonymous, OOR and efficient assignment rule in large economies, provided for instance by Random Serial Dictatorship.

1 Introduction

Centralization devices in object-assignment problems have been introduced as a way to provide fairer assignments. This is of paramount importance in school choice for instance, if one wants to give a chance for better education to children who live in unsafe neighborhoods with low-quality public schools. One of the purposes of school choice mechanisms is to make income differences among applicants irrelevant in the assignment process. In those assignment processes, there typically are outside options, that is, agents might choose to opt out of the assignment process. In the school choice case, one could think of access to good private schools as an outside option. This paper points out at an important source of unfairness, in assignment problems where agents have unequal access to outside options. The better access to outside options for richer families imply that they can take more risk in the application process and apply for the better schools, while poorer families choose to play safer. As a consequence, agents with better outside options have easier access to better placements inside the assignment process than worse-outside-option agents do (see for instance Calsamiglia, Martínez-Mora and Miralles, 2017.)

In this paper, we consider a new concept of fairness, namely Outside Option Robustness (OOR.) According to this concept, no agent should envy any...
other equal-priority agent’s assigned probabilities for objects the former agent prefers to the outside option. An OOR assignment rule is indeed ex-ante envy-free among agents with equal priority type. Therefore, OOR is a refinement of the Nash equilibrium implementation concept, in assignment problems with a continuum of agents and a rich support of preferences. Yet, what else can be learned from this refinement?

The most striking result we find is that, for the same case of dense economies with a rich support of preferences, OOR implies that any information regarding the cardinality of preferences can be safely ignored. In other words, every OOR assignment in large economies is ordinal. Correspondingly, and combining this with a recent result by Liu and Pycia (2016,) the set of OOR, anonymous, efficient assignments is unique, and can be provided for instance by random dictatorship.

OOR is a concept that heavily relies on the fact that every final outcome has an attached payoff with full, interpersonally comparable economic meaning beyond a mere tool to compare lotteries. This is the case, for instance, of the economic value of offsprings’ education in school choice. According to this, two affine preference types (where one is an affine transformation of the other) are not equivalent. For that reason, our concept does not accept the normalization of all outside options to zero, in which case OOR would just be equivalent to ex-ante envy-freeness (for equal-priority types.) Incidentally, this indeed explains that OOR is a refinement of envy-freeness.

However, two affine types share the same preferences over lotteries including the outside option as a final outcome. Consequently, these two types would receive the same (or payoff-equivalent) probability assignment, if there is no exogenous reason to treat them differently (e.g. priority criteria.) The need to consider these types differently and at the same time give them equal payoffs restricts the scope for cardinal mechanisms, those who take into account the intensity of preferences beyond their ordinal preference component. In the limit, in dense economies where every type is represented by at least one agent, the scope for cardinality disappears.

There has been some literature stressing the fact that outside options shall be studied in depth in assignment problems: for instance Kesten and Kurino (2016) and Pycia and Unver (2017.) According to this new strand, outside options are more than an object with infinite capacity. However, we are not aware of other papers considering this fact as the source of a fairness concept, except of for the cited paper by Calsamiglia et al.

There is also some literature on cardinal versus ordinal decision and assignment rules. Hylland (1980) and more recently Dutta, Peters and Sen (2007) show that the only strategy-proof cardinal decision scheme satisfying a weak unanimity property is the random dictatorship, hence eliminating cardinality. Quite recently, Ehlers, Majumdar, Mishra and Sen (2014) establish that under some continuity criteria, incentive-compatible cardinal mechanisms are ordinal. We contribute to this debate from the perspective of fairness, complementing these previous incentives and continuity approaches. Our approach is appealing since our model starts from a pure cardinal approach with no bias in favor of
ordinality. It then introduces the OOR concept and finally it shows how it make cardinal mechanisms collapse in large markets.

This debate is interesting since a significant set of papers (Hylland and Zeckhauser 1979, Miralles 2008, Abdulkadiroglu, Che and Yasuda 2011 and 2015, Ashlagi and Shi 2016, Miralles and Pycia 2014, He, Miralles, Pycia and Yan 2015) stresses the importance of taking cardinal utilities into account. The present paper constitutes a warning. If one takes outside options seriously, there will be unsourmountable trade-offs between cardinal efficiency and the robustness of the mechanism with respect to outside options, at least in large markets.

An immediate application of the present paper’s model is, as mentioned several times, school choice. Outside options, and how they affect the assignment of children to public schools, are a sensitive issue that nevertheless has received insofar insufficient attention. We are aware since Black (1999) that parents have cardinal preferences for the schools that can be expressed in monetary terms as willingness to pay through the residential market. YET CHOOSING WITH CARDINAL UTILITIES IS A MESS (LITERATURE)

COSAS QUE METER? VOUCHERS FOR PRIVATE SCHOOLS AND OTHER APPROACHES.

We hope that this contribution will foster further research in this important matter.

2 Notation and definition

There is a fixed set of agents $I$, each one to be assigned to exactly one of a set of objects $S \cup \{o\}$ where $o$ is the outside option. Notation for agents include $i, j$, whereas notation for objects include $s, t$. Each object $s$ has a positive number of copies $\eta_s$, and we assume $\eta_o = \infty$. The supply vector is also fixed and denoted as $\eta = (\eta_s)_{s \in S \cup \{o\}}$. Agents $i$ have valuations $v_i = (v_s^i)_{s \in S \cup \{o\}} \in \mathbb{R}^{[S]+1}$. We assume no indifferences between any two objects with non-zero valuation. $V = (v_i)_{i \in I}$ is a matrix summarizing the economy’s preference profile. Let $V = \mathbb{R}^{[I]\times([S]+1)}$ be the set of all preference profiles. The definition of ordinal references arising from valuations is standard.

A (random) assignment $Q$ is a matrix $Q = (q_{ij})_{i \in I} \in \Delta(S \cup \{o\})^{[I]}$. An assignment is feasible if it accomplishes with the condition $\sum_{i \in I} q_{i}(V, \Pi) \leq \eta$. Let $F$ denote the set of feasible assignments. Let $P = \{\Pi = (\pi_i)_{i \in I}\}$ be the set of possible public signals about each individual priority level $\pi_i$ in the economy. Two agents $i$ and $j$ are priority-equal, or priority-equivalent, if $\pi_i = \pi_j$. $\Pi$ is anonymous if $\pi_i = \pi_j$ for every pair $i, j \in I$. An assignment rule is a function $Q : V \times P \rightarrow F$ that maps agents’ preferences and a publicly observable signal into a feasible assignment.

We impose the following axioms or desiderata in the assignment rule. These axioms are imbedded in all the results we provide throughout this paper.

**Axiom 1** We assume that the rule is anonymous up to public signals:
any permutation in \((V, \Pi)\) across agents leads to an equivalent permutation in \(Q(V, \Pi)\). For any given permutation \(\rho : I \rightarrow I\) we have \(Q(\rho(V), \rho(\Pi)) = \rho(Q(V, \Pi))\). In other words, only the information gathered at \(\Pi\) might justify differences in the assignment between two agents with equal preferences.

**Axiom 2** We also impose the *efficiency condition* that no agent gets positive chances at an object that is weakly worse than her outside option.

**Axiom 3** We finally impose that the rule is invariant to affine transformations above the outside option. We say that \(v_i^s\) is an affine transformation of \(v_i\) above the outside option if there are numbers \(\alpha > 0, \beta \in \mathbb{R}\) such that for every \(s\) with \(v_i^s > v_i^o\) we have \(v_i^s = \alpha v_i^o + \beta\), and finally whenever \(v_i^s \leq v_i^o\) we also have \(v_i^s \leq v_i^o\). We say that an assignment \(Q\) is invariant to affine transformations above the outside option if for every pair \(i, j \in I\) such that \(\pi_i = \pi_j\) and \(v_i\) is an affine transformation of \(v_i\) above the outside option we have \(q_i = q_j\). We say that an assignment rule is invariant to affine transformations above the outside option if for every \(V'\) that is equal to \(V\) for all agents except for some agent \(i\), for which \(v_i^s\) is an affine transformation of \(v_i\) above the outside option, we have \(Q(V', \Pi) = Q(V, \Pi)\).

The reason for this last requirement is that, while affine transformations mutate types in economic sense, they do not alter agents’ preferences over lotteries. The last two properties project the idea of an assignment game behind each assignment, which is its Nash equilibrium outcome. In no game a best-responding agent would possibly end up with an object worse than the outside option. We resolve indifference between the outside option and some other object in favor of the outside option, giving other agents more available copies of objects. In such a game, again, two equal-priority agents with the same preferences over lotteries would obtain (payoff) equivalent individual assignments. Without loss of generality (WLOG) we can give the same allocation probabilities to both without violating feasibility, simply by properly averaging the two individual probability assignments.

**Definition 1** An assignment rule is Outside Option Robust (or is OOR) if for every \(V\) and \(\Pi\) and all \(i, j \in I\) such that \(\pi_i = \pi_j\) we have

\[
\sum_{v_i^s > v_i^o} q_i^s(V, \Pi)v_i^s \geq \sum_{v_i^s > v_i^o} q_j^s(V, \Pi)v_i^s
\]

This fairness concept states that no agent should envy the allocation of probabilities any other equal-priority agent obtains, for objects better than the former agent’s outside option. Or, in other words, no agent should envy any other equal-priority agent’s individual assignment, even if her outside option (and worse objects) were worthless.
3 Implications of OOR

We provide a couple of properties of OOR rules in finite economies. The first one simply states that OOR is a refinement of envy-freeness (among equal-priority agents.) The second property states that all agents with the same "relevant" ordinal preference must obtain the same chances at the outside option, under OOR.

Definition 2 An assignment rule is **envy-free for equal publicly observable types** if for every $V$ and $\Pi$ and all $i, j \in I$ such that $\pi_i = \pi_j$ we have

$$q_i(V, \Pi) \cdot v_i \geq q_j(V, \Pi) \cdot v_i$$

Proposition 1 Every OOR rule is envy-free for equal publicly observable types.

Proof. Fix $\Pi$. For every $V$ we can always make the transformation $\tilde{V}$ such that for each $i \in I$ and $s \in S \cup \{o\}$ we set $\tilde{v}_i^s = (v_i^s - v_o^s)_+ \equiv \max\{0, v_i^s - v_o^s\}$. By invariance to affine transformations above the outside option we have $Q(V, \Pi) = Q(V, \Pi)$. By OOR we have, for any $V$ and $\Pi$ and all $i, j \in I$ such that $\pi_i = \pi_j$: $\sum_{i^s \succ_i \tilde{i}^s} q_i^s(V, \Pi) \tilde{v}_i^s \geq \sum_{i^s \succ_i \tilde{i}^s} q_j^s(V, \Pi) \tilde{v}_i^s$. By the previous equality between assignments and by rewriting the transformation of vNM valuations we have $\sum_{i^s \succ_i v_i^s} q_i^s(V, \Pi)(v_i^s - v_o^s) \geq \sum_{i^s \succ_i v_i^s} q_j^s(V, \Pi)(v_i^s - v_o^s)$. Due to the efficiency condition we know that $\sum_{i^s \succ_i v_i^s} q_i^s(V, \Pi) + q_o^s(V, \Pi) = 1$. Then by adding $v_o^s$ to both sides of the previous inequality and rearranging we obtain

$$q_i(V, \Pi) \cdot v_i = \sum_{i^s \succ_i v_i^s} q_i^s(V, \Pi) v_i^s + q_o^s(V, \Pi) v_o^s$$

$$\geq \sum_{i^s \succ_i v_i^s} q_j^s(V, \Pi)(v_i^s - v_o^s) + v_o^s$$

$$\geq \sum_{i^s \succ_i v_i^s} q_j^s(V, \Pi)(v_i^s - v_o^s) + v_o^s$$

$$= \sum_{i^s \succ_i v_i^s} q_j^s(V, \Pi) v_i^s + v_o^s \left(1 - \sum_{i^s \succ_i v_i^s} q_j^s(V, \Pi)\right)$$

$$\geq \sum_{i^s \succ_i v_i^s} q_j^s(V, \Pi) v_i^s + q_o^s(V, \Pi) v_o^s + \sum_{i^s < v_i^s} q_j^s(V, \Pi) v_i^s$$

$$= q_j(V, \Pi) \cdot v_i$$

Notice that the converse is not always true. Consider the case in which $\pi_i = \pi_j$, $i$ does not envy $j$, yet $q_o^s(V, \Pi) > q_j^s(V, \Pi)$. Then, following the steps of the proof of next proposition, we can find a violation of OOR.

Proposition 2 Let $i, j \in I$ and $\Pi \in \mathcal{P}$ such that $\pi_i = \pi_j$. Then, under any OOR rule it must be the case, for any $V$, that $\sum_{i^s \succ_i v_i^s} q_i^s(V, \Pi) \geq \sum_{i^s \succ_i v_i^s} q_j^s(V, \Pi)$. 
Proof. Suppose that $i$ and $j$ are priority-equivalent ($\pi_i = \pi_j$) yet $\sum_{v_i^s > v_j^s} q_i^s(V, \Pi) < \sum_{v_i^s > v_j^s} q_j^s(V, \Pi)$. Consider the transformation $V'$ that keeps all agents’ valuations unchanged except for agent $i$’s, that is transformed to $v_i^s = \alpha v_i^s + (1 - \alpha) \max_{s' \in S \cup \{o\}} v_i^{s'}$ for $\alpha \in (0, 1)$ and for all $s \in S \cup \{o\}$. Notice that by invariance to affine transformations we must have $Q(V', \Pi) = Q(V, \Pi)$. But for $\alpha$ sufficiently close to zero we will obtain $\sum_{v_i^s > v_j^s} (q_i^s(V', \Pi) - q_j^s(V', \Pi)) v_i^s$ sufficiently close to $-\mu q_j^s(V', \Pi)$ $\max_{s' \in S \cup \{o\}} v_i^{s'}$ and hence strictly negative, a violation of OOR.

Corollary 1 Under any OOR rule, any two priority-equal agents with the same set of objects preferred to the outside option are given the same probabilities of obtaining the outside option.

Corollary 2 Under any OOR rule, the assignment rule only takes into account the ordinal component of agent’s preferences whenever she prefers at most two objects to the outside option.

Note as well that the envy-free property defined above is equivalent to Nash-implementation in direct-revelation games in large markets with a continuum of agents and a rich support of preferences. That is, OOR is a refinement of Nash equilibrium in large markets.

4 Characterization in large markets

For the purposes of this section, we consider the case in which $I = [0, 1]$ which is endowed with the Lebesgue (uniform) measure. We define an economy as a triple $(I, V, \Pi)$ where $V : I \rightarrow \mathcal{V}$ and $\Pi : I \rightarrow \mathcal{P}$ are now Lebesgue-measurable profiles instead of matrices, in order to accommodate for an infinite set $I$. Abusing notation, we will still use $v_i$ for $V(i)$ and $\pi_i$ for $\Pi(i)$, so as to keep it identical to the notation used so far.

Definition 3 An economy $(I, V, \Pi)$ is dense if $V$ is onto: for every preference vector $v \in \mathcal{V}$ there is an agent $i$ such that $v_i = v$.

Definition 4 An assignment $Q(V, \Pi)$ is ordinal if every two priority-equal agents with the same ordinal preferences among non-outside-option objects obtain the same allocation of probabilities for objects above the outside option.

Notice that the definition of ordinality here comes with a plus. We do not only require that agents with same ordinal preferences receive the same probability distribution over objects. We give a special treatment for agents who have the same ordinal preferences when the outside option is ignored.

\footnote{Alternatively we could have defined $I = [0, 1]$ and then a dense economy in a more standard sense: for every $v \in \mathcal{V}$ and every $\varepsilon > 0$ there is $i \in I$ with $\|v_i - v\| < \varepsilon$. The results would also follow in this other specification, with slightly longer proofs than under our main text specification.}
Theorem 1 Any OOR rule satisfies that for the corresponding assignment in each dense economy there is a payoff-equivalent ordinal assignment.

Proof. It is enough if we study the set of agents with identical priority status that prefer object 1 to object 2 to object 3 and so on. Their ordinal preferences differ only in the position the outside option occupies in agents’ ranking. We prove the following induction argument: if for the set of agents in which o occupies the \( n \)-th position in the agents’ ranking (note: putting o always above zero-valuated objects), all of them obtain same probabilities \( q^1, \ldots, q^{n-1} \) for the objects ranked above \( o \), then for the set of agents for which \( o \) occupies the \( (n+1) \)-th position we have that all of them obtain the same probabilities \( q^1, \ldots, q^{n-1}, q^0 \) for the objects ranked above \( o \), in a payoff-equivalent assignment.

Proof of the induction argument. Take an agent of type \( v \) in the latter set of agents and her assigned probabilities \( q^1, \ldots, q^n, q^0 \). Since all agents in this set obtain identical \( q_o \), due to proposition 2, it is WLOG by OOR that a type \( v' \) defined as \( v'^n = v^n - v^n + \varepsilon \), for arbitrarily small \( \varepsilon > 0 \), if \( s \leq n \), and \( v'^n = 0 \) otherwise, obtains the same assignment \( q^1, \ldots, q^n, q^0 \). (Proof: Let \( q'^1, \ldots, q'^n, q^0 \) be the assignment probabilities for type \( v'' \) defined as \( v'^n = v^n \), if \( s \neq o \), and \( v'^n = v^n - \varepsilon \). By OOR we have \( q'^1v^1 + \ldots + q'^nv^n = q^1v^1 + \ldots + q^n v^n \), thus it is WLOG that \( (q'^1, \ldots, q'^n) = (q^1, \ldots, q^n) \) - giving a properly weighed average of the two probability vectors to both types \( v \) and \( v'' \) would be payoff-equivalent and feasible redistribution. Now, \( v' \) is an affine transformation above the outside option of \( v'' \). Hence \( v' \) obtains the same assignment as \( v'' \) by invariance to affine transformations above the outside option.)

Since \( \varepsilon \) can be arbitrarily small, OOR imposes that \( q^1(v^1 - v^n) + \ldots + q^{n-1}(v^{n-1} - v^n) \leq q^1(v^1 - v^n) + \ldots + q^{n-1}(v^{n-1} - v^n) \). But for an agent of the former set of agents with type \( v'' \) defined as \( v''^n = (v^n - v^n) \), we have \( q^1(v^1 - v^n) + \ldots + q^{n-1}(v^{n-1} - v^n) \geq q^1(v^1 - v^n) + \ldots + q^{n-1}(v^{n-1} - v^n) \) by OOR again, thus \( q^1(v^1 - v^n) + \ldots + q^{n-1}(v^{n-1} - v^n) = q^1(v^1 - v^n) + \ldots + q^{n-1}(v^{n-1} - v^n) \). Now, for all types in the former set, the assignment probabilities \( q^1, \ldots, q^{n-1} \) must be (weakly) preferable to \( q^1, \ldots, q^{n-1} \), implying the FOSD condition \( \sum_{s=1}^{n} q^s \geq \sum_{s=1}^{n} q^s, n = 1, \ldots, n-1 \). Together with the former equality we must have \( (q^1, \ldots, q^{n-1}) = (\bar{q}^1, \ldots, \bar{q}^{n-1}) \).

This happens for all types in the latter set. Since \( q^0 \) is the same for all of them (by the first corollary to proposition 2), so is the probability of obtaining object \( n \), which we write as \( \bar{q}^n \). □

The proof is completed by noticing that all agents’ types ranking \( o \) in second position must obtain by OOR the same probability of obtaining object 1, which we write as \( \bar{q}^1 \).

We would complete the characterization by noticing that OOR implies that the assignment probabilities corresponding to the true ordinal preferences must first-order stochastically dominate those obtained if other preferences were declared. In other words, the assignment obeys to an Ordinal Bayesian Nash Equilibrium.² By a well-known result provided by Liu and Pycia (2016) we

²When we talk about an OOR assignment rule, that applies to any distribution of ordinal preferences in a dense economy, then we say that the rule is strategy-proof.
have

**Proposition 3** If $\Pi$ is agent-invariant (i.e. anonymous,) there is a unique OOR and efficient allocation in a dense economy, for instance the one provided by Random Dictatorship.

When the priority status differ among agents, both Deferred Acceptance and Top-Trading Cycles are recommendable, depending on the consideration we pay to the respect for priorities.

References


