Frequency Domain Estimation as an Alternative to Pre-Filtering Low-Frequency Cycles in Structural VAR Analysis

YULIYA LOVCHA1,2* ALEJANDRO PEREZ-LABORDA1

1Universitat Rovira-i-Virgili and CREIP.

Abstract: This paper shows that the frequency domain estimation of VAR models over a frequency band can be a good alternative to pre-filtering the data when some variables contain low-frequency cycles and the treatment of these part of the spectrum is required. As stressed in the econometric literature, pre-filtering destroys the low-frequency range of the spectrum, leading to substantial bias in the responses of the variables to structural shocks, especially if the model is identified with a long-run restriction. Our analysis shows that if the estimation is carried out in the frequency domain, but employing a sensible band to exclude from the likelihood (maybe part) of the frequencies containing the cycle, the resulting VAR estimates and the impulse responses to structural shocks do not present significant bias. This result is robust to several specifications of the low-frequency cycle. An empirical application studying the effect of technology shocks on hours worked is provided to illustrate the results.

Keywords: Impulse-response, band-pass, hodrick-prescott.

JEL Classification: C32, C51, E32, E37

* Address correspondence to Yuliya Lovcha. Universitat Rovira-i-Virgili, Department of Economics. Campus Bellisens. Av. Universitat 1, Reus. Spain. 43834. Email: yuliya.lovcha@gmail.com
1. INTRODUCTION

In this paper, we turn to the frequency domain to circumvent the problems associated with the estimation of the VAR in the presence of low-frequency cycles. Structural VARs are often employed to validate theoretical results since they represent flexible representations of a wide class of structural macroeconomic models.¹ The VAR is generally estimated by OLS, thus including information from all frequencies in the spectrum. The macroeconomic models, on the contrary, generate cross-frequency restrictions at some specific parts, typically at business cycle frequencies. The behavior at zero frequency is modeled only for accepted non-stationary variables, usually by means of unit root processes. For example, productivity and output in RBC models usually contain a unit root driven by a random walk in technology. In practice, however, even a priori stationary data presents significant movements at low frequencies, which often count for a much higher portion of the variance than is specified in theoretical models. In some cases, the existence of this low-frequency cycle is regarded as external to the process modeled, having nothing to do with the aim of the study.² In other cases, these fluctuations are considered a misspecification of the theoretical model, yet with the acknowledgment that we do not really have good models of low-frequency movements.

In either case, in order to compare the predictions of the theoretical models with those that emerge from the VAR, the influence of these low-frequency fluctuations is

¹ For example, Gali (1999) found a negative response of hours worked to a technology shock identified with long run restrictions in a structural VAR. This result has generated a strong debate in macro literature on the validity of Real Business Cycle models to explain economic fluctuation, given that they produce a positive response of hours.

² Francis and Ramey (2009), for example, analyze hours worked and argue that “there are significant (low frequency) demographic and institutional movements over the postwar period, that are features of the commonly used measure of hours worked per capita. Our premise is that these low frequency movements have nothing to do with the kinds of technology shocks typically modeled in RBC theory.”
reduced as much as possible to shorten the discrepancy between the models and the data. With this in mind, researchers routinely apply standard filters prior estimating the VAR, such as the Hodrick-Prescott (HP) or the Band-Pass (BP). Although the practitioner often believes that the filtered series conserve intact its business cycle properties being free from the low-frequency movements, the econometric literature has emphasized that this is not true (King and Rebelo 1993; Harvey and Jaeger, 1993; Cogley and Nason, 1995). In fact, the use of filtering may result just as unattractive for the structural VAR analysis as the presence of the low-frequency cycle itself. Pre-filtering destroys the low-frequency range of the spectrum, over-subtracting the variances at zero and neighboring frequencies producing a strong dip in the periodogram, which reduces significantly the persistence of the filtered series. In addition, this over-subtraction may change significantly the way the variables interact in the VAR by removing their low-frequency co-movement (Fernald, 2007; Gospodinov et al. 2011). In particular, Gospodinov et al. (2011) show that pre-filtering the data may lead to substantial biases in the impulse responses even if the low-frequency correlation is small. The bias is magnified when posterior analysis employs estimates of the spectrum at zero frequency, such as the use of long-run restrictions to identify the structural model, and the underlying data generating process (DGP) is persistent.

In this paper, we propose to estimate the reduced VAR maximizing the Whittle likelihood (Whittle, 1964) over a frequency band that excludes (part) of the low frequencies affected by the cycle to overcome the problems associated with pre-filtering. Note that, unlike filtering, the use of a band does not destroy the low-frequency range nor changes the underlying DGP. Simply put, although the method does not employ the information at the excluded frequencies, the VAR is not driven towards an artificial dip, which implicitly imposes zero variability (and co-movement)
at low-frequencies. We show that this simple approach is able to overcome most of the problems associated with pre-filtering in the presence of a low-frequency cycle.

The idea is motivated from the DSGE literature, in particular, from the work of Hansen and Sargent (1993). The authors were primarily concerned about the estimation of rational expectation models whose seasonal dynamic is misspecified. To reduce the estimation bias, they propose to estimate parameters of the model deleting a narrow band around seasonal frequencies. Building on this work, Cogley (2001) examines how the method performs in the case of a misspecified trend in Real Business Cycle (RBC) models. He concludes that damp the low-frequency component does not remove trend specification errors, thus being not sufficient for constructing robust estimators and tests in RBC because these errors are spread across the entire frequency domain. Recently, Sala (2015) estimates a DSGE model in the frequency domain employing different frequency ranges. He concludes that the forecasting performance and the model parameter estimates vary substantially with the frequency bands employed for estimation.

Our paper is different from this literature in several aspects. First, our concern is not the estimation of a DSGE model but the estimation of a structural VAR. In particular, we focus our study on the resulting impulse responses, which are usually the object of interest in post-estimation analysis. This is important because, as stressed in Gospodinov et al. (2011), pre-filtering the data may lead to biased VAR responses, especially if long-run restrictions are employed. Note also that, unlike Cogley (2001), we deal with variables believed stationary but that present a cycle in the low part of the spectrum, not with variables with trend. Yet, not accounting for the low-frequency cycle also produces a misspecification in the VAR. For example, consider that the true DGP can be correctly approximated by an autoregressive process that presents changes in the
mean of a variable at some points in time. This specification produces a cycle in the low part of the spectrum of this variable. The omission of these changes in the mean may lead to biased VAR inference. In this paper, we show that the estimation of the VAR in the frequency domain over a band that exclude (maybe just part) of the frequencies affected by the low-frequency cycle, does not produce significant bias in either the estimated parameters or in the impulse-responses even if a long-run restriction is employed. Nevertheless, we also show that, as far as a trend shocks are small relative to the business cycle innovations, the proposed methodology can also deal with a random walk trend. Thus, it is a good alternative to the use of filters if the treatment of this part of the spectrum is judged necessary.

We have organized the rest of the paper in the following way. In Section 2, we review the frequency domain estimation of the VAR. We set the Monte-Carlo experiment in Section 3. In Section 4, we present the simulation results and conduct robustness analysis. We also discuss the drawbacks of the proposed method at the end of this section. In Section 5, we provide an empirical example assessing the response of hours worked to a technology shock. Finally, Section 6 concludes. There is also a separate Appendix with the tables and the figures from the robustness section. In addition, we have made available a “user-friendly” code to estimate a structural VAR model in the frequency domain over a frequency band. The Appendix and the code may be downloaded from the author’s site.³

³ https://sites.google.com/site/ylovcha/Research
2. ECONOMETRIC FRAMEWORK

Frequency domain methods are rarely employed in VAR estimation. The reason is simple. In the absence of further complications, the OLS provides an easy and reliable framework for the estimation of these models.\(^4\) Only when complications arise and the estimation requires the maximization of the likelihood, the transition to the frequency domain may result conveniently. For instance, frequency domain methods have proven useful for the estimation of fractal exponents in long-range dependent VARs (Hosoya, 1996; Shimotsu, 2007; Nielsen, 2011; Lovcha and Perez-Laborda, 2015). In this work, we propose to turn to the frequency domain to circumvent the problems associated with the omission (or misspecification) of a low-frequency cycle that is affecting model’s variables.

Consider the MA(\(\infty\)) representation of a reduced-form VAR(p) of the \((N \times 1)\) vector \(Y_t\):

\[
Y_t = \left[I - F(L)\right]^{-1} u_t
\]

(1)

\(I\) is the \((N \times N)\) identity matrix, \(F(L)\) is a finite order lag polynomial matrix with roots strictly inside the unit circle, and \(p\) is the number of lags. Reduced form errors have zero mean and variance-covariance matrix \(\Omega\). The multivariate spectrum of the process (1) at frequency \(\omega\) is given by:

\[
f_j(\omega, \theta) = \left(2\pi\right)^{-1} \left(I - F_p(e^{i\omega})\right)^{-1} \Omega \left(I - F_p(e^{-i\omega})\right)^{-1}
\]

(2)

where \(i\) is the imaginary unit, \(F_p(e^{i\omega}) = F_p e^{i\omega} + \ldots + F_p e^{ni\omega}\) and the vector \(\theta\) contains all the parameters of the model.

\(^4\) A notable exception is Christiano et al. (2003), which emphasizes the advantages of the frequency-domain estimation of these models.
To estimate the process in (1), we propose to use the approximate frequency domain maximum likelihood (Whittle, 1963). To derive the likelihood function, we compute the finite Fourier transform of each series $y_{n,t}$ in the vector as:

$$x_n(\omega_j, y_{n,t}) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} y_{n,t} e^{-i\omega_j(t-1)}$$

(3)

for the Fourier frequencies: $\omega_j = \frac{2\pi j}{T}, j = 0, ..., T/2$. An approximate log-likelihood function of $\theta$ based on $Y_t$ is given (up to constant multiplication) by:

$$\ln L(\theta) = -\sum_{j=\tau}^{T/2} \ln \det f_y(\omega_j, \theta) + trf^{-1}_y(\omega_j, \theta) I_T(\omega_j, Y)$$

(4)

The $N \times N$ periodogram matrix $I_T(\omega_j, Y)$ in the previous formula is defined as

$I_T(\omega_j, Y) = x(\omega_j, Y) x(\omega_j, Y)^*$, where $x(\omega_j, Y)$ is a complex $N \times 1$ vector with entries given by (3), and $x(\omega_j, Y)^*$ is its complex conjugate. For each Fourier frequency $\omega_j$, the elements of the main diagonal of $I_T(\omega_j, Y)$ are the periodograms of the different series evaluated at this frequency, which are real. The off-diagonal elements are cross-periodograms, which are complex.

Note that when $\tau = 0$ the estimation is carried out over all frequencies. However, if $\tau > 0$, the estimation employs a frequency band that excludes the first $\tau - 1$ frequencies. Obviously, in the absence of further complications, excluding frequencies from estimation would result in a senseless loss of information. However, we show that if the treatment of the low frequencies is required, the use of a sensible frequency band is a simple and effective way to circumvent the problems associated with pre-filtering.
The choice of the frequency band is not straightforward. Ideally, the researcher knows, at least with approximation, which frequencies are affected by the cycle after the inspection of the periodogram. However, this may not be always the case in practice. In either case, she should be aware of the trade-off between the precision of the method and the bias that may cause the neglect of some of these frequencies. As an alternative procedure, the researcher may want to exclude the complete low-frequency range and focus the estimation of the VAR on the business cycle (and higher) frequencies only. This strategy is conservative in terms of the bias and is comparable to the standard parameterization of the BP (and HP) filters for business cycle analysis. Defining the 8 years period frequency as the low-frequency boundary of the business cycle, this strategy can be reached by setting $\tau = T/8s$ in the band, where $s$ is the number of data observations per year.\(^5\) We show that even this conservative approach performs remarkably better than pre-filtering.

### 3. THE MONTE CARLO EXPERIMENT

We design our simulation experiment to shed light on two specific questions. First, we assess how the misspecification (the omission or the subtraction) of a low-frequency cycle contaminates the OLS estimation of the structural VAR model. In particular, we study the short sample properties of two estimation alternatives: either ignoring the cycle or pre-filtering the data. These two alternatives are compared to the proposed frequency-domain method. Second, we analyze whether we can recover reliable estimates of the relation among variables, represented by the impulse responses to structural shocks.

\(^5\)Although there is not a single definition of the business cycle, most of the studies employ a cycle with periodicity that ranges between 1.5 and 8 years, which is also the definition we employ in this paper.
3.1 Model Specification

We restrict our attention to a simple bivariate VAR(1) model with a persistent (but stationary) first variable which presents significant low-frequency movements. With this parameterization, we mimic the typical characteristics of the data - macroeconomic series are usually persistent but the fist-difference operator generally clears out the low-frequency cycles. The second variable is assumed to be I(1) and enters the VAR in differences. This allows us to study the impulse responses to structural shocks recovered with a long-run restriction, motivated by the work of Gospodinov et al. (2011). Overall, the hours-productivity literature provides a good example of this specification and, consequently, it is employed in Section 5 to illustrate our results.

3.1.1 The Structural VAR model and identification restrictions

Consider a reduced VAR model of order $p = 1$ for the vector $Y_t = \begin{bmatrix} y_{1,t} & \nabla y_{2,t} \end{bmatrix}^\top$ as in (1).

The structural representation of the model can be written as:

$$
\begin{bmatrix}
y_{1,t} \\
\nabla y_{2,t}
\end{bmatrix} =
\begin{bmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{bmatrix}
\begin{bmatrix}
y_{1,t-1} \\
\nabla y_{2,t-1}
\end{bmatrix} +
A\begin{bmatrix}
e_{1,t} \\
e_{2,t}
\end{bmatrix},
A = \begin{bmatrix} a & b \\
c & d
\end{bmatrix}
$$

(5)

where the errors $e_{1,t}$ and $e_{2,t}$ are assumed to be orthogonal, with variances scaled to unity. The matrix $A$ relates the structural and reduced-form shocks in the following way: $u_t = Ae_t$ and, as a result, $\Omega = AA'$. Simple algebraic transformations lead to the MA representation of the model:

$$
\begin{bmatrix}
y_{1,t} \\
\nabla y_{2,t}
\end{bmatrix} = \frac{1}{\det(F)}
\begin{bmatrix}
(1 - F_{22}L)a + F_{12}Lc & (1 - F_{22}L)b + F_{12}Ld \\
F_{21}La + (1 - F_{11}L)c & F_{21}Lb + (1 - F_{22}L)d
\end{bmatrix}
\begin{bmatrix}
e_{1,t} \\
e_{2,t}
\end{bmatrix}
$$

(6)

The impulse responses to structural shocks are given by the coefficients of the polynomials in (6).
To recover the structural matrix $A$ from the reduced-form estimation, it is necessary to impose one additional restriction. The most popular schemes are the short-run (SR) and the long-run (LR) identification restrictions. The SR identification is a recursive scheme, usually imposing that the shocks to the second variable take at least one period to percolate to the first (i.e., it restricts $A$ to be lower triangular). This is usually attained by the Cholesky decomposition of the variance-covariance matrix of the reduced-form errors. On the other side, the LR identification assumes that, in the long-run, the level of the non-stationary variable $y_{2,t}$ is only affected by its own shock. Thus, the infinite sum of the responses to the first shock $\varepsilon_{1,t}$ of this variable in differences (as it enters the model) must be zero. The LR restrictions can be expressed from (6) as: $F_{21}a + (1 - F_{11})c = 0$, with $\det(F) \neq 0$.

The precision of the estimated impulse responses depends on the estimation accuracy of both the autoregressive coefficients and the structural parameters of the matrix $A$. The SR only requires $\Omega$ to recover $A$ from the reduced-form estimation. On the contrary, LR requires both $F$ and $\Omega$. Consequently, the LR restriction usually leads to less precisely estimated responses presenting wider confidence intervals.

For the Monte Carlo exercise, we choose the following parameterization for the structural VAR:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.5 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}$$

This parameterization satisfies the restrictions required for both SR and LR identification. In addition, the first variable is very persistent and there is a non-trivial

---

correlation among variables at low frequencies. These last two properties are clearly observable in Figure 1, which plots their spectral densities and the coherence. To help in interpretation, we have signaled the frequencies with 20-year and 8-year periods with vertical dotted lines. Thus, the beginning of the business cycle starts straight to the right of the second vertical line.

3.1.2 The process for the low-frequency cycle

We assume that we observe the stationary variable containing a low-frequency cycle:

\[ y_{1,t} = y_{1,t} + C_t \]  

(8)

where \( y_{1,t} \) is the observed value of the first variable and \( C_t \) is the value of the low-frequency cycle. In order to model this cycle, we employ four different specifications, which cover the different approaches found in the time series literature: a deterministic trigonometric cycle, a stochastic trigonometric cycle, a cycle resulting from changes in the mean, and a random walk trend. The four specifications, together with the parameterization employed for the simulations, are briefly described below.

The deterministic trigonometric cycle. We generate this cycle by a cosine wave with frequency \( \omega \):

\[ C_{d,t} = B \cos \omega \]  

(9)

where \( B \) is a parameter regulating the amplitude of the cycle. The trigonometric deterministic cycle contributes to the fluctuations of the first variable at frequency \( \omega \) only, not affecting the variable at other frequencies. Although this assumption is rather restrictive, this type of cycle is interesting from the instructive point of view, given that we can exclude its contribution from the estimation completely. For the Monte-Carlo study, we set the amplitude parameter \( B \) to 4 and the frequency \( \omega \) to \( 2\pi/40s \),
corresponding to a low-frequency cycle of 40 years period, being \( s \) the number of data observations per year.

**The stochastic trigonometric cycle.** Following Harvey (1989), we write the process for this cycle as:

\[
C_{s,t} = S_t,
\]

\[
\begin{bmatrix}
S_t \\
S_t^*
\end{bmatrix} = \rho \begin{bmatrix}
\cos \omega & \sin \omega \\
-\sin \omega & \cos \omega
\end{bmatrix} \begin{bmatrix}
S_{t-1} \\
S_{t-1}^*
\end{bmatrix} + \begin{bmatrix}
\xi_t \\
\xi_t^*
\end{bmatrix}
\]

where \( \xi_t \) and \( \xi_t^* \) are mutually independent zero-mean error terms with equal variance; \( \rho \) is a damping factor, lying between zero and one. If this parameter is equal to one, the cycle is non-stationary. This cycle contributes to the variance of the series at the selected frequency \( \omega \) mostly, but the adjacent frequencies also result affected due to its stochastic behavior. The influence of the cycle to the neighborhood frequencies is larger the closer is \( \rho \) to one. We parameterize the cycle with \( \omega = 2\pi t/40s \) (the same as in the deterministic cycle), and we set the parameter \( \rho = 1 \), which corresponds to the case where the cycle influences neighboring frequencies the most. Finally, we set the standard deviation of the errors terms \( \text{std} (\xi_t) = \text{std} (\xi_t^*) = 0.3\sqrt{\sigma_i^2} \), where \( \sigma_i^2 \) is the variance of the reduced form error of \( y_{1,t} \) (from the matrix \( \Omega \)).

**Changes in the mean.** Changes in the mean contribute to the low part of the spectrum and are often regarded as the cause of the existence of a low-frequency cycle in the data (see e.g. Fernald, 2007 or Canova et al., 2010). To generate this type of fluctuations, we divide the sample of \( T \) observations into three equal subsamples, and we assume different means for each part.
In particular, for the Monte-Carlo exercise, we specify:

\[
C_{m,t} = \begin{cases} 
4; & 1 \leq t < T/3 \\
-2; & T/3 \leq t < 2T/3 \\
3; & 2T/3 \leq t < T 
\end{cases} \tag{11}
\]

which generates a cycle with similar amplitude to the previous two trigonometric cycles.

The random walk trend. This specification does not actually generate a low-frequency cycle, but a zero frequency trend. However, given its prevalence in the literature and the fact that it contributes strongly to the variance of neighboring frequencies, its inclusion into the analysis seems reasonable. We model this specification as:

\[
C_{r,t} = C_{r,t-1} + \xi_{t,t} \tag{12}
\]

with \( std(\xi_{t,t}) = 0.3\sqrt{\sigma_1^2} \), where \( \sigma_1^2 \) is the variance of the reduced form error of \( y_{i,t} \) (from the matrix \( \Omega \)). Note that the RW trend is a special case of the stochastic trigonometric cycle in (10) for \( \omega = 0 \) and \( \rho = 1 \). As higher the variance of the trend’s error term relatively to the one of \( y_{i,t} \), the higher the trend’s contribution to the variance of \( y_{t,c} \) at non-zero frequencies.

Choosing the parameterizations for the models above, we have assumed that the shock in the low-frequency cycle is about three times smaller in magnitude than the one governing the BC fluctuations in (5). This generates an identifiable cycle in the low part of the spectrum. Yet, the low-frequency cycle increases the variability of the data substantially, especially with the stochastic cycle and the random walk, which are non-stationary processes.

Figure 2 depicts examples of the four specifications of the low-frequency cycles \( C_t \), parameterized as for the Monte-Carlo for a realization of 500 observations, together
with the corresponding series $y_{c,t}$ generated from (8). The figure also plots the periodogram of the data with and without the low-frequency fluctuations. As can be seen in the figure, the deterministic cycle contributes only to the selected frequency of 40 years period, with the periodograms of $y_{c,t}$ and $y_{t}$ coinciding at any other point. On the contrary, the other specifications contribute to several frequencies in the low-frequency range. In particular, the trigonometric stochastic cycle also affects the selected 40-years period frequency, but adjacent frequencies result influenced as well. Changes in the mean contribute to very low frequencies, although there is a significant presence of it in the whole low-frequency range. Finally, the parameterized random walk trend has infinite variance at zero frequency and contributes strongly to other frequencies in the low part of the spectrum.

3.2 Simulation Set-Up

To study the sample properties of the three estimation alternatives, we assume normality and employ a structural VAR parameterized as in (7) to generate $I=1000$ bivariate time series of the following lengths: i) $T = 500$ quarterly observations (125 years); ii) $T = 240$ quarterly observations (60 years) and; iii) 720 monthly observations (60 years). In each simulation, we add to the first generated variable a cycle, using for that the four different specifications described in the previous section: the trigonometric deterministic (9), the trigonometric stochastic (10), the changes in the mean (11) and the random walk trend (12). For each simulated dataset, we estimate the parameters of a reduced form VAR(1) with the three estimation alternatives: i) OLS applied to non-filtered data; ii) OLS, pre-filtering the first variable with the HP and BP filters, and; iii) the proposed “Whittle” estimation with non-filtered data. After we recover the structural matrix $A$ with both SR and LR restrictions and we compute the responses of the variables to structural shocks.
For the HP filter, we use a standard value for the penalty parameter ($\lambda = 1600$ for quarterly data, and $\lambda = 14400$ for monthly data). For the BP, we do not allow passing frequencies with a period longer than 8 years, as typically done in the business cycle literature. As noted before, filtering the data completely subtracts the variance at low frequencies, creating a dip in the periodogram. An example of these dips is observable in Figure 3, which plots the periodogram of the filtered by HP and BP filters series generated with the trigonometric deterministic cycle presented in Figure 2. To help in interpretation, the figure also depicts the periodogram of the series generated from the VAR.

For the proposed “Whittle” method, we employ two different bands for estimation. The first band excludes frequencies with periods longer than 8 years. Thus, it employs only the information contained at business cycle and higher frequencies, which is the same parameterization that we use in the BP filter. The second band excludes frequencies periods longer than 20 years. The beginnings of these bands are marked with the vertical dotted lines in Figure 2. Note that the 20-years period band employs more information but sometimes neglects frequencies that are significantly affected by the cycle. This allows us to study the trade-off between the precision of the method and the bias that may cause the neglect of these frequencies.

4. DESCRIPTION OF THE RESULTS

We organize the results as follows. We discuss below the results of the Monte-Carlo experiment for quarterly data of length $T=500$ with the trigonometric deterministic cycle. We have chosen this cycle specification as a benchmark because its contribution is restricted to one frequency only. That is, if this frequency is excluded from the analysis (either by filtering or by employing our proposed frequency domain method), the resulting bias (if any) can be attributed to the filtering or to the estimation method,
but not to the residual influence of the cycle. The results for the other three specifications for the cycle and simulations with 60 years of data (both quarterly and monthly) are discussed as a robustness analysis.

Table 1 and Figure 4 summarize the simulation results for data containing the deterministic trigonometric cycle. Table 1 presents the mean of estimated parameters across the 1000 replicas. The first half of the table (Table 1.a) reports results for the estimation of the autoregressive coefficients. Table 1.b provides the mean of the structural parameters of the matrix $A$ recovered with both SR and LR restrictions. The numbers in parenthesis correspond to the 2.5th and 97.5th estimated percentiles. For the OLS with filtered data, we present only the results for the BP filter, since the results obtained with the HP are virtually identical, but they are available upon request.

As can be seen in the table, neglecting the low-frequency cycle strongly influences the OLS estimates of the autoregressive parameters. The autoregressive process for the first variable has to adjust for the additional low-frequency variance, increasing the persistence of the variable. As a result, the estimates of the own-autoregressive term of the series affected by the cycle result upward biased. The presence of the cycle also affects the estimates of the cross-autoregressive term of the second variable, which is biased downwards. Concerning the estimation of the structural matrix $A$, given that the variance-covariance matrix $\Omega$ is relatively well estimated, the parameters recovered with SR identification do not present strong bias. The LR identification also recovers a reliable estimate of this matrix due to the fact that the strong positive bias in $F_{11}$ is compensated by the negative bias in $F_{21}$. On the other side, when the first variable is pre-filtered by HP or BP filters, it becomes substantially less persistent, given that the VAR is driven to an artificial dip created by the filter. Consequently, the OLS estimation with filtered data produces negatively biased
estimates of the autoregressive coefficients. The estimation of the structural matrix \( A \) is also biased, especially when it is recovered with the LR restrictions. Note that the bias in the contemporaneous effect of the second structural shock on the first variable is particularly strong. Finally, the table also reports the results from the proposed frequency domain method. The Whittle estimation over a frequency band performs very well in terms of the bias. Neither the autoregressive coefficients from the matrix \( F \) nor the structural parameters from \( A \) present significant short sample estimation bias, even if LR restrictions are employed for identification. As expected, the method becomes less precise if the estimation is produced employing the 8-year period band compared to the use of the 20-years period band, given that less non-affected frequencies are included for estimation (recall that the trigonometric deterministic cycle only influences the 40 years period frequency, which is completely removed by the two bands).

Figure 4 plots the Monte-Carlo results for the impulse responses. We depict here only the responses of the first variable (i.e., the one that is influenced by the low-frequency cycle) since these responses concentrate the larger disparities and have received more attention in the macroeconomic literature.\(^7\) Figure 4.a depicts the responses of the first variable to its own structural shock recovered with SR and LR restrictions. The response of this variable to the second shock is provided in Figure 4.b. In both cases, the solid lines represent the mean value of the response across the simulations and the dashed lines the 2.5 and the 97.5 percentiles. We collect the responses obtained from a VAR estimated with OLS (using both filtered and non-filtered data) in the first column of each figure, while the second column analyses the impulse responses from a VAR estimated with our proposed frequency-domain method.

\(^7\) None of the estimation methods shows very strong bias in the responses of the second variable. Yet, the proposed methodology still performs better than the other estimation methods in the Monte Carlo study. These responses are available upon request to the authors.
(using the two selected frequency bands). In order to interpret the results, we have included the true responses in each graph (dotted line).

Consistent with the analysis of the estimated coefficients, the OLS estimation with non-filtered data produces largely overstated responses of the first variable to system shocks that are considerably more persistent than the true responses due to the neglected influence of the low-frequency cycle. Both the own- and cross-estimated responses are upward biased irrespective of the restrictions employed to identify the shocks. The percentile bands recovered with the LR scheme are substantially wider, reflecting the larger uncertainty associated with this identification method. As can be seen in the figure, filtering does not improve the results at all. The responses are strongly biased downward, with the true response lying always above the percentile bands. Note that the sampling uncertainty in the estimated responses is substantially smaller than in OLS estimation with non-filtered data, especially in the responses recovered with LR restrictions. However, it is important to highlight that this reduction in uncertainty comes from the removal of a sizable portion of the true variance of the first variable by the filter. In addition, given that the responses are strongly biased, the narrow bands not containing the true values indicate that, with very high probability, the estimated response is going to be far from the true response regardless of the identification method applied. Besides, the analysis of the response to the second shock (Figure 4.b) identified with a LR restriction is especially relevant. Recall that this response is not only affected by a bias in the estimated autoregressive coefficients, but also by a strong bias in the estimated structural parameter governing the contemporaneous effect. As can be seen in the figure, the mean of the estimated responses becomes negative in the short-run, with the zero value lying outside the percentile bands. This result is especially important because it provides an example of how pre-filtering may lead towards
misleading inference in the VAR, not only on the magnitude and the persistence of the responses but also on their sign, in line with the results of Gospodinov et al. (2011). Finally, the right column of the two figures collect the mean responses from a VAR estimated in the frequency domain with the two selected bands (20-years and 8-years period). As can be seen in the figure, none of the estimated responses presents significant bias, neither the ones recovered with LR restrictions nor the ones using the SR scheme. In particular, note that there is no bias in the response to the second shock recovered with LR restrictions. As expected, the sampling uncertainty grows with the number of frequencies excluded, but even the 8-years period band performs very well in terms of the bias and appears precise enough for inference. More important, as far as the confidence intervals for impulse responses reflect the true uncertainty observed in simulations, the econometrician would not be misled in inference using the proposed method regardless the identification scheme she uses. Yet, our results suggest a close inspection of the periodogram in order to select a sensible band prior the estimation in order to increase the precision.

4.1 Robustness Analysis

We check for robustness along two different directions. First, we carry out additional simulations for the other three specifications for the low-frequency cycle. After that, we study if the results of the previous section are robust to changes in the sample size or the frequency the data is available. The tables and the figures from this section are provided in a separate appendix to this work to conserve space.

4.1.1. Alternative specifications of the low-frequency cycle

In the previous section, the first variable is affected by a trigonometric deterministic cycle, which only alters a single frequency in the spectrum (40 years period). In this way, the two selected bands for the frequency domain estimation contain only
frequencies not influenced by the cycle. We relax this assumption with the other specification for the low-frequency cycle: the trigonometric stochastic cycle, the changes in the mean and the random walk trend.

Overall, the results of the simulation study are very similar to those obtained with the trigonometric deterministic cycle. A summary is provided in Table A1 and Figures A1, A2, and A3 in the separate Appendix to this work. Again, the proposed frequency domain method performs remarkably better than the two OLS alternatives. Yet, now the use of an 8-years period band usually leads to slightly better results in terms of the bias, but at the cost of penalizing precision. Obviously, the larger a number of neglected frequencies, the larger the bias incurred by the method. Yet, our results seem to indicate that the proposed methodology is not extremely sensible to a small amount of contamination, which again advocates for the selection of a sensible band upon the study of the periodogram.

4.1.2. Modifying the length and the frequency of the artificial data

The empirical studies of the business cycle are usually based on quarterly or monthly data, which often leads to the use of relatively short datasets. To see how the proposed methodology behaves with the lengths found these studies, we carry out additional simulations for shorter datasets of T=240 quarterly observations (corresponding to 60 years of data). Results are summarized in Table A2 and Figure A4 in the Appendix. Overall, we find no significant differences with respect to the use of T=500 quarterly observations, except an expected small increase in the sampling uncertainty associated with all the estimation methods.

Finally, we study if the use of higher frequency data improves the precision of the proposed frequency domain estimates. A larger number of observations per year increase the resolution of the periodogram in a very particular way since it makes
available a larger set of Fourier frequencies situated at the medium, and the high-frequency range of the spectrum (thus not affected by the low-frequency cycle). To study this issue, we draw artificial datasets of T=720 monthly observations (which correspond to the short span of 60 years). Figure A5 and Table A2 (right) in the Appendix collect these results. Again, the OLS alternatives are penalized either by contamination or by filtering, while the frequency domain estimates do not show significant bias. However, the most interesting result is that, although the use of monthly data reduces the uncertainty of all the methods, the frequency domain alternatives result much more benefited from it. This is especially relevant for the estimation employing the 8-years period band, which now presents virtually unbiased and very precise estimated responses, comparable to those obtained with the 20-years band.

4.3 Overall Evaluation and Limitations

In the simulation study, we show that the proposed frequency domain method is a useful alternative to deal with omitted low-frequency cycle in structural VAR analysis. It outperforms significantly the OLS estimation with non-filtered data and overcomes the strong problems associated with pre-filtering. However, like all econometric methodologies, it has several drawbacks and limitations, which should be taken into account in empirical applications.

The first of them is that the method requires the selection of a frequency band for estimation. The choice of the band can sometimes be ambiguous, and there is a trade-off between the bias due to the possible neglected contaminated frequencies and the precision of the estimates. In most situations, a close inspection of the periodogram may help to select a sensible band for estimation. On the contrary, if the researcher wants to restrict the estimation to the business cycle and higher frequencies, the precision of the
method suffers, especially if the true process is very persistent. As it is shown in the robustness section, the use of monthly data may help a lot in these situations. Yet, the estimation shows no significant bias and, as far as the confidence intervals capture the true uncertainty, the researcher is not going to make mistakes inferring with the estimated model no matter the identification scheme she applies. On the other hand, standard filtering methods also require the selection of a band, and their use leads to strong estimation bias and erroneous VAR inference.  

The second limitation concerns the use of maximum likelihood estimators, which are more computationally intensive, especially if the problem is big and the use of bootstrap is required. In addition, unlike with OLS estimation, it may be necessary to enforce stationarity if the process is very persistent. Moreover, in the presence of quasi-unit roots, the precision of the estimated error variance may be affected. As shown in the literature, the use of a taper may alleviate this problem. We illustrate the application of these strategies in the empirical example of the next section. Besides, note that these problems are not exclusive of the proposed frequency-domain method, and arise as well using maximum likelihood estimation in the time domain.

---

8 With the BP filter, the choice of the band is done exactly in the same way than the proposed frequency domain methodology. The issue is more problematic using the HP filter since it is not always easy to establish a one to one relation between the choice of the penalty parameter and the frequencies removed.

9 Dahlhaus (1988) shows that tapering reduces the leakage effect of the periodogram as estimate of the true spectrum. He finds that the new estimate competes well with the Burg estimate for an AR(14) model where roots of the characteristic equation are complex and close to the unit circle. There are several other studies concerning the use of data tapers in univariate AR models (Pukkila and Nyquist, 1985; Kang, 1987; Hurvich, 1988; or Zhang, 1991). The general conclusion of these works is that tapering should be conducted. However, the specific taper and amount applied of it appear not to be of much importance.
5. EMPIRICAL EXAMPLE

In order to illustrate the results of the simulation study, we assess the response of hours to a (positive) technology shock. This response has been the subject of a strong debate in the literature since the work of Gali (1999), who cornered the real business cycle models (RBCs) finding a negative response of hours in a structural VAR identified with a LR restriction. The heart of the debate centers on how hours should enter the VAR. If hours are either differenced or filtered, and the technology shock is identified with a LR restriction (Gali, 1999; Canova et al., 2010), the response of hours is negative. However, when the hours enter in levels (Christiano et al., 2003), the response is positive even if the same LR identification is applied. Fernald (2007), Francis and Ramey (2009) and Canova et al. (2010), analyze the low-frequency movements in hours worked arguing that the series contains some noise at very low frequencies that must be treated prior VAR estimation. For Fernald (2007) and Canova et al. (2010), the noise is the consequence of some changes in the mean of the process. Francis and Ramey (2009) attribute these changes to demographic and sectoral movements affecting the US labor force. However, Gospodinov et al., (2011) call into attention that differencing or filtering this noise corrupts the application of the LR scheme, which may lead to erroneous inference.

To assess the response of hours, we construct a dataset similar to that of Christiano et al. (2003), collecting data from the Federal Reserve Bank of St. Louis (FRED). The dataset runs from 1948:1 to 2009:4, thus covering a similar period than other popular datasets in the literature and contains data from all sectors (including the farm sector). The total business productivity is defined as the log of the output per hour of all persons (OPHPBS), and the hours worked as the log of the ratio of the business

\[ \text{OPHPBS} \]
hours of all persons (HOABS) to the civilian non-institutional population over the age 16 (CNP16OV). The last series is converted to quarterly by taking the average of the monthly observations inside the quarter. Except for population, all the series are seasonally adjusted.

Figure 5 presents the estimated impulse responses up to 20 quarters (5 years) recovered from a VAR(4) identified with both SR and LR restrictions together with one standard deviation bootstrapped bands.\textsuperscript{11} We have included the SR responses in order to illustrate the results of the simulation study, however, although sometimes employed for this purpose, the recursive assumption implied by this scheme (i.e., that the productivity shock takes at least one quarter to percolate to hours) is often judged as too restrictive to recover technology shocks. We also include in the figure the periodogram of the first variable and its estimated spectrum from the VAR.

The results from OLS estimation with rough data are collected in the first row of the figure. Consistent with the previous analysis, the periodogram of the non-filtered hours presents a very strong peak at the very low part of the spectrum, corresponding to a cycle of about 60 years period. Consequently, the estimated process is very persistent (the highest eigenvalue of the companion form matrix equals 0.98). As can be seen in the figure, the responses recovered with SR and LR restrictions are both positive, but the response recovered with LR restrictions is much stronger presenting wider confidence bands.

As a second step of the analysis, we BP filter hours before OLS estimation.\textsuperscript{12} As standard in the business cycle literature, we filter all the frequencies with a period longer than 8 years, but the results are robust to the use of a 20 years period band in the

\textsuperscript{11} We employ 1000 nonparametric bootstrap replicas.

\textsuperscript{12} Again, the analyses of the HP and BP filtered series do not differ, so we present in the paper the results from the BP only to save space.
The responses estimated with filtered data are depicted in the second row of the figure. Filtering the data over-subtracts the variance at low frequencies, producing a strong dip in the periodogram, which is visible in the left plot. Consequently, the estimated model attempts to reproduce that dip. As a result, the model is significantly less persistent than was estimated with rough data. The SR identification returns positive responses, but smaller than those obtained without filtering the data. On the other hand, the LR identification returns negative and significant responses, in compliance to the well-known puzzle in the literature. Confidence bands for LR responses are of the same magnitude as SR responses since filtering removes a substantial part of the variance.

Finally, we employ the proposed frequency domain methodology for estimation of the VAR. We assume that the 60 years period cycle in hours is external (in the sense of Fernald, 2007; Canova et al., 2010; and Francis and Ramey, 2009) and employ the 20 years band used for simulations in order to exclude the lesser number of frequencies as possible. Due to the strong persistence of the data, we ensure stationarity and apply a cosine taper before estimation (Tukey, 1967). Results from the frequency domain estimation are depicted in the last row of Figure 5. The dots in the periodogram of tapered hours signal correspond to the excluded frequencies. Bootstrapped bands for impulse responses are computed with non-parametric bootstrap in the frequency domain (Berkowitz and Diebold, 1998). The estimated process is less persistent than the one

---

13 The periodogram shows another peak on the border of business cycle range, which may contain useful information for business cycle analysis. However, the results do not change significantly including or excluding this frequency with BP filter.

14 Although including more frequencies for estimation would be useful to reduce the variance associated to persistence, this is not feasible due to the small amount of observations.

15 The ratio of taper to constant sections (the parameter \( \alpha \)) is taken equal to 0.125. The results are robust to the choice of the taper and to other parameterizations.
estimated by OLS using raw data but substantially more than the one estimated with filtered hours. In particular, note that, unlike with the use of the filter, the spectrum is not driven towards zero in the low-frequency range. The responses of hours recovered with the SR and the LR schemes are positive and very similar to each other, smaller in magnitude than those obtained by OLS with non-filtered data. Furthermore, the confidence bands for LR contain the responses recovered with the SR scheme. Note that with the other estimation methods, the responses recovered with the LR and the SR schemes are not compatible with each other.

To sum up, the proposed methodology backs up positive and similar responses under LR and SR restrictions, smaller than those found neglecting the external noise. On the other hand, if the data is filtered, the response recovered with an LR scheme is negative and significant, even if the same 20-years cycle band used for frequency domain estimation is employed in the BP filter. In the view of the results of the Monte-Carlo study, the negative response found with filtered data under LR restrictions seems to be a direct consequence of the use of the filter, unless in our dataset.

5. CONCLUSIONS

Our main conclusion is that the proposed methodology is a good alternative to pre-filtering if the low frequencies have to be treated prior VAR estimation. In this situations, the use of filtering in combination with OLS can be even more harmful that the neglect of the cycle. Not only the estimates of the autoregressive parameters are strongly biased, but also the estimates of the structural parameters under the LR scheme. As a result, the true responses in the DGP and the one recovered from the model may diverge completely, in not only magnitude and persistence but also in sign. On the contrary, the proposed frequency domain method performs remarkably well overcoming most of the problems of filtering. If a sensible band is employed in estimation, neither
the model estimates nor the impulse responses present signs of significant bias. This result is robust to several specifications of the low-frequency cycle. We have shown the utility of the method in applied work assessing the response of hours to a productivity shock, which we have found positive, proposing a solution to a largely debated puzzle in the literature.

Like any empirical methodology, our approach suffers from several shortcomings that we have mainly discussed in the text. In our view, the most important drawback is the trade-off between the bias and the precision faced when selecting a frequency band for estimation. Yet, pre-filtering the data also requires the selection of a band. Moreover, we show that although the precision suffers, the proposed methodology works much better than pre-filtering even excluding the completely low-frequency range and that the researcher cannot go wrong no matter the identification scheme she applies. Nevertheless, it would be interesting to elaborate a testing methodic, based on the stability of the estimated parameters, to choose the cutting frequency. We leave this interesting question for further research.

References


## TABLES AND FIGURES

### Table 1 - Monte Carlo results: trigonometric deterministic; T=500-quarters

#### Tab. 1.a Autoregressive parameters from the matrix F

<table>
<thead>
<tr>
<th></th>
<th>True</th>
<th>OLS, non-filtered</th>
<th>OLS, BP</th>
<th>Whittle, 20</th>
<th>Whittle, 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{11}$</td>
<td>0.8</td>
<td>0.94 [0.92; 0.95]</td>
<td>0.61 [0.52; 0.69]</td>
<td>0.79 [0.72; 0.85]</td>
<td>0.79 [0.67; 0.91]</td>
</tr>
<tr>
<td>$F_{12}$</td>
<td>0.3</td>
<td>0.30 [0.22; 0.38]</td>
<td>0.19 [0.12; 0.26]</td>
<td>0.29 [0.23; 0.37]</td>
<td>0.30 [0.20; 0.40]</td>
</tr>
<tr>
<td>$F_{21}$</td>
<td>0.1</td>
<td>0.02 [-0.01; 0.05]</td>
<td>0.08 [0.00; 0.16]</td>
<td>0.10 [0.02; 0.17]</td>
<td>0.10 [-0.00; 0.21]</td>
</tr>
<tr>
<td>$F_{22}$</td>
<td>0.5</td>
<td>0.49 [0.40; 0.57]</td>
<td>0.52 [0.43; 0.60]</td>
<td>0.49 [0.40; 0.57]</td>
<td>0.49 [0.38; 0.60]</td>
</tr>
</tbody>
</table>

#### Tab 1.b Structural parameters from the matrix A

<table>
<thead>
<tr>
<th></th>
<th>True</th>
<th>OLS, non-filtered</th>
<th>OLS, BP</th>
<th>Whittle, 20</th>
<th>Whittle, 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>SR IDENTIFICATION</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>1.0</td>
<td>1.05 [0.97; 1.11]</td>
<td>0.95 [0.89; 1.01]</td>
<td>0.99 [0.93; 1.06]</td>
<td>1.00 [0.92; 1.07]</td>
</tr>
<tr>
<td>c</td>
<td>-0.5</td>
<td>-0.52 [-0.61; 0.43]</td>
<td>-0.55 [-0.64; -0.45]</td>
<td>-0.49 [-0.59; -0.40]</td>
<td>-0.49 [-0.59; -0.39]</td>
</tr>
<tr>
<td>d</td>
<td>1.0</td>
<td>0.99 [0.93; 1.05]</td>
<td>0.98 [0.91; 1.03]</td>
<td>0.99 [0.93; 1.06]</td>
<td>1.00 [0.92; 1.07]</td>
</tr>
<tr>
<td>LR IDENTIFICATION</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>1.0</td>
<td>1.02 [0.82; 1.10]</td>
<td>0.90 [0.81; 0.97]</td>
<td>0.98 [0.89; 1.05]</td>
<td>0.95 [0.71; 1.05]</td>
</tr>
<tr>
<td>b</td>
<td>0.0</td>
<td>-0.01 [-0.64; 0.37]</td>
<td>-0.31 [-0.48; -0.13]</td>
<td>-0.01 [-0.35; 0.33]</td>
<td>-0.02 [-0.50; 0.74]</td>
</tr>
<tr>
<td>c</td>
<td>-0.5</td>
<td>-0.50 [-0.84; 0.18]</td>
<td>-0.19 [-0.38; -0.00]</td>
<td>-0.48 [-0.79; -0.12]</td>
<td>-0.47 [-1.05; 0.03]</td>
</tr>
<tr>
<td>d</td>
<td>1.0</td>
<td>1.00 [0.74; 1.15]</td>
<td>1.10 [1.02; 1.18]</td>
<td>1.00 [0.78; 1.13]</td>
<td>1.00 [0.38; 1.14]</td>
</tr>
</tbody>
</table>

**Notes:** OLS - OLS estimation with non-filtered data; OLS, BP - OLS estimation data filtered with the BP; Whittle, 20 and Whittle, 8 denote the simulation results from the “Whittle” estimation of the VAR excluding frequencies with a period longer than 20 and 8 years, respectively.

### Figure 1 - Spectral densities and coherence of the parameterized VAR

**Notes:** a) the spectral densities and the coherence are depicted up to 1.5-year period frequency; b) the vertical dotted line signals the frequencies with periods 20-years (left) and 8-years (right)
Figure 2 - Examples of contamination by low-frequency cycles

SAMPLE OBSERVATIONS

PERIODOGRAM

Trigonometric Deterministic

Trigonometric Stochastic

Changes-in-Mean

Random walk trend

Notes: a) the periodograms are depicted up to the 1.5 years period frequency; b) the vertical dotted lines signal frequencies with periods 20 years (left) and 8 years (right); c) \( y_1 \) - first variable without low-frequency fluctuations, \( yC_1 \) – variable including the low-frequency cycle \( C \).

Figure 3 - Periodograms of the filtered series; trigonometric deterministic cycle

HP

BP

Notes: \( y_1 \): non-contaminated fist variable, \( y_1hp \), and \( y_1bp \): HP and BP filtered series respectively.
Figure 4 - Monte Carlo results: trigonometric deterministic, T=500 quarterly

**Fig. 4.a** Impulse response of the first variable to the first shock

**Fig. 4.b** Impulse response of the first variable to the second shock

**Notes:** a) impulse responses are depicted up to a 5 years horizon; b) solid lines depict the average response across simulations. Dashed lines the percentiles 2.5 and 97.5%.
Figure 5 Empirical example: the response of hours to a positive technology shock

Notes: CI denotes one standard deviation confidence intervals for impulse responses computed with non-parametric bootstrap.