Measuring $r^*$: An Alternative Specification
Banco de España

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The analysis and conclusions set forth are those of the authors and do not indicate concurrence by other members of the research staff or the Board of Governors.
"That’s based on our view that the neutral nominal federal funds rate...is currently quite low by historical standards. That means that the federal funds rate does not have to rise all that much to get to a neutral policy stance."

![Graph showing the evolution of the federal funds rate over time, with two shaded periods and a trend line representing the neutral nominal federal funds rate.](Image)
Main Results

We build on (Holston, Laubach and Williams (2003, 2015, 2016) (HLW) to study the effect of alternative specifications in measuring \( r^* \).

- **Contribution 1:** We obtain richer time-series dynamics of \( r^* \) by incorporating the uncertainty of estimating all the parameters in a single step using Bayesian methods with less informative priors.

- **Contribution 2:** Models that do not assume permanent shocks on the non-growth component of \( r^* \) produce an elevated level of the median \( r^* \) estimate after the great recession, about 1.5% higher than HLW in 2016Q3.

- **Caveat:** Our estimates of \( r^* \) are not statistically significantly different from HLW. However, they could lead to economically different policy advice.
The original HLW model

\[ \tilde{y}_t = a_{y,1} \tilde{y}_{t-1} + a_{y,2} \tilde{y}_{t-2} + a_r \frac{\sum_{j=1}^{2} (r_{t-j} - r^*_{t-j})}{2} + \varepsilon_{\tilde{y},t} \]

\[ \tilde{y}_t = y_t - y^*_t \]

\[ \pi_t = b_\pi \pi_{t-1} + (1 - b_\pi) \pi_{t-2,4} + b_Y \tilde{y}_{t-1} + \varepsilon_{\pi,t} \]

\[ y^*_t = y^*_{t-1} + g_{t-1} + \varepsilon_{y^*,t} \]

\[ g_t = g_{t-1} + \varepsilon_{g,t} \]

\[ r^*_t = g_t + z_t \]

\[ z_t = z_{t-1} + \varepsilon_{z,t} \]
We extend the HLW model to allow for stationarity of $g$ and $z$

\begin{align*}
\tilde{y}_t &= a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} + \frac{ar}{2} \sum_{j=1}^{2} (r_{t-j} - r_{t-j}^*) + \varepsilon_{\tilde{y},t} \\
\tilde{y}_t &= y_t - y_t^* \\
\pi_t &= b_{\pi}\pi_{t-1} + (1 - b_{\pi})\pi_{t-2,4} + b_{\gamma}\tilde{y}_{t-1} + \varepsilon_{\pi,t} \\
y_t^* &= y_{t-1}^* + g_{t-1} + \varepsilon_{y^*,t} \\
g_t &= \mu_g + \rho_g (g_{t-1} - \mu_g) + \varepsilon_{g,t} \\
r_t^* &= E_t [g_{t+1} + z_{t+1}] \\
z_t &= \rho_z z_{t-1} + \varepsilon_{z,t}
\end{align*}
Model estimation

- We use Bayesian methods to avoid the “pile-up” problem (DeJong and Whiteman 1993, Kim and Kim 2016) and allow for joint estimation of all parameters in a single step with loose priors.
- Original LW method from Stock and Watson 1998 involves imposing:

\[
\lambda_g \equiv \frac{\sigma_g}{\sigma_{y^*}}, \quad \lambda_z \equiv \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}.\]

LW Method:

- Step 1 Simplify model, estimate \( \lambda_g \).
- Step 2 Impose \( \lambda_g \) value, use alternative simplification, estimate \( \lambda_z \).
- Step 3 Impose \( \lambda_g \) and \( \lambda_z \), estimate remaining parameters.

Imposing LW estimates of \( \lambda_g \) and \( \lambda_z \) used in their third step, we can replicate HLW estimates under Bayesian methods.
We estimate the key parameters $\rho_g$, $\mu_g$ and $\rho_z$ of the extended model. We consider four alternatives:

- **Model I** $\rho_g = 1$, $\rho_z = 1$, (only different estimation technique)
- **Model II** $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
- **Model III** $\rho_g = 1$, $\rho_z$ estimated
- **Model IV** $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Model I: \( \rho_g = 1, \rho_z = 1 \)

Smoothed \( r^* \) Draws in Model I (Shaded 10\(^{th}\) to 90\(^{th}\) Percentile)
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model III: $\rho_g = 1$, $\rho_z$ estimated

Smoothed $r^*$ Draws in Model III
Headwinds

- Frequently, “headwinds” are cited as a reason for why the level of $r^*$ is still so low.
  - “...lingering sense of caution on the part of households and businesses in the wake of the trauma of the Great Recession.” (Yellen, 3/3/17)
- $z_t$ is the “special sauce” (Williams, 2015 Brookings), it is all the things that are not economic growth.
  - There is nothing that says the components have to be stationary or persistent.
  - In the current version of the model, there is no data specifically aimed at estimating $z_t$.
  - $z_t$ soaks up the variation in the rate gap that doesn’t appear to be linked to growth.
  - Headwinds seem like they should be temporary.
Model I: $\rho_g = 1, \rho_z = 1$
Model III: $\rho_g = 1$, $\rho_z$ estimated

Smoothed $z$ in Model III
Model III: $\rho_g = 1, \rho_z$ estimated

Posterior Distribution of $\rho_z$ in Model III
Model I vs. Model III

- The difference between the median path of $r^*$ in Model I and Model II doesn’t seem large.
- Model III looks different than Model I (in terms of the median path).
- The only difference between the models is that Model I has a degenerate prior on $\rho_z \equiv 1$.

Model Comparison

- Bayes Factor for nested models reduces to the Savage-Dickey density ratio (Dickey, 1971).

$$B_{III,I} = \frac{pr(Y|M_{III})}{pr(Y|M_I)} = \frac{p_{III}(\rho_z = 1)}{p_{III}(\rho_z = 1|Y)}$$
Savage-Dickey Density Ratio

Figure: Prior and Post. $\rho_z$

Figure: Area around $\rho_z = 1$

$$B_{III,I} = \frac{p_{III}(\rho_z = 1)}{p_{III}(\rho_z = 1|Y)} = 8.4$$
Connection to Theory

- The $r^*$ equation is a linearized euler equation, we can denote the stochastic discount factor (SDF) by $S$:

  $$e^{-r^*_t} = E_t [S_{t+1}]$$

- Consider an SDF that differs from that of log-utility by an extra term ($\tilde{Z}$) as in Campbell and Cochrane, Epstein-Zin, etc. Then we have

  $$r^*_t = \log E_t \left[ \frac{C_{t+1}}{C_t} \tilde{Z}_{t+1} \right] = \log E_t \left[ e^{g_{t+1} + \tilde{z}_{t+1}} \right] \approx E_t [g_{t+1} + z_{t+1}]$$

- $z_t$ can be interpreted as an asset pricing term that measures the separation from log-utility of the SDF. We can give $z$ this “headwinds” interpretation.
Future Work: Stochastic Discount Factor Interpretation

Define the SDF as follows

\[ S_{t+1} \equiv e^{-g_{t+1}-z_{t+1}-\frac{1}{2}V_t[g_{t+1}+z_{t+1}]} \]

This SDF give an interpretation to the \( g \) and \( z \) process of LW

\[ r^* = -\log E_t [S_{t+1}] = g_t + z_t \]

and we can use it to price financial assets. We add equations to price specific cashflows for financial assets, starting with aggregate equities.

\[
0 = \log E_t \left[ S_{t+1} \frac{P_{t+1} + D_{t+1}}{P_t} \right] = \log E_t \left[ S_{t+1} \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} \frac{D_t}{P_t} \right]
\]

\[
0 = \log E_t \left[ \exp \left[ -g_{t+1} - z_{t+1} + \log (1 + \exp [p d_{t+1}]) + \Delta d_{t+1} - p d_t \right] \right]
\]
Future Work: Additional equations

We close the system by log-linearizing the previous equation and adding an empirical model for the log price dividend ratio and the log-dividend growth

\[
(1 - \gamma^{pd}(L))pd_{t+1} = pd + \theta^{pd}(L)\varepsilon^{pd}_{t+1} + \theta^{pd,z}(L)\varepsilon^{z}_{t+1} + \theta^{pd,g}(L)\varepsilon^{g}_{t+1}
\]

\[
(1 - \gamma^{\Delta d}(L))\Delta d_{t+1} = \mu\Delta d + \theta^{\Delta d}(L)\varepsilon^{\Delta d}_{t+1} + \theta^{\Delta d,z}(L)\varepsilon^{z}_{t+1} + \theta^{\Delta d,g}(L)\varepsilon^{g}_{t+1}
\]

\[
pd_t \approx -E_t [g_{t+1}] - E_t [z_{t+1}] + E_t [\Delta d_{t+1}] + \log \left(1 + e^{E_t[pd_{t+1}]}\right) - \frac{1}{2} V_t [g_{t+1} + z_{t+1}] + \frac{1}{2} V_t [-g_{t+1} - z_{t+1} + \kappa_t dp_{t+1} + \Delta d_{t+1}]
\]

where \( \kappa_t \equiv \frac{e^{E_t[pd_{t+1}]} }{1 + e^{E_t[pd_{t+1}]} } \)
Conclusion

- Single-step Bayesian estimation with less informative priors shows deeper drops and subsequent recoveries after recessions, in contrast to multi-step MLE results.
- When $z$ is not assumed to be a random walk, we estimate a greater recovery of $r^*$ since the lows of the great recession, reaching closer to 2% at the end of 2016Q3.
- Our conclusion is that permanent shocks to $z$ (and thus, in our minds, the SDF) are needed to produce a persistent low level of $r^*$ after the great recession.
- The dynamics of $z$ are hard to estimate with this data.
  - This data does not strictly prefer one model over the other.
  - More structure around $z$ may be helpful.
APPENDIX
Laubach-Williams Methodology

Laubach-Williams 3-part Estimation

Step 1  Hold $g$ constant, drop real rate gap from model, then:
- Get estimate of potential output, $\hat{y}^*$, compute $\Delta \hat{y}^*$
- $\lambda_g$ is equal to Andrews and Ploberger (1994) exponential Wald statistic for the test of a structural break at unknown date in $\Delta \hat{y}^*$.

Step 2  Impose $\lambda_g$ value from Step 1, include real rate gap, but hold $z$ constant, then:
- Estimate the simplified model
- $\lambda_z$ is equal to Wald statistic for the test of a shift in the intercept of the IS equation.

Step 3  Impose $\lambda_g$ from Step 1 and $\lambda_z$ from Step 2, and estimate the remaining parameters by MLE.
HLW replication

One-sided Bayesian estimate (blue) and HLW results (red)
The “pile-up” problem

- Classical inference confronts the “pile-up” problem.
- Maximum-likelihood estimation of models with a small permanent component and a sizable transitory component tends to drive the estimated role of the permanent component toward zero, that is, the estimated variance of this component “piles-up” near zero. (Stock and Watson, 1996),
- The “pile-up” problem is severe in analyses of the equilibrium real interest rate (Laubach and Williams, 2003). DeJong and Whiteman (1993) (and also Kim and Kim 2013) suggest that the “pile-up” problem vanishes for the Bayesian approach.
Getting Around the “pile-up” problem

DeJong and Whiteman (1993)

Monte Carlo exercise where the true parameter value is 0.85, $T = 100$
Sample distribution of MLE estimate (left) and posterior distribution (right)
Our estimation technique does not suffer from the pile-up problem.

To illustrate this: Consider Stock and Watson 1998 local level model of log GDP growth.

\[ \Delta y_t = \beta_t + u_t \]
\[ \beta_t = \beta_{t-1} + \frac{\lambda}{T} \eta_t \]
\[ u_t = a_1 u_{t-1} + a_2 u_{t-2} + a_3 u_{t-3} + a_4 u_{t-4} + \varepsilon_t \]
Replication Stock and Watson 98

Histogram of draws of lambda
Getting Around the “pile-up” problem

Replication Stock and Watson 98

dlgdp and beta

Lewis & Vazquez-Grande (FRB)
## Prior Distributions

<table>
<thead>
<tr>
<th>Name</th>
<th>Domain</th>
<th>Density</th>
<th>Parameter 1</th>
<th>Parameter 2</th>
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<td>2</td>
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<td>$\sigma_z$</td>
<td>$[0, 5]$</td>
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</tbody>
</table>
Key Posterior Estimates From Each Model
Model I: $\rho_g = 1, \rho_z = 1$
Model I: $\rho_g = 1$, $\rho_z = 1$
Model I: $\rho_g = 1, \rho_z = 1$
Model I: $\rho_g = 1, \rho_z = 1$

Posterior of $\sigma_{r^*}$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$

Smoothed output gap

![Smoothed output gap graph](image-url)
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model II: \( \rho_g \) and \( \mu_g \) estimated, \( \rho_z = 1 \)
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$

Posterior of $\mu_g$ (Quarterly)
Model III: $\rho_g = 1$, $\rho_z$ estimated

Smoothed output gap
Model III: $\rho_g = 1$, $\rho_z$ estimated

Smoothed $g$ path
Model III: $\rho_g = 1$, $\rho_z$ estimated

Smoothed $z$ path
Model III: $\rho_g = 1$, $\rho_z$ estimated

Posterior of $\sigma_{r^*}$
Model III: $\rho_g = 1$, $\rho_z$ estimated

Posterior of $\rho_z$
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Smoothed $g$ path
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Smoothed $z$ path
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Posterior of $\rho_g$
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Posterior of $\rho_z$
Model IV: $\rho_g, \mu_g$ and $\rho_z$ estimated
Model IV: $\rho_g, \mu_g$ and $\rho_z$ estimated

Posterior of $b_Y$
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

![Posterior of $b_1$](image-url)