Multiproduct trading with a common agent under complete information: Existence and characterization of Nash equilibrium

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Abstract

This paper focuses on oligopolistic markets in which indivisible goods are sold by multiproduct firms to a continuum of homogeneous buyers, with measure normalized to one, who have preferences over bundles of products. Our analysis contributes to the literature on private, delegated agency games with complete information, extending the insights by Chiesa and Denicolò (2009) to multiproduct markets with indivisibilities and where the agent’s preferences need not be monotone. By analyzing a kind of extended contract schedules -mixed bundling prices- that discriminate on exclusivity, the paper shows that pure strategy efficient equilibria always exist in such settings. There may also exist inefficient equilibria in which the agent chooses a suboptimal bundle and no principal has a profitable deviation inducing the agent to buy the surplus-maximizing bundle because of a coordination problem among the principals. Inefficient equilibria can be ruled out by either assuming that all firms are pricing unsold bundles at the same profit margin as the bundle sold at equilibrium, or imposing the solution concept of subgame perfect strong equilibrium, which requires the absence of profitable deviations by any subset of principals and the agent. We provide a characterization of the equilibrium pure strategies and show that each principal’s set of equilibrium contracts of minimum cardinality may contain at least three offers. When the social surplus function is monotone and with constant costs only two offers are required at the equilibrium outcome.

Keywords: Multiproduct Price Competition, Delegated Agency Games, Mixed Bundling Prices, Subgame Perfect Nash Equilibrium, Strong Equilibrium, Coalition-proof equilibrium.

JEL Classification: C72, D21, D41, D43, L13.
1. Introduction

This paper focuses on oligopolistic markets in which indivisible goods are sold by multi-product firms to a continuum of homogeneous buyers, with measure normalized to one, who have preferences over bundles of products. In these settings linear pricing does not guarantee the existence of efficient subgame perfect Nash-equilibrium outcomes and, even worse, sometimes equilibrium (either efficient or inefficient) fails to exist. We wish to investigate whether a kind of nonlinear prices, \textit{mixed bundling prices}, restores equilibrium existence and efficiency. Mixed bundling refers to the practice of offering a consumer the option of buying goods separately or else packages of them at a special price.

Our analysis contributes to the literature of delegated agency games extending their insights to multiproduct markets with indivisibilities and where the agent’s preferences need not be monotone. In our model, the principals are multi-product oligopolists offering a menu of prices for the different bundles of their own indivisible products and the agent is the consumer. We tackle the general question of whether a kind of unrestricted offers—mixed-bundling equilibrium contracts—offered by multiple principals to an agent will be efficient and how the social surplus will be split among them. Obviously a single firm can achieve efficiency and extract all surplus in this setting when it can price nonlinearly. However, when prices charged by one firm impose some contractual externalities on other firms it is far from clear why equilibrium should be expected to be efficient.

When several firms sell (non-homogeneous goods) to the same consumer using some price scheme as a price discrimination strategy, the price schedule which arises can also be modeled as an equilibrium to a common agency game. It is natural to allow the consumer the option of purchasing exclusively from one firm, or from a set of firms, and so common agency is no longer intrinsic to the game but a choice variable that is delegated to the agent. Our model directly applies to the analysis of delegated agency games with complete information. More particularly, we extend the abstract model of trading of Chiesa and Denicolò (2009) to cover multiproduct markets with indivisibilities and an agent with preferences over bundles of products. Like theirs we also analyze unrestricted offers. In this setting we show the equilibrium existence and efficiency; we also characterize the set of equilibrium payoffs and strategies.

Under decentralized contracting, the \textit{Revelation Principle} characterizes the set of feasible allocations. Unfortunately, for multi-contracting environments, the applicability of the Revelation Principle comes into question. We take the path followed by Peters (2001, 2003), Martimort and Stole (2002) and Page and Monteiro (2003), who give up the Revelation Principle even in its generalized form. What really matters \textit{per se} is not the kind of communication that the principal uses with his agent but the set of options that this principal makes available to the agent. In particular, Martimort (2007) states that, in economic applications with quasi-linear payoff functions, the space of mechanism allowing a full description of all pure strategy equilibrium allocations is the space of nonlinear prices.

In multiproduct settings, where firms sell products of a very general nature, a firm has always the incentive to offer exclusive dealing contracts for subsets of its goods. Therefore a
realistic pricing schedule is that firms set prices for individual items as well as for subsets of their goods. Implementing this idea, we extend the principals’ strategy spaces and consider that each principal’s strategy space is the price vector of all its subsets of goods. In other words, each principal’s strategy space is enlarged to offer prices not only for separated items but also for all possible subsets of them. This means that at pure strategy equilibrium, prices could be nonlinear and their role in our model is twofold. On the one hand, they can be seen as an aggressive pricing policy for exclusive dealing equilibrium outcomes and, on the other, as out-of-equilibrium offers sustaining more collusive outcomes: the equilibrium consumption sets of individual components in delegated common agency allocations.

The rationale for these contracts is similar to that of Martimort and Stole (2003), who analyze the equilibrium set of a simple common agency game with direct externalities. They demonstrate the great importance of out-of-equilibrium choices under delegated agency. When principals are forced to use singleton contract offers (i.e., direct revelation mechanisms) rather than menus of offers in a delegated common agency, the only pure strategy equilibrium (when it exists) is for head-to-head competition for the right of exclusive agency; principals earn zero profits. With a more realistic extension of the strategy spaces to allow for nonlinear prices, they find that more collusive outcomes can be sustained where principals share the market. This suggests an important role of out-of-equilibrium offers: such offers change the strategic nature of the game, making it possible to move from intense (head-to-head) competition to more collusive outcomes. The multiplicity of equilibrium outcomes that exist under complete information arise because out-of-equilibrium offers are costless for a principal to include. Under complete information (and along the equilibrium path), only one price-output pair from each principal is chosen by the agent. The remaining portions of a principal’s nonlinear price schedule are irrelevant for the offering principal’s payoffs; they only impact the rival principal’s strategy.

In the context of non-direct, contractual externalities, Chiesa and Denicolò (2009) like us set no restrictions on feasible supply schedules. Given that they work with a homogeneous product, each principal’s maximum payoff is determined by the threat of being unilaterally replaced by one of his competitors, and since marginal costs are increasing, in equilibrium a principal can obtain more than his marginal contribution. Therefore, the equilibrium is not truthful and the equilibrium supply schedules must contain some contracts that will never be accepted. The natural extension to multiproduct firms is to allow firms to set prices for any possible subset of their products. Here, when the efficient bundle is a mixed bundle each principal’s maximum payoff is determined by the threat of being replaced by an exclusive dealing contract of any of his competitors or by a mixed bundle of them. Since the agent’s value function may be non-monotonic, then exclusive dealing contracts need not be linear in prices, the equilibrium is not truthful and therefore the equilibrium supply schedules must contain some contracts that will never be accepted. Thus, subgame perfection translates to specifying not only the equilibrium prices but also the prices of alternative bundles of products (the credible threats in the game theoretical parlance).\footnote{Common agency games are considered normal form games between principals with no active role for}
We also contribute to the understanding of the interaction between competition and nonlinear pricing schemes. Because most of the theoretical work in multi-principal contract games has restricted attention in large part to intrinsic settings, it remains unclear how competition affects the character of nonlinear prices among oligopolists. Clear exceptions, among others, are Armstrong and Vickers (2001) and Rochet and Stole (2002), who analyze nonlinear pricing applications assuming exclusive purchasing in which the consumer must buy from only one firm in equilibrium, and competition can only matter through the outside option; Ivaldi and Martimort (1994), who allow for purchasing from multiple vendors in equilibrium but restrict preferences such that full coverage arises in equilibrium; and Martimort and Stole (2009), who study how competition in nonlinear pricing between two principals (sellers) affects market participation by a privately-informed agent (consumer). In contrast, in our complete information model each firm produces a set of goods, the consumer has a value function over bundles of goods and both multiple vendors (common agency) and exclusivity are possible equilibrium outcomes. In our analysis, the strategy space of principals are the set of firms’ strategies in the complete information game of firms and the consumer. Thus, we consider extended contract schedules instead of only the equilibrium offers. There are at least two reasons for focusing on the complete information case. First, the literature on common agency games deals mainly with principals selling one or two substitute goods under continuity assumptions so that the general existence theorems for pure strategy equilibrium (when available) for such games cannot be easily extended to cover models with indivisible goods and where the agent’s preferences need not be monotone, even under complete information. Second, ruling out informational considerations allow us to isolate the effect on the outcome of the principals’ competition game from the effects stemming from private information. In particular, this permits us, in calculating the equilibrium payoff of the agent, to know the magnitude of the agent’s rent as the result of competition among principals.

The paper is also related to the literature of bundling and mixed bundling literature. The models on multi-product pricing focus on the use of bundling to extract a surplus from heterogeneous buyers or to price-discriminate (Adams and Yellen (1976); Schmalensee (1984); MacAfee et al. (1989)) or to apply monopoly leverage across markets (Whinston (1990); Choi (1996, 2004); Carabaj et al. (1990)) or to deter entry into the market (Whinston (1990); Nalebuff (1999, 2004)). In all these models, demands are assumed to be continuous, there is private information and consumers are heterogeneous. In contrast, in our setting with indivisible good, mixed bundling is profitable when there is a representative consumer

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2For existence results in common agency games under incomplete information see Carmona and Fajardo (2009), who show that general menus games satisfying enough continuity properties have subgame perfect equilibrium in mixed strategies. See also Page and Monteiro (2003) and Monteiro and Page (2008) who show Nash equilibrium existence in mixed catalog strategies.
and no opportunity to apply leverage across markets, even under complete information.\footnote{Mixed bundling also takes place with substitute goods, in contrast to the well-studied cases of bundling with independent or complementary goods. There are few general results for bundles of more than two goods. McAdams (1997) found that the existing analytical machinery for analyzing mixed bundling could not be easily generalized to even three goods, because of the interactions among sub-bundles. In general, price-setting for mixed bundling of many goods is an NP-complete problem requiring sellers to determine a number of prices and quantities that grows exponentially as the size of the bundle increases. Therefore, a general analysis is still lacking and, what is worse, it is not even known whether a Nash equilibrium may exist in such a general setting. In a related setup, Liao and Urbano (2002) and Liao and Tauman (2002) consider a duopoly and assume that each firm produces two complementary goods which are substitutes for the two corresponding goods produced by the other firm. Liao and Tauman (2002) find that mixed bundling strategies play a key role in stabilizing the market, although efficient and inefficient equilibria may exist. If the use of mixed bundling is not allowed, then Liao and Urbano (2002) show that subgame perfect linear pricing equilibria may fail to exist.}

Finally, let us remark that our model deals with indivisibilities. A large literature (Crawford and Knoer (1981); Quinzii (1984); Zhao (1992), among others) on markets with indivisibilities has grown following Shapley and Shubik’s house market (Shapley and Shubik (1972)). All these works assume price-taking behavior and study the equivalence between core outcomes and those of competitive equilibria. While many important goods are indivisible, technical barriers continue to limit our understanding of markets for trading these goods. As Scarf (1994) has stressed, indivisibilities render the calculus of limited value in characterizing allocation markets. The standard approaches to indivisibilities either assume linear utility or make assumptions that smooth away the discreteness. Later authors have addressed variations on the assignment game (Bikhchandani and Mamer (1997); Ma (1998); Bikhchandani and Ostroy (2002)). The central issue in these papers is that the competitive equilibrium supporting prices can be obtained when the underlying market is represented as an assignment problem. In these cases, there is a set of prices (dual variables) for the commodities that fall in the core of an assignment game, and there may be, and generally are, many pricing vectors that support a competitive equilibrium. An interesting paper is Bikhchandani and Ostroy (2002), who extend results for package bidding beyond the assignment model. They study assignment problems where individuals trade packages consisting of several, rather than single, objects. Although buyers’ reservation values are nonadditive (although they are monotone), efficient assignments can be formulated as a linear programming problem in which the pricing function expressing duality may be nonlinear in the objects constituting the packages. The point of departure of Bikhchandani and Ostroy (2002) is to consider pricing functions which are nonadditive over objects and also possibly nonanonymous, thus extending the price characterization of efficient allocations in the standard assignment problem to the package assignment model.\footnote{More recently, O’Neill et al. (2005) address the existence of competitive market clearing prices and the economic interpretation of strong duality for integer programs in the economic analysis of markets with non-convexities (indivisibilities). They present a method for constructing a set of linear prices that will support a Walrasian competitive equilibrium in markets with non-convexities that is based on mixed integer programming. The most recent paper in the field is Caplin and Leahy (2014), who introduce new mathematical structures (GA-structures) for analyzing competitive equilibria in non-transferable utility} Our model deals with strategic equilibrium and can also be formulated as a package assignment problem, with
the dual providing the non-linear prices (when needed). However, the lack of monotonicity of the agent’s preferences over bundle of products precludes the market clearing condition. In other words, the prices sustaining efficient equilibrium outcomes need not be Walrasian prices. In fact, out-of-equilibrium menu offers with non-linear prices are needed to deter principals’ deviations and sustain the efficient equilibrium outcome.

To the best of our knowledge, our paper is the first one dealing with multiproduct oligopolistic competition when goods are indivisible, of a very general nature and the agent’s preferences over bundles of goods need not be monotonic. For instance, in many common situations, agents have complementary preferences for objects in the marketplace. Consider an agent trying to construct a computer system by purchasing components. Among other things, the agent needs to buy a CPU, a keyboard and a monitor, and may have a choice over several models for each component. The agent’s valuations of a package depends on the components in any particular combination, involving products of either only one firm or several firms. This example is a general instance of allocation problems characterized by heterogeneous, discrete resources and complementarities in agents’ preferences. These kinds of models are probably close to many circumstances in real world markets, but they are also more difficult to analyze. With indivisibilities, it is well-known that many familiar properties of the profit functions may fail to ensure the existence of pure strategy Nash equilibrium prices. As already mentioned, the use of marginal calculus is precluded, and the applications of fixed point theorems based on continuity properties, while still possible in some cases, is certainly not straightforward.

Furthermore, equilibrium efficiency may require additional restrictions in our model. For instance, in a typical (not necessarily efficient) delegated common agency (subgame perfect) Nash equilibrium, the prices of the equilibrium bundles are well-specified, but any vector of sufficiently high prices for the individual components (out-of-equilibrium prices) would support any given equilibrium outcome. In this way, if a principal reduces the price of any of its unsold individual components, then the equilibrium consumption set is not upset since rivals’ prices for their individual components are also too high to induce the agent to choose an alternative bundle to that of the (not necessarily efficient) equilibrium. This leads to well-known inefficiencies, since efficiency may imply some coordination among firms (as stressed by Martimort (2007)), specially if the efficient consumption is a multi-firm bundle (common agency). Thus, we need a refinement of the (subgame perfect) Nash-equilibrium concept which would require the candidate equilibrium to remain so even if all firms reduced the prices of their unsold bundles to some degree, thus restoring coordination and efficiency.

The question of equilibrium efficiency was already addressed by Bernheim et al. (1987) and Bernheim and Whinston (1986b) and where each principal can contract on the whole array of actions of the agent. Under complete information the so-called truthful equilibrium implements the outcome which maximizes the aggregated payoff of the grand coalition made allocation markets. In their model buyers can derive utility from at most one element of the set of indivisible goods, but buyers’ utility function is not linear in wealth.
of the principals and the agent. The rationale for truthful menus is that they are coalition-proof, i.e., immune to deviations by subsets of principals which are themselves immune to deviations by sub-coalitions, etc. Coalition-proof equilibrium payoffs can be implemented with truthful schedules in environments with quasi-linear utility functions. Unfortunately, in more complex settings, where the social value function need not be monotonic and may even fail to be continuous, (subgame perfect) equilibria need not be truthful and the implementation of the efficient coalition-proof equilibrium is difficult to characterize. Moreover, as argued by some researchers in the field, despite the fact that the concept of Coalition-proof equilibrium is a reasonable way to incorporate the strategic role of coalitions when non-binding communication is possible, the notion itself is not immune to criticism. In fact, the validity of a coalitional deviation is only checked against further valid deviations of sub-coalitions of that coalition recursively. But, some members of a given coalition could deviate by convincing players outside their coalition provided they improve their payoffs. We impose instead the Strong equilibrium refinement, which takes into account any deviation by any set of firms and the player, exists and is easy to characterize in our model.

The Strong equilibrium (Aumann (1959)) is the equilibrium concept capturing the idea of stability against joint deviations that are mutually profitable for any subset of players. The extension to subgame perfection of such an equilibrium concept allows us to refine the equilibrium correspondence by selecting its efficient equilibria. In fact, we show that the set of subgame perfect strong equilibria of our game is the set of its efficient subgame perfect Nash-equilibria. Therefore inefficient equilibria are ruled out by imposing the solution concept of (perfect) strong equilibrium which requires the absence of profitable deviations by any subset of principals and the agent. More importantly, since the notion of Strong equilibrium allows for all possible coalitional deviations, whereas the definition of Coalition-proof equilibrium allows only credible deviations, the set of Strong equilibrium is always a subset of the set of Coalition-proof equilibria in common agency games –Konishi et al. (1999) and Konishi et al. (1997). Therefore this latter set is also non-empty.

The paper offers a positive existence result: with mixed-bundling contracts and subgame perfect strong equilibria, an efficient pure strategy equilibrium outcome always exists, no matter whether the efficient bundle is either exclusive dealing or common agency (involving either several principals or all of them). The paper extends the traditional wisdom of the delegated common agency literature to settings with multi-product principals producing indivisible goods and an agent with preferences not necessarily monotone over bundles of goods. By analyzing mixed-bundling contracts that discriminate on exclusivity and by extending the space of contract schedules beyond equilibrium offers the paper mainly provides 1) a proof of the existence of efficient pure strategy subgame perfect Nash-equilibria and hence that of perfect strong equilibria (or efficient perfect coalition proof equilibrium) in delegated agency games, 2) a characterization of such pure strategy equilibria, 3) a characterization of the set of the principals’ equilibrium rents by some projection of the core of such agency games. 4) Finally, we show that in delegated agency equilibrium outcomes, i.e., when the agent buys to a subset of principals, each principal’s set of contracts of minimum cardinality (Chiesa and Denicolò (2009)) may contain at least three offers to support the
equilibrium outcome.

Furthermore, specific results about the structure of equilibrium prices and payoffs for common agency outcomes are offered when the social surplus function is monotone and either submodular or supermodular. In the former case, principals are substitutes and their equilibrium rents are equal to their social marginal contributions with an agent’s positive rent, thus reflecting market competition. In the latter case, the agent’s rent is zero and then the core of the value function is always priced by the subgame perfect Nash-equilibrium rents. In both cases the equilibrium is sustained by two offers: each principal submitting his consumption contract and the null offer (TIOLI contracts). In fact, we show that monotonicity and constant unit costs are sufficient to sustain subgame perfect equilibrium outcomes by take it or leave offers.

The paper is organized as follows. The model is presented in Section 2. The characterizations of efficient subgame perfect Nash-equilibrium in terms of all possible deviations and hence that of subgame perfect strong equilibrium are provided in Section 3. The existence and efficiency of such equilibria are proven in Section 4. Section 5 offers specific results for common agency equilibrium outcomes when the social surplus function is monotone and either submodular or supermodular. Concluding remarks close the paper.

2. The model

Consider a set of the principals and a continuum of potential homogeneous buyers, with measure normalized to one (the agent). In our model the principals are $n$ firms and each of them produces a finite set of heterogeneous goods. Moreover each firm’s products can be different from or identical to those of any other firm. Let $N = \{1, 2, ..., n\}$ be the set of firms. Let $\Omega_i$ be firm $i$’s finite set of pure strategies and $\Omega = \Omega_1 \times \ldots \times \Omega_n$. Let $c_i(w)$ be the (constant) unit cost of production of firm $i$ for good $w \in \Omega_i$, where costs are additive, i.e. $c_i(T) = \sum_{w \in T} c_i(w)$, $T \subseteq \Omega_i$. Trade is modeled as a complete information first-price auction in which principals simultaneously submit a menu of contracts and the agent then chooses the set of products she will purchase from each principal.

A consumption set is a vector of subsets $S = (S_1, \ldots, S_n) \subseteq \Omega$, where $S_i \in 2^{\Omega_i}$ represents firm $i$ selling set $S_i$ in $S$, which can be the empty set if the agent does not buy anything from firm $i$. Let $c(S) = \sum_{i \in N} c_i(S_i)$ be the cost of the consumption set $S$ (where $c_i(\emptyset) = 0$). A firm is said to be active in a given consumption set if some of its products is consumed, and non-active otherwise. Let $F(S)$ be the set of active firms in $S$, i.e. $F(S) = \{i \in N | S_i \neq \emptyset\}$.

The agent is characterized by her value function over any subset $S \subseteq \Omega$, $v(S)$, which represents her total willingness to pay for consumption set $S$, with $v(\emptyset, \ldots, \emptyset) = 0$.\footnote{The value function can be derived by standard primitives. Suppose that the agent’s utility function is quasi-linear in money, that is, it is given by $u(x_1, \ldots, x_n, m) = f(x_1, \ldots, x_n) + m$, where $m$ is the monetary numerare and $(x_1, \ldots, x_n)$ is a consumption bundle. Then, $f(x_1, \ldots, x_n)$ measures the monetary value of the bundle $(x_1, \ldots, x_n)$. Let $S \subseteq N$ be a consumption set and let $e^S$ be the corresponding quantities consumed, namely, $e^S_k = 1$ if $k \in S$ and $e^S_k = 0$ if $k \notin S$. The value function $v$ is defined as $v(S) = f(e^S)$.} Initially
we do not impose any assumption on the value function. It is assumed that the agent has no endowment of goods but she has enough money to buy any bundle of products \( S \subseteq \Omega \).

To keep the model as general as possible, we set no restriction on feasible contracts by firms.\(^6\) Namely, in our setting we do not require principals to submit only one offer for each product of the equilibrium outcome but allow them to submit offers for each subset of their product sets. Thus, since the sets traded by the agent with other principals are not contractible, each generic firm \( i \) sets prices (offers contracts) for its products and may also offer subsets of them as bundles at a price that can be different from the sum of the prices of the bundle products. Notice that this implies that we do not consider singleton contracts (direct mechanisms) in the delegated agency game. Such mechanisms do not allow for any offer to remain unchosen in equilibrium; in other words, out-of-equilibrium messages are possible, to use the language of mechanism design.

A strategy of firm \( i \in \mathcal{N} \) is a \( 2^{\Omega_i} \)-tuple specifying the price of each subset of \( \Omega_i \). Let \( p_i(T) \) be the price of \( T \subseteq \Omega_i \) and set \( p_i(\emptyset) = 0 \). Let \( \mathcal{P}_i \) be the set of firm \( i \)'s strategies, i.e., the set of functions \( p_i : 2^{\Omega_i} \rightarrow \mathbb{R}_+ \) and \( \mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n \). The price of bundle \( T \), \( p_i(T) \), can be different from or equal to \( \sum_{w \in T} p_i(w) \) for all \( T \subseteq \Omega_i \). In the former case we say that firm \( i \) follows a \emph{mixed bundling strategy}.

After each firm \( i \) has chosen a price schedule \( p_i \in \mathcal{P}_i \), independently of and simultaneously to the other firms, in the second stage the agent observes the price vector \( p = (p_1, \ldots, p_n) \in \mathcal{P} \), and selects a consumption set \( S(p) \subseteq \Omega \) as a function of \( p \). Thus, the set of strategies of the agent is the set of functions \( S(p) \) from \( \mathcal{P} \) to \( \Omega \).

The payoff of each firm \( i \in \mathcal{N} \) is given by its profit function,

\[
\pi_i(S(p)) = (p_i - c_i)(S_i(p))
\]

where \( (p_i - c_i)(S_i(p)) \) means \( p_i(S_i(p)) - c_i(S_i(p)) \). When there is not ambiguity we will write \( S \) instead of \( S(p) \), so that the firm \( i \)'s profit is \( \pi_i(S) = (p_i - c_i)(S_i) \).

The agent’s payoff function when purchasing \( S \) at prices \( p \) is her consumer surplus,

\[
\text{cs}[S, p] = v(S(p)) - \sum_{i \in N} p_i(S_i(p)) = v(S) - \sum_{i \in F(S)} p_i(S_i).
\]

where the function \( v : \Omega \rightarrow \mathbb{R}_+ \) denotes the agent’s utility function, in monetary terms.

Hence, formally, we have a strategic game with an agent and a set \( N \) of firms. Let \( G^{MB}(N + 1, (\mathcal{P}_i)_{i \in N}, (\Omega_i)_{i \in N}, v, c) \) (where MB stands for mixed bundling pricing) denote such a game.

Since the prices set by each firm affect the profits of the other firms and the agent chooses whether to purchase from either all the firms, a subset of them or not purchasing at all, our model can be understood as a delegated agency game with contractual externalities,\(^6\) This is in contrast with settings with a perfectly divisible homogeneous product where is analytically convenient allowing principals not to submit any offer for certain output levels, as in multi-unit-auctions or markets for electricity, where principals offer a finite number of contracts.
where $n$ (multiproduct) principals offer contracts (price schedules) for any bundle of her own products and the agent (the consumer) chooses whether to accept all the contracts, a subset of them or none at all. Given the set of the agent’s strategies, she is considered as another player and therefore game $G^{MB}$ denotes such a delegated agency game, where the principals offer mixed-bundling contracts. Therefore, the strategy space of principals are the set of firms’ strategies in the complete information game of firms and the consumer. Thus, we consider the extended space of contract schedules instead of only the equilibrium offers. Since we are dealing with a two-stage game of complete information, it seems appropriate to employ the solution concept of subgame perfect Nash equilibrium.

A subgame perfect Nash equilibrium is a list of strategies, $(\tilde{S}, \tilde{p}_1, \ldots, \tilde{p}_n)$ one for each player, such that:

$$
\tilde{S} \in \arg\max_{S} \text{cs}[S, \tilde{p}] 
$$

and

$$
\tilde{p}_i \in \arg\max_{p_i \in P_i} \pi_i(\tilde{S}(\tilde{p}_{-i}, p_i)) \text{ for all } i \in N
$$

where $(\tilde{p}_{-i}, p_i) = (\tilde{p}_1, \ldots, \tilde{p}_{i-1}, p_i, \tilde{p}_{i+1}, \ldots, \tilde{p}_n)$.

A possible strategy for firm $i$ is not selling at all or, equivalently, setting prices high enough in order the buyer does not buy any of its bundles. The non-selling strategy guarantees firm $i$ a payoff equal to zero, and implies that a bundle price below its cost is a dominated strategy. Thus, by the equilibrium definition, any price satisfying $p_i(T) < c_i(T)$ cannot belong to the equilibrium price vector.

Let $SPE$ be the set of pure strategy subgame perfect equilibria of $G^{MB}$. If $(S, p)$ is an element in $SPE$, $p$ is called an $SPE$-price vector, $S$ is an $SPE$-consumption set and $(S, p)$ is denoted as an $SPE$-outcome. If bundle $S$ has goods of only a subset of firms, then the $SPE$-consumption set is a partial common agency allocation and the $SPE$-outcome is a partial common agency equilibrium, being a common agency equilibrium if bundle $S$ has goods of all the firms. Alternatively, when $S$ has goods of only one firm, then the $SPE$-consumption set is an exclusive dealing allocation and the $SPE$-outcome is an exclusive dealing equilibrium.

Let function $(v - c)(S) = v(S) - \sum_{i \in F(S)} c_i(S_i)$ be the social surplus function. Bundle $\tilde{S}$ is socially efficient if $\tilde{S} \in \arg\max_{S} (v - c)(S)$. It is assumed that the maximized social surplus function is always positive\(^7\). An $SPE$-outcome is efficient if its $SPE$-consumption set maximizes the social surplus.

Two questions deserve some clarification. The first one is concerned to the use of mixed bundling strategies as opposed to linear prices. Linear contracts do not guarantee in these

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\(^7\)Otherwise, if for every consumption set $S$, $v(S) < \sum_{i \in F(S)} c_i(S_i)$, then the model is degenerated. Hence, at every equilibrium point $(S, p)$, $S = (\emptyset, \ldots, \emptyset)$ must hold and therefore no production will take place.
settings, even under extended contract schedules, the existence of (subgame perfect) equilibrium outcomes, either in common agency or in exclusive dealing (see Liao and Urbano (2002), for technical details). The intuition is clear: suppose a market for systems, where two firms produce two goods each and the agent has preferences over systems (all the bundles of two goods) and that common agency is the efficient allocation. Starting from an efficient allocation where the agent buys from the two principals, any of them may find it profitable to deviate and exclusively deal with the agent. However, these deviations need not lead to an exclusive dealing equilibrium because both principals will compete fiercely and a deviation from a principal may be followed by other deviations from the rival and the sequence of deviations need not converge.\(^8\)

Example 1 below shows that linear prices do not guarantee the existence of SPE-outcomes. The second concerns refers to considering the extended space of contract schedules instead of only the equilibrium offers. Namely, as in Chiesa and Denicolò (2009) the subgame perfect equilibrium entails the inclusion of out-of-equilibrium offers. Example 1 (continuation) shows that equilibrium supply schedules must contain some contracts that will never be accepted. In Section 4 we will discuss how many of such contracts are needed to support an equilibrium.

**Example 1 (The market for systems): Linear Prices.** Let the set of firms (principals) be \(N = \{1, 2\}\), producing \(\Omega_1 = \{a, b\}\) and \(\Omega_2 = \{c, d\}\), respectively. Assume for simplicity that \(c_i(w) = 0\) for all \(i \in N\) and \(w \in \Omega_i\) and that firms set linear prices, i.e. the exclusive dealing bundles \(\{a, b\}\) and \(\{c, d\}\) are sold at prices \(p_a + p_b\), and \(p_c + p_d\), respectively. The agent’s value function is,

\[
v(S) = \begin{cases} 
4 & S = \{a, b\} \\
9 & S = \{a, d\} \\
5 & S = \{b, c\} \\
\delta & S = \{c, d\} \\
0 & \text{otherwise}
\end{cases}
\]

with \(0 < \delta < 9\).\(^9\) Think, for instance, of an agent trying to construct a computer system by purchasing two components, a CPU and a monitor, and having the choice of two models for each component. The agent’s valuations of a package or bundle depends on the components

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\(^8\)Alternatively, consider that an exclusive dealing bundle is the efficient allocation and that the common agency bundles are quite attractive to the agent as compared with the exclusive dealing ones. In this case, the other exclusive dealing allocation is not very attractive to the agent and then, because of competition, the agent may get most of the surplus. Then, the principal selling the efficient bundle may find it profitable to raise the price of one of the individual components of its own bundle and set the other one in such a way that the agent chooses now the common agency bundle thus giving more profits to this principal. Again, this deviation need not lead to a delegated common agency equilibrium since the other principal may find another profitable deviation and so on.

\(^9\)Notice that a more rigorous notation would be to denote by \(\{(a, b), \emptyset\}\) and \(\emptyset, \{c, d\}\) the one-firm bundles and by \(\{(a), \{d\}\}\) and \(\{b\}, \{c\}\) the two-firm bundles. To ease the notation and since no confusion will arise we follow the more simple notation of \{a, b\}, \{c, d\}, \{a, d\} and \{b, c\}, to respectively denote such sets.
in any particular combination, and then both common agency and exclusive dealing bundles are possible consumption sets.

The efficient consumption set is the common agency bundle $S = \{a, d\}$. Suppose that $\delta = 8$. In the Appendix it is shown that no linear prices support $S = \{a, d\}$ as the equilibrium consumption set. The principals’ incentives to deviate and exclusively deal with the agent cannot be overcome by linear pricing. The intuition is that every time the price of a good in a bundle is changed, the prices of the other bundles which contain that good are also changed. Firms can use this property to design profitable deviations by either increasing or decreasing subsets of prices to discourage the consumption of the (efficient) common agency consumption set while encouraging that of exclusive dealing. The above reasoning is applied to any other bundle to conclude that under linear pricing the equilibrium may fail to exist.

Given the above result, one question is whether another kind of extended contracts, mixed bundling prices, ensures the existence of equilibrium. The answer is affirmative. In the above market for systems example, mixed bundling contracts are conditional on exclusive dealing. In a more general model, with multi-product firms and an agent with preferences over any bundle of goods, mixed bundling contracts can be conditional on exclusive dealing for each bundle of two or more goods. Therefore, mixed-bundling contracts can be seen as either an aggressive pricing policy for exclusive dealing outcomes or as out-of-equilibrium offers sustaining the equilibrium consumption sets of individual components in delegated common agency allocations (involving either several principals or all of them). The discrimination on exclusivity helps principals set incentive-compatible contracts by both facilitating collusion on common agency outcomes and by representing a credible threat that avoids deviations by the principals. Thus, mixed bundling contracts make it easier to sustain equilibrium outcomes and are sufficient to guarantee the equilibrium existence.

Example 1 (continuation): Mixed bundling prices and the role of out of equilibrium offers. Consider example 1 where $\delta = 8$. Notice that in this case the common agency efficient bundle $S = \{a, d\}$ is not highly valued by the agent, in the sense that $v(a, d) < v(a, b) + v(c, d)$. This will imply that the agent will get some surplus. Suppose that firms do not precommit to linear pricing and let $p_{ab}$ and $p_{cd}$ be the prices of the exclusive dealing bundles $\{a, b\}$ and $\{c, d\}$, respectively.

Since the efficient bundle $S$ is one of common agency some particularly important deviations by principals are those to their exclusive dealing bundles. Although mixed-bundling contracts discriminate on exclusivity, if the out-of-equilibrium prices $p_{ab}$ and $p_{cd}$ are priced at unit costs, they become quite attractive to the agent and then the prices of the goods of the efficient common agency bundle $S$ have to be quite low to avoid the agent’s choice of an exclusive dealing bundle. Therefore, there is an incentive for each firm to increase the prices of its own bundles above unit costs, making them less attractive. However, equilibrium subgame perfection imposes some restrictions on firms’ exclusive dealing prices. In particular, each firm will set the price of its own bundle as to guarantee itself the same profit margin as under $S$ and then the equilibrium price vector will have to be immune to such an action.

In the Appendix we show that the efficient bundle $S = \{a, d\}$ is a $SPE$-consumption set,
supported by equilibrium prices verifying:

\[ 0 \leq p_a \leq 1, \quad 4 \leq p_d \leq 5, \quad 5 \leq p_a + p_d, \]
\[ p_{ab} = p_a + p_d - 5, \quad p_{cd} = p_a + p_d - 1, \]
\[ p_b \geq 0, \quad p_c \geq p_d - 4, \quad p_b + p_c \geq p_a + p_d - 4. \]

Thus, the out-of-equilibrium exclusive dealing prices \( p_{ab} \) and \( p_{cd} \) are set strategically by firms to make the exclusive dealing bundles as profitable for the consumer as the efficient bundle, and the prices for products \( b \) and \( c \) are set high enough to make the other common agency bundle \( \{b, c\} \) unprofitable for the agent. Notice also that the consumer surplus is positive and upper bounded, \( 3 \leq cs[S, p] = 9 - p_a - p_d \leq 4 \). Moreover, there is also an inefficient bundle \( T = \{c, d\} \) that can be supported at equilibrium by letting, for example, \( p_a = 1, \quad p_b = p_{ab} = 0, \quad p_c = 1, \quad p_d = 5 \) and \( p_{cd} = 4 \), with a consumer surplus of 4.

This example also illustrates that principals submitting only their consumption contract and the null contract, i.e., making a take-it-or-leave-it (TIOLI) offer, need not be a subgame perfect equilibrium.

3. Subgame perfect equilibrium and efficiency: Strong equilibria and Coalition-proof equilibria

In what follows, we characterize both the set of subgame perfect equilibria and the set of efficient subgame perfect equilibria of the delegated agency game \( G^{MB} \) where the principals (firms) might use mixed bundling contracts.

It is standard in agency games\(^{10}\) that every principal \( i \) plays a bargaining game with the agent and hence leaves her with a payoff that does not exceed her disagreement payoff, which is the maximum payoff she can obtain by trading optimality with the remaining \( N \setminus i \) principals (individual excludability). This means that given the \( N \setminus i \) principals’ equilibrium strategies, the agent equilibrium payoff does nor depend of firm \( i \)’s strategy. As a consequence, the strategy of firm \( i \) must maximize his joint payoff with the agent, given the other principals’ equilibrium strategies (bilateral efficiency). Individual excludability and bilateral efficiency characterize the equilibrium of the agency game in Chiesa and Denicolò (2009, see Lemma 1, pag. 301). This characterization result recurs in the common agency literature (see Bernheim and Whinston (1986a, Lemma 2) and the fundamental equations of Laussel and Le Breton (2001)).

The next Proposition characterizes the set of \( SPE \)-outcomes\(^{11}\) of our agency game and parallels the aforementioned Lemma 1 in Chiesa and Denicolò (2009). This characterization is equivalent to the translation of individual excludability and bilateral efficiency properties to our setting and in terms of principals’ deviations. Namely, condition BC below precludes

\(^{10}\)Where principals sell a homogeneous, perfectly divisible product and the agent’s utility function is strictly concave.

\(^{11}\)These conditions extend those of Arribas and Urbano (2005) and Tauman et al. (1997) to multiproduct principals.
unilateral deviations by the agent. Condition FC1 refers to individual excludability and guarantees that each active principal $j$ does not have an incentive to increase the equilibrium prices of its sold bundles (to increase his payoff), since there is at least a bundle of the other principals that leave the agent with the same equilibrium payoff. Bilateral efficiency or pairwise stability is given by conditions FC2 and FC3.

Since principals are multiproduct firms selling bundles of products and at equilibrium they sell only one, condition FC2 says that each active principal $j$ does not have an incentive to reduce the prices of her unsold bundles in order to sell any of them profitably, given the other principals’ equilibrium strategies. On the other hand, condition FC3 additionally guarantees that each non-active principal cannot benefit from price reductions to unit costs. Conditions FC2 and FC3 are equivalent to saying that the equilibrium joint payoff of the agent and any principal selling at equilibrium is the maximum joint payoff given the other principals’ equilibrium strategies. Bilateral efficiency or pairwise stability is given by conditions FC2 and FC3.

Conditions BC to FC3 do not preclude bundle prices to be below their cost. As already mentioned in the equilibrium definition, and to avoid the use of dominated strategies, we additionally impose that $p \geq c$ in the following characterization of the SPE.

**Proposition 1.** ($\tilde{S}, \tilde{p}$) is an SPE-outcome of $G^{MB}$, where $\tilde{S} = (\tilde{S}_1, \ldots, \tilde{S}_n) \subseteq \Omega$ and $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n)$, $\tilde{p}_i \in \mathcal{P}_i$ with $\tilde{p} \geq c$, if and only if

For all $S \subseteq \Omega$, $cs[\tilde{S}, \tilde{p}] \geq cs[S, \tilde{p}]$, i.e.,

$$v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) \geq v(S) - \sum_{i \in F(S)} \tilde{p}_i(S_i).$$

(BC)

For all $j \in F(\tilde{S})$ there is $S_j' \subseteq \Omega$ with $S_j' = \emptyset$ such that, $cs[\tilde{S}, \tilde{p}] = cs[S_j', \tilde{p}]$, i.e.,

$$v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) = v(S_j') - \sum_{i \in F(S_j')} \tilde{p}_i(S_j').$$

(FC1)

For all $S \subseteq \Omega$ such that $j \in F(S)$, $cs[\tilde{S}, \tilde{p}] \geq cs[S, (\tilde{p}_{-j}, \tilde{p}_j(\tilde{S}_j) - c_j(\tilde{S}_j) + c_j(S_j))]$, i.e.,

$$v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) \geq v(S) - [\tilde{p}_j(\tilde{S}_j) - c_j(\tilde{S}_j) + c_j(S_j)] - \sum_{i \in F(S) \setminus j} \tilde{p}_i(S_i).$$

(FC2)

For all $S \subseteq \Omega$ such that $j \notin F(\tilde{S})$ and for all $S \subseteq \Omega$ such that $j \in F(S)$, $cs[\tilde{S}, \tilde{p}] \geq cs[S, (\tilde{p}_{-j}, c_j(S_j))]$,
i.e.,
\[ v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) \geq v(S) - c_j(S_j) - \sum_{i \in F(S) \setminus j} \tilde{p}_i(S_i). \]  
(FC3)

Notice that BC-FC3 are implied by subgame perfection requirements: BC by the agent and (FC1-FC3 by the firms’ incentives. To see this, suppose that FC1 does not hold, then by BC there is \( j \in N \), such that for all \( S' \subseteq \Omega \) with \( S'_j = \emptyset \),
\[ v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) > v(S') - \sum_{i \in F(S')} \tilde{p}_i(S'_i). \]

The above inequality implies that firm \( j \) is better off charging a price \( \tilde{p}_j(\tilde{S}_j) + \varepsilon \), for a sufficiently small \( \varepsilon > 0 \), such that BC is still satisfied. Now, the agent observing the new price vector will again choose the consumption set \( \tilde{S} \), but firm \( j \) will obtain an extra profit of \( \varepsilon \). Hence FC1 must be verified if \( (\tilde{S}, \tilde{p}) \) is an SPE-outcome.

If FC2 does not hold, then for some firm \( j \in N \) there is a consumption set \( S \subseteq \Omega \) such that,
\[ v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) < v(S) - [\tilde{p}_j(\tilde{S}_j) - c_j(S_j)] - \sum_{i \in F(S) \setminus j} \tilde{p}_i(S_i). \]

Hence firm \( j \) can set a price \( p_j(S_j) = \tilde{p}_j(\tilde{S}_j) - c_j(S_j) + c_j(S_j) + \varepsilon \), for a sufficiently small \( \varepsilon > 0 \), such that still
\[ v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) < v(S) - p_j(S_j) - \sum_{i \in F(S) \setminus j} \tilde{p}_i(S_i), \]
which implies that the agent will now select the consumption set \( S \) and firm \( j \) will increase its profits.

Finally, if FC3 is not verified, then for some firm \( j \notin F(\tilde{S}) \) there is a consumption set \( S \subseteq \Omega \) with \( j \in F(S) \) such that,
\[ v(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{p}_i(\tilde{S}_i) < v(S) - c_j(S_j) - \sum_{i \in F(S) \setminus j} \tilde{p}_i(S_i). \]

Thus, similarly to the above, if firm \( j \) sets price \( p_j(S_j) = c_j(S_j) + \varepsilon \), for a sufficiently small \( \varepsilon > 0 \), then the agent will select set \( S \) and firm \( j \) will increase its profits.

Conversely, if BC, FC1, FC2 and FC3 are satisfied, then \( (\tilde{S}, \tilde{p}) \) is an SPE-outcome since \( \tilde{S} \) is a best choice for the agent and no firm has an incentive to either reduce or increase its prices. Notice that the set \( S^j \) in FC1 may be empty and in this case \( cs[\tilde{S}, \tilde{p}] = 0 \), and firms extract the entire consumer surplus.
In agency model where the agent’s utility function is strictly concave and principals sell a homogeneous product, then the social surplus function is globally concave and the subgame perfect Nash-equilibrium outcome is efficient, i.e., it maximizes the social surplus function. Thus, subgame perfection implies efficiency. In contrast, in our model with multiproduct firms and where the agent’s utility function over bundles need not be strictly concave, the Nash concept of stability (or bilateral efficiency) may not be enough to guarantee efficiency. This is particularly true when the equilibrium bundle pertains to several firms so that additional coordination between principals may be required. Here, subgame perfection does not rule out inefficient equilibria and therefore the conditions of Proposition 1 characterize both efficient and inefficient subgame perfect Nash-equilibrium as example 1 and the following example illustrate.

**Example 2. Efficient and inefficient equilibrium outcomes:** Consider again example 1, now with $0 < \delta < 1$. In this case both common agency bundles $\{a,d\}$ and $\{b,c\}$ are highly valuable to the agent in the sense that $v(a,d) > v(b,c) > v(a,b) + v(c,d)$. We prove that both efficient and inefficient equilibrium outcomes exist, where the consumer gets zero surplus and all the rent goes to principals.

By the same reasoning than in example 1 above, $p_{ab}$ and $p_{cd}$ cannot be equal to unit costs. However, since the two common agency bundles are highly valued with respect to the exclusive dealing ones, subgame perfection translates to setting $p_{ab} > v(a,b)$, $p_{cd} > v(c,d)$ and then the binding constraint is $p_a + p_d = v(a,d) = 9 > v(a,b) + v(c,d)$, with $p_a \geq v(a,b)$ and $p_d \geq v(c,d)$. This additionally avoids principals’ price deviation in order to sell its exclusive dealing bundle.

The high prices of products $b$ and $c$ and those of the exclusive dealing bundles $\{a,b\}$ and $\{c,d\}$ make it only attractive for the agent the efficient common agency bundle $S = \{a,d\}$. The agent’s surplus is zero and firm 1’s profit is $p_a = 9 - p_d$. But principal 1 could sell consumption set $\{a,b\}$ instead by setting $p_a$ and $p_b$ big enough and $p_{ab} = 4$. Hence, at the equilibrium it must be that $p_a \geq 4$ or $p_d \leq 5$; similarly, in order firm 2 does not deviate, $p_d \geq \delta$ or $p_a \leq 9 - \delta$. Hence, at the equilibrium $p_a + p_d = 9$ with $4 \leq p_a \leq 9 - \delta$ and $\delta \leq p_d \leq 5$.

Thus, a subgame perfect equilibrium is the **efficient common agency outcome** with consumption set $S = \{a,d\}$ and equilibrium prices satisfying:

\[
p_a + p_d = 9, \quad 4 \leq p_a \leq 9 - \delta, \quad \delta \leq p_d \leq 5, \quad \text{and}
\]

\[
p_{ab} \quad \text{big enough, \quad k} \in \{b, c, \{a,b\}, \{c,d\}\}.
\]

The efficient common agency bundle is very valued by the agent compared with any of the exclusive dealing bundles. Hence, principals can obtain all the surplus. Notice that there are multiple subgame perfect equilibrium prices sustaining the efficient common agency consumption set but in all of them the agent’s surplus is zero and the sum of the firms’ profits is a constant.

Nevertheless, it is also easy to show that the **inefficient common agency outcome** $\{b,c\}$, supported by the price vector: $p_b + p_c = v(b,c) = 5$; $p_b \geq v(a,b) = 4$; $p_c \geq v(c,d) = \delta$;
big enough for \( k \in \{a, d, \{a, b\}, \{c, d\}\} \), is also a subgame perfect equilibrium. The reason parallels the one above. There are also multiple subgame perfect equilibrium prices sustaining the inefficient consumption set.

Notice that the inefficient outcome is not perturbed even if either principal 1 reduces the price of its remaining unsold goods to the price of the sold one, i.e. \( p_{ab} = p_a = p_b \) or principal 2 sets \( p_{cd} = p_d = p_c \). It can be easily checked that all the \( \text{SPE} \)-outcomes satisfy conditions BC to FC3.

In conclusion, Proposition 1 characterizes the set of all \( \text{SPE} \)-outcomes, but both efficient and inefficient outcomes belong to the Nash equilibrium correspondence. Recalling that a deviation by a coalition of players is self-enforcing, if no sub-coalition has an incentive to initiate a new deviation, the two \( \text{SPE} \)-outcomes of example 2 are furthermore immune to self-enforcing deviations by any coalition of players. This is Bernheim et al. (1987) and Bernheim and Whinston (1986b)’s definitions of Coalition-proof Nash Equilibrium. Notice first that, in the two \( \text{SPE} \)-outcomes of example 2 neither individual deviations are profitable (because of the Nash equilibrium property) nor are deviations by one firm and the consumer, by subgame perfection. Also, by subgame perfection, the principals will deviate whenever they can induce the consumer to choose a new consumption bundle. Therefore, it remains to check whether the coalition of the two principals and the consumer have a self-enforcing profitable deviation. To do that, we have to clarify first the notion of a coalition deviation. In Bernheim et al. (1987), a group of players can deviate only if each of its members can be made better off (strict deviations) meanwhile in Bernheim and Whinston (1986b), it is only required that a group of players can deviate only if at least one of its members is better off while all other members are at least as well off (weak deviations)\(^{12}\). We stick to Bernheim et al. (1987)’s definition, and hence when considering joint deviations by coalitions of principals and the consumer we do not allow the formers to resolve the agent’s indifference between outcomes\(^{13}\) because the agent chooses in the last stage.

Coming back to example 2, it is easy to see that there is no self-enforcing deviation by the coalition of the two principals and the consumer under the efficient \( \text{SPE} \)-outcome, but we have to carefully check deviations from the inefficient one. Suppose that starting from the \( \{b, c\} \) outcome (where the \( cs=0 \)), the two principals change prices in such a way that the consumer’s new choice is \( \{a, d\} \). To achieve this outcome, the principals have to give the consumer some positive surplus. But, as long as the consumer surplus is positive, the two principals will further have an incentive to increase prices to reduce the consumer surplus to zero (as in the efficient \( \text{SPE} \)-outcome). Therefore, these deviations are not self-enforcing. As a consequence the two \( \text{SPE} \)-outcomes of example 2 are subgame perfect Coalition-proof equilibrium, denoted as \( \text{SPCP} \) (see the extension to subgame perfection of Bernheim et al. (1987)’s definition in the Appendix).\(^{14}\)

\(^{12}\)As a result, the two papers use different definitions of Coalition-proof Nash equilibrium. As Konishi et al. (1999) pointed out, a notion of strict deviation reduces a potential threat of deviation, which results in a weaker equilibrium notion: weakly Coalition-proof equilibrium, whereas weak deviations give rise to a strictly Coalition-proof equilibrium.

\(^{13}\)See, Bernheim and Whinston (1986b), footnote 11, page 16.

\(^{14}\)One can also check that under strictly perfect Coalition-proof equilibrium (Bernheim and Whinston
Among the set of not self-enforcing deviations, we are particularly interested in those deviations of out-of-equilibrium path prices: when both firms coordinate and make a price reduction of their unsold bundles without changing the prices of the sold goods. The problem with the inefficient equilibria of example 2 is that the two principals charge high prices for the goods of the efficient common agency bundle so that no individual firm can benefit from a price reduction of its part of the efficient bundle. Therefore a higher degree of coordination among principals is needed to achieve efficiency. To isolate efficient SPE-outcomes from inefficient ones, a new condition has to be imposed as illustrated below.

**Example 2. (continuation)** Suppose that in example 2 above both active principals simultaneously reduce the prices of their unsold goods to those of the sold ones at a given inefficient SPE. For instance, consider the inefficient SPE-outcome, \{b, c\}, and prices: \( p'_a = p'_b = p'_{ab} = 5 - \delta \), the price of the sold product b; and \( p'_c = p'_d = p'_{cd} = \delta \), the price of the sold product c. At these new prices:

\[
\text{cs}[\{a, d\}, p'] = v(a, d) - (p'_a + p'_d) = 9 - (5 - \delta + \delta) = 4 > 0 = v(b, c) - (p'_b + p'_c) = \text{cs}[\{b, c\}, p']
\]

and the inefficient SPE is ruled out. The same reasoning rules out any inefficient equilibria.\(^{15}\) Furthermore, it is also easily checked that all the efficient SPE’s are immune to these simultaneous price reductions.

With this idea in mind, we would like to consider only the subset of subgame perfect equilibrium outcomes which remains as equilibrium outcomes even if all non-active principals set unit cost prices and all active principals set prices for their unsold bundles equal to those of their sold ones adjusted by the cost-differential.\(^{16}\) In other words, we want FC3 to be satisfied for all subsets of non-active firms: for all \( A \subseteq N \setminus F(\tilde{S}) \) and FC2 to be satisfied for all subsets of active firms: for all \( B \subseteq F(\tilde{S}) \). Thus, the condition of bilateral efficiency or pairwise stability has to be extended to a notion of stability against joint deviations that are mutually profitable to any subset of principals and the agent. We denote these conditions as strong efficiency or strong stability, meaning that the equilibrium joint payoff of the agent and any subset of principals selling at equilibrium is the maximum joint payoff given all possible strategies of all other subsets of principals, even of those subsets not selling at equilibrium. These conditions remove the set of SPE-outcomes in which some firms charge unreasonably high prices so that no individual firm can benefit from a price reduction of its products only.

One way to define these constraints on set SPE is the following. Consider the price vector \( p = (p_1, \ldots, p_n) \in P_1 \times \cdots \times P_n \), and let \( S \subseteq \Omega \). Define vector \( p^S \) for all \( i \in N \),
\[ T_i \subseteq \Omega_i, \text{ as} \]
\[
p_i^{S}(T_i) = \begin{cases} 
p_i(S_i) & \text{if } i \in F(S), \ T_i = S_i \\
p_i(S_i) - c_i(S_i) + c_i(T_i) & \text{if } i \in F(S), \ T_i \neq S_i \\
c_i(T_i) & \text{if } i \notin F(S), \end{cases} \tag{5}
\]
i.e. all the non-active firms set prices equal to the marginal cost, and all active firms set prices for unsold bundles equal to those of their sold bundles adjusted by the cost-differentials.

**Definition 1.** For every triple \((N, v, c)\), the subset SPE* of SPE-outcomes of \(G^{MB}\) is defined as:

\[
\text{SPE}^* = \{(S, p) \in \text{SPE}|(S, p^{S}) \in \text{SPE}\}.
\]

Equivalently, \(\text{SPE}^*\) is the set of equilibrium outcomes satisfying BC, FC1, and FC4 (instead of FC2 and FC3), where FC4 is stated as:

For all \(A \subseteq N\setminus F(S)\), \(B \subseteq F(S)\) and for all \(S \subseteq \Omega\) such that \((A \cup B) \subseteq F(S)\),

\[
c^S[S, \tilde{p}] \geq c^S[S, ((\tilde{p}_i)_{i \in F(S)\setminus (A \cup B)}, (c_i(S_i))_{i \in A}, (\tilde{p}_i(S_i) - c_i(S_i) + c_i(S_i))_{i \in B})],
\]
i.e.,

\[
v(S) - \sum_{i \in F(S)} \tilde{p}_i(S_i) \geq v(S) - \sum_{i \in F(S) \setminus (A \cup B)} \tilde{p}_i(S_i) - \sum_{i \in A} c_i(S_i) - \sum_{i \in B} [\tilde{p}_i(S_i) - c_i(S_i) + c_i(S_i)]. \tag{FC4}
\]

Thus, we restrict the analysis to a certain subset \(\text{SPE}^*\) of SPE-outcomes. The next Proposition shows that FC4 selects the set of efficient consumption bundles of \(G^{MB}\).

**Proposition 2.** For every value function \(v\) and unit cost vector \(c\), if \((\tilde{S}, \tilde{p})\) is an SPE*-outcome of \(G^{MB}\), then \(\tilde{S}\) is socially efficient.

**Proof:** If \((\tilde{S}, \tilde{p})\) is an SPE*-outcome, then \((\tilde{S}, \tilde{p}^S) \in \text{SPE}^*\). By BC in Proposition 1, \(c^{S}[\tilde{S}, \tilde{p}^S] \geq c^S[S, \tilde{p}^S]\) for all \(S \subseteq \Omega\). Therefore,

\[
(v - c)(\tilde{S}) - (v - c)(S) \geq \sum_{i \in F(S)} (\tilde{p}_i^S - c_i(S_i)) - \sum_{i \in F(S)} (\tilde{p}_i^S - c_i)(S_i)
\]

\[
= \sum_{i \in F(S)} (\tilde{p}_i^S - c_i(S_i)) - \sum_{i \in F(S)\setminus F(S)} (\tilde{p}_i^S - c_i(S_i))
\]

\[
= \sum_{i \in F(S) \setminus F(S)} (\tilde{p}_i - c_i)(S_i) \geq 0,
\]
given that \(\tilde{p}_i^S(S_i) = c_i(S_i)\) for all \(i \notin F(S)\) and \(\tilde{p}_i^S(S_i) - c_i(S_i) = \tilde{p}_i^S(S_i) - c_i(S_i)\) for all \(i \in F(S) \setminus F(S)\). Thus, \((v - c)(\tilde{S}) \geq (v - c)(S)\) for every \(S \subseteq \Omega\). \(\blacksquare\)
The definition of \( SPE^* \) captures the idea of equilibrium stability against joint deviations that are mutually profitable to any subset of firms and the agent. As we saw in example 2, the concept of Coalition-proof equilibrium does not refine much the set of subgame perfect equilibrium, and, when the equilibrium is not truthful, to isolate the subset of efficient Coalition-proof equilibrium is quite difficult even in very simple examples. We take an alternative route and consider another refinement: an equilibrium concept behind conditions BC, FC1, and FC4, or set \( SPE^* \), refining the set of \( SPE \)-outcomes is that of Aumann (1959). He proposes the concept of Strong equilibrium as an equilibrium such that no subset of players has a joint deviation that strictly benefits all of them. While the Nash concept of stability defines equilibrium only in terms of unilateral deviations, Strong Nash equilibrium allows for deviations by every conceivable coalition. One of the criticism of the Strong equilibrium concept is that it might not exist, but it does it in our model and is easy to characterize. The extension of that concept to a Subgame Perfect Strong Equilibrium –SPSE– of our game \( G^{MB} \) is as follows,

**Definition 2.** \((\hat{S}, \hat{p})\) is an SPSE-outcome of \( G^{MB} \), where \( \hat{S} = (\hat{S}_1, \ldots, \hat{S}_n) \subseteq \Omega \) and \( \hat{p} = (\hat{p}_1, \ldots, \hat{p}_n) \), \( \hat{p}_i \in \mathcal{P}_i \) with \( \hat{p} \geq c \) if

i) \( cs[\hat{S}, \hat{p}] \geq cs[S, \hat{p}] \) for all \( S \subseteq \Omega \),

ii) For all \( M \subseteq N \) there is no \( (p'_j)_{j \in M} \) and \( S' \subseteq \Omega \) such that,

a) \( (p' - c)_j(S'_j) > (\hat{p} - c)_j(\hat{S}_j) \), for each \( j \in M \),

b) \( cs[S', ((\hat{p}_i)_{i \in N \setminus M}, (p'_i)_{i \in M})] > cs[S, \hat{p}] \) for all \( S \subseteq \Omega \)

Since the deviating coalition can be either an individual principal or the agent, this implies that an SPSE-outcome is therefore an SPE. Moreover, it is shown next that conditions BC, FC1, and FC4 are the only conditions characterizing SPSE-outcomes.

**Proposition 3.** For every value function \( v \), unit cost vector \( c \), and game \( G^{MB} \), the set of SPSE-outcomes coincides with the set of \( SPE^* \)-outcomes.

**Proof:** Let us first prove that any SPSE-outcome is an SPE*. Let \((\hat{S}, \hat{p})\) be an SPSE and suppose that it is not an \( SPE^* \). By FC4 of the definition of \( SPE^* \), there are sets \( A \subseteq N \setminus F(\hat{S}) \), \( B \subseteq F(\hat{S}) \) and \( S \subseteq \Omega \) such that,

\[
v(\hat{S}) - \sum_{i \in F(\hat{S})} \hat{p}_i(\hat{S}_i) < v(S) - \sum_{i \in F(S) \setminus (A \cup B)} \hat{p}_i(S_i) - \sum_{i \in A} c_i(S_i) - \sum_{i \in B} [\hat{p}_i(\hat{S}_i) - c_i(\hat{S}_i) + c_i(S_i)]
\]

and the consumer switches to \( S \). Thus, the coalition of firms in \( A \cup B \) has an incentive to jointly deviate, obtaining higher profits than those under \((\hat{S}, \hat{p})\), which contradicts \((\hat{S}, \hat{p})\) being an SPSE-outcome.
Now let us assume that \((\tilde{S}, \tilde{p})\) is an \(SPE^*\)-outcome, but not an \(SPSE\)-outcome. Since both equilibrium concepts are subgame perfect, then \((\tilde{S}, \tilde{p})\) would not verify condition ii) above: there exists \(M \subseteq N\), \((p'_j)_{j \in M}\) and \(S \subseteq \Omega\) such that for all \(j \in M\), \((p'_j - c'_j)(S_j) > (\tilde{p} - c)_j(\tilde{S}_j)\), and \(cs(S, ((p_j)_{j \in N \setminus M}, (p'_j)_{j \in M})) > cs(\tilde{S}, \tilde{p})\). Define the sets \(A = M \cap (N \setminus F(S))\) and \(B = M \cap F(S)\), then condition FC4 does not hold, which contradicts \((\tilde{S}, \tilde{p})\) being an \(SPE^*\).

Therefore set \(SPSE\) coincides with set \(SPE^*\), and hence any \(SPSE\)-consumption set is socially efficient.

4. Existence, efficiency and characterization of Subgame Perfect Strong Equilibria.

4.1. Existence and efficiency

Our main result establishes that given any agent’s value function \(v\), and unit cost vector \(c\), there is always a subgame perfect equilibrium of the delegated agency game \(G^{MB}\) verifying conditions BC, FC1 and FC4, i.e. set \(SPSE\) is non-empty (Theorem 1). Moreover, not only any \(SPSE\)-consumption set is efficient (Proposition 2), but also any socially efficient consumption set belongs to an \(SPSE\)-outcome (Corollary 1).

**Theorem 1.** For every value function, \(v\), and unit cost vector, \(c\), there is an equilibrium in \(SPSE\).

**Proof:** By Proposition 3, it suffices to prove that set \(SPE^*\) is non-empty. Let \(\tilde{S} \in \arg \max_S (v - c)(S)\). Define the set

\[
\Pi' = \{\pi \in \mathbb{R}_+^n | \pi_i = 0, \forall i \notin F(\tilde{S}) \text{ and } \sum_{i \in F(\tilde{S}) \setminus F(S)} \pi_i \leq (v - c)(\tilde{S}) - (v - c)(S), \forall S \subseteq \Omega\}.
\]

Set \(\Pi'\) is non-empty since \(0 \in \Pi'\) and it is bounded, since for every \(\pi \in \Pi'\) we have \(\pi_i \leq (v - c)(\tilde{S}) - (v - c)(\tilde{S} \setminus S_i)\) for all \(i \in F(\tilde{S})\), and \(\pi_i = 0\) for all \(i \in N \setminus F(\tilde{S})\).

Also observe that \(\Pi'\) is closed and hence compact. Thus \(\Pi'\) contains an element \(\tilde{\pi}\) which is maximal with respect to the lexicographical order on \(\Pi'\). Let \(\tilde{\mathbf{p}} = (\tilde{p}_1, ..., \tilde{p}_n) \in P_1 \times \cdots \times P_n\) be defined as \(\tilde{p}_i(T_i) = \tilde{\pi}_i + c_i(T_i)\) for all \(i \in N, T_i \subseteq \Omega_i\). Notice that \(\tilde{p}_i(T_i) = c_i(T_i)\) for all \(i \notin F(\tilde{S})\), and \(\tilde{p}_i(S_i) - c_i(S_i) = \tilde{\pi}_i\) for all \(i \in F(\tilde{S})\). This leads to,

\[
\tilde{p}_i(T_i) = \tilde{\pi}_i + c_i(T_i) = \tilde{p}_i(S_i) - c_i(S_i) = \tilde{\pi}_i \text{ for all } i \in F(\tilde{S}).
\]

for all \(i \in F(\tilde{S})\) and \(T_i \neq S_i\). Hence \(\tilde{p}_i(T_i) = \tilde{p}_i^S(T_i)\) as defined in (5).

We claim that \((\tilde{S}, \tilde{\mathbf{p}})\) is an \(SPE^*\)-outcome. Since \(\tilde{\mathbf{p}} = \tilde{p}^S\), it suffices to prove that \((\tilde{S}, \tilde{\mathbf{p}})\) is an \(SPE\)-outcome. Let \(S \subseteq \Omega\), since \(\tilde{\pi} \in \Pi'\), then \(\sum_{i \in F(\tilde{S}) \setminus F(S)} \tilde{\pi}_i \leq (v - c)(\tilde{S}) - (v - c)(S)\) or equivalently

\[
(v - c)(\tilde{S}) - \sum_{i \in F(\tilde{S})} \tilde{\pi}_i \geq (v - c)(S) - \sum_{i \in F(S)} \tilde{\pi}_i.
\]

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hence,
\[ v(\overline{S}) - \sum_{i \in F(\overline{S})} \left[ \overline{\pi}_i + c(\overline{S}_i) \right] \geq v(S) - \sum_{i \in F(S)} \left[ \overline{\pi}_i + c(S_i) \right], \]

which by the definition of \( \overline{p}_i \) is condition BC of Proposition 1.

Conditions FC2 and FC3 of Proposition 1 hold trivially, given that \( \tilde{\pi}_i(T_i) = \tilde{\pi}_i(T_i) \).

To prove condition FC1 of Proposition 1 suppose, on the contrary, that exists \( j \in F(\overline{S}) \) such that \( v(S) - \sum_{i \in F(\overline{S})} \tilde{p}_i(\overline{S}_i) > v(S) - \sum_{i \in F(S)} \tilde{p}_i(\overline{S}_i) \) for all \( S \subseteq \Omega \) with \( S_j = \emptyset \). Let
\[ \varepsilon = \min_{S \subseteq \Omega, S_j = \emptyset} \left\{ v(S) - \sum_{i \in F(\overline{S})} \tilde{p}_i(\overline{S}_i) - \left( v(S) - \sum_{i \in F(S)} \tilde{p}_i(\overline{S}_i) \right) \right\}, \]

then \( \varepsilon > 0 \). Let \( \tilde{\pi} \in \mathbb{R}_+^n \) be defined as
\[ \tilde{\pi}_i = \begin{cases} \overline{\pi}_i & i \neq j \\ \overline{\pi}_i + \varepsilon & i = j, \end{cases} \]

and let \( \tilde{p} = (\tilde{p}_1, ..., \tilde{p}_n) \) be defined as \( \tilde{p}_i(T_i) = \tilde{\pi}_i + c(T_i) \) for all \( i \in N, T_i \subseteq \Omega_i \). Notice that \( \tilde{p}_i(T_i) = \tilde{p}_i(T_i) \) for all \( i \in N \setminus j, T_i \subseteq \Omega_i \) and \( \tilde{p}_j(T_j) = \tilde{p}_j(T_j) + \varepsilon \). Since \( \tilde{\pi} \) is a maximal element of \( \Pi' \), then \( \tilde{\pi} \notin \Pi' \). However, we will show that \( \sum_{i \in F(\overline{S}) \setminus F(S)} \tilde{\pi}_i \leq (v - c)(\overline{S}) - (v - c)(S) \) for all \( S \subseteq \Omega \) and hence \( \tilde{\pi} \in \Pi' \), which is a contradiction. Given \( S \subseteq \Omega \), if \( j \in F(S) \), then
\[ \sum_{i \in F(\overline{S}) \setminus F(S)} \tilde{\pi}_i \leq (v - c)(\overline{S}) - (v - c)(S) \] since \( \tilde{\pi}_i = \tilde{\pi}_i \) for \( i \neq j \). Suppose next that \( j \notin F(S) \), by the definition of \( \varepsilon \)
\[ v(S) - \sum_{i \in F(S)} \tilde{p}_i(\overline{S}_i) - \varepsilon \geq v(S) - \sum_{i \in F(S)} \tilde{p}_i(\overline{S}_i), \]

but by the definition of \( \tilde{p} \) and \( \tilde{p} \) the left hand side of the above inequality can be written as,
\[ v(S) - \sum_{i \in F(S)} \tilde{p}_i(\overline{S}_i) - \varepsilon = v(S) - \sum_{i \in F(S)} \left[ \overline{\pi}_i + c_i(\overline{S}_i) \right] - \varepsilon = (v - c)(\overline{S}) - \sum_{i \in F(S)} \tilde{\pi}_i, \]

and the right hand side as
\[ v(S) - \sum_{i \in F(S)} \tilde{p}_i(\overline{S}_i) = v(S) - \sum_{i \in F(S)} \left[ \overline{\pi}_i + c_i(S_i) \right] = (v - c)(S) - \sum_{i \in F(S)} \tilde{\pi}_i. \]
Hence,

\[ \sum_{i \in F(S) \setminus F(S)} \pi_i \leq (v - c)(S) - (v - c)(\tilde{S}), \]

as claimed. Therefore \((\tilde{S}, \tilde{p})\) is an SPE*-outcome.

The proof of Theorem 1 also shows that efficiency is a sufficient condition to belong to an SPSE-outcome, i.e. if \(S \in \arg \max_S (v - c)(S)\), then there is a price vector \(\tilde{p}\) such that \((\tilde{S}, \tilde{p})\) is an SPSE-outcome. This, jointly with Proposition 2, allows us to assert that the subgame perfect Strong equilibrium concept selects the set of efficient SPE of \(G^{MB}\) from the SPE correspondence.

**Corollary 1.** For every value function \(v\) and unit cost vector \(c\), SPSE is the set of efficient SPE-outcomes of \(G^{MB}\).

Now, notice that the notion of Strong equilibrium allows for all possible coalitional deviations, whereas the definition of Coalition-proof equilibrium allows only credible deviations, and therefore the set of Strong equilibrium is always a subset of the set of Coalition-proof equilibria in common agency games –Konishi et al. (1999) and Konishi et al. (1997). Therefore this latter set is also non-empty. By Theorem 1 and Propositions 2 and 3, we have that,

**Corollary 2.** For every value function \(v\), unit cost vector \(c\), the set of efficient SPCP is the set of efficient SPE-outcomes of \(G^{MB}\).


SPSE-consumption sets have been characterized as the socially efficient ones. In this section, we characterize the set of firms’ profits (the principals’ rents) which comes from SPSE-outcomes in the delegated agency game \(G^{MB}\). The characterization is made in terms of the core of the game and the marginal contribution of each principal. By Definition 1 and Proposition 3, notice that if \((S, p)\) is an SPSE-outcome, then \((S, p^S)\) is also an SPSE-outcome and the principals and the agent obtain the same payoffs under such outcomes: the two equilibria are payoff-equivalent. Thus, any pair \((S, p^S)\) makes it possible to identify its payoff-equivalence class, and for any SPSE-price vector we can only consider \((S, p^S)\) as the representative element of its equivalence class. We will show that the vector of principals’ profits from SPSE-outcomes are their most preferred points in the core of the agency game \(G^{MB}\). First, let us define the marginal contribution of principal \(i\) as the difference between the maximum social surplus attainable, say \(V^* = \max_{S \subseteq \Omega} : (v - c)(S)\), and the maximum social surplus attainable when principal \(i\) is inactive, say \(V'_{-i}\), i.e.,

\[
mc_i = V^* - V'_{-i} = \max_{S \subseteq \Omega} : (v - c)(S) - \max_{\substack{S \subseteq \Omega \ni S_i = \emptyset}} : (v - c)(S).
\] (6)
Let us define the core as

$$\text{core}(v - c) = \{(\pi^b_i, (\pi_i)_{i \in N}) \in \mathbb{R}_+^{n+1} | \pi^b + \sum_{i \in N} \pi_i = V(N) \text{ and } \pi^b + \sum_{i \in F(S)} \pi_i \geq (v - c)(S), \forall S \subseteq \Omega\}$$

and let $\Pi^{PF}$ be the Pareto frontier of the projection of $\text{core}(v - c)$ on the $n$ last coordinates. Formally,

$$\Pi^{PF} = \{(\pi_i)_{i \in N} \in \mathbb{R}_+^n | \text{there is } \pi^b \geq 0 \text{ with } (\pi^b_i, (\pi_i)_{i \in N}) \in \text{core}(v - c)$$

and there is no other $(\pi^b_i', (\pi_i')_{i \in N}) \in \text{core}(v - c)$, such that $\pi_i' \geq \pi_i$, for all $i \in N$ with $\pi_i' > \pi_j$ for at least some $j$\}.

Notice that the core of the game is defined through linear inequalities and thus, it is a polytope and so is $\Pi^{PF}$. We will prove that $\Pi^{PF}$ is the set of the principals’ equilibrium rents. First, the following lemma states that principal $i$’s coordinate or payoff in the core is lower than or equal to his marginal contribution, which implies that if a principal is not selling in some efficient bundle, then his coordinate in the core will be zero.\(^{17}\)

**Lemma 1.** Let $v$ be a value function and $c$ a unit cost vector. If $(\pi^b_i, (\pi_i)_{i \in N}) \in \text{core}(v - c)$, then $\pi_i \leq mc_i$ for all $i \in N$. Therefore, $\pi_i = 0$ for all $i \notin F(S)$, $S \in \arg \max_T (v - c)(T)$.

**Proof:** Given $i \in N$, let $S \in \arg \max_T \{v(S) - (v - c)(T) : (v - c)(T) \leq 0\}$. If $(\pi^b_i, (\pi_i)_{i \in N}) \in \text{core}(v - c)$, then by definition $V^* = \pi^b + \sum_{j \in N} \pi_j$ and $V^*_i \leq \pi^b + \sum_{j \in F(S)} \pi_j$. Because $\pi_j \geq 0$ for all $j \in N$, then $V^*_i \leq \pi^b + \sum_{j \in (N \setminus i)} \pi_j$. Thus $V^* - V^*_i = mc_i \geq \pi_i$.

Now, if $i \notin F(S)$ for some $S \in \arg \max_T : (v - c)(T)$, then $V^* = V^*_i$ and $mc_i = \pi_i = 0$.

The following result states that there is a bijection between the core and the set of price vectors: first, each element $(\pi_i)_{i \in N} \in \Pi^{PF}$ has associated an equilibrium price vector (a contract vector) such that $\pi_i$ is firm $i$’s profit or rent; second, if $p$ is an SPSE-price vector, then the corresponding firms’ profit vector belongs to $\Pi^{PF}$. In the Appendix it is proven,

**Proposition 4.** For every value function $v$ and unit cost vector $c$ it is verified that

i) if $(\pi_i)_{i \in N} \in \Pi^{PF}$, then $(S, p) \in \text{SPSE}$, where $S$ is any socially efficient consumption set and $p_i(T_i) = \pi_i + c_i(T_i)$ for all $i \in N$, $T_i \subseteq \Omega_i$,

ii) if $(S, p)$ is an SPSE-outcome, then $(\pi_i)_{i \in N} \in \Pi^{PF}$, where $\pi_i = (p_i - c_i)(S_i)$, $i \in N$.

The above Proposition 4 and Lemma 1 characterize the principals’ equilibrium payoffs. First, the principals’ equilibrium payoffs are not unique and are characterized by the polytope formed by their most preferred points in the core of the agency game $G^{MB}$. Second, the

\(^{17}\)In Proposition 1 in Chiesa and Denicolo (2009) a principal’s maximum payoff can exceed his marginal contribution, as payoffs are defined by each principal’s pivotal competitor and marginal cost are increasing.
maximum payoff of a firm is bounded from above by its marginal contribution. Third, a firm would only obtain a positive profit if it were selling at least a component of every efficient bundle. Fourth, if none of the firms sells at least one component of every efficient bundle, then every equilibrium consumption set is offered at unit cost prices, firms’ payoffs are zero and the consumer extracts the entire social surplus. This is the case when principals are identical (they produce the same substitute products with the same technology as in the classical Bertrand Theory), or there are two or more socially efficient exclusive dealing bundles.

The intuition is clear, suppose two firms and the market for systems of our previous examples. If no principal sells at least one component of every efficient bundle, then there exist at least two efficient systems: the exclusive dealing bundles. By individual excludability (condition FC1) at equilibrium \((v - c)(a, b) = (v - c)(c, d)\). By Lemma 1, firms’ profits are zero and then by Proposition 4, \(p_{ab} = v(a, b) - \max\{(v - c)(c, d), 0\} = c_a + c_b,\) \(p_{cd} = v(c, d) - \max\{(v - c)(a, b), 0\} = c_c + c_d\) and therefore \(\pi_1 = \pi_2 = 0\) and the agent obtains all the surplus. Notice that \(mc_1 = mc_2 = 0\). Next, suppose that only one firm, say Firm 1, sells at least one component of every efficient bundle. This can be only if \(\{a, b\}\) is the efficient bundle but \(\{c, d\}\) is not. Then, again by condition FC1, the profit of Firm 1 is \(\pi_1 = (v - c)(a, b) - \max\{(v - c)(c, d), 0\},\) which will be positive as long as \((v - c)(a, b) > (v - c)(c, d),\) \(\pi_2 = 0\) and the agent obtains \((v - c)(c, d)\). Notice that \(mc_1 = (v - c)(a, b) - (v - c)(c, d) = \pi_1\) and \(mc_2 = 0 = \pi_2\). Therefore, if in all socially efficient consumption sets the agent chooses only products of the same firm (exclusive dealing), say \(i\), then that firm will obtain its marginal contribution, i.e., \(\pi_i = mc_i\), the products of any other firm will be offered at unit cost prices, and the agent will obtain a positive payoff equal to \(V^*_i\). The same results will be obtained if there is an exclusive dealing equilibrium, with products belonging to say firm \(i\), as well as another or several (partial) common agency equilibria. Then, all firms but \(i\) will obtain zero profits and firm \(i\)’s payoff will be its marginal contribution.

On the contrary, when the equilibrium consumption set is a common agency bundle (i.e., it contains products of two or more principals), then although the principals might offer their products as bundles at special prices, the agent selects a subset of products of each firm. For example, suppose that both Firm 1 and 2 sell at least one component of every efficient system. The efficient bundles are either \(\{a, d\}\) or \(\{b, c\}\), or \(\{a, d\}\) and \(\{b, c\}\). In each of them, by strong stability (condition FC4) \(p_{ab}\) and \(p_{cd}\) are higher than unit costs at equilibrium with either \(p_{ab} = p_a, p_{cd} = p_d\) or \(p_{ab} = p_b, p_{cd} = p_c\) or both, and principals’ profits are positive. By Lemma 1, it is interesting to notice that if the sum of all principals’ marginal contributions is lower than or equal to the maximum social surplus \(V^*\), then all equilibrium outcomes will be payoff equivalent and, in each equilibrium, the principals will obtain their marginal contribution and the agent a positive surplus. This is the case of Example 1, where \(V^* = 9\) and the marginal contributions are \(mc_1 = 1\) and \(mc_2 = 5\). In the efficient \(SPSE\) these values are firms’ payoffs and the consumer surplus is 3. On the contrary, if the sum of all principals’ marginal contributions is greater than \(V^*\), then there will be different price vector equilibria but in all of them the consumer surplus will be zero. In Example 2, \(V^* = 9, mc_1 = 9 - \delta\) and \(mc_2 = 5\), with \(0 < \delta < 1\). Recall that the efficient
bundle was \( S = \{a, d\} \) and there were infinite equilibria depending on the price vector

\[
p_a + p_d = 9, \quad 4 \leq p_a \leq 9 - \delta, \quad \delta \leq p_d \leq 5, \quad \text{and} \quad p_k \quad \text{big enough,} \quad k \in \{b, c, \{a, b\}, \{c, d\}\}.
\]

Thus, in all efficient equilibria the joint principals’ payoffs are 9, so that the consumer surplus is zero. However, depending on the equilibrium price vector, the principals’ payoffs will be different. Namely, if principal 1’s rent is his marginal contribution \( 9 - \delta \) (his most preferred point in the core of the game), then principal 2’ rent will be \( \delta < mc_2 = 5 \), and viceversa\(^{18}\).

The Pareto dominant equilibrium for the principals is denoted as the minimum rent equilibrium by Chiesa and Denicolò (2009), since it minimizes the agent’s payoff. Notice, however that when the sum of principals marginal contributions exceeds the maximum social surplus, the consumer’s surplus is always zero and each principal’s marginal contribution is his more preferred point in the core, then by Lemma 1 only one principal can get his preferred rent. By the above results we can conclude:

**Corollary 3.** 1) The minimum rent equilibria of \( G^{MB} \) when the sum of all principals’ marginal contributions is lower than or equal to the maximum social surplus is outcome equivalent to the truthful equilibrium. 2) When the sum of all principals’ marginal contributions is greater than the maximum social surplus, then there will be infinite equilibrium price vectors but in all of them the consumer surplus will be zero; only one principal will obtain its marginal contribution in his preferred minimum rent equilibria but the others will obtain less than their marginal contributions.

The results in Corollary 3 differs from those of the delegated agency literature and, in particular, from those of Chiesa and Denicolò (2009). They show that if the number of principals is higher than 2, then at the minimum rent equilibria every principal’s payoff will exceed his marginal contribution to the social surplus, where the existence of strictly increasing cost explains these authors’ results. In our case, given the heterogeneity of firms’ products, when the efficient consumption set is a mixed bundle and the other bundles are not too valued by the agent, the marginal contribution of each principal is very high and the sum of all of them may exceed the value of the social surplus. Since the agent’s rent is zero and all the marginal contributions are equal, then by Lemma 1, only one principal can get its preferred rent. Therefore, the product heterogeneity and the lack of monotonicity of the agent’s value function for bundles drive the above results.

The precise relationship between the maximum social surplus and the sum of the principals’ marginal contributions is difficult to obtain in general, unless we know some properties of the value function. Next, we explore a question that deserves some attention.

\(^{18}\)It is easy to show that the same results obtain with positive (constant) costs of production.
4.3. Structure of the principals’ equilibrium strategies: number of out of equilibrium offers

We have seen that when the equilibrium consumption set is a common agency bundle, mixed bundling contracts are out-of-equilibrium offers sustaining the equilibrium consumption sets of individual components. These contracts are the reason of the multiplicity of equilibrium price vectors. In view of the role that the out of equilibrium offers play sustaining the equilibrium schedules, an interesting question to discuss is the minimum number of them needed to support SPSE-outcomes.

We can distinguish two cases. When the equilibrium outcome is exclusive dealing, condition FC1, or individual excludability, will suffice to price the equilibrium bundle and strong stability or condition FC4 will guarantee the equilibrium condition for any subset of agents (no deviations). Thus, only the equilibrium offer and the null offer (TIOLI) offers are needed to support exclusive dealing equilibrium outcomes.

However, when the equilibrium outcome is delegated common agency, where several firms are active in the equilibrium consumption set, individual excludability and strong stability will require at least three offers. At equilibrium, each active principal could need to price its equilibrium bundle, at least another of its own bundles, and the null bundle. The pricing of out of equilibrium bundles by each principal avoids his deviation to exclusive dealing outcomes. Therefore out of equilibrium offers by each principal are needed to sustain equilibrium outcomes. For instance, the delegated common agency equilibria in the market for systems of Example 1 implies that the equilibrium consumption set is \( \{a, d\} \). To sustain the equilibrium each principal needs to offer three bundle prices. Namely, \( p_{ab} = p_a, p_{cd} = p_d \) and the null offer. In general, the out of equilibrium bundles priced by each principal will depend on the agent’s preferences as shown in the following example.

**Example 3**: Let the set of firms (principals) be \( N = \{1, 2, \ldots, n\} \), each one producing two products \( \Omega_i = \{a_i, b_i\} \) for all \( i \in N \). Assume for simplicity that production costs are zero. Let \( A = \{a_1, \ldots, a_n\} \) and for all \( i \in N \) let \( B_{-i} = \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n\} \). In bundle \( A \) all firms sell their first product, meanwhile in the other bundles all firms but one sell their second product. The agent’s value function is \( v(A) = \alpha \) and \( v(B_{-i}) = \beta \) where \( \alpha > \beta > 0 \) and \( v(S) = 0 \) otherwise. Thus, the agent’s most valuable bundle is \( A \); she values less the bundles \( B_{-i} \), and any other bundle is worthless for her.

The efficient consumption set is the common agency bundle \( A \), the equilibrium prices are \( p_{ai} = \alpha - \beta \), and the out-of-equilibrium prices are \( p_{bi} = \alpha - \beta \), for all \( i \in N \). The consumer surplus is \( v(A) - \sum p_{ai} = \alpha - n(\alpha - \beta) \), therefore we assume that \( \beta > \frac{n-1}{n} \alpha \) to guarantee a positive agent’s surplus. Since there are no costs, the marginal contribution of principal \( i \) is given by \( v(A) - v(B_{-i}) = \alpha - \beta \), which is equal to his payoff. In this simple example each principal needs to price his two products, the sold one, the unsold one and the null offer.

Notice that if each principal \( i \) priced only \( a_i \) as \( p_{ai} = \alpha - \beta \), and set \( b_i \) equal to marginal cost (which is zero), the agent would deviate to any of bundles \( B_{-i} \), since \( v(A) - \sum p_{ai} = \alpha - n(\alpha - \beta) < \beta = v(B_{-i}) \), for any \( i \).

Chiesa and Denicolo (2009) found (see Proposition 3, pag. 305) that in their model any equilibrium outcome can be supported by equilibrium supply schedules (pairs of product-price) such that two principals submit two offers (the empty set and one bundle), and the
remaining \( n - 2 \) principals submit three offers (the empty set and two different bundles). In our delegated common agency equilibrium, all principals may need to submit at least three offers. The reason behind this result is the deterrence of each principal’s deviations to his exclusive dealing outcomes. Given the non monotonicity of the agent’s value function for bundles, the precise number of offers will depend on the particular details of the model.


In this section we study the surplus sharing between the agent and the principals in delegated common agency, when the social surplus function is monotonic. In this context, exclusive dealing is never an equilibrium consumption set, and the agent will contract with either a subset of principals (partial delegated common agency) or with all of them (delegated common agency). We show that when the social surplus function is supermodular, then the agent’s surplus is zero and the set of principals’ rents is completely characterized by the convex hull of the vertex arising from their accumulative marginal contributions, which coincides with the Pareto frontier of \( \text{core}(v - c) \). We also prove that when the principals are substitutes (a more general condition than strong subadditivity), then at any equilibrium the principals’ rents are their marginal contributions (truthful equilibrium) and the agent obtains the difference between the social marginal contribution of her consumption set \( \Omega \) and the sum of the social marginal contributions of the principals \( \sum mc_i \). In both cases of monotonic social surplus functions TIOLI offers are able to sustain the SPSE outcomes.

Let us introduce some convention in notations. Let \( \Omega \setminus S \) be \((\Omega_1 \setminus S_1, ... , \Omega_n \setminus S_n)\) and let \( v(S_i) \) be \( v(\emptyset, ... , S_i, ... \emptyset) \) i.e., a consumption set where the agent only buys from principal \( i \). Given \( w \in \Omega_i \) and \( S \subseteq \Omega \), let \( S + w \) be \((S_1, ... , S_i + \{w\}, ... , S_n)\), with \( S_i + \{w\} = S_i \cup \{w\} \).

**Definition 3.**

i) \((v - c)\) is monotonic if and only if \((v - c)(S) \leq (v - c)(T)\) whenever \( S \subseteq T \subseteq \Omega \),

ii) \((v - c)\) is submodular if and only if \((v - c)(T + w) - (v - c)(T) \leq (v - c)(S + w) - (v - c)(S)\) whenever \( S \subseteq T \subseteq \Omega \setminus w \), and

iii) \((v - c)\) is supermodular if the opposite inequality holds.

Thus, the monotonicity of \((v - c)\) implies that the social surplus increases for larger consumption sets. If \((v - c)\) is monotonic, then, by Theorem 1, there is always an SPSE-outcome with \( \Omega \) as the equilibrium consumption set. Furthermore, if \((v - c)\) is strictly monotonic, then \( \Omega \) is the unique equilibrium consumption set. Thus under strict monotonicity of the social value function, \( G^{MB} \) is a delegated common agency game where the agent contracts with all the principals (common agency). Moreover, by Lemma 1 and Proposition 4 we know that,

**Lemma 2.** If \((\Omega, p) \in \text{SPSE-outcome set}\), then \( p_i(\Omega_i) - c(\Omega_i) \leq mc_i = (v - c)(\Omega) - (v - c)(\Omega \setminus \Omega_i) \) for all \( i \in N \).

Suppose now that the social surplus function is submodular. This is a condition that seems particularly attractive since it expresses, in the case of indivisible goods, the idea that
the marginal utility of an item decreases when the bundle of goods to which it is added gets larger. The submodularity of \((v - c)\) reflects a kind of substitution among products or bundles of products so that there is competition among principals and the agent will obtain some surplus.

Let us extend the concept of principal’s marginal contribution to a set of principals: the marginal contribution of a set of principals \(A \subseteq N\), say \(mc_A\), is the difference between the maximum social surplus attainable \(V^*\) and the maximum social surplus attainable when set \(A\) of principals is inactive \(V^*_A\), i.e.,

\[
mc_A = V^* - V^*_A, \quad \text{where } V^*_A = \max_{S \subseteq \Omega : S \neq \emptyset, i \in A} (v - c)(S).
\]

Following Shapley (1962), we say that principals are substitutes if the social marginal contribution of set \(A\) is bigger than or equal to the sum of the marginal contributions of firms in \(A\),

\[
mc_A \geq \sum_{i \in A} mc_i, \quad \forall A \subseteq N. \tag{FS}
\]

The next proposition states that the equilibrium prices of monotonic social surplus functions satisfying FS are equal to the social marginal contributions of the principals plus their corresponding marginal costs. In the Appendix it is proven.

**Proposition 5.** Let \((v - c)\) be a monotonic social surplus function and suppose that principals are substitutes (FS holds). Then \((S^*, p^*) \in SPSE\)-outcome set, with \((v - c)(S^*) = (v - c)(\Omega)\) and \(p^*_i(T_i) = mc_i + c(T_i)\), for all \(T_i \subseteq \Omega_i\), and \(i \in N\). The converse is also true.

If \((v - c)\) is strictly monotonic, then the unique equilibrium consumption set is \(\Omega\).

Therefore, each principal \(i\) sells \(S^*_i\) as a bundle and obtains its marginal contribution as its profits, \((p^*_i - c_i)(S^*_i) = mc_i\). Moreover, the agent surplus is positive, reflecting market competition under FS:

\[
\begin{align*}
    cs(S^*) &= v(S^*) - \sum_{i \in F(S^*)} p^*_i(S^*_i) = (v - c)(S^*) - \sum_{i \in F(S^*)} (p^*_i - c_i)(S^*_i) \\
    &= mc_{F(S^*)} - \sum_{i \in F(S^*)} mc_i \geq 0.
\end{align*}
\]

Two straightforward results are the following: 1) if \((v - c)\) is a submodular social surplus function, then principals are substitutes, i.e. FS is satisfied; thus the FS condition is more general than the concavity (or strong subadditivity) condition in the literature; and 2) if \((v - c)\) is a submodular value function and \(mc_i \geq 0\) for all \(i \in N\), then \((v - c)\) is monotonic. Then, trivially from Proposition 5 we obtain the next result.

**Corollary 4.** Let \((v - c)\) be submodular and \(mc_i \geq 0\) for all \(i \in N\). Thus, principals are substitutes, and for all \(i \in N\), principal \(i\)’s equilibrium rent in any SPSE-outcome is equal to his social marginal contribution.
By the above Proposition (5) principals are substitutes, and for all \( i \in N \), principal \( i \)'s price is \( p^*_i(\Omega_i) = v(\Omega) - v(\Omega \setminus \Omega_i) + c(\Omega_i) \) and his rent is his social marginal contribution. The agent is indifferent between buying \( \Omega \) or \( \Omega \setminus \Omega_i \). Therefore, each principal’s strategy of selling his set of products as an indivisible bundle is an equilibrium outcome and there is not need to set prices out of the equilibrium offer. Thus, TIOLI is an efficient equilibrium outcome.

**Corollary 5.** Let \((v-c)\) be submodular and \(mc_i \geq 0\) for all \( i \in N \). Then, the minimum rent equilibrium is outcome equivalent to the truthful equilibrium. Moreover only the equilibrium offer and the null offer (TIOLI) are needed to sustain the SPSE outcome.

Two questions deserve some comments. First, since the minimum rent equilibrium is a truthful equilibrium, it can be easily implemented as a SPCP, i.e., immune to self-enforcing coalitional deviations. In other words, sets SPCP and SPSE coincide, i.e., all possible deviations are credible deviations. Second, our results here are due to the fact that we are in a class of environments in which pure strategy equilibrium can all be supported with simple take or leave it offers. In other words, menu offers have no role to play. One way menus are used is to exploit random behavior of principals or the agents as a correlating device. But here all players use pure strategies, so that this role for menus disappears. However, this still leaves open the role of menus to deter deviations. By here again, the agent’s ranking of of the option of any principal’s menu is independent of what other principals are doing, and therefor, menus are needless. These environments are, as denoted in Peters (2003), no-externality environments, where 1) each principal’s price (his marginal contribution) is independent of the other principals’ prices (this is condition (i) in Peters (2003) or D1 in Han (2012)); and 2) the agent has a weak preference ordering over each principal’s prices and her choice is independent of the other principals’ prices for the products (this is condition (ii) in Peters (2003) or D2 in Han (2012)).

Now suppose that the social surplus function is supermodular. The supermodularity of \((v-c)\) reflects complementarities among products or bundles of products and hence among principals. Therefore, it induces only weak market competition so that principals can extract the entire agent surplus. The higher degree of complementary among principals translates into a higher marginal contribution of each of them, so that their sum is bigger than the social surplus (and property FS does not hold). In this framework, principals extract all the social surplus, but not all of them can obtain at equilibrium their social marginal contributions.

As the next proposition states, the set of principals’ equilibrium rents is a polytope, whose corners are the vectors of the principals’ equilibrium rents verifying that at least one principal obtains his social marginal contribution. The intuition is as follows. Assume that principals are ordered, so that the first principal is principal 1, the second principal is principal 2 and so on. A market with only a consumer has a null social surplus; if principal 1 enters in that market, then the social surplus will increase in \( V^*_{-(N \setminus \{1\})} - V^*_{-(N \setminus \{1\})} \); if now principal 2 enters, then the social surplus will increase in \( V^*_{-(N \setminus \{1,2\})} - V^*_{-(N \setminus \{1\})} \); in general, after the entry in the market of principal \( i \), the social surplus will increase in \( V^*_{-(N \setminus \{1,\ldots,i\})} - V^*_{-(N \setminus \{1,\ldots,i-1\})} \); the entry of the last principal, principal \( n \), increases the social surplus in \( V^* - V^*_{-n} \).
Notice that in the above process only principal $n$ generates an increase in the social surplus equal to his marginal contribution, and that the sum of all these increases is equal to $V^*$, the social surplus. The next proposition states that the vector of the social surplus increases is a corner in the polytope of the set of principals’ equilibrium rents. Thus, the different orders in the set of principal generates the different corners, and the convex hull of these corners are also feasible equilibrium rents.

Formally, let $\Sigma$ be the set of permutations (orderings) of principals (permutation of $N$) and let $\sigma \in \Sigma$ be any of its elements. Let $P^\sigma_i$ be the set of principals which precede principal $i$ with respect to permutation $\sigma$, i.e., for all $i \in N$ and $\sigma \in \Sigma$, $P^\sigma_i = \{ j \in N | \sigma(j) < \sigma(i) \}$. Define, following Shapley (1971), the marginal contribution vector $x^\sigma(v-c) \in \mathbb{R}^n$ of $(v-c)$ with respect to ordering $\sigma$ by,

$$x^\sigma_i(v-c) = V^*_N \setminus (P^\sigma_i + i) - V^*_N \setminus P^\sigma_i,$$

so that, $x^\sigma_i(v-c)$ is the principal $i$’s social contribution when he enters in a market where only principals in $P^\sigma_i$ were active (the ones preceding him with respect to ordering $\sigma$). Notice that these vectors are not the marginal contributions since their sum exceed the social surplus. In addition, it is straightforward to prove that if $(v-c)$ is nonnegative and supermodular, then $(v-c)$ is monotonic. In the Appendix it is proven,

**Proposition 6.** Let $(v-c)$ be a supermodular social surplus function, such that $mc_i \geq 0$ for all $i \in N$. Then the agent’s surplus is zero and the principals’ equilibrium profits in any SPSE-outcome are $\text{conv}\{x^\sigma(v-c) | \sigma \in \Sigma\}$.

By Proposition 6, the principals’ equilibrium rents are characterized by set $\text{conv}\{x^\sigma(v-c) | \sigma \in \Sigma\}$, which turns out to be the Pareto frontier of $\text{core}(v-c)$. Thus, when $(v-c)$ is supermodular, then the core of the value function is always priced by subgame perfect strong equilibrium payoffs.

Under monotonic and supermodular social surplus functions, again TIOLI offers sustain the equilibrium outcome. Now, the sum of principals’ marginal contributions exceed the social surplus and each principal’s price depend on the other principals’ prices. This implies that condition (i) in Peters (2003) is not satisfied although condition (ii) is still verified. Therefore, this environment in not one of no-externalities, but still TIOLI offers will support pure strategy subgame perfect equilibrium. The reason is that the consumer surplus is zero and thus she is indifferent between buying all products and buying none. Hence, each principal, say $i$, has only to price his own bundle $\Omega_i$ and the null offer. Furthermore, given monotonicity the no-rent property ($cs=0$), as in Konishi et al. (1999), ensures that sets SPCP and SPSE coincide.

The above results make us wonder whether monotonicity and constant unit costs are sufficient to sustain subgame perfect equilibria by TIOLI offers. Certainly, they are not necessary conditions as we already showed: when at the efficient equilibrium there is only one active firm (exclusive dealing), then condition FC1 (or individual excludability) will suffice to price the equilibrium bundle and strong stability (or condition FC4) will guarantee
are sufficient conditions to sustain subgame equilibrium outcomes by TIOLI offers.

However, it is straightforward to prove that monotonicity and constant unit costs are sufficient conditions to sustain subgame equilibrium outcomes by TIOLI offers\textsuperscript{19}.

**Proposition 7.** Let \((v - c)\) be a monotonic value function, with constant unit costs, and let \((\Omega, p) \in SPSE\)-outcome set. Then \((\Omega, (p_1(\Omega_1), \ldots, p_n(\Omega_n)))\) is a TIOLI offer.

**Proof:** Without loss of generality assume that costs are zero. Suppose that \((\Omega, p)\) is a SPSE-outcome supporting a TIOLI offer. We have to prove that no firm \(i\) has an incentive to profitably deviate, i.e., for all firm \(i\) and bundle \(S_i \subseteq \Omega_i\) with price \(p_i(S_i) > p_i(\Omega_i)\), and for all set of firms \(M \subset N \setminus i\), it is verified that

\[
v(\Omega) - \sum_{j \in N} p_j(\Omega_j) \geq v\left( \bigcup_{j \in M} \Omega_j \right) - \sum_{j \in M} p_j(\Omega_j) - p_i(S_i).
\]  

(7)

Thus firm \(i\)’s attempt to profitably sell bundle \(S_i\) gives the buyer a lower surplus than under the TIOLI offer, and she will never switch.

**Case 1:** \(cs = 0\). First let us see that inequality (7) holds when the consumer surplus is zero, \(v(\Omega) - \sum_{j \in N} p_j(\Omega_j) = 0\). In this case we have that,

\[
v(\Omega) - \sum_{j \in N} p_j(\Omega_j) = 0 \geq v\left( \bigcup_{j \in M} \Omega_j \right) - \sum_{j \in M} p_j(\Omega_j)
\]

\[
\geq v\left( \bigcup_{j \in M} \Omega_j \right) - \sum_{j \in M} p_j(\Omega_j) - p_i(S_i)
\]

where the first inequality holds by BC, the second by monotonicity and the third one by assumption \(p_i(S_i) > p_i(\Omega_i)\). In this case, firm \(i\)’s attempt to profitably sell bundle \(S_i\) gives the buyer a negative surplus, and she will never switch.

**Case 2:** \(cs > 0\). Let us see that inequality (7) also holds when the consumer surplus is positive, \(v(\Omega) - \sum_{j \in N} p_j(\Omega_j) > 0\). If \((\Omega, p) \in SPSE\), then by Proposition 3 \((\Omega, p) \in SPE^*\) and, then by definition 1 \((\Omega, p^\Omega) \in SPE\). Recall that \(p^\Omega_i(T_i) = p_i(\Omega_i)\).

Since \((\Omega, p^\Omega) \in SPE\), then by FC1 there is \(S_i^t \in \Omega\) with \(S_i^t = \emptyset\) such that

\[
v(\Omega) - \sum_{j \in N} p_j(\Omega_j) = v(S^t) - \sum_{j \in F(S^t)} p_j(\Omega_j).
\]

Moreover, by BC

\[
v(\Omega) - \sum_{j \in N} p_j(\Omega_j) \geq v\left( \bigcup_{j \in F(S^t)} \Omega_j \right) - \sum_{j \in F(S^t)} p_j(\Omega_j).
\]

\textsuperscript{19}We thank an anonymous referee for suggesting us to prove this result.
The last two equations imply that \( v(S^i) \geq v(\bigcup_{j \in F(S^i)} \Omega_j) \), and by monotonicity this means that \( v(S^i) = v(\bigcup_{j \in F(S^i)} \Omega_j) \) and \( v(\Omega) - \sum_{j \in N} p_j(\Omega_j) = v(\bigcup_{j \in F(S^i)} \Omega_j) - \sum_{j \in F(S^i)} p_j(\Omega_j) \). Therefore, again

\[
v(S^i) = v(\bigcup_{j \in F(S^i)} \Omega_j) - \sum_{j \in F(S^i)} p_j(\Omega_j) \geq v(\bigcup_{j \in F(S^i)} \Omega_j) - \sum_{j \in M+i} p_j(\Omega_j)
\]

\[
\geq v((\bigcup_{j \in M} \Omega_j) \cup S_i) - \sum_{j \in M+i} p_j(\Omega_j)
\]

\[
> v((\bigcup_{j \in M} \Omega_j) \cup S_i) - \sum_{j \in M} p_j(\Omega_j) - p_i(S_i).
\]

In this case, the firm \( i \)'s attempt to profitably sell bundle \( S_i \) gives the buyer a lower surplus than under any offer \( (\bigcup_{j \in F(S^i)} \Omega_j) \) where firm \( i \) is inactive.

In view of the Propositions of this section, we can assert that, given constant costs, the lack of monotonicity of the agent’s preferences over bundles of goods and hence the inherited lack of monotonicity of the social value function is what drives our results on the number of out of equilibrium offers and on the principals’ rents in our general model.

6. Concluding Remarks

This paper has contributed to the literature on delegated common agency and complete information by extending the insights by Bernheim and Whinston (1986a,b), Martimort and Stole (2003) and Chiesa and Denicolò (2009), among others, to multi-product markets with indivisibilities and no necessarily monotone agent’s value function for bundles.

First, we have characterized the equilibrium outcomes in these settings and, by considering a kind of extended contracts—mixed bundling contracts—that stresses the role of out-of-equilibrium offers, we have shown the equilibrium existence. Under mixed bundling contracts the agent has the option of buying bundles of goods from a firm at special prices over the single good prices. In our model, with multi-product firms and an agent with preferences over each bundle of two or more goods, mixed bundling contracts are conditional on exclusive dealing for each bundle of two or more goods. Therefore, these contracts can be seen as either an aggressive pricing policy for exclusive dealing outcomes or as out-of-equilibrium offers sustaining the equilibrium consumption sets of individual components in delegated common agency allocations. The discrimination on exclusivity both facilitates collusion on common agency outcomes and represents a credible threat that avoids deviations by the principals, thus helping them set incentive-compatible contracts. We have also found that equilibrium need not be unique in the sense that many equilibrium price vectors may sustain the same equilibrium allocation. This is due to the fact that each principal offers contracts for its products and also offers subsets of them as bundles at a special price. Notice that this implies that we have not considered singleton contracts (direct mechanisms) in the delegated agency game which do not allow for any offer to remain unchosen in equilibrium. Furthermore, efficient and inefficient equilibria may belong to the subgame Nash correspondence.
The lack of coordination among the principals is the reason behind inefficient equilibria, in which the agent chooses a suboptimal bundle and no principal has a profitable deviation inducing the agent to buy the surplus-maximizing bundle.

Second, we have ruled out inefficient equilibria by either assuming that all firms are pricing unsold bundles at the same profit margin as the bundle sold at equilibrium, or imposing the solution concept of subgame perfect Strong equilibrium, which requires the absence of profitable deviations by any subset of principals and the agent. Given the lack of monotonicity of our general model, an the possible lack of existence of truthful equilibria, the Strong equilibrium concept is easier to characterize and show existence than that of Coalition-proof equilibrium, which requires the equilibrium immunity to deviations by subsets of principals which are themselves immune to deviations by sub-coalitions, etc. This shortcut allows us to prove the existence of efficient Coalition-proof equilibrium.

Third, we have shown that when the equilibrium consumption set is a common agency bundle, mixed bundling contracts are out-of-equilibrium offers sustaining the equilibrium consumption sets of individual components. Then, we have analyzed the minimum number of them needed to support SPSE-outcomes. In our delegated common agency equilibrium, all principals may need to offer at least three supply schedules. The reason behind this results was the deterrence of each principal’s deviations to his exclusive dealing outcomes or even to another mixed bundle.

Finally, we have analyzed the specific structure of equilibrium prices and payoffs for common agency outcomes when the social surplus function is monotone and either submodular or supermodular. In the former case, principals are substitutes and their equilibrium rents are equal to the principals’ social marginal contributions with an agent’s positive rent, thus reflecting market competition. In the latter case, the agent’s rent is zero and then the core of the value function is always priced by the subgame perfect Nash-equilibrium rents. In both cases, the set of subgame perfect Strong equilibrium and that of Coalition-proof equilibrium coincide and equilibrium outcomes are sustained by TIOLI offers. Finally, we have proven that monotonicity and constant unit costs are sufficient conditions for such offers. Therefore, the lack of monotonicity of the agent’s preferences over bundles of goods in our model drives our results on the number of out of equilibrium offers and the principals’ rents of the general model.

An interesting extension of our analysis is to consider both multiple principals and agents. Our intuition is that a new kind of extended contracts, non-anonymous and mixed bundling contracts, is needed to ensure the equilibrium existence. This is left for future research.

7. Appendix.

Example 1: Non-existence of equilibrium linear prices. The efficient consumption set is the common agency bundle $S = \{a, d\}$. Suppose that $\delta = 8$. Let us show that there are not linear prices (contracts) guaranteeing the existence of any equilibrium outcome. Traditional arguments suggest how to set prices for, say, the efficient consumption set $S = \{a, d\}$. The proof of the remaining cases is similar.
Whether the agent’s participation constraint is binding will help define the equilibrium prices. Suppose that the agent’s participation constraint is not binding, i.e. \( cs[S, p] = 9 - p_a - p_d > 0 \), or \( p_a + p_d < 9 \). Since \( S \) is a two-firm bundle, then firms’ prices have to first guarantee the incentive compatibility constraints avoiding exclusive dealing, i.e., the agent’s surplus associated to the consumption of set \( S = \{a, d\} \), has to be bigger than or equal to the agent’s surplus if either firm 2 or firm 1 were removed from the market,

\[
\begin{align*}
  cs[S, p] &= 9 - p_a - p_d \geq 4 - p_a - p_b = cs[\{a, b\}, p], \\
  cs[S, p] &= 9 - p_a - p_d \geq 8 - p_c - p_d = cs[\{c, d\}, p].
\end{align*}
\]

Notice, however, that both (8) and (9) have to be binding since otherwise either firm 1 could deviate by raising \( p_a \) or firm 2 by increasing \( p_d \) and be better off, without changing the agent’s choice. Thus, we have that \( p_d = 5 + p_b \) and \( p_a = 1 + p_c \) and hence that \( p_a + p_d = 6 + p_b + p_c \).

The multiproduct nature of the model includes another incentive constraint dealing with the agent’s switching to the other common agency bundle, i.e., \( cs[S, p] = 9 - p_a - p_d \geq 5 - p_b - p_c = cs[\{b, c\}, p] \), which implies that \( p_a + p_d \leq 4 + p_b + p_c \) that contradicts the previous result. Hence the agent’s participation constraint has to be binding and thus \( p_a + p_d = 9 \).

Since, \( p_a + p_d = 9 \) the agent’s surplus is zero and firms 1 and 2’s profits are \( p_a = 9 - p_d \) and \( p_d = 9 - p_a \), respectively. However, firm 1 could deviate by setting \( p_b = 0 \) and \( p_a = 4 \) and trying to sell its own bundle \( \{a, b\} \). Therefore, at equilibrium it must be that \( p_a \geq 4 \); similarly, to avoid firm 2 deviation by setting \( p_c = 0 \) and \( p_d = 8 \), and selling its own bundle \( \{c, d\} \), it must be that \( p_d \geq 8 \). Hence \( p_a + p_d \geq 12 \) which contradicts the assumption that \( p_a + p_d = 9 \).

Therefore, no linear prices support \( S = \{a, d\} \) as the equilibrium consumption set. The above reasoning can be applied to any other bundle to conclude that under linear pricing the equilibrium may fail to exist.

**Equilibrium mixed-bundling prices in Example 1 (continuation).** The following steps prove that the efficient bundling \( S = \{a, d\} \) is supported by equilibrium prices

\[
\begin{align*}
  0 &\leq p_a \leq 1, \quad 4 \leq p_d \leq 5, \quad 5 \leq p_a + p_d, \\
  p_{ab} &= p_a + p_d - 5, \quad p_{cd} = p_a + p_d - 1, \\
  p_b &\geq 0, \quad p_c \geq p_d - 4, \quad p_b + p_c \geq p_a + p_d - 4.
\end{align*}
\]

as a SPE-outcome.

1) Suppose that the agent’s participation constraint is binding: \( p_a + p_d = 9 \). Since

\[
\begin{align*}
  cs(S, p) &= 9 - p_a - p_d \geq 4 - p_{ab} = cs(\{a, b\}, p), \\
  cs(S, p) &= 9 - p_a - p_d \geq 8 - p_{cd} = cs(\{c, d\}, p),
\end{align*}
\]

then \( p_{ab} \geq 4 \) and \( p_{cd} \geq 8 \). If \( p_a < 4 \), then firm 1 is better off first setting \( p_a \) and \( p_b \) high enough and then setting \( p_{ab}' = 4 - \varepsilon > p_a \), for \( \varepsilon \) sufficiently small so that the agent will
switch to firm 1’s exclusive dealing bundle \( \{a, b\} \) obtaining a profit equal to \( \varepsilon \). Therefore, \( p_a \geq 4 \). We can reproduce the same argument for \( p_d < 8 \) to obtain that \( p_d \geq 8 \). But this contradicts our statement that \( p_a + p_d = 9 \) and therefore the agent’s participation constraint is not binding: \( p_a + p_d < 9 \).

2) Let us see that the incentive compatibility constraints avoiding exclusive dealing have to be binding,

\[
\begin{align*}
\text{cs}(S, p) &= 9 - p_a - p_d = 4 - p_{ab} = \text{cs}(\{a, b\}, p), \quad (10) \\
\text{cs}(S, p) &= 9 - p_a - p_d = 8 - p_{cd} = \text{cs}(\{c, d\}, p), \quad (11)
\end{align*}
\]

because if the left-hand side of either 10 or 11 was strictly bigger than its corresponding right-hand side, then some firm would have an incentive to profitably raise its price. More precisely, notice that if both 10 and 11 were satisfied with strict inequality, then both firms will have an incentive to rise the prices of \( p_a \) and \( p_d \). On the other hand, if only 10 were satisfied with strict inequality, then firm 2 will have an incentive to rise \( p_d \) until inequality 10 becomes in an equality, rising simultaneously \( p_{cd} \) to maintain 11 as an equality. Similar argument could be applied if only 11 were satisfied with strict inequality.

3) Suppose now that \( p_{ab} > p_a \). Then, by equation 10 firm 1 could be better off by setting \( p_a \) and \( p_b \) high enough and decreasing the price of \( \{a, b\} \) to \( p_{ab} - \varepsilon > p_a \) and making the agent switch from \( \{a, d\} \) to \( \{a, b\} \). The same reasoning applies to \( p_{cd} > p_d \). Therefore, at equilibrium \( p_a \geq p_{ab} \) and \( p_d \geq p_{cd} \).

4) Consider then that \( p_a \geq p_{ab} \) and \( p_d \geq p_{cd} \); by equations 10 and 11,

\[
\begin{align*}
p_a + p_d &= 5 + p_{ab} \text{ or } p_d = 5 + p_{ab} - p_a \leq 5, \\
p_a + p_d &= 1 + p_{cd} \text{ or } p_a = 1 + p_{cd} - p_d \leq 1.
\end{align*}
\]

Notice, that the first equality also implies that \( p_{ab} = p_a + p_d - 5 \) and then \( p_a + p_d \geq 5 \).

5) Finally, suppose that \( p_d < 4 \). Then, firm 2 could be better off by setting \( p_c \) and \( p_d \) high enough and then setting the price of \( \{c, d\} \) to \( p_{cd} = 4 - \varepsilon > p_d \) and making the agent switch from \( \{a, d\} \) to \( \{c, d\} \). Thus, \( p_d \geq 4 \).

Therefore any equilibrium price vector is such that:

\[
\begin{align*}
0 &\leq p_a \leq 1, \quad 4 \leq p_d \leq 5, \quad 5 \leq p_a + p_d, \\
p_{ab} &= p_a + p_d - 5, \quad p_{cd} = p_a + p_d - 1, \\
p_b \text{ and } p_c \text{ bigger enough.}
\end{align*}
\]

With a little additional work we can obtain the precise lower bound for \( p_b \) and \( p_c \).

In similar fashion, we can find the conditions for the price vector supporting \( T = \{c, d\} \) as a subgame perfect equilibrium outcome.

**Definition of Subgame Perfect Coalition-Proof Equilibrium:** Let us introduce some additional notation. For every coalition \( M \subseteq N \), \( S \subseteq \Omega \) and strategies \( (\tilde{p}_j)_{j \in N \setminus M} \) of principals outside \( M \), define the restriction of the game \( G^{MB} \) to players in \( M \) and the buyer, where the strategies \( (\tilde{p}_j)_{j \in N \setminus M} \) are held fixed. Formally, the game \( G^{MB}/(\tilde{p}_j)_{j \in M} \) is defined
as
\[ G^{MB}/(\vec{p}_j)_{j \in M} = (M + 1, v, (c_j)_{j \in M}, (P_i)_{j \in M}) \]
where for every player \( i \in M \) and strategy \((p_j)_{j \in M}\)

\[ \pi_i(S((p_j)_{j \in M}, (\vec{p}_j)_{j \in N \setminus M})) = (p_i - c_i)(S((p_j)_{j \in M}, (\vec{p}_j)_{j \in N \setminus M})) \]  \hspace{1cm} (12)

and

\[ cs[S, ((p_j)_{j \in M}, (\vec{p}_j)_{j \in N \setminus M})] = v(S(p)) - \sum_{i \in F(S) \cap M} p_i(S_i) \geq \sum_{i \in F(S) \cap (N \setminus M)} \vec{p}_i(S_i). \]  \hspace{1cm} (13)

**Definition 4.** A subgame perfect coalition-proof NE (SPCP) of \( G^{MB} \) is defined recursively.

1. In a game \( G^{MB} \) with a buyer and a single principal \((n = 1)\), \((\vec{S}, \vec{p}_1)\) is a SPCP if and only if it is a SPNE.

2. In a game \( G^{MB} \) where \( n > 1 \), \((\vec{S}, \vec{p}_1, \ldots, \vec{p}_n)\) is a self-enforcing profile of strategies if for all \( M \subseteq N \), \((\vec{p}_j)_{j \in M}\) is a subgame perfect coalition-proof NE in the game \( G^{MB}/(\vec{p}_j)_{j \in M}\).

3. A profile \((\vec{S}, \vec{p}_1, \ldots, \vec{p}_n)\) is a SPCP of the game \( G^{MB} \) if it is self-enforcing and there is no other self-enforcing profile \((S', \vec{p}')\) that yields a higher payoff to each principal in \( N \), i.e., \( i \) \( cs[\vec{S}, \vec{p}] \geq cs[S, \vec{p}] \) for all \( S \subseteq \Omega \),

ii) There is no \((p'_j)_{j \in N}\) and \( S' \subseteq \Omega \) such that,

\[ a) (p' - c)_j(S'_j) \geq (\vec{p} - c)_j(\vec{S}_j), \text{ for each } j \in N, \text{ and} \]

\[ b) cs[S', (p'_j)_{j \in N}] \geq cs[S, \vec{p}], \text{ for all } S \subseteq \Omega \]

**Proof of Proposition 4:** The proof of i) closely follows the steps of the proof of Theorem 1. Let \((S, p)\) be an \( SPE^*\)-outcome, then \((S, p^S) \in SPE^*\). We define \( \pi_i = (p^S_i - c_i)(S_i) \), \( i \in N \) and \( \pi^b = (v - c)(S) - \sum_{i \in N} \pi_i \). First let us prove that \((\pi^b, (\pi_i)_{i \in N}) \in core(v - c)\). By the definition of \( p^S \), if \( j \notin F(S) \), then \( \pi_j = (p^S_j - c_j)(S_j) = 0 \), which implies that \( \pi^b = (v - c)(S) - \sum_{i \in F(S)} \pi_i \).

Since \( V^* = (v - c)(S) \), we have that \( \pi^b + \sum_{i \in N} \pi_i = V^* \). Given \( T \subseteq \Omega \) by BC

\[ v(S) - \sum_{i \in F(S)} p^S_i(S_i) \geq v(T) - \sum_{i \in F(T)} p^S_i(T_i), \]

which implies that

\[ (v - c)(S) - \sum_{i \in F(S)} (p^S_i - c)(S_i) \geq (v - c)(T) - \sum_{i \in F(T)} (p^S_i - c)(T_i) \]
and hence,

\[ \pi^b + \sum_{i \in F(T)} (p^i_T - c)(S_i) \geq (v - c)(T) \]

and \((\pi^b, (\pi_i)_{i \in N}) \in core(v - c)\).

Now let us prove that \((\pi_i)_{i \in N}\) belongs to \(\Pi^{PF}\). Suppose, on the contrary, that \((\pi_i)_{i \in N}\) is Pareto dominated by an element \((\pi'_i) \in \Pi^{PF}\), i.e. \(\pi'_j \geq \pi_j, j \in N\) and \(j_0\) is such that \(\pi'_{j_0} > \pi_{j_0}\). W.l.o.g. we can assume that \(\pi'\) is in \(\Pi^{PF}\). Recall \(S \in \arg\max_{T \subseteq \Omega} \{(v - c)(T)\},\) then by Lemma 1 we have that \(j_0 \in F(S)\), otherwise \(\pi'_{j_0} = 0 > \pi_{j_0}\) which is a contradiction. Let \(p'\) be defined as \(p'_i(T_i) = \pi'_i + c_i(T_i),\) for all \(i \in N, T_i \subseteq \Omega_i\), so that \(p'_j(S_{j_0}) = \pi'_j + c_j(S_{j_0}) > \pi_{j_0} + c_j(S_{j_0}) = p_{j_0}(S_{j_0})\). By i), already proven, we have that \((S, p')\) is an \(SPE^*\)-outcome, then firm \(j_0\) has incentives to raise its equilibrium price vector up to \(p\), which contradicts that \((S, p)\) is a \(SPE^*\)-outcome. This implies that \((\pi_i)_{i \in N}\) belongs to \(\Pi^{PF}\), as claimed.

**Proof of Proposition 5:** Assume w.l.o.g. that unit costs are zero. Given that \(v\) is monotonic \(v(\Omega) \geq v(S)\) for all \(S \subseteq \Omega\) and \(mc_i = v(\Omega) - v(\Omega \setminus \Omega_i) \geq 0\). Let \(p^*_i(T_i) = p^*_i(\Omega_i) = v(\Omega) - v(\Omega \setminus \Omega_i)\). Thus, prices are all positive and \(mc_i = p^*_i(\Omega_i)\).

First we prove that \((\Omega, \mathbf{p}^*) \in SPE^*\)-outcome set. By the monotonicity of \(v\), Corollary 1 and that \(p^*_i(T_i) = p^*_i(\Omega_i)\) it suffices to prove that \((\Omega, \mathbf{p}^*) \in SPE^*\)-outcome set.

Step 1: Consumer surplus is non-negative. By FS, \(mc_N = v(\Omega) \geq \sum_{i \in N} mc_i = \sum v(\Omega) - v(\Omega \setminus \Omega_i) = \sum_{i \in N} p^*_i(\Omega_i)\), thus \(cs = v(\Omega) - \sum_{i \in N} p^*_i(\Omega_i) \geq 0\).

Step 2: Condition BC in Proposition 1 is verified. Given \(S \subseteq \Omega\), by definition, \(mc_{N \setminus F(S)} \leq v(\Omega) - v(S)\) and by FS, \(mc_{N \setminus F(S)} \geq \sum_{i \in N \setminus F(S)} mc_i\). Thus,

\[ v(\Omega) - v(S) \geq \sum_{i \in N \setminus F(S)} mc_i = \sum_{i \in N \setminus F(S)} p^*_i(\Omega_i) = \sum_{i \in F(S)} p^*_i(\Omega_i) - \sum_{i \in F(S)} p^*_i(\Omega_i) = \sum_{i \in F(S)} p^*_i(\Omega_i) - \sum_{i \in F(S)} p^*_i(S_i) \]

then \(v(\Omega) - \sum_{i \in F(S)} p^*_i(\Omega_i) \geq v(S) - \sum_{i \in F(S)} p^*_i(S_i)\).

Step 3: Condition FC1 in Proposition 1 is verified. Given \(j \in N\), by definition \(p^*(\Omega_j) = v(\Omega) - v(\Omega \setminus \Omega_j)\), then \(v(\Omega) - p^*(\Omega_j) = v(\Omega \setminus \Omega_j)\) which implies that \(v(\Omega) - \sum_{i \in N} p^*(\Omega_i) = v(\Omega \setminus \Omega_j)\). Thus, \(\Omega_j = \Omega \setminus \Omega_j\) is the one which verifies FC1 for all \(j \in N\).

Step 4: Condition FC2 in Proposition 1 is verified. This condition holds trivially since \(p^*(S_i) = p^*(\Omega_i)\) for all \(i \in N, S_i \subseteq \Omega_i\).

Condition FC3 in Proposition 1 does not apply given that \(N = F(\Omega)\).

Thus \((\Omega, \mathbf{p}^*) \in SPE^*\)-outcome set.

Now, let us prove the converse. Consider \((\mathbf{S}^*, \mathbf{p}^*) \in SPE\)-outcome set, so that \(v(\mathbf{S}^*) = \sum_{i \in F(T)} \mathbf{p}^i - c(S_i) = (v - c)(T)\) for all \(T \subseteq \Omega\).
\( v(\Omega) \) and \( mc_{N \setminus F(S^*)} = 0 \). Then by FS, for all \( T_i \subseteq \Omega_i \) and \( i \in N \setminus F(S^*) \)

\[
0 = mc_{N \setminus F(S^*)} \geq \sum_{i \in N \setminus F(S^*)} mc_i \geq \sum_{i \in N \setminus F(S^*)} p_i^*(T_i) \geq 0,
\]

which implies that \( p_i^*(T_i) = 0 \).

Moreover, by BC \( v(\Omega) - \sum_{i \in N} p_i^*(\Omega_i) \geq v(S^*) - \sum_{i \in F(S^*)} p_i^*(S_i^*) \). Therefore,

\[
0 = v(\Omega) - v(S^*) \geq \sum_{i \in F(S^*)} p_i^*(\Omega_i) - p_i^*(S_i^*).
\]

Thus, \( p_i^*(\Omega_i) = p_i^*(S_i^*) \) for all \( i \in F(S^*) \).

**Proof of Proposition 6:** Let \( \Sigma \) be the set of permutations (orderings) of \( N = \{1, 2, ..., n\} \) and let \( \sigma \in \Sigma \) be any of its elements. Let \( P_i^\sigma \) be the set of firms which precede firm \( i \) with respect to permutation \( \sigma \), i.e. for all \( i \in N \) and \( \sigma \in \Sigma \), \( P_i^\sigma = \{ j \in N | \sigma(j) < \sigma(i) \} \).

Define, following Shapley (1971), the marginal contribution vector \( x^\sigma(v - c) \in \mathbb{R}^n \) of \( (v - c) \) with respect to ordering \( \sigma \) by, \( x_i^\sigma(v - c) = V(P_i^\sigma + i) - V(P_i^\sigma) \), for all \( i \in N \). If \( (v - c) \) is super modular, then the marginal contribution vector \( x^\sigma(v - c) \) will be positive.

The equality between the sets \( \text{conv}\{ x^\sigma(v - c) | \sigma \in \Sigma \} \) and \( \text{core}(v - c) \) is given in Driessen (1993), but this equality implies that \( \pi^b = 0 \) and hence that \( \text{core}(v - c) = \Pi^{PF} \). Thus, by Proposition 4 consumer surplus is zero and the equilibrium prices are the \( \text{core}(v - c) \). ■

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