The Role of Learning for Asset Prices, Business Cycles and Monetary Policy

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Abstract

Two of the challenges of macroeconomics today are that credit frictions seem to be more important for the business cycle than standard models imply; and that these standard models also have disappointing asset pricing properties. I propose a model that jointly addresses these two challenges. Firms’ access to credit depends on the market value of their assets, but agents cannot infer the true dynamics of asset prices and form expectations via learning. The production side of the economy interacts with learning to form a feedback loop between beliefs, asset prices and fundamentals. This adds substantial amplification and endogenous persistence to the real side of the economy while also improving asset price characteristics. Expectations are consistent with forecast error predictability in survey data. The optimal monetary policy rule in the model includes a reaction to asset prices not present under rational expectations.

Keywords: Learning, Credit Constraints, Business Cycles, Monetary Policy, Survey Data

JEL Classification: D83, E32, E44, E52
1 Introduction

I think financial factors in general, and asset prices in particular, play a more central role in explaining the dynamics of the economy than is typically reflected in macroeconomic models, even after the experience of the crisis. - Andrew Haldane, 30 April 2014

How come that the Chief Economist of the Bank of England expresses such a concern more than 15 years after the publication of “Credit Cycles” (Kiyotaki and Moore, 1997)? Credit frictions have since been regarded as a central mechanism by which asset prices can interact with the wider economy, and a wealth of research has been dedicated to it. But we still have an incomplete understanding of this interaction, in part also because our understanding of asset prices themselves remains incomplete.

In this paper, I want to address two particular challenges faced by modern macroeconomics. The first challenge is that models of credit frictions typically do not generate much amplification and propagation, especially for shocks that originate outside of the financial system itself (Quadrini, 2011). This observation might be partly responsible for comments such as the one above. The second challenge is that asset prices are not explained well by these models either. Many empirical regularities of asset prices pose a challenge to simple and rational asset pricing models and have therefore been called “puzzles”, including the excess volatility puzzle (Shiller, 1981), the return predictability puzzle (Fama and French, 1988), or most famously the equity premium puzzle (Mehra and Prescott, 1985). To be sure, it is possible to make the theory compatible by modifying the utility function until it generates the right variation in discount factors, and this approach has been very popular. But another possible interpretation of these puzzles is that agents might not be able to form fully rational expectations about complex objects such as asset prices. If this interpretation is true, it has potentially important implications for the business cycle and for policy. Understanding these implications is the main goal of this paper.

I propose a model of firm credit frictions in which agents are unable to form rational expectations about a particular asset price - the price of equities in the stock market. Instead, they have to learn from past observations to form subjective beliefs in an extrapolative fashion. This allows to attach a precise meaning to notions of “optimism” and “pessimism” in financial markets. Otherwise, agents are fully rational and forward-looking. The borrowing constraint of firms in turn is such that equity valuations matter for their access to credit. It slightly differs from the usual practice of having borrowing limits depend on the liquidation value of a firm’s assets, but emerges naturally if defaulting firms can be restructured and resold instead of being liquidated, similar to the procedure under Chapter 11 of the US Bankruptcy Code.

There are four main results of the paper. First, the learning mechanism improves asset price properties such as price and return volatility and return predictability without relying on complex preferences. This is due to the self-referential nature of beliefs and prices.
When beliefs become more “optimistic” today, this increases asset demand, raising the price-dividend ratio. Observing this rise, agents will be even more optimistic tomorrow. This interaction between beliefs and prices creates volatility similar to the one observed in the data, whereas under rational expectations stock prices would be too smooth. At the same time, expectations are mean-reverting, so that when agents are most optimistic, the P/D ratio is high but future returns will be low, just as in the data. This first result parallels the analysis of an endowment economy by Adam, Marcet and Nicolini (2013).

Second, a positive feedback loop emerges between beliefs, asset prices and the production side of the economy. The fluctuations in asset prices that come from adjustments in investor beliefs affect firms’ access to credit and lead to considerable amplification and propagation of shocks. Thus, what Bernanke et al. (1999) have called the “financial accelerator” becomes a much more powerful mechanism. When investor beliefs are more optimistic, credit conditions ease which allows firms to move closer to their profit optimum. Provided counteracting general equilibrium forces are not too strong, they will also be able to pay higher dividends to their shareholders. This in turn raises asset prices further and propagates the optimism among stock market investors. As a result, the endogenous asset price volatility induced by learning is higher than in an endowment economy.

Third, I find that the subjective beliefs agents hold in the model are consistent with evidence from survey data. I document that forecast errors on macroeconomic aggregates (from the US Survey of Professional Forecasters) as well as on stock returns (the Duke-Fuqua CFO Survey) can be predicted positively by their own lags, forecast revisions, and changes in the price-dividend ratio; and (to some extent) negatively predicted by the level of the P/D ratio. All of these relationships hold for the subjective beliefs in the model. This holds despite the fact that I only relax the rational expectations assumption on stock prices, but not on any other variable in the model.

Fourth, the model allows me to study some of the normative implications of asset price fluctuations under learning. A recurring question in monetary economics is whether policy should react to asset price “misalignments”. Gali (2014) writes that justifying such a reaction requires “the presumption that an increase in interest rates will reduce the size of an asset price bubble” for which “no empirical or theoretical support seems to have been provided”. The present model does establish at least some theoretical support. I find that under learning, the optimal interest rate rule (within a class of interest rate rules) reacts negatively to asset price growth. By raising interest rates when asset prices are rising, monetary policy curbs the build-up of optimistic investor beliefs, thus reducing the size of both asset price volatility and business cycle volatility. Under rational expectations, this channel of monetary policy disappears and the optimal policy only reacts to inflation and output, in line with results in the existing literature.

The remainder of this paper is structured as follows. Section 2 reviews the related literature. Section 3 presents some empirical evidence on the relationship between stock market valuation, investment and credit constraints, as well as the plausibility of replacing ra-
tional expectations with learning in stock market pricing. Section 4 presents a simple
version of the model. This version is highly simplified and only serves as an illustration
of the amplification mechanism at work. The full, quantitative model is then presented
in Section 5. Section 6 contains the quantitative exploration as well as the comparison
of model-implied beliefs with survey data. Section 7 contains the analysis of monetary
policy. Section 8 concludes and discusses avenues for future research.

2 Related literature

This paper draws on insights from the literature on asset pricing on the one hand and
on business cycles with financial frictions on the other hand. On the asset pricing side,
I follow the approach laid out in a series of papers by Klaus Adam and Albert Marcet
(Adam and Marcet, 2011; Adam et al., 2014, 2013). They show that parsimonious models
of adaptive learning about asset price growth succeed in explaining key aspects of observed
stock price data such as the excess volatility, equity premium, and return predictability
puzzles. They also match survey data on investor expectations, which are hard to reconcile
with rational expectations (this has been pointed out as well by Greenwood and Shleifer
(2014)). They study asset pricing in endowment economies. I embed their approach in
a model with full endogenous production that enables to study the interactions between
financial markets and the real economy and policy implications.

There are at least two other approaches to asset pricing that are able to explain asset price
puzzles. The first one is due to Campbell and Cochrane (1999) and relies on a particular
non-linear form of habit formation combined with high risk aversion. The second, so-called
“long-run risk” approach due to Bansal and Yaron (2004), introduces small, predictable
and observable components to long-run growth of consumption and dividend processes
combined with Epstein-Zin preferences. There is a small but growing literature which
embeds these alternative approaches in production economies: Boldrin et al. (2001) for
habit formation, Tallarini Jr. (2000) and Croce (2014) for long-run risk. These papers are
considering models without financial frictions and are mainly concerned with endogenising
consumption and dividend streams in a production economy while preserving the asset
price implications in endowment economies. Furthermore, asset price fluctuations are
interpreted as efficient in these approaches. There can be no price “misalignments”. This
leads to markedly different policy implications.

When setting up the quantitative model, I use a structure that is close to the seminal
“financial accelerator” paper by Bernanke et al. (1999). They aimed to show that finan-
cial frictions could lead to sizeable endogenous amplification and propagation of business
cycles. However, subsequent research has produced mixed evidence on their quantitative
importance. Most prominently, Kocherlakota (2000) and Cordoba and Ripoll (2004) have
documented the fragility of the amplification mechanism. Instead, the literature seems to
have moved on to incorporate additional shocks that directly influence the degree of credit
frictions in the economy, so-called “financial shocks”. Studies including Christiano et al. (2010) and Jermann and Quadrini (2012) find that exogenous shocks to credit conditions are powerful in explaining fluctuations at business cycle frequency. This paper instead re-examines the issue of amplification of conventional supply and demand shocks. My main finding in this respect is that the apparent weakness of amplification through credit frictions is in fact a problem of counterfactually low asset price volatility.

This point is also made in two other recent papers. Liu et al. (2013) construct a model in which firms are borrowing against the value of their land, while households can also use land for housing. The price of land is then made to fluctuate as much as in the data through shocks to households’ marginal utility of housing. This shock directly affects firms’ ability to borrow and is further amplified by competing demand for land by firms and households. It is found to account for a large part of fluctuations at business cycle frequency. Xu et al. (2013) obtain similar results in a model where firms borrow against the market value of their assets, similar to the credit friction in my model. They prove the existence of rational liquidity bubbles in the spirit of Farhi and Tirole (2012) and introduce a shock that governs the size of this bubble. In both cases, the new shock is essentially an exogenous shock to asset prices. I do not rely on such a shock, but instead study the implications of endogenous asset price volatility for business cycles with credit frictions.

The present paper also relates to the literature on adaptive learning in DSGE models. There have been attempts to assess the interaction between learning and credit frictions (Caputo, Medina and Soto, 2010; Milani, 2011; Gelain, Lansing and Mendicino, 2013). The approach taken to learning in these papers consists of two steps: deriving the linearised equilibrium conditions of the economy under rational expectations, and then replacing all terms involving expectations of future variables with a forecast function, the parameters of which are updated every period. I find several problems with this approach. First, it implies suboptimal decisions. In any model with an Euler equation, for example, the stochastic discount factor needs to be parametrised. Thus, agents are assumed to forecast an object that depends entirely on their own choice. Second, it does not lead to a consistent system of beliefs. Parametrising the expectation of the discount factor in the Euler equation could be interpreted as a belief about, say, $E_t [u'(C_{t+1})]$, but it does not imply anything about, say, $E_t [C_{t+1}]$, which if it appeared in another model equation would need to be parametrised with a separate function. Third, because of the need to parametrise every single occurrence of the expectations operator in the equilibrium conditions, the dynamics of such models become prohibitively complex and intransparent. For example, it is nearly impossible to distinguish whether a certain dynamic response is due to learning about asset prices, future consumption, inflation etc., since everything needs to be learned about simultaneously. I therefore abandon this approach completely. In the model of this paper, agents make optimal decisions conditional on their beliefs about the law of motion for stock prices, and they also know the structural relations of the economy, including
their own belief updating. This is a smaller and more parsimonious deviation from rational expectations, and allows me to focus exclusively on the role of learning about asset prices.

3 Some empirics

The purpose of this section is to document three sets of facts that motivate my modelling strategy. First, movements in the aggregate stock market have sizeable effects on investment and credit constraints. Second, stock price movements exhibit characteristics which are not easy to reconcile with simple asset pricing models (as typically used in macroeconomics). Third, measures of expectations from survey data, both on stock prices and macro variables, reveal systematic deviations from rational expectations. Most of these observations have been documented previously in separate strands of the literature. I present them here because the model developed later on aims at providing a joint explanation for them.

3.1 Effect of the stock market on investment and credit constraints

It has been known for some while that the stock market has predictive content for investment (Barro, 1990). One hypothesis is that the stock market is just a reflection of expectations of future economic “fundamentals”. For example, when information about profitable investment opportunities arrives, this gives rise to higher equity valuations as well as a subsequent increase in investment, without a direct causal link between the two. Another hypothesis is that stock market movements have a direct effect on investment, even when they do not reflect changing expectations about fundamentals. Blanchard et al. (1993) constructed a measure of expected fundamentals⁴ and regressed investment on both stock prices and this measure. They essentially found that both have a positive effect, although fundamentals seem to have somewhat higher explanatory power. Using modern VAR/VECM techniques, Beaudry and Portier (2006) later found that stock prices have considerable explanatory power in forecasting long-run changes in aggregate productivity, thus emphasising the information content of stock prices. In general though, identifying market movements due to expected fundamentals is impossible without spelling out a structural model that specifies what fundamentals are and how expectations are formed.

I do not want to tackle this identification problem either, but rather document the effects of a “stock price shock” on investment, dividends, and credit frictions. To this end, using quarterly US data, I construct a VAR in five variables: Investment, total factor productivity, dividends, a corporate credit spread and the price-dividend-ratio of the stock market. Investment is real private non-residential fixed investment. Productivity is adjusted for

⁴Essentially, they used the predicted values from a regression of future discounted dividend profits on a set of instruments.
Figure 1: Stock price shock in a VECM.

Five-variable VECM with lag length 2 (selected by Akaike Information Criterion), cointegration rank 2 (selected by Johansen trace test) and shock identification by Cholesky decomposition in the ordering Investment, TFP, Dividends, Spread, P/D ratio. Quarterly US data 1962Q1-2012Q4. Investment is real private non-residential fixed investment. TFP is the capacity utilisation-adjusted series produced by Kimball et al. (2006). Dividends and P/D ratio from the S&P Composite index. Credit spread is the baa-aaa Moody’s corporate spread. Displayed units are basis points for the spread and log*100 for all other series. Bootstrapped confidence bands at the 90% level.

capacity utilisation as in Kimball et al. (2006). I include it in order to see whether the stock market shock I identify is a “news shock” forecasting future TFP. Dividends are four-quarter moving averages from the S&P Composite index. The corporate credit spread is Moody’s baa-aaa corporate bond spread, serving as a proxy for credit market conditions. The P/D ratio is again from the S&P composite index. While the spread and the P/D ratio are stationary, the three real variables are not, and there is good reason to believe that they are cointegrated. Therefore, I estimate the model in its error-correction form with a cointegration rank of two and two lags.²

I identify a “stock price shock” as one that has an immediate effect on the P/D ratio but no contemporaneous effect on the real variables (investment, TFP and dividends) as well as the credit spread.³ This shock accounts for more than 85% of the conditional variance in the P/D ratio at all horizons up to 10 years. Figure 1 presents the estimated impulse

²The rank choice is supported by the Johansen trace test which can only reject a cointegration rank smaller than two at the 5% level. The lag length is chosen by the Akaike information criterion.
³The results are robust to ordering the credit spread after the P/D ratio.
response functions. A positive stock price shock leads to a very persistent rise in the P/D ratio. It significantly increases investment and dividends while reducing credit spreads. At the same time, it does not look like a “news shock”, since the effect on future TFP is negative and insignificant. At the 8-quarter horizon, the shock accounts for only 1% of the conditional variance in TFP, but for almost all (85%) of the conditional variance in the P/D ratio, as well as 33% of that in investment, 21% in dividends, and 36% in the credit spread.

Interpreting this shock is difficult without imposing additional structure, but it suggests that higher stock prices are systematically linked to higher levels of investment, higher dividends and easier access to credit. This is consistent with the type of credit constraint developed in the model.

### 3.2 Asset price “puzzles”

It is well known that asset prices, and stock prices in particular, exhibit characteristics that are not easy to reconcile with basic asset price models. For the aggregate stock market, the most prominent are excess volatility, return predictability and the equity premium, summarised in Table 1 (quarterly US data).

The first row shows the ratio of the standard deviation of the cyclical components of stock prices and dividends. By this measure prices are 2.6 times more volatile than dividends. Additionally, the coefficients of variation for the price-dividend ratio (second row) and log stock returns (third row) are also high. Shiller (1981) showed that this degree of volatility is at odds with an asset pricing model based on rational expectations and constant discount rates. This finding leaves two options: either discount rates must vary a lot, or expectations are not rational.4

The fourth and fifth row of the table document return predictability at the one- and five-year horizon, respectively. A high P/D ratio reliably predicts low future returns at these horizons, even if short-run stock returns are almost unpredictable. Cochrane (1992) showed that the variance of the P/D ratio can be decomposed into its covariance with future returns plus its covariance with future dividend growth. Since dividend growth is not very volatile and the degree of predictability by the P/D ratio is not very high, and the P/D ratio is volatile, it follows that returns must be predictable. Cochrane defines the predictable component of returns as “discount rates”, without imposing a structural interpretation on them. The challenge is then to see what can explain their variation - preferences, fads, or something else? I don’t pretend to know the answer to this question. But the survey data presented below provide some support for a “fads”-type interpretation such as the one in this paper.

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4 A third option is of course that equilibrium asset prices do not equal expected discounted cash flows at all, a stance taken by behavioural finance.
Table 1: Stock market statistics.

<table>
<thead>
<tr>
<th>statistic</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>excess volatility</td>
<td>2.632</td>
</tr>
<tr>
<td>$\sigma \left( \text{hp} (\log P_t) / \text{hp} (\log D_t) \right)$</td>
<td></td>
</tr>
<tr>
<td>$\sigma \left( \log P_t / \sigma (\log P_t) \right)$</td>
<td>.114</td>
</tr>
<tr>
<td>$\sigma (\log R_t) / E (\log R_t)$</td>
<td>6.03</td>
</tr>
<tr>
<td>return predictability</td>
<td>-0.197</td>
</tr>
<tr>
<td>$\rho \left( \log P_t, \log R_{t+4} \right)$</td>
<td></td>
</tr>
<tr>
<td>$\rho \left( \log P_t, \log R_{t+4} \right)$</td>
<td>-0.370</td>
</tr>
<tr>
<td>equity premium</td>
<td>3.56%</td>
</tr>
<tr>
<td>$E [\log R_t]$</td>
<td></td>
</tr>
</tbody>
</table>


The last row of the table documents the equity premium puzzle: Returns on stocks are several percentage points higher than on bonds. Mehra and Prescott (1985) showed that an asset pricing model based on power utility can only produce such a large premium for an implausibly large degree of risk aversion. While Adam et al. (2013) have made great advances in explaining this most famous of asset price puzzles with learning, I do not attempt to do so here, instead focusing on volatility and predictability.

### 3.3 Survey data on expectations

The rational expectations hypothesis is a fundamental building block of modern macroeconomics, so much so that attempts to call it into question are met with much scepticism. In the words of Sargent (2008), the hypothesis asserts that “outcomes do not differ systematically [...] from what people expect them to be”. Put differently, any agent’s forecast error should not be predictable by information available to the agent at the time of the forecast. The hypothesis is testable based on survey measures of expectations. In fact, it is often rejected. I will document some tests for expectations on stock prices and also on macroeconomic aggregates.

Expectations on stock prices and returns have been collected in various surveys. All of them point in a similar direction, which is that expectations are positively correlated with past returns and/or the P/D-ratio, whereas the best statistical prediction would call for a zero or negative correlation (the most recent and very systematic study confirming this is Greenwood and Shleifer, 2014). I illustrate this with an example from the CFO survey by John Graham and Campbell Harvey at Duke University. The survey asks CFOs of major US corporations about various aspects of their own business as well as the economy in
Figure 2: Measured expectations and actual outcomes.


general. Since 2000, it includes a question on stock market return expectations (“Over the next year, I expect the average annual S&P 500 return will be:”). Figure 2 compares the survey expectations with realised returns. The left panel plots the mean survey response against the value of the P/D ratio in the month preceding the survey. The correlation is strongly positive: return expectations are more optimistic when stock valuations are high. However, high stock valuations actually predict low future returns, as documented above and illustrated again in the right panel of the figure. Such a pattern cannot be reconciled with rational expectations, as it implies that agents’ forecast errors are predictable by the P/D ratio, a publicly observable statistic.

Tests of forecast error predictability can be applied to other variables of macroeconomic significance. Table 1 describes tests on data from the Federal Reserve’s Survey of Professional Forecasters (SPF) as well as on the CFO survey data above. Each row and column corresponds to a univariate regression of a mean forecast error on a variable which is observable to respondents at the time the survey is conducted. Under rational expectations, the correlation coefficients should all be approximately zero. The first column reports autocorrelation of forecast errors. For stock returns, there is no correlation of this and next year’s (4 quarter ahead) forecast error in the CFO survey. But for macroeconomic variables in the SPF, the current forecast error positively predicts the next (3 quarter ahead) forecast error. Column (2) reports results of a particular test devised by Coibion and Gorodnichenko (2010). Since for any variable \(x_t\), the SPF asks for forecasts at one-through four-quarter horizon, it is possible to construct a measure of agent’s revision of the change in \(x_t\) as \(\hat{E}_t [x_{t+3} - x_t] - \hat{E}_{t-1} [x_{t+3} - x_t]\). Forecast errors are positively predicted by this revision measure, which Coibion and Gorodnichenko take as supportive evidence
Table 2: Forecast error predictability.

<table>
<thead>
<tr>
<th>correlation of $\hat{\epsilon}_{t+1}$ with</th>
<th>(1) $\hat{\epsilon}_t$</th>
<th>(2) forecast revision</th>
<th>(3) log $PD_t$</th>
<th>(4) $\Delta \log PD_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{t+4}^{stock}$</td>
<td>-.02</td>
<td>n/a</td>
<td>-.47***</td>
<td>.07</td>
</tr>
<tr>
<td></td>
<td>(-.13)</td>
<td></td>
<td>(-3.70)</td>
<td>(.52)</td>
</tr>
<tr>
<td>$Y_{t+3}$</td>
<td>.26***</td>
<td>.29***</td>
<td>-.21*</td>
<td>.22**</td>
</tr>
<tr>
<td></td>
<td>(2.91)</td>
<td>(3.83)</td>
<td>(-1.78)</td>
<td>(2.42)</td>
</tr>
<tr>
<td>$I_{t+3}$</td>
<td>.31***</td>
<td>.31***</td>
<td>-.20*</td>
<td>.25***</td>
</tr>
<tr>
<td></td>
<td>(5.38)</td>
<td>(3.79)</td>
<td>(-1.74)</td>
<td>(2.88)</td>
</tr>
<tr>
<td>$C_{t+3}$</td>
<td>.44***</td>
<td>.23***</td>
<td>-.19*</td>
<td>.21**</td>
</tr>
<tr>
<td></td>
<td>(3.58)</td>
<td>(2.67)</td>
<td>(-1.85)</td>
<td>(2.37)</td>
</tr>
<tr>
<td>$u_{t+3}$</td>
<td>.29***</td>
<td>.43***</td>
<td>.05</td>
<td>-.27***</td>
</tr>
<tr>
<td></td>
<td>(2.33)</td>
<td>(6.07)</td>
<td>(.12)</td>
<td>(-3.07)</td>
</tr>
</tbody>
</table>

Correlation coefficients for mean forecast errors $\hat{\epsilon}_t$ on one-year ahead stock returns (Graham-Harvey survey) and three-quarter ahead real output growth, investment growth, consumption growth and the unemployment rate (SPF). t-statistics for corresponding univariate regressions in parentheses. Regressors: Column (1) is the lagged forecast error. Column (2) is the forecast revision as in Coibion and Gorodnichenko (2010). Column (3) is the S&P 500 P/D ratio and Column (4) is its first difference. Data from Graham-Harvey covers 2000Q3-2012Q4. Data for the SPF covers 1981Q1-2012Q4.

for sticky information models in which information sets are gradually updated over time. As we will see, this observation is also consistent with models of learning.

Columns (3) and (4) relate more specifically to the possibility of predicting forecast errors using stock market valuations. Column (3) shows that the price-dividend is negatively correlated with forecast errors: When stock prices are high, people tend to be systematically too optimistic about the future. This holds in particular for stock returns, but also for macroeconomic aggregates. Column (4) shows that the change in the price-dividend ratio is also a predictor of forecast errors, but in the opposite direction: When stock prices are rising, people tend to be systematically too pessimistic about the future. Since the stock market itself positively predicts economic activity, this suggests that agents underestimate the size of an expansion in its beginning, but overestimate it when it is about to end.

Of course, survey data are only an imperfect measure of people’s true expectations. When answering the questions, respondents might wilfully misstate their true expectations, answer carelessly or misunderstand the question. When being asked for a point estimate, they might not report the mean of their belief distribution but some other moment. They might be influenced by behavioural biases that would not show if they needed to make actual decisions, and so forth. Still, survey data are the best available test for the rational
expectations hypothesis. The fact that the hypothesis is so often rejected underlines the need for macroeconomic models that are consistent with the expectations patterns observed in the data.

4 Understanding the mechanism: A stylised model

In this section, I construct a simple model which illustrates the interaction between equity valuations and credit frictions in a stylised way. I impose several simplifying assumptions which mean that the model admits a closed-form solution. Quantitative analysis will require a richer model, the development of which is relegated to the next section.

4.1 Model setup

Time is discrete at \( t = 0, 1, 2 \ldots \). The model economy consists of a representative household and a representative firm. The representative household is risk-neutral and inelastically supplies one unit of labour. Its utility maximisation program is as follows:

\[
\max_{C_t, S_t, B_t} E_P^0 \sum_{t=0}^{\infty} \beta^t C_t
\]

s.t. \( C_t + S_t V_t + B_t = w_t + S_{t-1} (V_t + D_t) + R_{t-1} B_{t-1} \)

\( S_t \in [0, S], S_{t-1}, B_{t-1} \)

\( C_t \) is the amount of non-durable consumption goods purchased by the household in period \( t \). The consumption good is traded in a competitive spot market and serves as the numéraire. \( w_t \) is the real wage. Labour is also traded in a competitive spot market, so that \( w_t \) is taken as given by the household. Moreover, the household can trade two financial assets, again in competitive spot markets: one-period bonds, denoted by \( B_t \) and paying gross real interest \( R_t \) in the next period, and stocks \( S_t \) which trade at price \( V_t \) and entitle their holder to dividend payments \( D_t \). The holdings for stocks are assumed to be exogenously bounded between 0 and some positive constant \( S > 1 \).

The household maximises the expectation of discounted future consumption under the measure \( P \). This probability measure is the subjective belief system held by agents in the model economy at time \( t \). When it coincides with the true probability measure in equilibrium, then expectations are rational. When expectations are only boundedly rational, however, it is important to distinguish between subjective (or “perceived”) expectations \( E_P^0 \) and objective (or “actual”) expectations \( E^0 \).

\(^5\)This is necessary to guarantee existence of the learning equilibrium later on.
The first order conditions describing the household’s optimal plan under $\mathcal{P}$ are

$$R_t = R = \beta^{-1}$$  \hspace{1cm} (4.1)

$$S_t \begin{cases} = 0 \\ \in [0, \bar{S}] \text{ if } V_t \leq \beta E_t^{P} [V_{t+1} + D_{t+1}] \\ = \bar{S} \end{cases}$$  \hspace{1cm} (4.2)

Let us now turn to the firm. It engages in the production of a good which can be used both for consumption and investment. It is produced using capital $K_{t-1}$, which the firm owns, and labour $L_t$ according to the constant returns to scale technology

$$Y_t = K_{t-1}^\alpha (A_t L_t)^{1-\alpha}$$  \hspace{1cm} (4.3)

where $A_t$ is its productivity. To keep the system free from state variables, I assume only permanent shocks to productivity:

$$\log A_t = \log G + \log A_{t-1} + \varepsilon_t, \varepsilon_t \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right) \text{ iid}$$  \hspace{1cm} (4.4)

In particular, the expected growth rate of productivity is constant at $E_t A_{t+1}/A_t \equiv G$. The capital stock is predetermined and owned by the firm. It depreciates at the rate $\delta$ at the end of each period. The firm can also issue shares and bonds as described above. Thus, its period budget constraint reads as follows:

$$Y_t + (1 - \delta) K_{t-1} + B_t + S_t V_t = w_t L_t + K_t + S_{t-1} (V_t + D_t) + RB_{t-1}$$  \hspace{1cm} (4.5)

In the absence of financial constraints, the Modigliani-Miller theorem would render financial instruments redundant and the model would collapse to a standard stochastic growth model. Here, I impose constraints on both the equity and debt instruments.

On the equity side, the firm is not allowed to change the quantity of shares outstanding, fixed at $S_t = 1$. Also, it is not allowed to use retained earnings to finance investment. Instead, all earnings net of interest and depreciation have to be paid out to shareholders:

$$D_t = Y_t - w_t L_t - \delta K_{t-1} - (R - 1) B_{t-1}$$  \hspace{1cm} (4.6)

Combined with equation (4.5), this assumption implies that the firm’s capital stock must be entirely debt-financed: $K_t = B_t$ at all $t$. In other words, the firm’s book value of equity after dividend payouts is constrained to be zero. This does not mean, however, that the market value of equity is also zero (shown below).

On the debt side, the level of debt that can be acquired by the firm is limited to a fraction

\footnote{This holds under the suitable initial condition $K_{-1} = B_{-1}$, e.g. the firm starts with zero book value of equity on its balance sheet. This assumption is relaxed in the full model.}
\( \xi \in [0, 1] \) of the total market value of its assets, i.e. the sum of debt and equity:

\[
B_t \leq \xi (B_t + V_t) \quad \iff \quad K_t \leq \frac{\xi}{1 - \xi} V_t
\]

(4.7)

Equation (4.7) is a simple constraint on leverage, i.e. debt divided by value of total assets. The departure from most of the existing literature lies in the definition of the value of total assets as the market value of the firm instead of its book value (which equals \( K_t \)). This captures the idea that a firm which is more highly valued by financial markets will have easier access to credit. This could be because high market value acts as a signal to lenders for the firm’s ability to repay, or because the funds lenders can recover in the event of default depends on the price at which a firm can be resold to other financial market participants. The latter motive is used in the next section to formally derive (4.7) from an incomplete contract problem.

The firm maximises the presented discounted sum of future dividends, using the household discount factor:

\[
\max_{K_t, L_t} E_0^P \sum_{t=0}^{\infty} \beta^t D_t
\]

subject to:

\[
D_t = Y_t - w_t L_t + (1 - \delta - R) K_{t-1}
\]

\[
K_t \leq \frac{\xi}{1 - \xi} V_t
\]

\[
K_t \geq 0, \; K_{t-1}
\]

In particular, it makes its decisions under the same belief system \( P_t \) as the household. The first order condition for the labour choice implies

\[
\max_{L_t} Y_t - w_t L_t = \alpha \left( \frac{(1 - \alpha) A_t}{w_t} \right)^{\frac{1-\alpha}{\alpha}} K_{t-1}
\]

which is linear in \( K_{t-1} \) due to the constant returns to scale in production. We can hence define an internal return rate on capital and write dividends as follows

\[
\max_{L_t} D_t = \left( R^*_t - R \right) K_{t-1}
\]

(4.8)

where

\[
R^*_t = \alpha \left( \frac{(1 - \alpha) A_t}{w_t} \right)^{\frac{1-\alpha}{\alpha}} + 1 - \delta
\]

(4.9)

The firm’s choice in period \( t \) can then be summarised by

\[
K_t \begin{cases} 
0 & \text{if } E_0^P R^k_{t+1} \geq R \\
\frac{\xi}{1 - \xi} V_t & \text{if } \frac{\xi}{1 - \xi} V_t \leq E_0^P R^k_{t+1} < R \\
\frac{\xi}{1 - \xi} V_t & \text{if } E_0^P R^k_{t+1} < \frac{\xi}{1 - \xi} V_t
\end{cases}
\]

(4.10)
Before solving for the equilibrium, market clearing needs to be imposed. The market clearing condition for bonds is just $R = \beta^{-1}$. That for equity is $S_t = 1$, which is equivalent to the Euler equation:

$$V_t = \beta E_t^P [V_{t+1} + D_{t+1}]$$  \hspace{1cm} (4.11)

Goods market clearing requires

$$Y_t + K_t = C + (1 - \delta) K_{t-1}$$  \hspace{1cm} (4.12)

Finally, labour market clearing requires $L_t = 1$, which leads to the following expression for the internal rate of return on capital:

$$R_k^t = \alpha \left( \frac{A_t}{K_{t-1}} \right)^{1-\alpha} + 1 - \delta$$  \hspace{1cm} (4.13)

### 4.2 Rational expectations equilibrium

The equilibrium under rational expectations admits a closed form solution. As productivity has a stochastic trend, it is convenient to make it stationary by dividing the level variables by the level of productivity. In particular, let the effective capital stock be $k_t = K_t / A_t$ and effective expected dividends be

$$d(k_t) = E_t \left[ \frac{D_{t+1}}{A_t} \right] = \alpha \tilde{G} k_t + (1 - \delta - R) k_t$$  \hspace{1cm} (4.14)

where $\tilde{G} = G \exp \left( -\alpha (1 - \alpha) \sigma^2 \right)$. The effective stock price under rational expectations is simply the discounted sum of future dividends:

$$v_t = \frac{V_t}{A_t} = \beta \sum_{s=0}^{\infty} (\beta G)^s d(k_{t+s})$$  \hspace{1cm} (4.15)

First, let us consider the case $\xi = 1$. In this case, it can be verified that in equilibrium, $k_t$ is constant across time and states and is at its efficient level $k^*$ such that $E_t R_k^t = R$. By consequence, expected dividends are zero: $d(k^*) = 0$. By consequence, $v_t = 0$.

Intuitively, when the firm can borrow up to the total amount of its market value, it effectively faces no financial friction any more. Book and market value of the firm coincide. The expected return on capital equals the interest rate due to risk neutrality of households, and since all capital is financed by debt and the production function has constant returns to scale, in expectation all profits are paid out as interest payments to debt holders. The residual equity claims trade at a price of zero. This result is the expectation version of the zero-profit condition under constant returns to scale.

Next, let us turn to the case in which $\xi$ strictly smaller than one. Here as well, in equilibrium the effective capital stock $k_t$ and stock price $v_t$ turn out to be constant. The

---

7 The borrowing constraint (4.7) is still satisfied: $K_t \leq K_t + V_t = K_t + 0$. 

---
The equilibrium is characterised by two equations:

\[ v = \frac{d(k)}{R - G} \]  \hspace{1cm} (4.16)  
\[ k = \frac{\xi}{1 - \xi} v \] \hspace{1cm} (4.17)

The first equation pins down the stock market value of the firm as a function of its capital stock, while the second determines the capital stock that can be reached by exhausting the borrowing constraint that depends on the stock market value. In particular, the internal rate of return is always greater than the return on debt \((E_t R_{t+1}^k > R)\) and so the borrowing constraint is always binding. The equilibrium is depicted graphically in Figure 3. Equation (4.17) is represented by a straight line of slope \((1 - \xi)/\xi\), while the market value of equity (4.16) is hump-shaped. The equilibrium lies at some point A where the two lines intersect. Firms pay positive dividends (in expectation) and the market value of equity is positive. The firm, taking the wage as given, borrows the maximum possible amount as long as \(E_t R_{t+1}^k > R\). The capital stock \(k\) remains inefficiently low: \(k < k^*\).

While the capital stock and expected output are an increasing function of maximum leverage \(\xi\), expected dividends are non-monotonous and hump-shaped. Why is that so? There are two opposing forces affecting expected dividends, as can be seen from the following decomposition:

\[ \frac{d}{d\xi} d(k) = \begin{cases} E_t R_{t+1}^k - R & \text{if } \frac{d}{d\xi} E_t R_{t+1}^k > 0 \\ \frac{d}{d\xi} E_t R_{t+1}^k & \text{if } \frac{d}{d\xi} E_t R_{t+1}^k < 0 \\ \frac{d}{d\xi} R_{t+1}^k & \text{if } \frac{d}{d\xi} R_{t+1}^k > 0 \end{cases} \] \hspace{1cm} (4.18)

The first term in brackets captures a partial equilibrium effect. When a firm is financially constrained, its internal rate of return is higher than the return it has to pay to debt holders. By borrowing more, it can increase its scale of production and make more profit.
The second term, however, captures a general equilibrium effect, as more borrowing and investment lowers the marginal product of capital. When $\xi$ is small (financial frictions are severe) then the scale effect dominates, while for large $\xi$ the general equilibrium effect dominates.

Most importantly however, financial frictions do not lead to any amplification or propagation of shocks in the rational expectations equilibrium. They have a level effect on output, capital etc., but the dynamics of the model are identical for any value of $\xi$. This can be seen by looking at the variances of log stock price and output growth which do not depend on $\xi$:

$$\text{Var}[\Delta \log V_t] = \sigma^2$$

$$\text{Var}[\Delta \log Y_t] = \left(1 - 2\alpha + 2\alpha^2\right)\sigma^2$$

Intuitively, with financial frictions, a shock to productivity raises asset prices just as much as to allow the firm to instantly adjust the capital stock proportionately.

At the same time, the model cannot replicate many of the stylised facts on stock price data. Up to a first-order approximation, the relative volatility of asset price growth with respect to dividend growth is bounded from below:

$$\frac{\sigma(\Delta \log V_t)}{\sigma(\Delta \log D_t)} < \left(1 - 2\alpha + 2\alpha^2\right)^{1/2} \leq \sqrt{2}$$

This is a direct consequence of the fact that dividends are at least as volatile as output due to leverage. The asset price volatility observed in the data can therefore not be matched. The volatility of the price-dividend ratio can also not be matched: in fact, the forward price dividend ratio is even constant:

$$PD_t = \frac{V_t}{E^P_t D_{t+1}} \equiv \frac{1}{R - G}$$

Furthermore, excess returns are unpredictable: $E_t[(V_{t+1} + D_{t+1})/V_t] - R = 0$, contrary to the data. And finally, by definition of rational expectations, forecast errors are unpredictable, which is also at odds with the data.

### 4.3 Learning equilibrium

I now describe the equilibrium under learning. Qualitatively, this alteration to the expectation formation process can address many of the issues encountered in the last section: it will increase the volatility of stock prices, account for return and forecast error predictability, and most importantly, induce endogenous amplification and propagation on the production side of the model economy.

Expectations are governed by the measure $\mathcal{P}$. I impose the following assumptions: Under $\mathcal{P}$,
1. agents have the exact knowledge about the distribution of the exogenous productivity shock $\varepsilon_t$ and the filtration $\mathcal{F}_t$ induced by it;

2. all optimality conditions and equilibrium conditions are common knowledge, except the market-clearing conditions for stocks (4.11) and final goods (4.12);

3. agents believe that stock prices $V_t$ evolve according to

$$\log V_t - \log V_{t-1} = \mu_t + \eta_t$$

$$\mu_t = \mu_{t-1} + \nu_t$$

where

$$\begin{pmatrix} \eta_t \\ \nu_t \end{pmatrix} \sim \mathcal{N} \left( -\frac{1}{2} \begin{pmatrix} \sigma_\eta^2 \\ \sigma_\nu^2 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\nu^2 \end{pmatrix} \right) iid$$

and this belief is itself common knowledge. The variable $\mu_t$ and the disturbances $\eta_t$ and $\nu_t$ are unobserved and the prior about $\mu_t$ in period 0 is given by

$$\mu_0 | \mathcal{F}_0 \sim \mathcal{N} \left( \hat{\mu}_0, \sigma_\mu^2 \right)$$

where $\sigma_\mu^2 = -\frac{\sigma_\nu^2 + \sqrt{\sigma_\nu^4 + 4\sigma_\eta^2 \sigma_\nu^2}}{2}$

4. when agents make choices in period $t$, their beliefs about $\mu_t$ must be $\mathcal{F}_{t-1}$-measurable.

These assumptions satisfy the notion of internal rationality introduced in Adam and Marcket (2011). In addition, the degree of common knowledge leaves $\mathcal{P}$ very close to rational expectations. For example, I impose that the labour market clearing condition $L_t = 1$ is common knowledge. This effectively requires both households and firms to know that each household inelastically supplies one unit of labour, and thus goes beyond the level of rationality assumed, for example, in Eusepi and Preston (2011). The only deviation from rational expectations lies in the belief system (4.23)-(4.26), which replaces the market clearing condition for stocks. Agents do not know that the aggregate supply of stocks is fixed at unity. By consequence, they must also be unaware of market clearing in at least one other market, since otherwise they could infer stock market clearing from Walras’ law. This is why I also assume that they are unaware of goods-market clearing.

Intuitively, agents can be thought of as believing that there exist some “residual traders” in the stock market, the precise behaviour of which they are unable to discern. Their best guess for the evolution of equilibrium stock prices is the belief system (4.23)-(4.26). They then behave as if they lived in a small open economy in which the two assets traded with the “rest of the world” are consumption goods and stocks with the latter in elastic supply.

---

8Internal rationality essentially requires that each agent know their own optimal choice behaviour in the future and also their belief updating process. This is in contrast to a practice, still common in the literature, to solve the system of model equations with all expectations replaced by some parametrised object. Internal rationality prohibits this practice, effectively imposing a lower bound on the rationality of agents’ choices. As an example, suppose we imposed an arbitrary belief on, say, the expected return on capital $E_t R_{t+1}^f$. This object is the endogenous outcome of the firm’s optimal plan, and thus parameterising this expectation violates internal rationality.
The belief system about $V_t$ posits that asset prices are a random walk which has a small, unobservable, but predictable drift $\mu_t$. The fact that $\mu_t$ is unobserved means that (4.23)-(4.26) is a linear state-space system. Under $P$, agents’ beliefs about $\mu_t$ at time $t$ are normally distributed with stationary variance $\sigma^2_\mu$ and mean $\hat{\mu}_{t-1}$. This belief about the mean of $\mu_t$, which I will simply refer to as “the belief”, evolves according to the updating equation

$$\hat{\mu}_t = \hat{\mu}_{t-1} - \frac{\sigma^2_\mu}{2} + g \left( \log V_t - \log V_{t-1} + \frac{\sigma^2_\nu + \sigma^2_\eta}{2} - \hat{\mu}_{t-1} \right) \quad (4.27)$$

The parameter $g$ is called the “learning gain”. It governs the speed with which agents move their prior in the direction of the last forecast error. When $g$ is high, agents are confident that observed changes in the growth rate of asset prices are due to changes in the trend $\mu_t$ rather than the noise $\eta_t$. The gain is not decreasing in time: Agents believe that the drift in asset prices is itself time-varying, so that even after a long period of time it remains difficult to forecast it.

It is important to keep in mind that the disturbances $\eta_t$ and $\nu_t$ are objects that exist only in the subjective belief system $P$. The actual equilibrium under learning does not contain any shock process other than the productivity shock $\varepsilon_t$. Instead, the equilibrium still contains the actual market clearing conditions (4.11) and (4.12), even if they are unknown to the agents. By Walras’ law, it is sufficient to impose stock market clearing. Under $P$, it (4.11) reads:

$$V_t = \frac{E^P_t V_{t+1} + E^P_t D_{t+1}}{R} \quad (4.28)$$

$$= \frac{V_t \exp \left( \hat{\mu}_{t-1} + \frac{1}{2}\sigma^2_\mu \right) + E^P_t D_{t+1}}{R} \quad (4.29)$$

$$= \frac{E^P_t D_{t+1}}{R - \exp \left( \hat{\mu}_{t-1} + \frac{1}{2}\sigma^2_\mu \right)} \quad (4.30)$$

$$= A_t \frac{d (k_t)}{R - \exp \left( \hat{\mu}_{t-1} + \frac{1}{2}\sigma^2_\mu \right)} \quad (4.31)$$

The second line is obtained by substituting in agents’ beliefs about the evolution of the future stock price. The third line is just algebra. The fourth line makes use of the fact that under $P$, agents know how to forecast future dividends accurately. Since they only depend on future productivity and capital $k_t$, the same $k_t$ leads to the same expectation of time $t + 1$ dividends as under rational expectations.

9The gain is related to the variances of the disturbances by the formula $g = \frac{1}{2} \left( \frac{\sigma^2_\nu}{\sigma^2_\mu} + \sqrt{\frac{\sigma^2_\nu}{\sigma^2_\mu} + 4 \frac{\sigma^2_\eta}{\sigma^2_\mu}} \right)$ and is strictly increasing in the signal-to-noise ratio $\sigma^2_\nu/\sigma^2_\mu$.

10Due to the fourth assumption about $P$, the belief about $V_{t+1}$ can be conditioned about $V_t$, but not about $\hat{\mu}_t$. This “lagged belief updating” rule is essential to obtain the serial correlation in stock returns observed in the data. For a further discussion see Adam et al. (2014).
Figure 4: Stock price dynamics under learning.

In sum, the learning equilibrium is the solution to

\[ v_t = \frac{d(k_t)}{R - \exp(\hat{\mu}_{t-1} + \frac{1}{2}\sigma^2)} \quad (4.32) \]

\[ k_t = \frac{\xi}{1 - \xi} v_t \quad (4.33) \]

\[ \hat{\mu}_t = \hat{\mu}_{t-1} - \frac{\sigma^2}{2} + g \left( \log \frac{v_t}{v_{t-1}} + \log G + \varepsilon_t - \hat{\mu}_{t-1} + \frac{\sigma^2_G + \sigma^2}{2} \right) \quad (4.34) \]

The first two equations are static and the third is dynamic. It also depends on the productivity innovation \( \varepsilon_t \), so that \( v_t \) and \( k_t \) are not constant any more as under rational expectations. The resulting stock price dynamics after a positive innovation at \( t = 1 \) are depicted in Figure 4.\( ^{11} \) The initial shock at \( t = 1 \) raises stock prices proportionally to productivity. In the rational expectations equilibrium, this would be all that happens - the path of stock prices are depicted by the grey line in the figure. Learning investors however, are not sure whether the rise in \( V_1 \) it is indicative of a transitive or permanent increase in the growth rate of stock prices. They therefore revise upwards their beliefs about future asset price growth. In \( t = 2 \) then, their demand for stocks is higher and stock prices need to rise further to clear the market. Beliefs continue to rise as long as observed asset price growth (the dashed black line in Figure 4) is higher than the current belief \( \hat{\mu}_t \) (the solid red line). The differences between observed and expected price growth are the forecast errors, marked by dotted red lines. In the figure, the increase in prices and beliefs ends already in \( t = 3 \), when the forecast error is zero. There is no need for a further belief revision. Now, by equations (4.32)-(4.33), in the absence of subsequent shocks \( V_t \) just comoves with beliefs \( \hat{\mu}_{t-1} \), so when there is no belief revision in \( t = 4 \), realised asset price growth is also zero. Thus, investors observe stalling asset prices at the

\( ^{11} \)The figure depicts the case in which beliefs start at their rational expectations value \( \hat{\mu}_t = \log G \) and subjective uncertainty is vanishing in the sense that \( \sigma^2_G, \sigma^2 \rightarrow 0 \) while \( g \) remains constant.
peak of their optimism. This causes a subsequent downward revision of beliefs $\hat{\mu}_t$. The stock price endogenously reverses, until it returns to its steady-state level in the absence of further shocks.

Figure 4 can be used to illustrate the key properties of beliefs and prices under learning. First, the described dynamics do not depend on time: Beliefs do not converge to the rational expectations equilibrium. Still, the system of $k_t, v_t, \hat{\mu}_t$ is stationary (for $g$ and $\sigma^2$ sufficiently small). Second, asset prices are more volatile than under rational expectations. Third, (excess) returns are predictable even though agents are risk-neutral and the discount factor is constant. To see this, it is again convenient to look at the forward price dividend ratio:

$$PD_t = \frac{1}{R - \exp \left( \hat{\mu}_{t-1} + \frac{1}{2} \sigma^2 \right)}$$

The forward P/D ratio is directly related to the belief $\hat{\mu}_t$. A high P/D ratio is realised at the peak of asset prices in periods 3 and 4 of Figure 4, just after which stock prices start declining. Therefore, the P/D ratio negatively predicts future returns. Furthermore, forecast errors are also predictable: By Equation (4.34), forecast errors are a linear function of $\hat{\mu}_t - \hat{\mu}_{t-1}$. Since a high P/D ratio implies declining beliefs in the future, forecast errors are predictable in the same way as future returns, as in the data.

The aforementioned asset pricing implications are present even when dividends are completely exogenous, as in Adam et al. (2013). But the model considered here also contains a link between asset prices, output and dividends. The effective capital stock $k_t$ is directly related to equity valuations $v_t$ through Equation (4.33). Thus, the fluctuations in the stock market translate into corresponding fluctuations in investment, the capital stock and hence output. Thus, the presence of financial frictions in combination with learning leads to amplification of productivity shock, whereby under rational expectations amplification was zero.

It is also possible that this amplification mechanism is further enhanced by positive feedback from capital to expected dividends. This additional feedback, however, depends on the slope of the function $d(k_t)$. As demonstrated above, expected dividends $d(k_t)$ are increasing in $k_t$ if financial frictions are sufficiently severe. This case is depicted in Panel (a) 5. When the degree of financial frictions is high, the credit constraint line is steep. Assume the initial equilibrium in Period 0 is at $(v_0, \hat{\mu}_0)$. Now consider the effect of a positive productivity shock in Period 1 as before. The immediate effect will be a proportionate rise of stock prices and capital which leaves $v_1$ and $k_1$ unchanged, but raises beliefs from $\hat{\mu}_0$ to $\hat{\mu}_1$. This leads to higher stock prices in $t = 2$ and allows the firm to invest more and increase its expected profits $d(k)$. But this adds further to the rise in stock prices, further relaxing the borrowing constraint and so forth. Stock prices, beliefs, investment and output rise more compared to a situation in which $d$ is constant.\footnote{To my knowledge, this paper is the first to establish a positive feedback from fundamentals to beliefs under learning. Adam et al. (2012) also model economies with endogenous fundamentals. Their learning}
However, this additional amplification channel only works when $\xi$ is sufficiently low. In Panel (b), $\xi$ is large and the firm is operating in the downward-sloping bit of the profit curve $d(k)$. A relaxation of the borrowing constraint due to a rise in $\hat{\mu}$ still allows the firm to invest and produce more, but dividends fall in equilibrium. This is due to the general equilibrium forces mentioned earlier one: The marginal product of capital has to fall in equilibrium, which in practice comes about through an increase in real wages, effectively reducing the firm’s profits. In this situation, the endogenous response of dividends dampens rather than amplifies the dynamics of investment and asset prices.

In sum, the learning equilibrium can qualitatively account for a number of asset pricing facts and predictability of forecast errors. At the same time, the larger endogenous asset price volatility induces corresponding fluctuations in the slackness of the firm’s borrowing constraint. The presence of financial frictions thus magnifies productivity shocks, while there is no amplification under rational expectations. When financial frictions are sufficiently severe, a two-sided positive feedback loop emerges between beliefs, asset prices and firm profits, which further amplifies the dynamics. The presence of positive feedback from assets prices to profits depends on the relative strength of general equilibrium forces, which in the model of this section operate through the real wage.

This very stylised model was already successful in explaining the role of learning for both asset prices and the amplification and propagation of shocks in the presence of financial frictions. I now turn to the development of a richer model that embeds the same mechanism, but can also be taken to the data to study its quantitative implications.
5 Full model for quantitative analysis

This section develops a New-Keynesian business cycle model along the lines of the Bernanke et al. (1999) financial accelerator model, with three key differences. First, instead of a costly state verification problem, I use a limited commitment problem to introduce financial frictions. This means that firms do not face credit spreads but credit rationing. The borrowing limit arises endogenously out of an incomplete contracting problem and incorporates the kind of stock-market dependence that was analysed in the last section. Second, I replace rational expectations about stock prices with learning. Third, I allow for nominal rigidities in both price- and wage-setting as in Erceg et al. (2000). While it is not essential to the amplification mechanism at the centre of the model, allowing for wage rigidities improves its quantitative properties.

5.1 Model setup

The economy is closed and operates in discrete time. There are a number of different agents:

1. *Intermediate goods producers* (or simply “firms”) are at the heart of the model. They combine capital and and differentiated labour to produce a homogeneous intermediate good. They are owned by entrepreneurs but are financially constrained. They borrow funds from households.

2. *Entrepreneurs* only consume differentiated final goods. They trade shares in intermediate goods producers and receive dividend payments.

3. *Households* consume differentiated final goods and supply homogeneous labour to labour agencies. They lend funds to intermediate goods producers.

4. *Labour agencies* transform homogeneous household labour into differentiated labour services, which they sell to intermediate goods producers. They are owned by households.

5. *Final good producers* transform intermediate goods into differentiated final goods. They are owned by households.

6. *Capital goods producers* combine depreciated capital and differentiated final goods to form new capital, subject to an adjustment cost.

7. *The fiscal authority* sets certain tax rates to offset steady-state distortions from monopolistic competition.

Since most elements of the model are standard, I focus on the financially constrained firms, entrepreneurs, households and the central bank. Additional details are provided in Appendix A.

5.1.1 Households

A representative household with time-separable preferences maximises utility as follows:

$$\max_{C_t, L_t, B_{jt}} \mathbb{E}^{P} \sum_{t=0}^{\infty} \beta^t u (C_t, L_t)$$

s.t. $$C_t = \tilde{w}_t L_t + B_{it}^{na} - (1 + i_{t-1}) \frac{P_{t-1}}{P_t} B_{it-1}^{g} + \int_0^1 (B_{jt} - R_{jt-1} B_{jt-1}) dj + \Pi_t$$

The utility function $u$ satisfies standard concavity and Inada conditions and $\beta \in (0, 1)$. Further, $\tilde{w}_t$ is the real wage received by the household and $L_t$ is the amount of labour supplied. $B_{it}^{g}$ are real quantities of nominal one-period government bonds (in zero net supply) that pay a nominal interest rate $i_t$ and $P_t$ is the price level, defined below. Households also lend funds $B_{jt}$ to intermediate goods producers indexed by $j \in [0, 1]$ at the real interest rate $R_{jt}$. These loans are the outcome of a contracting problem described later on. $\Pi_t$ represents lump-sum profits and taxes. Finally, consumption $C_t$ is itself a composite utility flow from of a variety of differentiated goods that takes the familiar CES form:

$$C_t = \max_{C_{it}} \left( \int_0^1 \left( \frac{P_{it}}{C_{it}} \right)^{\frac{\sigma - 1}{\sigma}} dt \right)^{-\frac{1}{\sigma}}$$

s.t. $$P_t C_t = \int_0^1 P_{it} C_{it} di$$

As usual, the price index $P_t$ of composite consumption consistent with utility maximisation and the demand function for good $i$ is given by:

$$P_t = \left( \int_0^1 (P_{it})^{1-\sigma} dt \right)^{\frac{1}{1-\sigma}} ; \quad C_{it} = \left( \frac{P_{it}}{P_t} \right)^{-\sigma} C_t$$

Consequently, the inflation rate is given by $\pi_t = P_t / P_{t-1}$. The first order conditions of the household are also standard and given by:

$$\tilde{w}_t = - \frac{u_{L_t}}{u_{C_t}}$$

$$1 = \mathbb{E}_t^{P} \beta \frac{u_{C_t+1}}{u_{C_t}} \frac{1 + i_t}{\pi_t}$$
5.1.2 Central bank

Like most of the New Keynesian literature, the model is cashless, with the central bank affecting allocations in the presence of nominal rigidities by setting the nominal interest rate. In the baseline version of the model, I assume that the central bank conducts monetary policy through the use of a simple Taylor rule with interest rate smoothing:

\[
\log (1 + i_t) = \rho_m \log (1 + i_{t-1}) + (1 - \rho_m) \left( \log \tilde{R} + \log \pi^* + \phi_\pi (\log \pi_t - \log \pi^*) + \epsilon_{mt} \right)
\]

where \( \tilde{R} \) is the steady-state value of the real interest rate, \( \pi^* \) is the central bank’s inflation target, \( \phi_\pi > 1 \) and \( \epsilon_{mt} \) is a monetary policy shock.

5.1.3 Intermediate good producers (“firms”)

The production of intermediate goods is carried out by a continuum of firms, indexed \( j \in [0, 1] \). Firm \( j \) enters period \( t \) with capital \( K_{jt-1} \) and a stock of debt \( B_{jt-1} \) which needs to be repaid at the gross real interest rate \( R_{jt-1} \). First, capital is combined with a labour index \( L_{jt} \) to produce output

\[
Y_{jt} = (K_{jt-1})^\alpha (A_t L_{jt})^{1-\alpha}
\]

where \( A_t \) is aggregate productivity. The labour index is a CES combination of differentiated labour services parallel to the differentiated final goods bought by the household:

\[
L_{jt} = \max_{L_{jht}} \left( \int_0^1 (L_{jht})^{\frac{\sigma_w}{\sigma_w - 1}} dh \right)^{\frac{\sigma_w}{\sigma_w - 1}}
\]

s.t. \( w_t P_t L_{jt} = \int_0^1 W_{jht} L_{jht} dh \)

The firm’s problem can then be treated as if the labour index was acquired in a competitive market at the real wage index \( w_t \).\(^{13}\) Output is sold competitively to final good producers at the price \( q_t \). During production, the capital stock depreciates at the rate of \( \delta \). This depreciated capital can be traded by the firm at the price \( Q^d_t \).

At this point, the net worth of the firm is the difference between the value of its assets and its outstanding debt:

\[
N_{jt} = q_t Y_{jt} - w_t L_{jt} + Q^d_t (1 - \delta) K_{jt-1} - R_{jt-1} B_{jt-1}
\]

I assume that the firm exits with a probability \( \gamma \). This probability is exogenous, independent across time and across firms. As in Bernanke et al. (1999), exit prevents firms from becoming financially unconstrained. If a firm does not exit, it can decide to pay a dividend

\(^{13}\)This real wage index does not necessarily equal the wage \( \tilde{w}_t \) received by households due to wage dispersion.
$D_{jt} \geq 0$ to its shareholders (defined below).\footnote{The firm is not allowed to pay negative dividends, or to change the amount of shares outstanding. Without this constraint, the firm would be financially unconstrained even if it could not raise any debt at all.} If it exits, it must pay out its entire net worth as dividends. It is subsequently replaced by a new firm which receives the index $j$. I assume that this new firm gets endowed with a fixed number of shares, normalised to one, and is able to raise an initial amount of net worth. This amount equals $\omega N_t$ where $\omega \in (0,1)$ and $N_t = \int_0^1 N_{jt} dj$ is the aggregate net worth. Thus, aggregate equity issuance in the economy will be procyclical, as documented by Covas and den Haan (2011).

The net worth of firm $j$ after equity changes, entry and exit is given by:

$$
\tilde{N}_{jt} = \begin{cases} 
N_{jt} - D_{jt} & \text{for continuing firms} \\
\omega N_t & \text{for new firms} 
\end{cases}
$$

This firm then decides on the new stock of debt $B_{jt}$ and the new capital stock $K_{jt}$. Its balance sheet must satisfy:

$$
Q_t K_{jt} = B_{jt}^t + \tilde{N}_{jt} \tag{5.9}
$$

where the price of capital, $Q_t$, can vary in the presence of adjustment costs.

Firms maximise the present discounted value of their dividend payments using the discount factor of their owners. In doing so, they face financial constraints. Before describing these constraints though, I first turn to the description of the firms’ owners.

### 5.1.4 Entrepreneurs

Entrepreneurs differ from households in their capacity to own intermediate firms. The representative entrepreneur is risk-neutral and discounts future income at the rate $\beta = \beta G - \theta < \beta$ ($G$ is the growth rate of consumption in the non-stochastic steady state). He can buy shares in firms indexed by $j \in [0,1]$. As described above, when a firm exits it pays out its net worth $N_{jt}$ as dividends, and is replaced by a new firm raises a limited amount of equity $\omega N_t$. Let the set of exiting firms in each period $t$ be denoted by $\Gamma_t \subset [0,1]$. Then the entrepreneur’s utility maximisation problem is given by:

$$
\max_{C_t^e, S_t^j} \mathbb{E}^P_0 \sum_{t=0}^{\infty} \beta^t C_t^e \\
\text{s.t. } C_t^e = \int_{j \notin \Gamma_t} S_{jt-1} (V_{jt} + D_{jt}) dj + \int_{j \in \Gamma_t} [S_{jt-1} N_{jt} - \omega N_t + V_{jt}] dj - \int_0^1 S_{jt} V_{jt} dj \tag{5.10} dt \\
S_t^j \in [0, \bar{S}] \tag{5.11}
$$

"
for some $\tilde{S} > 1$. Here, entrepreneurial consumption $C^e_t$ is the same aggregator of differentiated final goods as for households.

The first term in the budget constraint deals with continuing firms and is standard: Each share in firm $j$ pays a dividend $D_{jt}$ and continues to trade, at price $V_{jt}$. The second term deals with firm entry and exit. If the household owns a share in the exiting firm $j$, he receives its entire net worth as dividend payment. The firm is then delisted in the stock market, so $S_{jt-1}V_{jt}$ does not appear. At the same time, a new firm $j$ appears which is able to raise a limited amount of equity $\omega N_t$ from the entrepreneur in exchange for a unit amount of shares that can be traded at price $V_{jt}$. In addition, upper and lower bounds on traded stock holdings are introduced to make entrepreneurs’ demand for stocks finite under arbitrary beliefs, as in the stylised model of the previous section. In equilibrium, they are never binding.

The first order conditions of the entrepreneur are:

$$S_{jt} = \begin{cases} 
0 & \text{if } V_{jt} \preceq \tilde{E}_t^P \left[ \beta E_t^P \left[ \gamma N_{jt+1} + (V_{jt+1} + D_{jt+1})1_{\{j \notin \Gamma_{t+1}\}} \right] \right] \\
\tilde{S} & \text{if } V_{jt} > \tilde{E}_t^P \left[ \beta E_t^P \left[ \gamma N_{jt+1} + (V_{jt+1} + D_{jt+1})1_{\{j \notin \Gamma_{t+1}\}} \right] \right]
\end{cases}$$

(5.12)

5.1.5 Borrowing constraint

In choosing their debt holdings, firms are subject to a borrowing constraint. The constraint is the solution to a particular limited commitment problem described here. Essentially, I set up an incomplete contract problem in which the outside option for the lender in the event of default depends on equity valuations.

Each period, lenders (households) and borrowers (firms) meet to decide on the loaning of funds. Pairings are anonymous, so repeated interactions are ruled out. The incompleteness of contracts imposed is that repayment of loans cannot be made contingent. Only the size $B_{jt}$ and the interest rate $R_{jt}$ of the loan can be contracted in period $t$. Both the lender (a household) and the firm have to agree on a contract $(B_{jt}, R_{jt})$. Moreover, there is limited commitment in the sense that at the end of the period, but before the realisation of next period’s shocks, firm $j$ can always choose to enter a state of default. In this case, the value of the debt repayment must be renegotiated. The outside option of this renegotiation process is the seizure of the firm by the lender.

The lender, a household, does not have the ability to run the firm though. The usual assumption in the literature is that she has to liquidate the firm’s asset in this case. In this model, the lender can always liquidate as well. In this case, all debt and a fraction $1 - \xi$ of the firm’s capital is destroyed. The remaining capital can be sold in the next period, resulting in a total recovery value of $\xi Q_{t+1}K_{jt}$. On top of this, with some probability $x$ (independent across time and firms) the lender gets the opportunity to “restructure” the
firm. Restructuring means that, similar to Chapter 11 bankruptcy proceedings, the firm gets partial debt relief but remains operational. I assume that the lender has to sell the firm to another entrepreneur, retaining a fraction $\xi$ of the initial debt. Denoting the resale price be $\tilde{V}_{jt}$, the recovery value after restructuring is then $\tilde{V}_{jt} + \xi B_{ij}$.

5.1.6 Further model elements and market clearing

Final good producers, indexed by $i \in [0, 1]$, combine the homogeneous intermediate good into a differentiated final good using a one-for-one technology. Their revenue is subsidised by the government at the rate $\tau$. Per-period profits of producer $i$ are $\Pi^Y_{it} = (1 + \tau) \left( \frac{P_{it}}{P_t} \right) Y_{it} - q_t Y_{it}$. They are subject to a Calvo price setting friction: Every period, each final-good producer can change their price only with a probability $1 - \kappa$, independent across time and producers. Similarly, labour agencies (indexed $h \in [0, 1]$) combine the homogeneous labour provided by households into differentiated labour goods which they sell on to intermediate good producers. Their revenue is subsidised at the rate $\tau_w$, per-period profits of agency $h$ are $\Pi^L_{ht} = (1 + \tau) \left( \frac{W_{ht}}{P_t} \right) L_{ht} - \tilde{w}_t L_{ht}$ and each agency can change its nominal wage $W_{ht}$ only with a probability $1 - \kappa_w$. The government collects subsidies as lump sum taxes from households and runs a balanced budget each period. Thus, the total amount of lump-sum payments $\Pi_t$ received by the household is the sum of the profits of all final good producers and labour agencies minus the sum of all subsidies. The government sets the subsidy rates such that, under flexible prices, the markup over marginal cost is zero in both the labour and output market.

Capital goods producers combine depreciated capital with differentiated final goods into new capital goods subject to standard capital adjustment costs.

Finally, the exogenous stochastic processes are productivity and the monetary shock:

$$\log A_t = t (1 - \rho) \log G + \rho \left( \log A_{t-1} + \log G \right) + \log \varepsilon_{At} \quad (5.13)$$

$$\varepsilon_{At} \sim \mathcal{N} \left( -\frac{\sigma_A^2}{2}, \sigma_A^2 \right) \quad (5.14)$$

$$\varepsilon_{mt} \sim \mathcal{N} \left( -\frac{\sigma_m^2}{2}, \sigma_m^2 \right) \quad (5.15)$$

Market clearing needs to take into account distortions from price and wage dispersion. All market clearing conditions are listed in Appendix A.

5.2 Rational expectations equilibrium

I first describe the equilibrium under rational expectations. Equilibrium is a set of stochastic processes for prices and allocations, as well as a set of strategies in the limited commitment game, and an expectation measure $\mathcal{P}$ such that, for all states and time
periods: markets clear; allocations solve the optimisation programs of all agents given prices and expectations \( \mathcal{P} \); the strategies in the limited commitment game are a subgame-perfect Nash equilibrium for all lender-borrower pairs; and the measure \( \mathcal{P} \) coincides with the actual probability measure induced by the equilibrium.

Under appropriate parameter restrictions, there exists a rational expectations equilibrium characterised by the following properties (proofs and characterisation of the restrictions are relegated to Appendix B):

1. All firms choose the same capital-labour ratio \( K_{jt}/L_{jt} \). This allows to define an aggregate production function and an internal rate of return on capital:
   \[
   Y_t = \alpha K_{t-1}^\alpha \left(A_t L_t \right)^{1-\alpha} \tag{5.16}
   \]
   \[
   R^k_t = q t \alpha \frac{Y_t}{K_t} + Q^d_t (1 - \delta) K_{t-1} \tag{5.17}
   \]

2. The expected return on capital is higher than the internal return on debt: \( E_t R^k_{t+1} > R_{jt} \). As a consequence, firms always reinvest all their net worth and do not pay any dividends until they exit: \( D_{jt} = 0 \).

3. At any time \( t \), the stock market valuation \( V_{jt} \) of a firm \( j \) is proportional to its net worth after entry and exit \( \tilde{N}_{jt} \). This permits to write an aggregate stock market index as
   \[
   V_t = \int_0^1 V_{jt} = \tilde{\beta} E_t \left[ \gamma N_{t+1} + \frac{1 - \gamma}{1 - \gamma + \omega} V_{t+1} \right] \tag{5.18}
   \]
   The “correction” term in the continuation value \( V_{t+1} \) can be understood as follows: The stock market index \( V_t \) is the value of all currently existing firms, not including firms that are not yet born. In \( t + 1 \), a fraction \( \gamma \) of firms exit and pay a dividend. The remaining firms are left with net worth \((1 - \gamma) N_{t+1}\), but a mass \( \gamma \) of new firms also enters, each endowed with initial net worth \( \omega \gamma N_{t+1} \).

4. Borrowers never default on the equilibrium path and borrow at the risk-free rate
   \[
   R_{jt} = R_t = (E_t \beta u_{Ct+1}/u_{Ct})^{-1} \tag{5.19}
   \]
   The lender only accepts debt payments up to a certain limit \( \bar{B}_{jt} \). The firm always exhausts this limit: \( B_{jt} = \bar{B}_{jt} \), which is proportional to the firm’s net worth \( \tilde{N}_{jt} \). If the firm defaulted and the lender seized the firm, she would always prefer restructuring to liquidation. Intuitively, this is because capital is more valuable inside the firm than outside of it. Because acquiring capital is difficult due to financial frictions, entrepreneurs will always willing to pay the lender a higher price for a restructured, operational firm than for its capital stock alone.

5. As a consequence of the previous properties of the equilibrium, all firms can be aggregated. Aggregate debt, capital and net worth are sufficient to describe the
intermediate goods sector and evolve as

\[ N_t = R_t^k K_{t-1} - R_{t-1} B_{t-1} \]  \hspace{1cm} (5.20)
\[ Q_t K_t = (1 - \gamma + \gamma \omega) N_t + B_t \]  \hspace{1cm} (5.21)
\[ B_t = x E_t \beta \frac{u_{Ct+1}}{u_{Ct}} Q_{t+1} \xi K_t + (1 - x) \xi (V_t + B_t) \]  \hspace{1cm} (5.22)

I solve for this rational expectations equilibrium up to a second-order approximation around the non-stochastic steady state.

### 5.3 Learning equilibrium

I introduce learning about stock market valuations as in the simple model of Section 4. For the belief system \( P \) under learning, I first retain the rational expectations belief that agents still understand that the stock price of an individual firm is proportional to firm size, and by consequence relates to the aggregate stock market value:

\[ V_{jt} = \frac{N_{jt}}{N_t} V_t \]  \hspace{1cm} (5.23)

But while investors are able to price individual stocks, they are uncertain about the evolution of aggregate stock market outcomes. Just as in the simple model of the previous section, I posit that agents hold the following subjective beliefs about the aggregate stock market under \( P \):

\[ \log V_t = \log V_{t-1} + \mu_t + \eta_t \]  \hspace{1cm} (5.24)
\[ \mu_t = \rho \mu_{t-1} + (1 - \rho) \log G + \nu_t \]  \hspace{1cm} (5.25)
\[ \begin{pmatrix} \varepsilon_t \\ \nu_t \end{pmatrix} \sim \text{i.i.d.} N \left( \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_\theta \\ \sigma^2_\nu \\ \sigma^2_\nu \end{pmatrix} \right) \]  \hspace{1cm} (5.26)

where \( \eta_t, \nu_t \) and \( \mu_t \) are unobserved. I also impose the other assumptions of the last section, namely that all agents know all shocks and their distribution, and that they know all model equations, optimality and market clearing conditions except those in the final goods market (A.10) and in the stock market (A.15). Thus, agents’ behaviour satisfies the internal rationality principle.

The model under learning must be solved in a two-stage procedure. The first stage is to solve the economy under \( P \), i.e. replacing stock and final goods market clearing conditions with the Kalman filtering problem corresponding to the state-space model above, and adding the prediction error of the filter as an additional exogenous shock. In the second stage, the realisations of the prediction error need to be backed out by imposing market
clearing. The method for solving the model up to a second-order approximation around the non-stochastic steady-state is described in Appendix (C).

5.4 Frictionless benchmark

Because my aim is to gauge the importance of financial frictions for business cycle analysis, I need a frictionless benchmark to which the model can be compared to. The benchmark model is identical to the rational expectations model above, except that intermediate firms are now owned by households, and face no frictions in accessing external finance. Details can be found in Appendix A.

5.5 Calibration

I calibrate the model at quarterly frequency to match certain moments in postwar US data.

The capital share in production is set to $\alpha = .33$, implying a labour share in output of two thirds. The depreciation rate $\delta = .025$ corresponds to 10% annual depreciation. The non-stochastic trend productivity growth rate $G$ is set to its post-war average of 1.85% annually. The persistence of the temporary component of productivity is set to .95. The variance of the TFP shock is set such that the variance of HP-detrended output matches exactly the one in the data (.0135) when the TFP shock is the only one affecting the economy.

The discount factor of the household is set such that the steady-state interest rate matches the average interest rate on Treasury bills of 2% annually, implying a rather high discount factor ($\beta = .9996$). The entrepreneurial discount factor is set such that, in the non-stochastic steady state, bonds and stocks have the same return ($\tilde{\beta} = .995$). The parameter $\phi$ is the inverse Frisch elasticity of labour supply. There is no real agreement of what this elasticity should be. Peterman (2012) reports that microeconomic estimates range from 0 to .5, while macroeconomic models typically calibrate to values between 2 and 4. I set $\phi = .5$, corresponding to a Frisch elasticity of 2, at the lower end of what is used in the macro literature.

Next, the parameters describing the financial constraints are $\xi$, the tightness of the borrowing constraint; $x$, the probability of restructuring after default; $\omega$, the equity received by new firms relative to average equity; and the rate of firm exit and entry. Reliable statistics on bankruptcy outcomes are hard to come by. About 30% of all corporate US bankruptcies in the last ten years were filed under Chapter 11 as opposed to Chapter 7; of these, the DOJ has estimated the rate of successful restructuring at 25%.\footnote{Data on bankruptcies by chapter are available at http://www.uscourts.gov/Statistics/BankruptcyStatistics.aspx; data on Chapter 11 outcomes are available at http://www.justice.gov/ust/eo/public_affairs/index.htm.} This leads to
a value of $x = .075$. The remaining three parameters are chosen to jointly match the US average investment share in output of 20%, debt-to-equity ratio of 1:1 (as recorded in the Flow of Funds), and quarterly price-dividend ratio of 139 (as in the S&P500). Intuitively, a large value of $\xi$ relaxes the borrowing constraint, leading to high leverage. A large value of $\omega$ means that new firms bring a lot of fresh equity, increasing aggregate net worth and capital. Since returns on capital are diminishing, this raises the investment share in output. And a higher rate of exit $\gamma$ makes firms shorter-lived, reducing the firm’s value relative to its current dividend payments. So a higher value of $\gamma$ induces a lower P/D ratio. The parameter values thus chosen are $\gamma = 0.053$, $\xi = 0.40$ and $\omega = 0.12$.

Turning to the nominal rigidity side of the model, I set the degree of price and wage stickiness to $\kappa = \kappa_w = 0.75$. This corresponds to an average time between price and wage adjustments of 3 quarters, a commonly used value. The elasticity of substitution between varieties of the final consumption good is set to $\sigma = \sigma_w = 10$. The strength of monetary policy reaction to inflation is set to $\phi_\pi = 1.5$, while the degree of nominal rate smoothing is set to $\rho_m = 0.5$. The inflation target is set to $\pi^* = 0$. Finally, the parameter for capital adjustment costs is set to $\psi = 1$. There is no authoritative guidance on what this parameter should be. I choose it to strike a balance between a realistic volatility of investment (for which $\psi$ needs to be high) and a realistic volatility of asset prices (for which the value needs to be low).

The only additional parameter relevant for the learning equilibrium is the learning gain $g$. In principle, it could be calibrated to match data on expectations about stock price growth. Since the aim of the paper is to assess the impact of endogenous asset price volatility on credit frictions, I choose to calibrate the learning gain such that the volatility of the stock

<table>
<thead>
<tr>
<th>Param.</th>
<th>Value</th>
<th>Target</th>
<th>Param.</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>.33</td>
<td>capital share</td>
<td>$\sigma$</td>
<td>9</td>
<td>markup 33%</td>
</tr>
<tr>
<td>$\delta$</td>
<td>.025</td>
<td>annual depreciation 10%</td>
<td>$\kappa$</td>
<td>0.75</td>
<td>avg price duration 3q.</td>
</tr>
<tr>
<td>$G$</td>
<td>1.0045</td>
<td>annual trend growth 1.58%</td>
<td>$\sigma_w$</td>
<td>9</td>
<td>markup 33%</td>
</tr>
<tr>
<td>$\beta$</td>
<td>.9995</td>
<td>annual risk-free real rate 2%</td>
<td>$\kappa_w$</td>
<td>0.75</td>
<td>avg wage duration 3q.</td>
</tr>
<tr>
<td>$\tilde{\beta}$</td>
<td>.995</td>
<td>equality of discount factor</td>
<td>$\phi_\pi$</td>
<td>1.5</td>
<td>Taylor (1993)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>.5</td>
<td>Frisch elasticity = 2</td>
<td>$\rho_m$</td>
<td>.5</td>
<td>??</td>
</tr>
<tr>
<td>$x$</td>
<td>.075</td>
<td>rate of successful restructuring after default</td>
<td>$\psi$</td>
<td>1</td>
<td>??</td>
</tr>
<tr>
<td>$\xi$</td>
<td>.40</td>
<td>$B/K = 50%$</td>
<td>$\rho_a$</td>
<td>0.95</td>
<td>Kydland &amp; Prescott (1982)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>.12</td>
<td>$I/K = 20%$</td>
<td>$\sigma_a$</td>
<td>0.0047</td>
<td>$\sigma(Y)$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-0.053</td>
<td>$P/D = 139$</td>
<td>$g$</td>
<td>.0055</td>
<td>$\sigma(P)/\sigma(D) = 2.59$</td>
</tr>
</tbody>
</table>
market matches that of the data. More precisely, I match the volatility of stock market prices relative to the volatility of dividends to be 2.59, which is the corresponding post-war value of the S&P1500.

Table 3 summarises the calibration.

6 Results

6.1 Dynamic response to shocks

The model under learning delivers a large amount of endogenous amplification of shocks due to the behaviour of asset prices. A look at impulse response functions makes this clear. In Figure 6, I plot the impulse responses to a standard, persistent technology shock. Green dashed lines represent the responses in the rational expectations equilibrium, while red solid lines depict the responses under learning. The frictionless benchmark economy is represented by black lines. One can see quite clearly from the figure that the learning solution amplifies the productivity shock substantially more than in the frictionless benchmark. Output $Y_t$, consumption $C_t$, investment $I_t$ and employment $L_t$ all rise substantially more in response to the shock. The path of consumption is hump-shaped although there is no habit formation in the model. By contrast, under rational expectations the responses of output, employment and investment are below their counterpart in the
frictionless comparison, except for the first quarter. Thus, adding financial frictions with rational expectations does not lead to substantial amplification of productivity shocks.

This difference is due to the different response of the aggregate stock market value $V_t$, despite the fact that the borrowing constraint only depends in a very limited way on stock prices - the probability of being able to sell a defaulting firm in the market was calibrated to only 7.5%. The learning mechanism leads to volatility in asset prices that is an order of magnitude larger than under rational expectations, so that even for a small degree of dependence on stock prices, the amplification of the shock is substantial. Also, net worth $N_t$ and therefore dividends rise substantially more under learning as well. This indicates that the behaviour of dividends further amplifies asset price learning.

The feedback loop between beliefs in financial markets and fundamentals in the economy is summarised in a diagram in Figure 7. Consider any positive shock to net worth $N_t$. This increases the internal resources of the firm available for investment, so $K_t$ rises. This can now raise or lower net worth in the next period $N_{t+1}$, depending on whether the general equilibrium effects of rising wages and interest rates outweigh the increased profits from reduced financial frictions or not. I depict the case of rising net worth as is the case in the calibrated model. As dividends are proportional to net worth, firm value rises, credit constraints relax and firms can borrow more. By consequence, investment rises further. This forward-looking channel is the standard financial accelerator effect. It is also present under rational expectations.

Learning introduces a second amplification channel, stemming from changes in subjective beliefs. Agents are unsure to what extent the increase in $V_t$ is temporary (is due to $\eta_t$) or signals an increase in the long-run growth of the market (is due to $\nu_t$). They update their belief $\hat{\mu}_{t+1}$ upwards in the next period. In order for the equity market to clear, the price $V_{t+1}$ rises in next period as well. Hence investors’ expectations $\hat{\mu}_{t+2}$ rise even further etc. Gradually, expectations and prices rise. Each period in which the stock market valuation increases, firms experience easier borrowing, can invest more and become more profitable, endogenously paying out higher dividends. This reinforces the upswing in the stock market, and it is in this sense that beliefs and fundamentals form a positive feedback
Figure 8: Impulse responses to a monetary shock.

loop. As in the simple model of Section 4, beliefs overshoot, and mean reversion means that the “boom” is eventually reversed. This then leads to a drop in asset prices and also in economic aggregates, as the impulse responses of investment in Figure 6 undershoots its steady-state level.

The feedback loop is also present after a shock to monetary policy. Figure 8 plots the response to a temporary reduction in the nominal interest rate (shock to $\varepsilon_{mt}$). Again, all macroeconomic aggregates rise substantially more under learning than under both rational expectations and the frictionless benchmark. Consumption is hump-shaped. Net worth and hence dividends also rise more under learning than under rational expectations.

As the interest rate falls, the inflation rate $\pi_t$ (labelled “pi” in the figure) sharply increases. But it then falls quickly, and is below its steady state-level at the peak of the stock price boom. This pattern is in line with the observation that stock price booms are often periods of low, not high inflation (Christiano et al., 2010). In the model, high asset prices relax credit constraints and expand productive capacity, which acts like a disinflationary positive supply shock.

6.2 Business cycle and asset price statistics

To get a better understanding of the quantitative properties of the model, I review key business cycle and asset price statistics. The results are summarised in Table 4. The first row of the table compares the standard deviation of detrended output in the data
Table 4: Comparing moments in the data and across model specifications.

<table>
<thead>
<tr>
<th>Moment</th>
<th>Data</th>
<th>learning</th>
<th>RE</th>
<th>frictionless</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma (Y_t)$</td>
<td>1.44%</td>
<td>1.44%</td>
<td>.92%</td>
<td>1.07%</td>
</tr>
<tr>
<td>$\sigma (C_t) / \sigma (Y_t)$</td>
<td>0.60</td>
<td>.34</td>
<td>.51</td>
<td>.28</td>
</tr>
<tr>
<td>$\sigma (I_t) / \sigma (Y_t)$</td>
<td>2.90</td>
<td>3.78</td>
<td>3.41</td>
<td>2.60</td>
</tr>
<tr>
<td>$\sigma (L_t) / \sigma (Y_t)$</td>
<td>1.13</td>
<td>1.03</td>
<td>.89</td>
<td>.90</td>
</tr>
<tr>
<td>$\sigma (\pi_t) / \sigma (Y_t)$</td>
<td>0.19</td>
<td>.015</td>
<td>.032</td>
<td>0.013</td>
</tr>
<tr>
<td>$\sigma (i_t) / \sigma (Y_t)$</td>
<td>.20</td>
<td>.016</td>
<td>.037</td>
<td>.016</td>
</tr>
<tr>
<td>$\rho (\log \frac{V_t}{D_t}, \text{hp} (\log Y_t))$</td>
<td>0.094</td>
<td>0.070</td>
<td>-0.655</td>
<td>n/a</td>
</tr>
<tr>
<td>$\sigma (\text{hp} (\log V_t))/\sigma (\text{hp} (\log D_t))$</td>
<td>2.63</td>
<td>2.63</td>
<td>0.53</td>
<td>n/a</td>
</tr>
<tr>
<td>$\sigma (\Delta D_t)$</td>
<td>1.75%</td>
<td>.21%</td>
<td>.12%</td>
<td>n/a</td>
</tr>
<tr>
<td>$\sigma (\log \frac{V_t}{D_t})/E (\log \frac{V_t}{D_t})$</td>
<td>.114</td>
<td>0.032</td>
<td>0.015</td>
<td>n/a</td>
</tr>
<tr>
<td>$\sigma (\log R_t)/E (\log R_t)$</td>
<td>6.03</td>
<td>1.05</td>
<td>0.19</td>
<td>n/a</td>
</tr>
<tr>
<td>$\rho (V_t - D_t, R_{1y,t+1y}')$</td>
<td>-.20</td>
<td>-0.29</td>
<td>0</td>
<td>n/a</td>
</tr>
</tbody>
</table>

Quarterly US data 1948Q2-2012Q4. All figures of the top panel refer to moments of HP-filtered time series, while in the bottom panel filtering is indicated explicitly where applied.

and under the three model specifications: without financial frictions, with frictions and rational expectations, and under learning. By construction of the calibration, the volatility under learning matches that of the data. For the same volatility of shocks, the standard deviation is reduced by 63% to .92% under rational expectations. What is more, the volatility of output under rational expectations is actually lower than in the frictionless benchmark. By this measure, the financial accelerator is actually a “decelerator” (with TFP shocks) under rational expectations and low asset price volatility, but accounts for a significant amount of endogenous business cycle volatility with high asset price volatility and learning.

The rest of the top panel of the table examines the relative volatilities of macroeconomic time series (all standard deviations refer to HP-detrended simulated time series). The relative volatility of investment and consumption is roughly in line with the data. Investment is in fact more volatile than in the data while consumption is not volatile enough. In this respect, the rational expectations version performs slightly better. Employment is more volatile than under rational expectations as well, getting it closer to the corresponding moment in the data. What the model does not capture is the relative volatility of inflation and the nominal interest rate. This is an obvious shortcoming of the calibration.

Introducing learning considerably improves the asset price properties of the model. This is documented in the bottom panel of Table 4. The price-dividend ratio is almost uncor-
related with output, as in the data. This is in stark contrast to the rational expectations solutions. There, dividends typically fluctuate more than prices, leading to a countercyclical P/D ratio. Indeed, under rational expectations, the ratio of the standard deviation of stock prices and dividends is less than one, while under learning, it is possible to exactly match this ratio by choice of the learning gain $g$. The large Sharpe ratio of stock returns and the P/D-ratio observed in the data is not attained by any of the specifications. This is mainly due to the fact that dividend volatility itself is an order of magnitude lower than in the data. Still, the learning equilibrium comes much closer to the volatility in the data than the rational expectations counterpart. What is remarkable is the degree of return predictability under learning, which is very close to the data despite not being targeted by the calibration.

### 6.3 Relation to survey evidence on expectations

Agents in the learning equilibrium make systematic, predictable forecast errors. This applies not only to forecasts of stock prices, but also for almost all other model variables even though agents only have incomplete knowledge about the law of motion for stock prices, This is because a systematic mistake in predicting stock prices will translate into systemic mistakes in predicting borrowing constraints, and hence investment, output, inflation, interest rates and so forth. This creates a wealth of testable implications. Patterns of forecast error predictability from surveys about, say, investment, can be compared with the corresponding patterns in the model. Because the model does not have any degrees of freedom for specifying beliefs other than those about the stock market, this is a potentially tough test for the model.

For the patterns described in Section 3, the model passes most of the tests surprisingly well. In fact, the statistics for real variables are often better aligned with the survey data
than those for the stock market. Column (1) of Table 5 looks at the first autocorrelation of forecast errors on stock returns as well as output, investment, consumption and unemployment (taken to be one minus employment in the model). For stock returns, the autocorrelation is very high, whereas in the data it is essentially zero. This is a consequence of the still relatively low realised return volatility in the model, so that the predictable component of forecast errors is too large. However, the serial correlation is of the same order of magnitude as in the data for all the macroeconomic aggregates.

The model does very well in terms of the belief statistics devised by Coibion and Gorodnichenko (2010). As documented in Column (2), forecast errors on macro aggregates are predictable by the direction of the forecast revision to a similar extent as in the data, reflecting slow adjustment of beliefs. The model also predicts a negative correlation of return forecast errors with the revision, defined here as $\hat{E}_t R_{t,t+4} - \hat{E}_{t-1} R_{t,t+4}$. Unfortunately, the CFO survey does ask for quarterly estimates as the SPF does, so the corresponding statistic in the data is not available. It would be interesting to see whether a negative correlation would obtain.

Column (3) reports predictability based on the P/D ratio. The P/D ratio predicts returns as well as their forecast errors both in the model and in the data, and also forecast errors on economic activity: When the stock market is up, people are on average too optimistic about future returns and the economic outlook in general. Column (4) repeats the exercise for the growth rate of the P/D ratio. This measure positively predicts forecast errors: When the stock market is rising, people are on average not optimistic enough about returns and the economic outlook. This holds in the SPF data as well, but not for return expectations in the CFO survey. Again, because there is much more volatility in the growth rate of the P/D ratio in the data in the model, which is why it is a counterfactually good predictor in the model.

### 7 Implications for monetary policy

Shocks are amplified under learning through their impact on asset prices. Financial markets are not efficient and their fluctuations translate into generally inefficient endogenous business cycle fluctuation. A natural question is therefore to ask is whether monetary policy should react to asset price growth for stabilisation purposes.

This question has been discussed for a long time by policymakers, but generally answered in the negative by monetary economists. The most famous study in this respect is probably Bernanke and Gertler (2001). They examine the question in the standard Bernanke et al. (1999) framework and conclude that an asset price term in the Taylor rule gives only a modest improvement in terms of output gap and inflation volatility at best. Faia and Monacelli (2007), employing a more rigorous welfare criterion in a similar model, find that reacting to asset prices only improves welfare for low coefficients on inflation in the Taylor
Table 6: Optimal policy rules.

<table>
<thead>
<tr>
<th></th>
<th>policy parameters</th>
<th>Y (s.d.)</th>
<th>(\pi) (s.d.)</th>
<th>welfare loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>baseline</td>
<td>learning</td>
<td>(\phi_\pi = 1.5, \phi_{\Delta Y} = 0, \phi_{\Delta V} = 0)</td>
<td>4.12%</td>
<td>0.55%</td>
</tr>
<tr>
<td></td>
<td>RE</td>
<td>(\phi_\pi = 1.5, \phi_{\Delta Y} = 0, \phi_{\Delta V} = 0)</td>
<td>2.55%</td>
<td>0.18%</td>
</tr>
<tr>
<td>optimal</td>
<td>learning</td>
<td>(\phi_\pi = 3.00, \phi_{\Delta Y} = 0.24, \phi_{\Delta V} = 2.01)</td>
<td>2.51%</td>
<td>0.12%</td>
</tr>
<tr>
<td></td>
<td>optimal</td>
<td>(\phi_\pi = 2.80, \phi_{\Delta Y} = 0.44, \phi_{\Delta V} = 0)</td>
<td>2.30%</td>
<td>0.06%</td>
</tr>
</tbody>
</table>

Standard deviations of output and inflation (unfiltered) and welfare loss under learning and RE; baseline calibration vs. optimal rule.

rule, and that the optimal reaction to high asset prices is to lower interest rates, not raise them.

In the present model, letting the nominal interest rate respond systematically to price growth in the stock market turns out to be highly beneficial in terms of reducing output and inflation volatility. Consider extending the interest rate rule (5.4) as follows:

\[
i_t = \frac{1}{\beta} + \pi^* + \phi_\pi (\pi_t - \pi^*) + \phi_{\Delta Y} (\log Y_t - \log GY_{t-1}) + \phi_{\Delta V} (\log V_t - \log GV_{t-1}) + v_{mt}
\]  

In addition to raising interest rates when inflation is above its target level, the central bank will raise interest rates by \(\phi_{\Delta Y}\) percentage points when real GDP growth is above long-run TFP growth; and by \(\phi_{\Delta V}\) percentage points when stock market growth is above long-run TFP growth.\(^{16}\) I explicitly do not include the levels of output or asset prices or output gap measures. From a theoretical perspective, this would imply that the central bank has more knowledge than the private sector under learning, since the long-run level of asset prices and output is believed to be essentially a random walk. From a practical perspective, imposing a level target for asset prices is an even more audacious measure than a target for price growth as well.

Table 6 summarises the key findings of this section. The top two rows show the volatility of output and inflation as well as the welfare loss of business cycle fluctuations in the baseline calibration of the model under learning and rational expectations. Both output and inflation are more volatile under learning. The welfare loss, measured in the welfare-equivalent percentage reduction in steady-state consumption, is over four times higher under learning. This suggests a considerable role for policy.

I compute the welfare-maximising monetary policy within the set of rules given by (7.1), subject to non-negativity of all coefficients. The results are in the bottom two rows of Table 6 and are striking. Under learning, and only then, the optimal policy reacts to rising asset prices by strongly raising interest rates (\(\phi_{\Delta V} > 0\)). Intuitively, the amplification of business cycles under learning works to a great extent through asset price growth and expectations thereof. By systematically reacting to asset price growth, the central bank is

\(^{16}\)This rule does not imply that the policy rate should adjusted to daily changes in asset prices, but only to changes at the quarterly frequency.
Standard deviations of output and inflation and welfare loss under learning for Taylor rules including reaction terms to output growth and stock price growth. Theoretical moments of non-filtered variables in logarithms. The sensitivity of interest rates to inflation is kept constant at $\phi_{\pi} = 3.00$. The black dashed line traces the level of the respective standard deviation under strict inflation targeting. The black dot represents the welfare-maximising rule.

effectively mitigating this amplification channel. This highlights how sensitive normative policy implications are to the way asset prices are modelled. Earlier studies such as Faia and Monacelli (2007) typically find no or little benefit for a reaction to asset prices in the Taylor rule even under financial frictions. When financial market expectations are rational, this conclusion also holds in my model ($\phi_{\Delta V} = 0$ at optimum).

Figure (9) can provide additional insight into the effects of varying the coefficients in the interest rate rule. It shows the metrics from Table 6 for different values of $\phi_{\Delta Y}$ and $\phi_{\Delta V}$, holding the inflation coefficient at its optimum level under learning ($\phi_{\pi} = 3$). From the leftmost panel, one can see that both a reaction to output growth and asset price growth reduces output volatility. But the middle panel reveals that inflation volatility, which is an important determinant of welfare with Calvo price rigidity, can be reduced more effectively by targeting asset price growth. This means that welfare is maximised at a point where $\phi_{\Delta V} > 0$, the black dot in the rightmost panel of the figure. This optimal policy achieves a higher level of welfare than even strict inflation targeting, which is indicated by the black dashed line.

8 Conclusion

This paper analysed the effects of learning in the stock market for a business cycle model with financial frictions. When firms borrow against the market value of their assets, learning in the stock market generates realistic stock market dynamics which in turn lead to procyclical variations in firms’ access to credit. As a consequence, the amplification and propagation of typical business cycle shocks such as productivity and monetary shocks is greatly increased. In addition, a two-sided feedback loop emerges between beliefs, stock
prices and firm profits that amplifies the asset price dynamics.

I have embedded the mechanism in a dynamic stochastic general equilibrium model with nominal rigidities. Unlike most of the literature on adaptive learning, I retain high degree of rationality and internal consistency of beliefs. In the baseline calibration of the model, introducing learning considerably improves the model’s asset price properties, notably volatility and return predictability, while still matching standard business cycle statistics. At the same time, it leads to a large amount of propagation and amplification of both supply and demand shocks, endogenising up to one third of the volatility of the business cycle.

A natural criticism of a theory based on beliefs other than rational expectations is that it has many degrees of freedom to adjust for almost any fact of choice. But the present model only introduces a single degree of freedom, the learning gain $g$, which is calibrated to the relative volatility of stock prices over dividends. Moreover, the predictable biases in forecasts made by agents in the model correspond surprisingly well to the patterns found in actual data from surveys.

The model could also be used to study normative implications of learning. In particular, I revisited the question of whether monetary policy should react systematically to asset prices. I found that a strong reaction to stock price growth is desirable from a welfare perspective when investors in financial markets are learning. By contrast, with rational expectations such a reaction is not desirable, in line with most previous findings of the literature.

An asset pricing theory based on learning has important normative implications. The model gives rise to a natural notion of asset price misalignments which are responsive to policy actions. This suggests that there is scope for policy intervention in order to mitigate the waves of optimism and pessimism which partially drive business cycle fluctuations. I compute the optimal monetary policy within a certain class of rules that do not assume a superior informational advantage of the central bank over financial markets. I find that the optimal interest rate rule reacts strongly to the growth rate of stock prices under learning. By contrast, when financial markets are fully rational, the desirability of such a reaction disappears.

Several extensions seem worthwhile pursuing. The present structure of beliefs successfully captures key properties of aggregate stock market data, but it might not be the only one. Exploring different belief specifications will likely lead to additional insights about the interactions between beliefs and the real economy. Second, the analysis would be considerably enhanced by a costly state verification framework which allows to study credit spreads. Third, on the asset pricing side, a shortcoming of the model is that dividend volatility is low, which limits the amount of return volatility the learning mechanism can generate to a value that is still much lower than that found in the data. Finding a better way of modelling dividends will likely improve upon this.
The analysis in this paper also suggests two broader avenues for future research. First, it will be interesting to apply the methods developed here to models with a more explicit treatment of the financial sector. This will hopefully lead to further insights for financial economic stability of the absence of rational expectations. Second, the analysis can be easily transferred to the many other applications of credit frictions with a collateralised asset, whether it be housing, land or other financial assets, and whether the constrained agents is a firm, a household or a financial institution.
References


A Further details on the full model

Retailers

Retailers transform a homogeneous intermediate good into differentiated final consumption goods using a one-for-one technology. The intermediate good trades in a competitive market at the real price $q_t$ (expressed in units of the composite final good). Each retailer enjoys market power in her output market though, and sets a nominal price $p_i^t$ for her production. A standard price adjustment friction à la Calvo means that a retailer cannot adjust her price with probability $\alpha_p$, which is independent across retailers and across time. Hence, the retailer solves the following optimisation:

$$\begin{align*}
\max_{P_i^t} \sum_{s=0}^{\infty} \alpha_p^s Q_{t,t+s} ((1 + \tau) P_{it} - q_{t+s} P_{t+s}) Y_{it+s} \\
\text{s.t. } Y_{it} = \left( \frac{P_{it}}{P_t} \right)^{-\sigma} \tilde{Y}_t
\end{align*}$$

where $Q_{t,t+s}$ is the nominal discount factor of households between time $t$ and $t + s$ and $\tilde{Y}_t$ is aggregate demand for the composite final good. Since all retailers that can re-optimise at $t$ are identical, they all choose the same price $P_{it} = P_i^*$. Since I want to evaluate welfare in the model, I cannot log-linearise the first-order conditions of this problem. Their derivation is nevertheless standard (for example Maussner, 2010) and I only report the final equations here:

$$\begin{align*}
\frac{P_i^*}{P_t} &= \frac{1}{1 + \tau} \frac{\sigma}{\sigma - 1} \frac{\Gamma_{1t}}{\Gamma_{2t}} \\
\Gamma_{1t} &= q_t + \alpha E_t Q_{t,t+1} \tilde{Y}_{t+1} \pi_{t+1}^{\sigma} \\
\Gamma_{2t} &= 1 + \alpha E_t Q_{t,t+1} \tilde{Y}_{t+1} \pi_{t+1}^{\sigma-1}
\end{align*}$$

I assume that the government sets subsidies such that $\tau = 1/(\sigma - 1)$ so that the steady-state markup over marginal cost is zero. Inflation and the reset price are linked through the price aggregation equation which can be written as:

$$1 = (1 - \alpha_p) \left( \frac{P_i^*}{P_t} \right)^{1-\sigma} + \alpha_p \pi_t^{\sigma-1}$$

and the Tak-Yun distortion term is

$$\Delta_t = (1 - \alpha_p) \left( \frac{\Gamma_{1t}}{\Gamma_{2t}} \right)^{-\sigma} + \alpha_p \pi_t^{\sigma} \Delta_{t-1}$$

This term $\Delta_t \geq 1$ is the wedge due to price distortions between the amount of intermediate goods produced and the amount of the final good consumed.
Labour agencies

Similarly to retailers, labour agencies transform the homogeneous household labour input into differentiated labour goods at the nominal price \( \tilde{w}_t P_t \) and sell them to intermediate firms at the price \( W_{ht} \). Labour agency \( h \) solves the following optimisation:

\[
\max_{p_{it}} \sum_{s=0}^{\infty} \alpha_w^{\sigma_w} Q_{t,t+s} ((1 + \tau_w) W_{ht} - \tilde{w}_{t+s} P_{t+s}) L_{ht+s} \\
\text{s.t. } L_{ht} = \left( \frac{W_{ht}}{W_t} \right)^{-\sigma_w} \tilde{L}_t
\]

Since all labour agencies that can re-optimise at \( t \) are identical, they all choose the same price \( W_{ht} = W_t^* \). The first-order conditions are analogous to those for retailers. Again, I assume that the government sets taxes such that \( \tau = 1/(\sigma_w - 1) \) so that the steady-state markup over marginal cost is zero. Wage inflation \( \pi_{wt} \) and the Tak-Yun distortion \( \Delta_{wt} \) are defined in the same way as for retailers. Finally, the real wage that intermediate producers effectively pay is

\[
w_t = \frac{W_t}{P_t} = w_{t-1} \frac{\pi_{wt}}{\pi_t} \tag{A.6}
\]

Capital good producers

Capital good producers internalise capital adjustment costs in the model. They operate competitively in input and output markets, producing new capital goods using old (depreciated) capital goods and final consumption goods. For the latter, they have a CES aggregator just like households. Their maximisation program is entirely intratemporal:

\[
\max_{I_t, K_t^d} Q_t \left( K_t^d + I_t \right) - Q_t^d K_t^d - I_t - \frac{\psi}{2} \left( \frac{I_t}{K_t^d} - \frac{I}{K^d} \right)^2 K_t^d
\]

Here, \( I/K^d \) is the steady state ratio of investment to depreciated capital. Due to constant returns to scale, these producers make zero profits in equilibrium. Their first-order conditions define the price for new and depreciated capital goods:

\[
Q_t = 1 + \psi \left( \frac{I_t}{K_t^d} - \frac{I}{K^d} \right) \tag{A.7}
\]

\[
Q_t^d = Q_t + \frac{I}{K^d} (Q_t - 1) + \frac{\psi}{2} \left( \frac{Q_t - 1}{\psi} \right)^2 \tag{A.8}
\]
Market clearing

The market clearing conditions are summarised below. Supply stands on the left-hand side, demand on the right-hand side.

\[ Y_t = \int_0^1 Y_{jt} dj = \int_0^1 Y_{it} di \]  
\[ \dot{Y}_t = \frac{Y_t}{\Delta_t} = C_t + I_t - \frac{\psi}{2} \left( \frac{I_t}{K_t} - \frac{I}{K^d} \right) K_t^d + C^e_t \]  
\[ L_t = \int_0^1 L_{ht} dh \]  
\[ \ddot{L}_t = \frac{L_t}{\Delta_{wt}} = \int_0^1 L_{jt} dj \]  
\[ K_t = \int_0^1 K_{jt} dj = (1 - \delta) K_{t-1} + I_t \]

\[ 1 = S_{jt}, \ j \in [0, 1] \]
\[ 0 = B^d_t \]

B Properties of the rational expectations equilibrium

The rational expectations equilibrium considered here has the following properties that need to be verified. All statements are local in the sense that for each of them, there exists a neighbourhood of the non-stochastic steady-state in which the statement holds.

1. All firms choose the same capital-labour ratio \( K_{jt}/L_{jt} \).

2. The expected return on capital is higher than the internal return on debt: \( E_t R^k_{t+1} > R_t \).

3. At any time \( t \), the stock market valuation \( V_{jt} \) of a firm \( j \) is proportional to its net worth after entry and exit \( \tilde{N}_{jt} \) with a slope that is strictly greater than one.

4. Firms always reinvest all their net worth and do not pay any dividends until they exit: \( D_{jt} = 0 \).

5. Borrowers never default on the equilibrium path and borrow at the risk-free rate, and the lender only accepts debt payments up to a certain limit.

6. If the firm defaults and the lender seizes the firm, she always prefers restructuring to liquidation.

7. The firm always exhausts the borrowing limit.
8. All firms can be aggregated. Aggregate debt, capital and net worth are sufficient to
describe the intermediate goods sector.

I take the following steps to prove existence of this equilibrium is as follows. After setting
up the firm value functions, Property 1 just follows from constant returns to scale. I then
take Properties 2 and 3 as given and prove 4 to 7. I verify that 3 holds. The aggregation
property 8 is then easily verified. I conclude by establishing the parameter restrictions for
which 2 holds.

Value functions

An operating firm $j$ enters period $t$ with a predetermined stock of capital and debt. It is
convenient to decompose its value function into two stages. The first stage is given by:

$$
\Upsilon_1 (K, B) = \max_{N, L, D} \gamma N + (1 - \gamma) (D + \Upsilon_2 (N - D))
$$

s.t. $N = qY - wL + (1 - \delta) Q^d K - RB$

$$
Y = K^\alpha (AL)^{1-\alpha} \\
D \geq 0
$$

(I suppress the time and firm indices for the sake of notation.) After production, the firm
exits with probability $\gamma$ and pays out all net worth as dividends, otherwise it can choose
to pay an additional dividend. The second stage of the value function consists in choosing
a debt level as well as a strategy in the default game:

$$
\Upsilon_2 (N) = \max_{K', B', \text{strategy in default game}} \beta E \left[ \Upsilon_1 (K', B') , \text{ no default} \right] \\
+ \beta E \left[ \Upsilon_1 (K', B^*) , \text{ debt renegotiated} \right] \\
+ \beta E \left[ 0, \text{ lender seizes firm} \right]
$$

s.t. $K' = N + B'$

A firm which only enters in the current period starts directly starts with an exogenous net
worth endowment and the value function $\Upsilon_2$.

Characterising the first stage

The first order conditions for the first stage with respect to $L$ equalises the wage with the
marginal revenue: $w = q (1 - \alpha) (K/L)^\alpha A^{1-\alpha}$. Since there is no firm heterogeneity apart
from capital $K$ and debt $B$, this already implies Property 1 that all firms choose the same
capital-labour ratio. Hence the internal rate of return on capital is also common across firms. This permits to write the evolution of net worth more conveniently:

\[ R^k = \alpha q \left( 1 - \alpha \right)^{\frac{qA}{w}} + (1 - \delta) Q^d \]  
\[ N = R^k K - RB \] 

(B.1) 
(B.2)

Next, the first-order condition on dividends are:

\[ D \left( \gamma (N - D) - 1 \right) = 0 \]  
\[ D \geq 0 \] 
\[ \gamma (N - D) \geq 1 \] 

(B.3) 
(B.4) 
(B.5)

Now taking Property 3 as given (and verifying it later on), \( \gamma \) is a linear function with slope strictly greater than one. Thus \( \gamma > 1 \) for all firms and \( D = 0 \), which establishes Property 4. Furthermore, the first-stage value function \( \gamma_1 \) is linear in its argument \((K, B)\) with slope greater than one as well:

\[ \gamma_1 (K, B) = \gamma N + (1 - \gamma) \gamma_2 (N) \]
\[ = \left( \gamma + (1 - \gamma) \gamma_2 \right) \left( R^k K - RB \right) \]
\[ = \gamma_1 \left( 1, \frac{B}{K} \right) K \] 

(B.6)

This property will be used repeatedly in the next step of the proof.

**Characterising the second stage**

The second stage involves solving for the subgame-perfect equilibrium of the default game between borrower and lender. Pairings are anonymous, so repeated interactions are ruled out. Also, only the size \( B \) and the interest rate \( \tilde{R} \) of the loan can be contracted (I omit primes for ease of notation and separate \( \tilde{R} \) from the risk-free rate \( R \)). The game is played sequentially:

1. The firm (F) proposes a borrowing contract \((B, \tilde{R})\).
2. The lender (L) can accept or reject the contract.

   - A rejection corresponds to setting the contract \((B, \tilde{R}) = (0, 0)\).
     Payoff for L: 0. Payoff for F: \( \tilde{\beta} E[\gamma_1 (N, 0)] \).
3. F acquires capital and can then choose to default or not.
• If F does not default, it has to repay in the next period.
  Payoff for L: $E_t Q_{t,t+1} \bar{R} B - B$. Payoff for F: $\hat{\beta} E \left[ \Upsilon_1 \left( K, \frac{\bar{R}}{R} B \right) \right]$.

4. If F defaults, the debt needs to be renegotiated. F makes an offer for a new debt level $B^*$.\(^{17}\)

5. L can accept or reject the offer.

  • If L accepts, the new debt level replaces the old one.
    Payoff for L: $E_t Q_{t,t+1} \hat{R} B^* - B$. Payoff for F: $\hat{\beta} E \left[ \Upsilon_1 \left( K, \frac{\hat{R}}{R} B^* \right) \right]$.

6. If L rejects, then she seizes the firm. A fraction $1 - \xi$ of the firm’s capital is lost in the process. Nature decides randomly whether the firm can be “restructured”.

  • If the firm cannot be restructured, or it can but the lender chooses not to do so, then the lender has to liquidate the firm.
    Payoff for L: $Q_t \xi K_t - B$. Payoff for F: $0$.

  • If the firm can be restructured and the lender chooses to do so, she retains a debt claim of present value $\xi B$ and sells the residual equity claim in the firm to another investor.
    Payoff for L: $\xi B + \hat{\beta} E \left[ \Upsilon_1 \left( \xi K, \xi B \right) \right] - B$. Payoff for F: $0$.

Backward induction leads to the (unique) subgame-perfect equilibrium of this game. Start with the possibility of restructuring. The lender L prefers this to liquidation if

$$\xi B + \hat{\beta} E \left[ \Upsilon_1 \left( \xi K, \xi B \right) \right] \geq Q_t \xi K_t \quad (B.7)$$

This holds true at the steady state because $R^k > R$ (Property 2), $Q = 1$ and $\hat{\beta} = Q_{t,t+1}$:

$$\xi B + \hat{\beta} E \left[ \Upsilon_1 \left( \xi K, \xi B \right) \right] > \xi B + \hat{\beta} E \left[ R^k \xi K - R \xi B \right] = \hat{\beta} E \left[ R^k \xi K \right] > \xi K \quad (B.8)$$

Since the inequality is strict, the statement holds in a neighbourhood around the steady-state as well. This establishes Property 6.

Next, the lender L will accept an offer $B^*$ if it gives him a better expected payoff (assuming that lenders can diversify among borrowers so that their discount factor is invariant to the outcome of the game). The probability of restructuring is given by $x$. The condition for accepting $B^*$ is therefore that

$$E_t Q_{t,t+1} \hat{R} B^* \geq x \left( \xi B + \hat{\beta} E \left[ \Upsilon_1 \left( \xi K, \xi B \right) \right] \right) + (1 - x) Q_t \xi K_t \quad (B.9)$$

\(^{17}\)That the interest rate on the repayment is fixed is without loss of generality.
Now turn to the firm F. Among the set of offers $B^*$ that are accepted by L, the firm will prefer the lowest one, i.e. that which satisfies (B.9) with equality. This follows from $\Upsilon_1$ being a decreasing function of debt. This lowest offer will be made if it leads to a higher payoff than expropriation: $\tilde{\beta}E \left[ T_1 \left( K, \frac{R}{R} B^* \right) \right] \geq 0$. Otherwise, the firm offers zero and the lender seizes the firm.

Going one more step backwards, the firm has to decide whether to declare default or not. It is preferable to do so if the $B^*$ that the lender will just accept is strictly smaller than $B$ or if expropriation is better than repaying, $\tilde{\beta}E \left[ \Upsilon_1 \left( K, \frac{R}{R} B \right) \right] \geq 0$.

What is then the set of contracts which the lender L accepts in the first place? From the perspective of L, there are two types of contracts: those that will not be defaulted on and those that will. If the firm does not default ($B^* \geq B$), the lender will accept the contract simply if it pays at least the risk-free rate, $\tilde{R} \geq R$. If the firm does default ($B^* < B$), then the lender accepts if the expected discounted recovery value exceeds the size of the loan, i.e. $E_t Q_{t,t+1} \tilde{R} B^* \geq B$.

Finally, let’s consider the contract offer by the firm. Firm F can offer a contract on which it will not default. In this case, it is optimal to offer just the risk-free rate $\tilde{R} = R$. Also note that the payoff from this strategy is strictly positive since

$$\tilde{\beta}E \left[ \Upsilon_1 \left( K, B \right) \right] = \tilde{\beta}E \left[ R^k K - RB \right] = \tilde{\beta}E \left[ R^k N + \left( R^k - R \right) B \right] > 0$$

(B.10)

The payoff is also increasing in the size of the loan $B$. So conditional on not defaulting, it is optimal for F to take out the maximum loan size $B = B^*$, and this is preferable to default with expropriation. However, it might also be possible for F to offer a contract that only leads to a default with debt renegotiation. The optimal contract of this type is the solution to the following problem:

$$\max_{\tilde{R}, B, B^*} \tilde{\beta}E \left[ T_1 \left( N + B, \frac{\tilde{R}}{R} B^* \right) \right]$$

s.t. $E_t Q_{t,t+1} \tilde{R} B^* \geq B$

$$E_t Q_{t,t+1} \tilde{R} B^* = x \left( \xi B + \tilde{\beta}E \left[ T_1 \left( \xi \left( N + B \right), \xi B \right) \right] \right) + \left( 1 - x \right) Q_{t} \xi \left( N + B \right)$

The first thing to note is that only the product $\tilde{R} B^*$ appears, so the choice of the interest rate $\tilde{R}$ is redundant. Further, $B = B^*$ and $\tilde{R} = R$ solve this problem, and this amounts to the same as not declaring default. This choice solves the maximisation problem above
if the following condition is satisfied at the steady state:

\[ \xi \left( 1 + x \mathcal{Y}'_1 \left[ \frac{R^k}{R} - 1 \right] \right) < 1 \]  

(B.11)

For the degree of stock price dependence \( x \) sufficiently small, this condition is satisfied. This establishes Properties 5 and 7.

**Linearity of firm value**

Since firms do not default and exhaust the borrowing limit \( B^* \), the second-stage firm value can be written as follows:

\[
\mathcal{Y}_2(N) = \hat{\beta} E[\mathcal{Y}_1(N + B, B)]
\]

(B.12)

where

\[
B = x \left( \xi B + \hat{\beta} E[\mathcal{Y}_1(\xi (N + B), \xi B)] \right) + (1 - x) Q\xi (N + B)
\]

(B.13)

We already know that if \( \mathcal{Y}_2 \) is a linear function, then \( \mathcal{Y}_1 \) is also linear. The converse also holds: The constraint above together with linearity of \( \mathcal{Y}_1 \) imply that \( B \) is linear in \( N \), and thus \( \mathcal{Y}_2 \) is linear, too.

To establish Property 3, it remains to show that the slope of \( \mathcal{Y}_2 \) is greater than one. This is easy to see in steady state:

\[
\mathcal{Y}'_2 = \frac{\hat{\beta}}{N} \frac{\mathcal{Y}_1(K, B)}{N} = \frac{\gamma (R^k K - RB)}{N} (1 - \gamma) \mathcal{Y}_2 \left( \frac{R^k K - RB}{N} \right)
\]

\[
= \hat{\beta} (\gamma + (1 - \gamma) \mathcal{Y}'_2) \left( \frac{R^k K - RB}{N} \right)
\]

\[
= (\gamma + (1 - \gamma) \mathcal{Y}'_2) \frac{R^k + \left( \frac{R^k - R}{N} \right) B}{N} \]

\[
= \frac{\gamma c_0}{1 - (1 - \gamma) c_0} > 1
\]

(B.14)

Finally, the aggregated law of motion for capital and net worth can be established (Property 8). Denoting again by \( \Gamma_t \subset [0, 1] \) the indices of firms that exit and are replaced in
period $t$, we have:

\[
K_t = \int_0^1 K_{jt}dj = \int_{j\in\Gamma_t} (N_{jt} + B_{jt}) dj + \int_{j\in\Gamma_t} (\omega N_t + B_{jt}) dj \\
= (1 - \gamma + \gamma \omega) N_t + B_t
\]  

(B.15)

\[
N_t = \int_0^1 N_{jt}dj = R^k_t K_{t-1} - R_{t-1} B_{t-1}
\]  

(B.16)

\[
B_t = \int_0^1 B_{jt}dj = x\xi (B_t + V_t) + (1 - x) \xi Q_t K_t
\]  

(B.17)

So far then, all model properties are established except for $R^k > R$.

Return on capital

It can now be shown under which conditions the internal rate of return is indeed greater than the return on debt. From the steady-state versions of equations (B.15) and (B.16), it follows that

\[
R^k = R + (G - R (1 - \gamma + \gamma \omega)) \frac{\bar{N}}{\bar{K}}
\]  

(B.18)

Provided that $\bar{N}/\bar{K}$ is positive, the condition $R^k > R$ holds if:

\[
\gamma > \frac{R - G}{G (1 - \omega)}
\]  

(B.19)

But $\bar{N}/\bar{K}$ is positive indeed. Combining the borrowing constraint (B.17) and stock market pricing equation (5.18) in steady-state leads to:

\[
\left( \frac{\bar{N}}{\bar{K}} \right)^{-1} = \frac{1 - \xi x}{1 - \xi} \left( 1 - \gamma + \gamma \omega + x \frac{G\gamma}{R - \frac{1 - \gamma}{1 - \gamma + \gamma \omega} G} \right)
\]  

(B.20)

which is strictly positive when constraint (B.19) is satisfied and $\xi < 1$. This concludes the proof of model properties.

C Approximation method for the learning equilibrium

The second order perturbation method for the learning equilibrium follows Schmitt-Grohe and Uribe (2004), but has to be adapted to allow for relaxation of the rational expectations assumption. A rational expectations equilibrium can generally be described as a solution $(y_t)_{t\in\mathbb{N}}$ to

\[
E_t [f (y_{t+1}, y_t, x_t, u_t)] = 0
\]  

(C.1)

where $E_t$ is the expectations operator with respect to the probability measure and filtration induced by exogenous stochastic disturbances $u_t$. These disturbances are of dimensionality
\(n_u\), independent and identically distributed, of zero mean and variance \(\sigma^2\Sigma_u\). The solution \(y_t\) is of dimensionality \(n\), as is the image of \(f\). \(x_t\) denotes a vector of predetermined state variables of dimensionality \(n_x < n\): \(x_{i,t} = y_{\iota(t),t-1}\) for an injective \(\iota: \{1..n\} \rightarrow \{1..n_x\}\), or simply \(x_t = Cy_{t-1}\) for an appropriate matrix \(C\). One is interested in finding a policy function that generates solutions of the form

\[
y_t = g(x_t, u_t, \sigma)
\]

(C.2)

Perturbation methods for approximating the policy function to arbitrary order are straightforward to compute in standard software packages such as Dynare. They compute Taylor expansions of \(g\), typically around a non-stochastic steady state of the model, i.e. a constant solution \(\bar{y}\) for \(\sigma = 0\) such that \(f(\bar{y}, \bar{y}, \bar{x}, 0) = 0\) and hence \(g(\bar{y}, 0, 0) = \bar{y}\).

In a learning equilibrium, (C.1) does not fully characterise the equilibrium because the probability measure used by agents to form expectations does not coincide with the actual probability measure of the model. The stock price in the model of this paper is determined by the usual market-clearing condition, but agents think it is determined by random unforecastable shocks that are not necessarily related to the rest of the economy. A model described by (C.1) cannot contain a shock that is perceived as exogenous but at the same time determined endogenously.

The model in this paper belongs to a class that can be written as follows:

\[
E_t^P \left[ f \left( y_{t+1}, y_t, x_t, u_t, z_t \right) \right] = 0 \quad \text{(C.3)}
\]

\[
E_t^P \left[ \phi \left( y_{t+1}, y_t, x_t, u_t, z_t \right) \right] = 0 \quad \text{(C.4)}
\]

Here, the probability measure \(P\) denotes beliefs for which the disturbances \(z_t\) (of dimensionality \(n_z\)) are perceived as exogenous, independent and identically distributed with zero mean and variance \(\sigma^2\Sigma_z\). They are also perceived as independent of \(u_t\), although this can be relaxed. These disturbances have the interpretation of forecast errors. The iid assumption then amounts to imposing that agents holding the belief \(P\) think their forecasts cannot be improved upon. As before, \(f\) is of dimensionality \(n\). The system (C.3) is assumed to have a unique solution for each initial condition \(x_t\) and path of disturbances \(u_t\) and \(z_t\) which can be described by a subjective policy function

\[
y_t = h(x_t, u_t, z_t, \sigma)
\]

(C.5)

Contrary to agents’ beliefs, \(z_t\) is not an exogenous disturbance, but determined endogenously by the second set of equilibrium conditions (C.4). The function \(\phi\) is of dimension \(n_z\). This set of conditions is not known to agents. The actual probability measure \(P_0\), induced by (C.3)-(C.4) and the disturbances \(u_t\), is thus different from \(P\). Under \(P_0\), \(z_t\) is
a function of the state and the fundamental disturbances
\[ z_t = r(x_t, u_t, \sigma) \] (C.6)

which leads to the objective policy function:
\[ g(x_t, u_t, \sigma) = h(x_t, u_t, r(x_t, u_t, \sigma), \sigma) \] (C.7)

All functional forms are assumed to be such that the functions \( \tilde{g} \) and \( r \) are uniquely determined.

In the case of the model of this paper, lagged belief updating requires two pseudo-disturbances, \( n_z = 2 \). Agents cannot update their beliefs about future stock prices at the same time as they observe current prices, yet by observing the price they can infer the current forecast error. This implies the following subjective belief equations that are part of (C.3):
\[
\log V_t = \log V_{t-1} + \hat{\mu}_{t-1} - \frac{\sigma_p^2 + \sigma_v^2}{2} + z_{1t} \] (C.8)
\[
\hat{\mu}_t = \hat{\mu}_{t-1} - \frac{\sigma_v^2}{2} + g_{z2t} \] (C.9)

The conditions (C.4) that pin down the values for the forecast errors \( z_t \) in equilibrium are then described as follows:
\[
V_t - \beta E^P_t \left[ D_{t+1} + V_{t+1} \right] = 0 \] (C.10)
\[
z_{2t} - z_{1t-1} = 0 \] (C.11)

Going back to the general case, the goal is to derive an accurate second-order approximation of the objective policy function \( g \) around the non-stochastic steady state:
\[
g(x_t, u_t, \sigma) \approx g(\bar{x}, 0, 0) + g_x (x_t - \bar{x}) + g_u u_t + g_{xx} [(x_t - \bar{x}) \otimes (x_t - \bar{x})] + g_{xu} [(x_t - \bar{x}) \otimes u_t] + g_{uu} [u_t \otimes u_t] + g_{\sigma\sigma} \sigma^2 \] (C.12)

The first step in deriving the approximation consists in calculating this approximation first for the subjective policy function \( h \). This can be done using standard methods as described in Schmitt-Grohe and Uribe (2004) and implemented e.g. in Dynare. The second step consists in finding the derivatives of the function \( r \) in (C.6). Substituting it into the
equilibrium conditions (C.4) gives:

\[ 0 = \Phi (x, u, \sigma) = \mathbb{E}_t^P [\phi (y', y, x, u, z)] = \mathbb{E}_t^P \left[ \phi \left( \begin{array}{c} h (Ch (x, u, z, \sigma), u', z', \sigma), \\ h (x, u, z, \sigma), x, u, z \end{array} \right) \right] \]

Here, I drop time subscripts and denote by a prime variables at \( t + 1 \). Note that the term \( z' \) must not be substituted out when the expectation is taken under \( \mathbb{P} \). Doing so would imply that agents know the true relationship between \( z_{t+1} \) and the model variables instead of taking it as an exogenous disturbance. Total differentiation at the non-stochastic steady state leads to the following first-order derivatives:

\[ 0 = \frac{d\Phi}{dx} (\bar{x}, 0, 0) = (\phi_y h_x C + \phi_y) (h_x + h_z r_x) + \phi_x + \phi_z r_x \quad \text{(C.14)} \]
\[ 0 = \frac{d\Phi}{du} (\bar{x}, 0, 0) = (\phi_y h_x C + \phi_y) (h_u + h_z r_u) + \phi_u + \phi_z r_u \quad \text{(C.15)} \]
\[ 0 = \frac{d\Phi}{d\sigma} (\bar{x}, 0, 0) = (\phi_y h_x C + \phi_y) (h_{\sigma} + h_z r_{\sigma}) + \phi_z r_{\sigma} \quad \text{(C.16)} \]

Since the existence of a unique solution for \( r \) is assumed, the first derivatives can be solved for. I also assume that the equilibrium conditions imply that \( \bar{z} = 0 \) at the steady-state. This means that in the absence of shocks, agents make no forecast errors under learning.

Define the matrix \( A = (\phi_y h_x C + \phi_y) h_z + \phi_z \). Then the first-order derivatives of \( r \) are given by:

\[ r_x = -A^{-1} ((\phi_y h_x C + \phi_y) h_x + \phi_x) \quad \text{(C.17)} \]
\[ r_u = -A^{-1} ((\phi_y h_x C + \phi_y) h_u + \phi_u) \quad \text{(C.18)} \]
\[ r_{\sigma} = 0 \quad \text{(C.19)} \]

Up to first order, the existence and uniqueness of the function \( r \) is equivalent to invertibility of the matrix \( A \). The first-order derivatives of the actual policy function are now readily calculated:

\[ g_x = h_x + h_z r_x \quad \text{(C.20)} \]
\[ g_u = h_x + h_z r_u \quad \text{(C.21)} \]
\[ g_{\sigma} = 0 \quad \text{(C.22)} \]

The certainty-equivalence property holds for the subjective policy function \( h \), hence \( h_{\sigma} = 0 \). This implies that \( r_{\sigma} = 0 \) and \( g_{\sigma} = 0 \) as well, so certainty equivalence also holds under
learning.

The second-order calculations are similar, if more tedious. The second-order derivative of $\Phi$ with respect to $x$ is:

$$0 = \frac{d^2 \Phi}{dx^2} (\bar{x}, 0, 0) = (\phi_y g_x C + \phi_y) (h_{xx} + 2h_{xz} [I_{n_x} \otimes r_x] + h_{zz} [r_x \otimes r_x])$$

$$+ \phi_y h_{xx} [C g_x \otimes C g_x] + B_{xx} + Ar_{xx} \quad (C.23)$$

This equation isn't-dimensional and linear in $r_{xx}$, and thus can be solved easily. As in first order, only invertibility of the matrix $A$ is required for a unique local solution under learning. I have collected all cross-derivatives of $\phi$ inside the matrix $B_{xx}$ (of size $n_z \times n_x^2$), which contains only first-order derivatives of the policy functions:

$$B_{xx} = \phi_y g_x \begin{bmatrix} [h_x C g_x \otimes h_x C g_x] + \phi_y g_z \otimes g_z + \phi_xz \otimes r_x[\otimes r_x] \\
+ 2\phi_y g_z \overline{C g_x \otimes g_x} + 2\phi_y g_z \otimes \left[ h_x C g_x \otimes I_{n_z} \right] + 2\phi_y g_z \otimes \left[ h_x C g_x \otimes r_x \right] \\
+ 2\phi_y g_x \otimes \left[ h_x C g_x \otimes r_x \right] + 2\phi_y g_x \otimes \left[ I_{n_x} \otimes r_x \right] \end{bmatrix} \quad (C.24)$$

The formulae to solve for $r_{xx}$ and $u_{xx}$ are analogous. It remains to look at the derivatives involving $\sigma$. This simplifies considerably because the first derivatives of the policy functions $g$ and $h$ with respect to $\sigma$ are zero. The cross-derivative of $\Phi$ with respect to $x$ and $\sigma$ thus reads:

$$0 = \frac{d^2 \Phi}{dx d\sigma} (\bar{x}, 0, 0) = (\phi_y g_x C + \phi_y) (h_{x\sigma} + h_z r_{x\sigma}) + \phi_z r_{x\sigma} \quad (C.25)$$

But because $h_{x\sigma} = 0$ as under rational expectations, $r_{x\sigma} = 0$ holds as well. The same applies to $r_{u\sigma} = 0$. Finally, the second derivative with respect to $\sigma$ involves the variance of the disturbances:

$$0 = \frac{d^2 \Phi}{d\sigma^2} (\bar{x}, 0, 0) = \phi_y g_x \begin{bmatrix} h_{\sigma\sigma} + h_{u\sigma} \text{vec} (\Sigma_u) + +h_{x\sigma} \text{vec} (\Sigma_z) \\
+ \phi_y g_x \left( \text{vec} (h'_u \Sigma_u h_u) + \text{vec} (h'_z \Sigma_z h_z) \right) \\
+ (\phi_y g_x C + \phi_y) (h_{\sigma\sigma}) + Ar_{\sigma\sigma} \end{bmatrix} \quad (C.26)$$

Again, this can be solved for $r_{\sigma\sigma}$ when $A$ is invertible. Note that the perceived variance $\Sigma_z$ appears in the calculation because it matters for expectations for the future (unless $\phi_y \phi_y' = 0$). This variance is not necessarily equal to the objective variance of $z$.

The second-order derivatives of the actual policy function $g$ are calculated without great effort once those of $r$ are known:

$$g_{xx} = h_{xx} + 2h_{xz} [I_{n_x} \otimes r_x] + g_{zz} [r_x \otimes r_x] + g_{r_xx} \quad (C.27)$$

and analogously for $g_{uu}$, $g_{uu}$ and $g_{\sigma\sigma}$. The cross-derivatives $g_{u\sigma}$ and $g_{x\sigma}$ are zero, just as under rational expectations.