Leadership Bias and Incentives in Political Parties*

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Abstract

We develop a novel model of intraparty competition in order to study how the strength of the party leadership impacts on the different party members’ incentives to provide costly effort for the party. Consistently with casual observation on current and past party leaders, we allow the party leader to be either strong and biased towards their own views or weak and unbiased. Party members are driven by either ideologic or opportunistic motives. We show that the party may face a crucial trade off between its desire to have a strong and thus electorally successful leader and the need to ensure all party members do their share of the necessary day to day party work. Among other things, we show that having opportunistic party members may be a plus when the party has a strong leader and it wishes to incentivize especially the party members that are closer to the leader. If parties wish to have all party members exert (some) effort (even when the leader is strong and biased), then having ideologic politicians is desirable.

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1 Introduction

Political parties are organizations that require the work of all their labor force to thrive in their political market. Whereas commentators and citizens alike typically focus on the decisions and actions of parties and their leaders at the time of elections, party members need to work for their party even and, perhaps, especially between elections, when their party and leading members are less in the spotlight of the media. Obviously, when parties select their leaders at the party congress, the leader’s expected capacity to lead successfully the party in the election race is very prominent on the mind of every party congress participant. Thus, if the party has the choice between two equally competent candidates but one is viewed as more charismatic than the other, we should expect parties to select the charismatic one. Yet, casual observation suggests that the more a leader is strong, charismatic and visionary, the more this leader is unwilling to allow alternative views to be discussed, let alone develop and burgeon fully within the party. In a nutshell, casual observation forcefully suggests that stronger leader are typically more biased towards their own views and vision.  

This desire by strong leaders to control their party machine notwithstanding, parties, even the most cohesive and disciplined ones, are not unitary, monolithic machines that support and defend blindly the vision and strategy of the party leader. Rather, parties are typically composed of factions of members which, if they share a broad common goal and vision and are typically willing to stand behind their leader, each have their own specificities and individual views and preferences. Also, these factions typically pursue and develop, besides the general party goals and activities, their own, faction-specific activities and strategies. Obviously, this implies that factions may have to trade off the available time and energy between their faction-specific activities and the general party work. In most advanced democracies, parties receive public funds to help them finance their activities and strategies. A large part of these funds is typically distributed by the party leadership to the different factions and members. Factions and members thus compete for these funds, and it is only too logical to conclude that such competition can and should be expected to be used by the party leadership to shape the trade off the different party members face when they decide how to allocate their effort between the party work and their own individual goals and activities. Consistently with the above accounts, we wish to develop a model that allows us to analyze the trade off a party may face when it has to pit its desire for a strong (and biased) leader with the need to ensure that party members and factions are provided the necessary incentives to work for the party and not only for their individual specialized interests. As politicians are typically viewed as being driven by either ideologic or opportunistic motives, we analyze this trade off under these two scenarios. In our model, party members have to provide costly effort or party work and the party, to incentivize its members, offers a part of the available party funds in return for this effort. To model the mapping between effort and the share of the funds each member receives, we rely on a standard (imperfectly discriminating) contest success function. To model leadership bias, we let the contest success function be biased against one or more members.

Notice that this association between leaders’ strength or vision and biases is also generally observed in the population of the CEOs of firms, see for example Rotemberg and Saloner (2000) for more on this.
competing in the contest.
Turning to our results,

2 Related Literature
To Be Added

3 The Model
The game we consider is a game between the party leadership and the other party members. The leadership needs party members to exert effort, to participate in the day-to-day party work. As party members also wish to dedicate (some of) their scare time and resources to their personal activities, the party needs to offer them a reward in return for their participation in the general party activities.
We consider the following timing:

1. Party learns the ideal location of leader for electoral purposes and then selects actual leader (we do not model this stage as it is not necessary for the analysis of the trade off we wish to focus on)
2. Members choose their effort level independently and simultaneously given their position and the leader’s biases (if there is a strong leader)
3. Budget shares allocated and utilities of all players are realized

The equilibrium concept is Nash equilibrium.
As we just said, in between each election, the party selects a leader, \(L\). This leader can be weak or strong. Strong leaders are better for electoral purposes but their strength implies that they will bias the reward, the fund allocation process across members to the advantage of the members they prefer.
There are three members \(A, B, C\). Each member prefers to see the available funds flow to him or her rather than to another player. Each player can exert costly effort to increase his share of this budget. Effort is costly (this cost is linear in effort). Members can be opportunistic, in which case they only care about their own welfare, or ideologic, in which case they attribute some utility to seeing members who are close to them ideologically win a share of the funds too.

The available budget, \(V\), is independent of the current effort choices (and the current leader type). Indeed, a common characteristic of the public funding of political parties is that the exact sums that go to each party are determined by the party score (and that of the other parties!) at the previous election. Also, even if the party were to be in a position to make credible promises\(^2\) to its party members and link their current effort decisions to the future allocation of funds, as the size of such funds is uncertain (it depends on the score of all the parties running at the

\(^2\)but this would require a commitment device on the side of the party leadership as such promises are clearly time inconsistent!
next election), it is much easier (and credible) for the party to link the allocation of current funds to the members’ choices in terms of current effort provision. In a nutshell, the party can choose between a contest which links current funds and efforts and one which links current effort to future, uncertain funds. As the former is not tainted by uncertainty and time consistency issues, it seems reasonable to assume that the party will choose the first type of contest. Also, it may seem intuitive to assume that strong leaders should be associated to a larger budget than weak ones, but as these funds are determined by the electoral scores of all parties involved in each election and that these other parties also choose their leaders strategically, a basic general equilibrium reasoning implies that, in any (symmetric) equilibrium, the budget available to each party is constant and independent of the type of leader each party chooses.

An opportunistic politician \(i = A, B, C\) solves

\[
\max_{e_i} \frac{b_i e_i}{\sum_j b_j e_j} V - e_i
\]

whereas an ideologic politician \(i = A, B, C\) solves

\[
\max_{e_i} \frac{b_i e_i}{\sum_m b_m e_m} V + \frac{b_i e_j}{\sum_m b_m e_m} V(1 - d_{ij}) + \frac{b_k e_k}{\sum_m b_m e_m} V(1 - d_{ik}) - e_i
\]

where \(d_{AC} = 1\), \(d_{AB} = B\), \(d_{BC} = 1 - B\).

Suppose the three players are located in positions \(A, B\) and \(C\) on \([0, 1]\) segment of the line of real numbers. This line represents the party’s and its different players’ bliss points (viewpoints or ideology for example). The leader’s position is also on this segment and determines the biases in the selection process. We make the following assumptions:

- \(d_{AC} = 1 \Rightarrow d_{AB} = B\) and \(d_{BC} = 1 - B\)
- \(L \in [0, 1/2]\)
- \(A\) is located at 0, and \(C\) is located at 1
- Biases are linear in the distance between the leader and a player, thus:
  - If \(B \geq L\), then:
    \[
    b_A = 1 - L \\
    b_B = 1 - B + L \\
    b_C = L
    \]
  - If \(B < L\), then:
    \[
    b_A = 1 - L \\
    b_B = 1 - L + B \\
    b_C = L
    \]
4 Weak Leader

As weak leaders are also leaders who are too weak within their party to manage to bias the allocation of the party funds, we model them as if they were altogether absent. Thus suppose there is no leader, hence there are no biases in the selection procedure. This is equivalent to assuming that $b_i = 1$ for all $i = A, B, C$. In such case, winning probabilities are exactly proportional to individual effort.

4.1 Opportunistic Politicians

Suppose that there are no externalities, that is, individuals only get positive valuation if their proposal is selected. Hence, individuals only differ in that they face a different bias. This is equivalent to assuming that $a = 1/d_{ij}$ for all $i \neq j$. In such case, we get the following equilibrium:

**Proposition 1.** For all $i = A, B, C$:

- $e^*_i = \frac{2V}{9}$
- $p^*_i = \frac{1}{3}$

The case without leader and externalities reduces to a standard Tullock contest in which the three individuals exert the same level of effort and thus have the same probability of being selected. Furthermore, individual effort is increasing in the prize valuation.

4.2 Ideologic Politicians

Suppose now that individuals value to some extent the use of the funds made by other players, according to (1). In such case, we get the following equilibrium:

**Proposition 2.**

- $e^*_A = e^*_C = \frac{aV}{4}$ and $e^*_B = 0$
- $p^*_A = p^*_C = \frac{1}{2}$ and $p^*_B = 0$

If there is no leader and party members are ideologically driven, only the extremists exert positive effort. Furthermore, they exert the same level of effort, hence they receive the same budget shares. Furthermore, individual effort is increasing in the prize valuation.

The presence of externalities gives incentives to the moderate individual to free-ride on the effort of the extremists. As it turns out, this effect is sufficiently strong to make the moderate individual inactive in equilibrium. This is consistent with Osborne et al. (2000), who show that moderate individuals tend to not participate to costly meetings when there are externalities among individuals.
5 Strong Leader

Suppose now that there is a leader whose position is given by \( L \in [0, 1/2] \), which in turn determines the biases \( b_i = 1 - d_{Li} \) for \( i = A, B, C \).

As it turns out, whenever there is one individual inactive in equilibrium, and even though there are biases in the selection procedure, the two active individuals exert the same level of effort. In turn, the individual who receives the largest budget share is the one whose location is the closest to the one of the leader.

Proposition 3. At any equilibrium at which only individuals \( i \) and \( j \) exert positive effort, it holds that:

- \( e_i^* = e_j^* \)
- \( p_i^* > p_j^* \) if and only if \( d_{Li} < d_{Lj} \)

5.1 Opportunistic Politicians

Suppose that there are no externalities, that is, individuals only value the funds accruing to them. Hence, individuals only differ in that they face a different bias. This is equivalent to assuming that \( a = 1/d_{ij} \) for all \( i \neq j \).

The following proposition provides the number and identity of active individuals in equilibrium given that \( L \in [0, 1/2] \), conditional on the position of \( B \).

Proposition 4. Let \( L \in [0, 1/2] \).

1. If \( L \leq \frac{1}{3} \)
   
   (a) If \( B < \frac{L^2 + 2L - 1}{2L - 1} \), \( C \) is inactive
   
   (b) If \( B > \frac{L^2 + 2L - 1}{2L - 1} \), the three individuals are active

2. If \( \frac{1}{3} < L < \frac{1}{2}(3 - \sqrt{5}) \)
   
   (a) If \( B < \frac{3L^2 - 4L + 1}{2L - 1} \), the three individuals are active
   
   (b) If \( \frac{3L^2 - 4L + 1}{2L - 1} < B < \frac{L^2 + 2L - 1}{2L - 1} \), \( C \) is inactive
   
   (c) If \( B > \frac{L^2 + 2L - 1}{2L - 1} \), the three individuals are active

3. If \( \frac{1}{2}(3 - \sqrt{5}) < L \leq \frac{1}{2} \), the three individuals are active

Therefore, individual \( C \) may be inactive in equilibrium. Figure 1 depicts the number of active individuals and the ranking in probabilities when there are no externalities, as a function of \( B \) and \( L \). Roughly speaking, for low values of \( L \), that is, when the leader is very close to \( A \), individual \( C \) is active provided that \( B \) is very close to him. Then, as the leader gets closer to 0.38, \( C \) is active for an increasing range of values of \( B \). Once the leader is above 0.38, the three individuals are active regardless of the location of \( B \).
The next proposition provides the ranking of individual efforts and winning probabilities when the three individuals are active:

**Proposition 5.** Let $L \in [0, 1/2]$. At an interior solution:

1. If $B < 2L$, then $e_B^* > e_A^* > e_C^*$ and $p_B^* > p_A^* > p_C^*$
2. If $B > 2L$, then $e_A^* > e_B^* > e_C^*$ and $p_A^* > p_B^* > p_C^*$

Without externalities and at any interior solution, the ordering of efforts and budget shares between the three players coincide. That is, the leader always offers the largest share of the funds to the party member closest to him. Furthermore, the ranking of efforts corresponds to the ranking of the distances between individuals and the leader. That is, the individual exerting the highest effort is the one closest to the leader, while the one exerting the lowest effort is the one farthest from the leader.

**Figure 1: No Externalities and Incentives**

### 5.2 Ideologic Politicians

Suppose now that there is a leader and individuals are ideologically motivated. The following proposition provides the number and identity of active individuals in equilibrium given that $L \in [0, 1/2]$, conditional on the position of $B$.

**Proposition 6.** Let $L \in [0, 1/2]$.

1. If $B < 2L$, the three individuals are active
2. If $B \geq 2L$, $C$ is inactive

As for the case without externalities, in equilibrium, individual $C$ may be inactive. More specifically, individual $C$ is inactive whenever the leader is located closer to $A$ than to $B$ (in turn, for this to happen, $B$ must be between the leader and $C$). Intuitively, if $B > 2L$, it is not worth for $C$ to exert effort as (i) the leader is very far away and (ii) $B$ is not that far and thus a victory by $B$ is valuable.
Proposition 7. Let $L \in [0, 1/2]$. At an interior solution:

1. If $B < 4L - 2L^2 - 1$, then $e^*_C > e^*_B > e^*_A$ and $p^*_B > p^*_A > p^*_C$
2. If $4L - 2L^2 - 1 < B < L$, then $e^*_C > e^*_B > e^*_A$ and $p^*_B > p^*_A > p^*_C$
3. If $L < B < 2L$, then $e^*_A > e^*_B > e^*_C$ and $p^*_B > p^*_A > p^*_C$

Contrary to the case without externalities, the orderings between budget shares and efforts no longer coincide. In particular, the individual exerting the highest level of effort never has the largest budget share. As it turns out, it is always an extremist who exerts the highest level of effort (except when $L = B$, in which case the three individuals exert the same level of effort). However, it is always individual $B$ who has the highest probability of being selected.

Figure 2 depicts the number of active individuals and the ranking in probabilities when there are externalities among individuals, as a function of $B$ and $L$. While in the absence of externalities, $C$ is always the individual exerting the smallest effort, when individuals value others’ ideas, $C$ is the one exerting the highest effort whenever $B$ is on the left of the leader. Whenever the three individuals are active, $B$ exerts an intermediate amount of effort, while the extremist exerting the highest effort depends on whether $B$ is on the left or the right of $L$.

Figure 2: Externalities and Incentives

6 Comparison Leader versus No Leader

In this section, we analyze the effect of the presence of a leader on

1. The number of active individuals (i.e., diversity)
2. The identity of the individual who receives the largest budget share
3. The level of effort of the individual who receives the largest budget share (i.e., quality and efficiency)
The number of active individuals may be interpreted as a measure of diversity inside the party.

The identity of the individual who receives the largest budget share provides a measure of the representativity of the budget allocation system. Indeed, whenever the distribution of individuals is not uniform, one may consider as desirable that the majority group receives the majority of the budget. Further, another criterion may be the relative extremism of the individual who receives the largest budget share. For instance, favoring a moderate may be desirable.

6.1 Opportunistic Politicians

We know that if there is no leader and individuals are opportunistically motivated, the three individuals exert the same level of effort \((2V/9)\) and receive the same budget shares \((1/3)\) regardless of their location.

First, the presence of a leader alters the number of active individuals, as individual \(C\) may be inactive in equilibrium. In this sense, the presence of a leader may decrease the diversity in a party. Second, it may be either individual \(A\) or \(B\) who receives the largest budget share, hence the introduction of a leader allows to alter the distribution of funds toward more moderate individuals. In turn, this means that the presence of a leader can potentially increase the cohesiveness of a party.

Third, as the individual who receives the largest budget share is also the one exerting the highest level of effort, it means that the selection procedure is efficient. Furthermore, the level of individual effort may exceed the level exerted when there is no leader.

6.2 Ideologic Politicians

We know that if there is no leader and individuals are ideologically motivated, only the two extremists exert a positive (and the same) level of effort \((V/4)\) and receive the same budget share \((1/2)\).

First, the presence of a leader may increase the number of active individuals, as well as alter their identity: \(B\) is always active, while \(C\) may be inactive in equilibrium. In this sense, the presence of a leader may increase the diversity of ideas produced.

Second, it is always the moderate individual who receives the largest budget share. This is remarkable, as when there is no leader this individual is always inactive.

Third, as the individual who receives the largest budget share is not the one exerting the highest level of effort, it means that the selection procedure is not efficient. Furthermore, the level of individual effort of the extremists may be lower than the one when there is no leader.
7 More than three party members

7.1 Opportunistic Politicians

Let five individuals \( i = 1, \ldots, 5 \) subsequently located on the segment \([0, 1]\) such that individual 1 is located at 0 and individual 5 is located at 1. Suppose there is no leader. In such case, in the absence of externalities, each individual exerts an amount of effort \( e^* = 4V/25 \) and faces an equal probability of winning \( p^* = 1/5 \).

Suppose first that the five individuals \( i = 1, \ldots, 5 \) are uniformly distributed on the segment \([0, 1]\). Introducing a centrist leader \( L = 1/2 \) alters the equilibrium in the following way:

\[
e_1^* = 0 \quad e_2^* = \frac{24V}{125} \quad e_3^* = \frac{30V}{125} \quad e_4^* = \frac{24V}{125} \quad e_5^* = 0
\]

\[
p_1^* = 0 \quad p_2^* = \frac{3}{11} \quad p_3^* = \frac{5}{11} \quad p_4^* = \frac{3}{11} \quad p_5^* = 0
\]

Suppose now that the leader is a non-centrist, and let \( L = 1/4 \). we then have:

\[
e_1^* = \frac{24V}{125} \quad e_2^* = \frac{30V}{125} \quad e_3^* = \frac{24V}{125} \quad e_4^* = 0 \quad e_5^* = 0
\]

\[
p_1^* = \frac{3}{11} \quad p_2^* = \frac{5}{11} \quad p_3^* = \frac{3}{11} \quad p_4^* = 0 \quad p_5^* = 0
\]

Whether centrist or not, the presence of a leader when individuals are uniformly distributed has the effect of making two individuals inactive, while increasing the level of individual effort of the ones who remain active.

Suppose now that individuals \( i = 1, \ldots, 4 \) are uniformly distributed on the segment \([0, 1/2]\), while individual 5 is located at 1. Introducing a centrist leader \( L = 1/2 \) alters the equilibrium in the following way:

\[
e_1^* = \frac{210V}{1369} \quad e_2^* = \frac{312V}{1369} \quad e_3^* = \frac{340V}{1369} \quad e_4^* = 0 \quad e_5^* = 0
\]

\[
p_1^* = \frac{7}{40} \quad p_2^* = \frac{13}{40} \quad p_3^* = \frac{17}{40} \quad p_4^* = 0 \quad p_5^* = 0
\]

If \( L = 1/4 \), we have:

\[
e_1^* = \frac{231V}{1600} \quad e_2^* = \frac{351V}{1600} \quad e_3^* = \frac{351V}{1600} \quad e_4^* = \frac{231V}{1600} \quad e_5^* = 0
\]

\[
p_1^* = \frac{7}{40} \quad p_2^* = \frac{13}{40} \quad p_3^* = \frac{13}{40} \quad p_4^* = \frac{7}{40} \quad p_5^* = 0
\]
7.2 Ideologic Politicians

Let five individuals \( i = 1, \ldots, 5 \) subsequently located on the segment \([0, 1]\) such that individual 1 is located at 0 and individual 5 is located at 1. Suppose there is no leader. In such case, only the two extreme individuals exert a positive (and the same) level of effort, while all moderates are inactive. That is, with externalities we have \( e_1^* = e_5^* = aV/4 \) and \( e_i^* = 0 \) for \( i = 2, 3, 4 \), while \( p_1^* = p_5^* = 1/2 \) and \( p_i^* = 0 \) for \( i = 2, 3, 4 \).

Suppose first that the five individuals \( i = 1, \ldots, 5 \) are uniformly distributed on the segment \([0, 1]\). Introducing a centrist leader \( L = 1/2 \) alters the equilibrium in the following way:

\[
e_1^* = \frac{aV}{12}, \quad e_2^* = \frac{aV}{18}, \quad e_3^* = \frac{aV}{12}, \quad e_4^* = \frac{aV}{18}, \quad e_5^* = \frac{aV}{12}
\]

\[
p_1^* = \frac{1}{6}, \quad p_2^* = \frac{1}{6}, \quad p_3^* = \frac{1}{3}, \quad p_4^* = \frac{1}{6}, \quad p_5^* = \frac{1}{6}
\]

If \( L = 1/4 \), we have:

\[
e_1^* = \frac{21aV}{200}, \quad e_2^* = \frac{27aV}{400}, \quad e_3^* = \frac{9aV}{200}, \quad e_4^* = \frac{9aV}{100}, \quad e_5^* = 0
\]

\[
p_1^* = \frac{7}{20}, \quad p_2^* = \frac{3}{10}, \quad p_3^* = \frac{3}{20}, \quad p_4^* = \frac{1}{5}, \quad p_5^* = 0
\]

Suppose now that individuals \( i = 1, \ldots, 4 \) are uniformly distributed on the segment \([0, 1/2]\), while individual 5 is located at 1. Introducing a centrist leader \( L = 1/2 \) alters the equilibrium in the following way:

\[
e_1^* = \frac{aV}{10}, \quad e_2^* = \frac{9aV}{100}, \quad e_3^* = \frac{9aV}{400}, \quad e_4^* = \frac{aV}{10}, \quad e_5^* = \frac{aV}{8}
\]

\[
p_1^* = \frac{1}{8}, \quad p_2^* = \frac{3}{20}, \quad p_3^* = \frac{3}{20}, \quad p_4^* = \frac{2}{5}, \quad p_5^* = \frac{1}{4}
\]

If \( L = 1/4 \), we have:

\[
e_1^* = \frac{aV}{11}, \quad e_2^* = \frac{27aV}{968}, \quad e_3^* = \frac{27aV}{968}, \quad e_4^* = \frac{aV}{11}, \quad e_5^* = 0
\]

\[
p_1^* = \frac{4}{11}, \quad p_2^* = \frac{3}{22}, \quad p_3^* = \frac{3}{22}, \quad p_4^* = \frac{4}{11}, \quad p_5^* = 0
\]

8 Conclusion

To Be Written
Proof of Proposition 1. Individual $i = A, B, C$ chooses his level of effort by maximizing

$$\frac{e_i}{\sum_j e_j} V - e_i$$

The FOC yields the best response

$$e_i = \sqrt{(e_j + e_k)V - e_j - e_k}$$

and thus

$$e_i^* = \frac{2V}{d} \text{ and } p_i^* = \frac{1}{3} \text{ for all } i = A, B, C$$

Proof of Proposition 2. Individual $i$ chooses his level of effort by maximizing

$$\frac{e_i}{\sum_m e_m} V + \frac{e_j}{\sum_m e_m} V(1 - ad_{ij}) + \frac{e_k}{\sum_m e_m} V(1 - ad_{ik}) - e_k$$

The FOC yields the best response

$$e_i = \sqrt{a(d_{ij}e_j + d_{ik}e_k)V - e_j - e_k}$$

and thus

$$e_i^* = \frac{2ad_{ij}d_{ik}(d_{ij} + d_{ik} - d_{jk})d_{jk}^2V}{[d_{ij} + (d_{ik} - d_{jk})^2 - 2d_{ij}(d_{ik} + d_{jk})]^2}$$

Given that $d_{AB} = B$, $d_{BC} = 1 - B$ and $d_{AC} = 1$, this yields:

$$e_A^* = e_C^* = \frac{aV}{4} \text{ and } e_B^* = 0$$

and the winning probabilities are given by

$$p_A^* = p_C^* = \frac{1}{2} \text{ and } p_B^* = 0$$

Proof of Proposition 3. At any corner solution, only two individuals are active and thus the game reduces to a contest with two players. Individual $i$ chooses his level of effort by maximizing

$$\frac{b_i e_i}{b_i e_i + b_j e_j} V + \frac{b_j e_j}{b_i e_i + b_j e_j} V(1 - ad_{ij}) - e_i$$

The FOC yields the best response

$$e_i = \frac{1}{b_i} \sqrt{ab_i b_j d_{ij} e_j V - b_j e_j}$$
and thus
\[ e^*_i = e^*_j = \frac{ab_i b_j d_{ij} V}{(b_i + b_j)^2} \]

If there are no externalities, we have that \( a = 1/d_{ij} \) and individual efforts reduce to
\[ e^*_i = e^*_j = \frac{b_i b_j V}{(b_i + b_j)^2} \]

Finally, the winning probabilities are given by
\[ p^*_i = \frac{b_i}{b_i + b_j} \text{ and } p^*_j = \frac{b_j}{b_i + b_j} \]

For \( b_i = 1 - d_{Li} \) and \( b_j = 1 - d_{Lj} \), we have that \( p^*_i > p^*_j \) if and only if \( d_{Li} < d_{Lj} \).

\( \square \)

**Proof of Proposition 4.** Individual \( i = A, B, C \) chooses his level of effort by maximizing
\[ \frac{b_i e_i}{\sum_j b_j e_j} V - e_i \]

The FOC yields the best response
\[ e_i = \frac{1}{b_i} \left[ \sqrt{b_i (b_j e_j + b_k e_k) V - b_j e_j - b_k e_k} \right] \]

and thus
\[ e^*_i = \frac{2b_j b_k [b_i (b_j + b_k) - b_j b_k] V}{[b_j b_k + b_i (b_j + b_k)]^2} \]
\[ p^*_i = \frac{b_i (b_j + b_k) - b_j b_k}{b_i (b_j + b_k) + b_j b_k} \]

where \( b_i = 1 - d_{Li} \). If \( B < L \), we have the following equilibrium levels of effort:
\[ e^*_A = \frac{2(1 + B - L) L [(1-L)^2 + B(1-2L)] V}{(1 + B - L^2)^2} \]
\[ e^*_B = \frac{2 [B + (1-L)^2] (1-L) V}{(1 + B - L^2)^2} \]
\[ e^*_C = \frac{2(1 + B - L)(1-L) [(4-3L) L - 1 - B(1-2L)] V}{(1 + B - L^2)^2} \]

We have that \( e^*_A > 0 \) and \( e^*_B > 0 \), while \( e^*_C > 0 \) if and only if \( B < \frac{3L^2 - 4L + 1}{2L - 1} \). Observe that the latter condition is never satisfied for \( L \leq 1/3 \), while it is always satisfied for \( L \geq \frac{1}{2}(3 - \sqrt{5}) \).

If \( B > L \), we have the following equilibrium levels of effort:
\[ e^*_A = \frac{2(1 - B + L)[1 - 3L^2 - B(1 - 2L)] V}{[B - 1 - (2 - L)L]^2} \]

\[ e^*_B = \frac{2(1 - L)L(1 - B + L^2) V}{[B - 1 - (2 - L)L]^2} \]

\[ e^*_C = \frac{2(1 - B + L)(L - 1)[1 - L(2 + L) - B(1 - 2L)] V}{[B - 1 - (2 - L)L]^2} \]

We have that \( e^*_A > 0 \) and \( e^*_B > 0 \), while \( e^*_C > 0 \) if and only if \( B > \frac{L^2 + 2L - 1}{2L - 1} \). Observe that the latter condition is always satisfied for \( L \geq \frac{1}{2}(3 - \sqrt{5}) \).

Proof of Proposition 5. Consider the interior solutions, that is, we are in one of the following situations:

1. \( L \leq \frac{1}{3} \) and \( B > \frac{L^2 + 2L - 1}{2L - 1} \)
2. \( \frac{1}{3} < L < \frac{1}{2}(3 - \sqrt{5}) \) and
   (a) \( B < \frac{3L^2 - 4L + 1}{2L - 1} \) or
   (b) \( B > \frac{L^2 + 2L - 1}{2L - 1} \)
3. \( \frac{1}{2}(3 - \sqrt{5}) < L \leq \frac{1}{2} \)

If \( B < L \), we have

\[ e^*_A - e^*_B = \frac{2BL[1 + B(1 - 2L) + L(3L - 4)] V}{(1 + B - L^2)^2} < 0 \]

\[ e^*_A - e^*_C = \frac{2[1 - (1 - L)^2](1 + B - L)(1 - 2L) V}{(1 + B - L^2)^2} > 0 \]

\[ e^*_B - e^*_C = \frac{2(1 + B - 2L)[1 - L][1 - L] V}{(1 + B - L^2)^2} > 0 \]

If \( B > L \), we have

\[ e^*_A - e^*_B = \frac{2(2L - B)L[1 - L(2 + L) - B(1 - 2L)] V}{[B - 1 - (2 - L)L]^2} > 0 \text{ if and only if } B > 2L \]

\[ e^*_A - e^*_C = \frac{2(1 - B + L)(1 - 2L)(1 - B + L^2)}{[B - 1 - (2 - L)L]^2} > 0 \]

\[ e^*_B - e^*_C = \frac{2(1 - B)(1 - L)[1 - B(1 - 2L) - 3L^2] V}{[B - 1 - (2 - L)L]^2} > 0 \]

and thus:
1. If \( B < L \), we have \( e^*_B > e^*_A > e^*_C \)

2. If \( B > L \), we have:

   (a) \( e^*_A > e^*_B > e^*_C \) if \( B > 2L \)

   (b) \( e^*_B > e^*_A > e^*_C \) if \( B < 2L \)

We know that for \( i = A, B, C \), the equilibrium winning probability is given by

\[
p_i^* = \frac{b_i(b_j + b_k) - b_jb_k}{b_i(b_j + b_k) + b_jb_k}
\]

Hence, it follows directly that \( p_i^* > p_j^* \) if and only if \( b_i > b_j \). Therefore, as the biases are linear in the distance:

1. If \( B < L \), we have \( p_B^* > p_A^* > p_C^* \)

2. If \( B > L \), we have:

   (a) \( p_A^* > p_B^* > p_C^* \) if \( B > 2L \)

   (b) \( p_B^* > p_A^* > p_C^* \) if \( B < 2L \)

Proof of Proposition 6. Individual \( i \) chooses his level of effort by maximizing

\[
\frac{b_i e_i}{\sum_m b_m e_m} V + \frac{b_j e_j}{\sum_m b_m e_m} V(1 - a d_{ij}) + \frac{b_k e_k}{\sum_m b_m e_m} V(1 - a d_{ik}) - e_i
\]

The FOC yields the best response

\[
e_i = \frac{1}{b_i} \left[ \sqrt{ab_i(d_{ij}b_j e_j + d_{ik}b_k e_k)V - b_j e_j - b_k e_k} \right]
\]

and thus

\[
e_i^* = \frac{2ab_i b_k d_{ij}d_{ik}d_{jk}^2 (b_i b_j d_{ij} + b_k b_k d_{ik} - b_j b_k d_{jk}) V}{[b_i(d_{ij} - d_{ik})(b_j d_{ij} - b_k d_{ik}) - (b_j b_i + b_k d_{ij} + b_k (b_i + b_j) d_{ik}) d_{jk} + b_j b_k d_{jk}^2]^2}
\]

\[
p_i^* = \frac{d_{jk} [b_j b_k d_{jk} - b_i (b_j d_{ij} + b_k d_{ik})]}{b_i (d_{ij} - d_{ik})(b_j d_{ij} - b_k d_{ik}) - (b_j (b_i + b_k) d_{ij} + b_k (b_i + b_j) d_{ik}) d_{jk} + b_j b_k d_{jk}^2}
\]

If \( B < L \), we have the following equilibrium levels of effort:

\[
e_A^* = \frac{a(1 + B - 2L)LV}{2(1 + B - L)}
\]

\[
e_B^* = \frac{a(1 - L)L [1 + B(1 - 2L) - 2(1 - L)L] V}{2(1 - B)(1 + B - L)^2}
\]
We have that $e_i^* > 0$ for all $i = A, B, C$.

If $B > L$, we have the following equilibrium levels of effort:

\[
e_A^* = \frac{a(1 - L)[2(1 - L)L - B(1 + B - 2L)]V}{2(1 - B)(1 + B - 2L)}
\]

\[
e_B^* = \frac{a(1 - L)[B - 2L(B - L)]V}{2B(1 - B + L)^2}
\]

\[
e_C^* = \frac{a(2L - B)(1 - L)V}{2(1 - B + L)}
\]

We have that $e_A^* > 0$ and $e_B^* > 0$, while $e_C^* > 0$ if and only if $B < 2L$.

\[\]

**Proof of Proposition 7.** Consider the interior solutions, hence $B < 2L$. From the equilibrium effort levels in the proof of Proposition 6, if $B < L$ we have:

\[
e_A^* - e_B^* = \frac{a(B - L)[2L(1 + B - L) - B(1 + B)]V}{2(1 - B)(1 + B - L)^2} < 0
\]

\[
e_A^* - e_C^* = \frac{a(L - B)[2L(1 + B - L) - (1 + B)]V}{2(1 - B)(1 + B - L)} < 0
\]

\[
e_B^* - e_C^* = \frac{a(1 + B)(1 + B - 2L)(L - B)(1 - L)V}{2(B - 1)(1 + B - L)^2} < 0
\]

If $B > L$ we have:

\[
e_A^* - e_B^* = \frac{a(2 - B)(2L - B)(B - L)LV}{2B(1 - B + L)^2} > 0
\]

\[
e_A^* - e_C^* = \frac{a(B - L)[B(1 - 2L) + 2L^2] V}{2B(1 + L - B)} > 0
\]

\[
e_B^* - e_C^* = \frac{a(B - L)(1 - L)[B(1 - B) + 2L(B - L)]V}{2B(1 - B + L)^2} > 0
\]

and thus:

1. If $B < L$, we have $e_C^* > e_B^* > e_A^*$
2. If $B > L$, we have $e_A^* > e_B^* > e_C^*$

Further, if $B < L$, the equilibrium winning probabilities are given by
\[ p^*_A = \frac{1 + B - 2L}{2(1 + B - L)} \]
\[ p^*_B = \frac{(B + 1) - 2L(1 + B - L)}{2(1 - B)(1 + B - L)} \]
\[ p^*_C = \frac{2L(1 + B - L) - B(1 + B)}{2(1 - B)(1 + B - L)} \]

and thus we have

\[ p^*_A - p^*_B = \frac{2L(2B - L) - B(B + 1)}{2(1 - B)(1 + B - L)} < 0 \]
\[ p^*_A - p^*_C = \frac{2L(2B - L) - B(1 + B)}{2(1 - B)(1 + B - L)} > 0 \]

if and only if \( B > 4L - 2L^2 - 1 \)

\[ p^*_B - p^*_C = \frac{(1 + B - 2L)^2}{2(1 - B)(1 + B - L)} > 0 \]

Finally, if \( B > L \), the equilibrium winning probabilities are given by

\[ p^*_A = \frac{2L(B - L) + B(1 - B)}{2B(1 - B + L)} \]
\[ p^*_B = \frac{B - 2L(B - L)}{2B(1 - B + L)} \]
\[ p^*_C = \frac{2L - B}{2(1 - B + L)} \]

and thus we have

\[ p^*_A - p^*_B = -\frac{(B - 2L)^2}{2B(1 - B + L)} < 0 \]
\[ p^*_A - p^*_C = \frac{B - 2L^2}{2B(1 - B + L)} > 0 \]
\[ p^*_B - p^*_C = \frac{B(1 + B) - 2L(2B - L)}{2B(1 - B + L)} > 0 \]

Therefore:

1. If \( B < 4L - 2L^2 - 1 \), we have \( p^*_B > p^*_C > p^*_A \)
2. If \( B > 4L - 2L^2 - 1 \), we have \( p^*_B > p^*_A > p^*_C \)
10 References


