The accumulation of wealth along the life cycle under aspirations *

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Abstract

We analyze the effects of the introduction of aspirations both on the pattern of wealth accumulation along the life cycle of individuals displaying a "joy-of-giving" motive for bequests and on the evolution of wealth within a dynasty. We will show that the introduction of aspirations at different ages display different effects on the amount of saving of workers. However, both adult and old aspirations dampen the positive effect on wealth accumulation brought about by warm-glow altruism. Therefore, under aspirations, both bequest motivated and non-bequest motivated individuals will behave more similarly than when aspirational concerns are absent. Finally, we show that aspirations raise the speed of convergence to the dynastic steady state.

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1. Introduction

In this paper, we analyze the effects of the introduction of aspirations both on the pattern of wealth accumulation along the life cycle of individuals displaying a "joy-of-giving" motive for bequests and on the evolution of wealth within a dynasty. When aspirations are present the utility of individuals depends on the consumption experience of their parents. We will consider a general framework where adult individuals compare their level of consumption with that of their parents when they were also adults. Similarly, old individuals will take into account the consumption of their parents when they were also old.

There is empirical evidence about the existence of aspirations associated with the involuntary transmission of tastes from one generation to the next. For instance, Cox et al. (2004) estimate that parental preferences explain between 5 to 10 percent of the preferences of their children after controlling for their respective incomes. Cavalli-Sforza and Feldman (1981) and Boyd and Richerson (1985) provide surveys about the evidence on intergenerational transmission of tastes. Among the theoretical studies on the macroeconomic implications of aspirations, we could mention those of de la Croix (1996, 2001), de la Croix and Michel (1999, 2001), and Alonso-Carrera et al. (2007).

Our analysis will be conducted in the framework of an overlapping generations (OLG) economy where preferences of individuals display "joy of giving" or "warm-glow" altruism. This means that individuals' utility will be an increasing function of the amount of bequest left to their children, like in Yaari (1965) and Abel (1986). Several alternative motives leading to intergenerational transfers have been proposed in the literature. Among them, and in addition to joy of giving, we could mention strategic behavior (Bernheim et al., 1985), existence of incomplete annuity markets (Abel, 1985), and pure intergenerational altruism (Barro, 1974). However, the empirical evidence is not conclusive about the reasons why individuals make intergenerational transfers and probably a combination of all those motives lies in the core of the mechanism governing the intergenerational transmission of wealth.

We will show that aspirations affect saving and both the size of bequests and the level of saving in the economy in a direction similar to the one obtained by de la Croix and Michel (2001), Jellal and Wolff (2002) and Alonso-Carrera et al. (2007), who conducted the analysis under the assumption of altruistic preferences à la Barro (1974). Moreover, these authors focus their analysis on the effect of aspirations on the operativeness of the bequest motive and on the implications of the interaction between altruism and aspirations to induce self-restraint in parental consumption. We will focus instead on the implications for the pattern of wealth accumulation along the life cycle brought about by aspirations.

Our analysis is inspired by the work of Hurd (1987), who was the first who conducted an empirical analysis showing that the patterns of saving and consumption of altruistic individuals do not differ substantially from those who are not altruistic. Hurd also shows through a simple model that the amount of wealth accumulation in all ages should be higher for altruistic individuals. In order to detect non-altruistic individuals, Hurd classifies as non-altruistic those individuals having no living children. In this paper we will show analytically that, when a bequest motive is introduced in our otherwise standard OLG economy, the implications for accumulation of wealth are attenuated by
the presence of aspirations. Therefore, the presence of aspirations makes the profile of consumption and saving of bequest motivated and non-bequest motivated individuals more similar than when no aspirational concerns are taken into account.

When parents wish to leave a state to their children they rise the amount of saving in all ages of their lives. We will see that this increase in the accumulation of wealth is dampened if either aspirations when individuals are adult/workers (adult aspirations) or when they old/retired (old aspirations) are present. This dampening effect occurs even if the introduction of aspirations at different ages has not the same effect on wealth accumulation. In particular, the presence of adult aspirations results in a smaller amount of saving for workers as they want to mimic the amount of consumption of their parent when they were also workers. However, the introduction of old consumption raises the amount of saving of workers as they want to shift consumption to the age were they are retired to imitate the standard of living of their parents when they were also old.

Another result of our analysis is that aspirations raise the speed of convergence to the dynastic steady state. This is so because aspirations introduce a sluggish response in consumption which implies in turn that all the adjustment is made through the amount of bequest, which is the main factor driving the convergence within a dynasty.

In our analysis we also perform some steady-state comparative statics concerning the interaction of aspirations with warm-glow altruism for families that have reached their stationary level of consumption, saving, and bequest. We check mainly trough a numerical analysis that the dampening effect of aspirations also hold when we compare two stationary equilibria with different configurations of parameter values.

The paper is organized as follows. Section 2 presents the general model with warm-glow altruism and aspirations. Section 3 analyzes the effect of bequest motive and aspirations on saving and inheritance. In Section 4, we analyze the effect of aspirations on the speed of convergence. In Section 5 and 6, we conduct the comparative statics analysis to characterize the effects of aspirations and bequest motive and of its interaction for a given individual with exogenously given values of aspirations and received inheritance. In Section 7 we make the numerical comparison between stationary equilibria. We conclude the paper in Section 8.

2. The Model

Let us consider a small open OLG economy, where individuals live for three periods and a new generation is born in each period. Each individual has offspring in the second period of his life and the exogenous number of children per parent is $n \geq 1$. We assume that agents make economic decisions only during the last two periods of their lives. Each agent inelastically supplies one time unit of labor in the second period of his life and is retired in the third period. We index each generation by the period in which its members work (i.e. when they are adults).

There is a single commodity, which can be devoted to either consumption or saving. An adult individual of generation $t$ distributes his net labor income and inheritance between consumption and saving. The budget constraint faced by this worker $i$ in period $t$ is

$$w_t + b_t = c_t + s_t,$$  \hspace{1cm} (2.1)
where $w_t$ is the wage compensation of this worker, $c_t$ is his amount of consumption (hereinafter, adult consumption), $b_t$ is the amount of inheritance he has received from parents and $s_t$ is the amount of saving.

When individuals are old, they receive a return on their savings, which is distributed between own consumption and bequests for their children. Therefore, the budget constraint of an old individual $i$ belonging to generation $t$ will be

$$R_{t+1}s_t = x_{t+1} + nb_{t+1},$$  \hspace{0.5cm} (2.2)

where $R_{t+1}$ is the gross rate of return on saving, $b_{t+1}$ is the amount of bequests the individual leaves to each of his descendants (who where born in period $t$) and $x_{t+1}$ is the amount of consumption of the old individual $i$ in period $t + 1$ (hereinafter, old consumption). Note that we are implicitly making an equal-treatment assumption so that all the direct descendants of the same individual receive the same amount of inheritance.

We will assume that in each period individuals derive utility from the comparison of their consumption with a consumption reference. Note that during their first period of life individuals neither work nor consume. However, as in de la Croix (1996), the member $i$ of the generation born in period $t - 1$ inherits a certain level of aspirations $a_t^i$ in period $t$. These aspirations are based on the standard of living achieved by his parents. More precisely, we assume that the inherited aspiration of an adult individual of generation $t$ is

$$a_t^i = c_{t-1},$$  \hspace{0.5cm} (2.3)

where $a_t^i$ is his parent’s amount of consumption when the parent was an adult (second period of life). We posit the following additive specification for the aspiration adjusted consumption $\tilde{c}_t^i$ of an adult individual $i$ belonging to generation $t$:

$$\tilde{c}_t^i = c_t^i - \delta a_t^i,$$  \hspace{0.5cm} (2.4)

with $\delta \in [0, 1)$. The adult individuals who have acquired higher aspirations due to their parents’ experience of consumption will require a larger amount of consumption in order to achieve the same level of utility. The aspirations arising when an individual is adult/worker will be dubbed adult aspirations.

Similarly, old individuals might derive utility from the comparison between their consumption when old and the consumption of their parents' old consumption. Therefore, the aspiration adjusted consumption $\tilde{x}_{t+1}^i$ of an old individual $i$ in period $t + 1$ is given by the following additive function:

$$\tilde{x}_{t+1}^i = x_{t+1}^i - \gamma a_{t+1}^i,$$  \hspace{0.5cm} (2.5)

where

$$a_{t+1}^i = x_t$$  \hspace{0.5cm} (2.6)

and $a_t^i$ is his parent’s amount of consumption when the parent was old (third period of life). The aspirations occurring when individuals are old/retired will be dubbed old aspirations.
The individual belonging to generation $t$ derives utility from both aspiration adjusted adult consumption and aspiration adjusted old consumption. We posit the following separable utility function representing the preferences of the individual $i$ belonging to generation $t$:

$$U(c_t^i, x_{t+1}^i, b_{t+1}^i) = u(c_t^i) + \beta u_x(x_{t+1}^i) + \rho v(b_{t+1}) ,$$

(2.7)

where both $\beta$ and $\rho$ are positive and the functions $u$, $u_x$, and $v$ are twice-differentiable, strictly increasing, strictly concave, and satisfy the Inada conditions at zero and at infinity. Note that we are generating positive bequests through a "joy-of-giving" motivation (as in Yaari, 1965; or Abel, 1986) so that the amount of bequests enters directly as an argument in the utility function. There are other motives for intergenerational transfer, such as altruistic preferences à la Barro (1974) and Becker (1981) where individuals derive utility from their children’s indirect utility function, or through paternalistic preferences where individuals care about their offspring’s level of consumption (Pollak, 1988). Under altruistic preferences, the last term in the utility (2.7) would be replaced by the indirect utility function of one’s children, which is an increasing function of the amount of inheritance received by descendants. If preferences were paternalistic, the last term in the utility (2.7) would be replaced by the offspring’s adult consumption, which in turn would be an increasing function of the amount $b_{t+1}$ bequest. In both cases, the results would be qualitatively similar to those obtained under a joy-of-giving specification. However, a problem posed by these two alternative types of preferences is the potential existence of corner solutions when the bequest motive is not operative, i.e. when the amount of bequest in equilibrium is equal to zero. We will avoid this problem by assuming joy-of-giving preferences displaying an Inada condition when the amount $b_{t+1}$ of bequest tends to zero.

Let us assume that the good of this economy is produced by means of a production function displaying constant returns to scale in capital and efficient labor. In our small open economy, capital is fully mobile and labor is not mobile. Under competitive input markets this implies that the rental price of a unit of capital is constant and equal to its international level $r$. Therefore, the gross rate of return on savings satisfies $R_{t+1} = 1 + r \equiv R$ for all $t$. We will assume throughout the paper that the interest rate is strictly positive, i.e., $R > 1$. Moreover, the equilibrium capital to labor ratio becomes constant and, thus, the marginal productivity of a unit of labor (which is equal to the competitive real wage per unit of labor) is also constant, $w_t = w$ for all $t$.

3. Aspirations, saving, and bequests

Individuals maximize (2.7) with respect to $\{c_t, x_{t+1}, b_{t+1}, s_t\}$ subject to (2.1), (2.2), (2.4) and (2.5), taking as given $\alpha_t^c$, $\alpha_t^{x_{t+1}}$, $b_t$, $w_t$ and $R_{t+1}$. If we plug the aspiration formation equations (2.3) and (2.6), and the competitive rental prices of inputs $w_t = w$ and $R_{t+1} = R$ into the solution of this individual problem, we obtain the following first order conditions of the individual belonging to generation $t$ are

$$u'_c(c_t) = \beta R_{t+1} u'_x(x_{t+1})$$

(3.1)

and

$$n\beta u'_x(x_{t+1}) = \rho v'(b_{t+1}) ,$$

(3.2)
where equation (3.1) give us the optimal allocation of consumption along the life cycle and equation (3.2) give us the optimal allocation of resources of an old individual between his own old consumption and the amount of bequest left to each of his direct descendants. Taking into account that in equilibrium

\[ c_t = w + b_t - s_t - \delta_c c_{t-1} \]  

and

\[ \hat{x}_{t+1} = R s_t - n b_{t+1} - \delta_x x_t, \]  

we can compute the effect on the values \( s_t \) and \( b_{t+1} \) of saving and bequest of changes in the bequest intensity \( \rho \), and the aspirations intensities \( \delta_c \) and \( \delta_x \) when adult and old, respectively. Making use of (3.3) and (3.4) and implicitly differentiating the system of equations (3.1) and (3.2) with respect to the parameters \( \rho, \delta_c \) and \( \delta_x \), we get the following partial derivatives, where we have suppressed the arguments of the functions to ease the notation:

\[
\frac{d s_t}{d \rho} = -\frac{\left( n \beta R u''_x v' \right)}{n^2 \beta u''_c u''_x + \rho u''_c v'' + \beta \rho R^2 u''_x v''} > 0, \quad (3.5)
\]

\[
\frac{d s_t}{d \delta_c} = -\frac{u''_c \left( n \beta u''_x + \rho v'' \right) c_{t-1}}{n^2 \beta u''_c u''_x + \rho u''_c v'' + \beta \rho R^2 u''_x v''} < 0, \quad (3.6)
\]

\[
\frac{d s_t}{d \delta_x} = \frac{\beta \rho R u''_x v' x_t}{n^2 \beta u''_c u''_x + \rho u''_c v'' + \beta \rho R^2 u''_x v''} > 0, \quad (3.7)
\]

\[
\frac{d b_{t+1}}{d \rho} = -\frac{\left( u''_c + \beta R^2 u''_x \right) v'}{n^2 \beta u''_c u''_x + \rho u''_c v'' + \beta \rho R^2 u''_x v''} > 0, \quad (3.8)
\]

\[
\frac{d b_{t+1}}{d \delta_c} = -\frac{n R u''_x c_{t-1}}{n^2 \beta u''_c u''_x + \rho u''_c v'' + \beta \rho R^2 u''_x v''} < 0, \quad (3.9)
\]

and

\[
\frac{d b_{t+1}}{d \delta_x} = -\frac{n R u''_c x_t}{n^2 \beta u''_x u''_c + \rho u''_c v'' + \beta \rho R^2 u''_x v''} < 0. \quad (3.10)
\]
Clearly, when the intensity $\delta_c$ of adult aspirations increases, the utility associated with adult consumption diminishes while its marginal utility rises. Therefore, the optimal reaction of the individual is to increase his adult consumption $c_t$ and reduce the other values of his other arguments of his utility function, namely, old consumption $x_{t+1}$ and bequests $b_{t+1}$. Moreover, the shift from old to adult consumption results in a lower amount of saving. However, when $\delta_c$ rises, the utility associated with old consumption diminishes while its marginal utility rises so that individuals optimally react by augmenting his old consumption, which implies in turn an increase in saving and a decrease in the amount of bequest. Finally, as Bossmann et al. (2007) and Hurd (1987) pointed out, under joy-of giving and pure altruism, respectively, an increase in the intensity of warm-glow altruism, parameterized by the value of $\rho$, raises the amount of bequest $b_{t+1}$ and, moreover, raises also the amount of saving $s_t$ as a result of the shift of resources from adult consumption to the next generation.

From now on, for simplicity we will assume that the function $U(c_t, x_{t+1}, b_{t+1})$ is not only additive separable but homothetic as in Abel (1986). Then, according to Katzner (1970, Theorem 2.4-4), the utility functions $u_c$, $u_x$ and $v$ must be isoelastic, i.e.,

\[
  u_c(z) = u_x(z) = v(z) = \begin{cases} 
    z^{1-\sigma} & \text{if } \sigma \neq 1 \\
    \ln z & \text{if } \sigma = 1,
  \end{cases}
\]

with $\sigma > 0$. Under this parametric assumption we can obtain the explicit equilibrium values of adult and old consumption, saving and bequest:

\[
  c_t = \frac{1}{H} \left\{ \left[ R (w + b_t) - \delta_c x_t \right] + \left[ (\beta R)^{\frac{1}{n}} + n \left( \frac{\rho R}{n} \right)^{\frac{1}{n}} \right] \delta_c c_{t-1} \right\},
\]

\[
  x_{t+1} = \frac{1}{H} \left\{ R \left( (\beta R)^{\frac{1}{n}} (w + b_t - \delta_c c_{t-1}) + R + n \left( \frac{\rho R}{n} \right)^{\frac{1}{n}} \right] \delta_x x_t \right\},
\]

\[
  s_t = \frac{1}{H} \left\{ \left[ (\beta R)^{\frac{1}{n}} + n \left( \frac{\rho R}{n} \right)^{\frac{1}{n}} \right] (w + b_t - \delta_c c_{t-1}) + \delta_x x_t \right\},
\]

\[
  b_{t+1} = \frac{1}{H} \left\{ R \left( \frac{\rho R}{n} \right)^{\frac{1}{n}} (w + b_t - \delta_c c_{t-1}) - \left( \frac{\rho R}{n} \right)^{\frac{1}{n}} \delta_x x_t \right\},
\]

where

\[
  H = R + (\beta R)^{\frac{1}{n}} + n \left( \frac{\rho}{n\beta} \right)^{\frac{1}{n}}.
\]

Note that the linearity of the previous functions with respect to the state variables $b_t$, $c_{t-1}$, and $x_t$ faced by an individual belonging to generation $t$ is a direct consequence of the homotheticity of the utility function $U$ and the assumed linearity of aspiration formation given in (2.4) and (2.5).
4. Transitional Dynamics

The evolution of consumption, saving and intergenerational transfers of the dynasty under consideration is entirely governed by the system of difference equations formed by (3.11), (3.12), and (3.14). The steady-state (or stationary) values of adult consumption, old consumption, and bequest, can be found easily by just making \( c_t = c_{t-1} = c \), \( x_{t+1} = x_t = x \), and \( b_{t+1} = b_t = b \) in that dynamic system and the solving for the steady-state value of adult consumption \( c \), old consumption \( x \), and bequest \( b \). Those steady-state values are the following:

\[
\begin{align*}
    c &= \frac{R (1 - \delta_c) w}{J} \quad (4.1) \\
    x &= \frac{(1 - \delta_c) R (R \beta)^{\frac{1}{2}} w}{J} \quad (4.2) \\
    b &= \frac{R (R \beta)^{\frac{1}{2}} (1 - \delta_c) (1 - \delta_x) \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} w}{J} \quad (4.3)
\end{align*}
\]

where

\[
J = R (1 - \delta_x) + (\beta R)^{\frac{1}{2}} (1 - \delta_c) + \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} (n - R) (1 - \delta_c) (1 - \delta_x) \quad (4.4)
\]

**Lemma 4.1.** If

\[
\frac{R \left( \frac{\rho R}{n} \right)^{\frac{1}{2}}}{R + (\beta R)^{\frac{1}{2}} + n \left( \frac{\rho R}{n} \right)^{\frac{1}{2}}} < 1 \quad (4.5)
\]

and the aspirations intensities and the aspiration intensity \( \delta_c \) and \( \delta_x \) are sufficiently small, then the dynamic system formed by equations (3.11), (3.12), and (3.14) converges monotonically to the steady state for adult consumption, old consumption and bequest given by (4.1), (4.2) and (4.3), respectively.

Note that under the assumption of the previous lemma the steady-state values \( c \), \( x \), and \( b \) are all strictly positive. To see this just observe that the numerators of (4.1), (4.2) and (4.3) are all strictly positive for sufficiently small values of \( \delta_c \) and \( \delta_x \), whereas \( J \) defined in (4.4) tends to

\[
R + (\beta R)^{\frac{1}{2}} + \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} (n - R)
\]

when \( \delta_c \) and \( \delta_x \) tend to zero. The previous expression is positive under the condition (4.5). This stability condition becomes simply

\[
\frac{\rho R}{n (1 + \beta + \rho)} < 1
\]

when the utility functions are logarithmic (\( \sigma = 1 \)). In this case, the stability condition has a direct interpretation since the return to capital \( R \) and the bequest motive \( \rho \)
should not be very high in order to prevent the dynasty from accumulating wealth per capita unboundedly. Obviously, a high rate of population growth will also prevent this excessive accumulation of wealth as the initial wealth of individuals will be small when the family state has to be divided among many children.

Another natural issue related with the transition towards the steady state of the endogenous variables of the model refers to the speed at which the steady-state values are approached. We will consider the speed of convergence around the steady state. If we write the system formed by linear difference equations (3.11), (3.12), and (3.14) in vector form (see the proof of Lemma (4.1) in the appendix), we see that the coefficient matrix $P$ defined in (A.1) has three-eigenvalues, $\lambda_1$, $\lambda_2$, and $\lambda_3$. Therefore, the solution of the linear dynamic system will have the form

$$z_t = A_{z,1} \lambda_1^t + A_{z,2} \lambda_2^t + A_{z,3} \lambda_3^t + z, \quad t = 0, 1, \ldots$$

for $z_t = c_t, x_{t+1}, b_{t+1}$, and where $z$ is the stationary value of $z_t$ and the transpose vector $(A_{c,j}, A_{x,j}, A_{b,j})'$, $j = 1, 2, 3$, takes the form $\kappa_j (m_{c,1}, m_{x,2}, m_{b,3})'$, where $(m_{c,1}, m_{x,2}, m_{b,3})$ is an eigenvectors associated with the eigenvalue $\lambda_j$ of the matrix $P$. The constants $\kappa_j$, $j = 1, 2, 3$, are pin down by the initial values of $c_{-1}, x_0$ and $b_0$. Then, the speed of convergence of the economy could be measured by the fraction of the distance between the value of the generic variable $z_t$ in period $t$ and the stationary value $z$ that the system travels in one period,

$$\frac{z_{t+1} - z_t}{z - z_t}.$$ 

Let us assume without loss of generality that $\lambda_1$ is the largest eigenvalue. It is straightforward to see that

$$\lim_{t \to \infty} \frac{z_{t+1} - z_t}{z - z_t} = 1 - \lambda_1,$$

so that the value of the largest eigenvalue of the matrix $P$ is inversely related with the speed of convergence around the steady-state.

**Lemma 4.2.** The speed of convergence around the steady-state increases when aspirations are introduced.

The introduction of aspirations decrease the importance of received inheritance to determine the transmission of wealth to the future generation. As the utility function is isoelastic, and thus displays decreasing absolute risk aversion, the utility functions for consumption become more concave in equilibrium. Therefore, as aspirations increase, individuals do not desire to change their profile of consumption. This means that when the economy is off its stationary equilibrium, the adjustment relies mainly on the intergenerational transfers through bequests. Since these intergenerational transfers constitute the main factor explaining the convergence towards the dynastic stationary equilibrium, the speed of convergence to the steady state increases.

From the expression of the derivatives of the largest eigenvalue $\lambda_1$ of the system with respect to the parameters representing the two types of aspirations, (A.3) and (A.4), we see that

$$\left| \frac{d\lambda_1}{dc} \right|_{\delta_c = 0, \delta_x = 0} \leq \left| \frac{d\lambda_1}{dx} \right|_{\delta_c = 0, \delta_x = 0}.$$
if and only if $\beta R^{1-\sigma} \geq 1$. Note that the stronger is the preference for old consumption $\beta$ the stronger will be the effect on the speed of convergence, $1 - \lambda_1$, of the introduction of old aspirations relative to that of adult aspirations.

As discussed in Caballé and Moro-Egido (2014), de la Croix and Michel (1999), and de la Croix (1996), the existence of a sufficiently high intensity of aspirations might be a source of endogenous fluctuations around the steady state. In particular, in our model, if we assume logarithmic utilities $\sigma = 1$ and only adult aspirations are present (i.e., $\delta_x = 0$) it can be proved that the eigenvalues of the matrix $P$ are real and positive if and only if

$$\delta_c \in \left[0, \frac{\rho R}{n(\beta + \rho)^2} \left(2 + \beta + \rho - 2\sqrt{1 + \beta + \rho}\right)\right].$$

Similarly, when $\sigma = 1$ and only old aspirations are present (i.e., $\delta_c = 0$), the eigenvalues of the matrix $P$ are real and positive if and only if

$$\delta_x \in \left[0, \frac{\rho R}{n(1+\rho)^2} \left(1 + 2\beta + \rho - 2\sqrt{\beta(1 + \beta + \rho)}\right)\right].$$

Therefore, oscillations could arise in the economy under strong aspirations. The models of de la Croix and Michel (1999), and de la Croix (1996) do not display transmission of wealth through bequests but endogenous rental prices of labor and capital. Moreover, they only consider adult aspirations. In this case, individuals subject to strong adult aspirations consume a lot when adults an thus save very little so that the next generation will receive a small labor income due the small stock of capital installed in the economy. This will result in a lower level of consumption for the next generation and this will give rise to consumption oscillations. In our setup with bequests and exogenous rental prices, the mechanism for oscillations under strong aspirations is more straightforward. Consider a generation that consumes a lot for an aspirational motive (i.e., to achieve the same standard of living of their parents either when adult or when old). This will result in a reduction in the amount left as bequests, which in turn will reduce the life-time income, and thus the consumption, of the next generation. Therefore, intergenerational oscillations of consumption will arise naturally.

5. Aspirations and wealth accumulation

We are going to study the presence of aspirations and bequest motives modifies the pattern of accumulation of wealth along the life cycle of an individual belonging to generation $t$ who has received a given amount of inheritance $b_t$ and who is exposed to the level of aspirations $c_{t-1}$ and $x_t$. We will consider two stages in the accumulation process: first, the individual’s accumulation of wealth until the end of the adult period, i.e., until the end of his participation in the labor market, which is summarized by the difference $s_t - b_t$ (adult accumulation) and, second, the wealth accumulation that takes place while the individual is retired, which is collected by the difference $nb_{t+1} - s_t$ (old accumulation). Finally, we can consider the total accumulation of wealth along the life cycle, which will be given by the difference between asset holdings at the final and the beginning of the individual’s life $nb_{t+1} - b_t$. 
As we have seen in Section 3 under general utility functions, saving increases with the intensity \( \rho \) of the warm-glow altruism, decreases with the intensity \( \delta_c \) of adult aspirations, and increases with the intensity \( \delta_x \) of old aspirations. Hence, these comparative statics effects are automatically translated into the pattern of adult accumulation \( s_t - b_t \) as the initial inheritance \( b_t \) of the individual is given. Moreover, since the amount of bequest left to each descendant \( b_{t+1} \) is decreasing in the intensities of both types of aspirations, \( \delta_c \) and \( \delta_x \), and increasing in the bequest motive \( \rho \), these comparative statics effects are automatically translated into the pattern of total accumulation \( nb_{t+1} - b_t \) for a given initial inheritance \( b_t \) received by the individual.

The analysis of the effect of aspirations and bequest motives on wealth accumulation by an old individual is more complex since the two terms of the difference \( nb_{t+1} - s_t \) are endogenous. Concerning the effect of the intensity \( \rho \) of warm-glow altruism, we can compute the following partial derivatives

\[
\frac{\partial (nb_{t+1} - s_t)}{\partial \rho} = \frac{(R\beta)^{\frac{1}{2}} \left( \frac{\rho}{n^\beta} \right)^{\frac{1}{2} + \sigma} \left[ R^{1-\rho} \left( R\beta \right)^{\frac{1}{2}} \right]}{\sigma \beta H^2} (R (w_t + b_t) - \delta_x x_t - R \delta_c c_{t-1}) > 0.
\]

where \( H \) is defined as in (3.15). Therefore, even if the bequest motive raises both the total amount of bequests \( nb_{t+1} \) and the amount \( s_t \) saved, the effect on \( nb_{t+1} \) is a direct effect that dominates the side effect on saving. Note, however, that the sign of the previous derivative relies crucially on the existence of a positive net return on saving, \( R - 1 > 0 \), as we have assumed. If the net return on saving were negative, larger bequests could only be achieved by increasing saving by a larger amount than the aimed increase in total bequests. Therefore, if \( R < 1 \) then old accumulation will decrease with the intensity of warm-glow altruism.

The effects of aspirations on adult consumption on the accumulation of wealth by a generic old individual is summarized by the following derivative:

\[
\frac{\partial (nb_{t+1} - s_t)}{\partial \delta_c} = \frac{(R\beta)^{\frac{1}{2}} \left( 1 + (1 - R) n \left( \frac{\rho}{n^\beta} \right)^{\frac{1}{2}} \right)}{H} c_{t-1} \geq 0.
\]

Note that the intensity \( \delta_c \) of aspirations on adult consumption decrease both the amount of bequests and the amount of saving. The net effect is ambiguous in general. If the net return on saving were negative \( R \leq 1 \) then effect of \( \delta_c \) on adult accumulation will be clearly negative. However, if the natural condition \( R > 1 \) holds, then we have the following:

\[
\frac{\partial (nb_{t+1} - s_t)}{\partial \delta_c} \geq 0 \quad \text{if and only if} \quad \frac{\beta}{\rho} \geq \frac{(R - 1)^{\sigma}}{n^{1-\sigma}}.
\]

In particular, if we consider the ratio \( \beta/\rho \) as a measure of the importance of old consumption relative to bequests in individual preferences, we see that the higher (lower) the value of this ratio, the higher (lower) will be the impact on old accumulation. If old consumption is very important relative to bequests then the amount of saving will be much larger than the amount of bequest. Therefore, the negative impact of \( \delta_c \) on saving will be of larger size than the negative impact on the small amount bequests. This results in a positive net effect on the accumulation of wealth \( nb_{t+1} - s_t \) by an old individual. The converse result will hold when the ratio \( \beta/\rho \) is small, i.e., when bequest
motives are more important than the appetite for consumption when old. In particular, note that
\[
\frac{\partial (nb_{t+1} - s_t)}{\partial \delta_c} \bigg|_{\rho=0} > 0,
\]
so that old accumulation increases with the intensity of old aspiration when the bequest motive is absent. Obviously, in this case \( b_{t+1} = 0 \) and saving decreases with the intensity \( \delta_c \) of adult aspirations.

Finally, the intensity \( \delta_x \) of old aspirations decrease the level of wealth accumulation of an old individual. This is a direct consequence of the induced increase in the amount of saving and the decrease in the amount of bequests given in (3.7) and (3.10). Obviously, as the level of old aspirations rises, old individuals want to consume more and this results in a lower value of \( nb_{t-1} - s_t \).

Table 1 summarizes all the effects of warm-glow altruism and aspirations on the accumulation of capital along the life cycle.

[Insert Table 1]

6. Bequest motive under aspirations

In this section we consider the introduction of a bequest motive when aspirations are present. We have already seen in the previous section that warm-glow altruism tends to increase the accumulation (or to decrease the disaccumulation) of wealth for both adult and old agents. According to the empirical work of Hurd (1987), there is no significative difference between the individuals with living children (who are assumed to display some bequest motive) and individuals with no living children (who cannot have any desire to leave bequest). We will see that the presence of aspirations weaken the effect of altruism so that individuals exhibiting aspirations are less sensitive to the presence of altruism. Therefore the presence of aspirations make the pattern of accumulation of bequest motivated individuals and non-bequest-motivated individuals more similar. Thus, aspirations may explain the result of Hurd concerning the insensitivity of wealth accumulation with respect to warm-glow altruism.

In order to obtain how aspirations contribute to the effect of the introduction of altruism we need to compute the second cross derivative of the endogenous variable under consideration with respect to the intensity \( \rho \) of bequest motive and the intensity of aspirations parameterized by the value of either \( \delta_c \) or \( \delta_x \). Note that we take as given the values of initial wealth \( b_t \) and aspirations \( c_{t-1} \) and \( x_t \) of a generic member the generation \( t \). Let us consider first the effects of adult aspirations, which are summarized by the following cross derivatives:

\[
\frac{\partial s_t}{\partial \rho \partial \delta_c} \bigg|_{\delta_c=0,\delta_x=0} = - \frac{R(R\beta)^{\frac{1}{2}} \left( \frac{\mu}{\sigma} \right)^{\frac{1}{2}(1-\sigma)}}{\sigma \beta H^2} c_{t-1} < 0,
\]

\[
\frac{\partial b_{t+1}}{\partial \rho \partial \delta_c} \bigg|_{\delta_c=0,\delta_x=0} = - \frac{R(R\beta)^{\frac{1}{2}} \left( R + (R\beta)^{\frac{1}{2}} \right) \left( \frac{\mu}{\sigma} \right)^{\frac{1}{2}(1-\sigma)}}{n \sigma \beta H^2} c_{t-1} < 0,
\]
\[
\frac{\partial (s_t - b_t)}{\partial p \delta_c} \bigg|_{\delta_c=0, \delta_x=0} = -\frac{n R^2 \left( \frac{\rho R}{n} \right)^{\frac{1}{\sigma}} c_{t-1}}{\sigma \rho H^2} < 0, \quad (6.1)
\]

\[
\frac{\partial (n b_{t+1} - s_t)}{\partial p \delta_c} \bigg|_{\delta_c=0, \delta_x=0} = \frac{R (R \beta)^{\frac{1}{\sigma}} \left( \frac{\rho}{n^2} \right)^{\frac{1}{\sigma} (1-\sigma)} \left( R - 1 + (R \beta)^{\frac{1}{\sigma}} \right) c_{t-1}}{\sigma \beta H^2} < 0, \quad (6.2)
\]

and

\[
\frac{\partial (n b_{t+1} - b_t)}{\partial p \delta_c} \bigg|_{\delta_c=0, \delta_x=0} = \frac{R (R \beta)^{\frac{1}{\sigma}} \left( \frac{\rho}{n^2} \right)^{\frac{1}{\sigma} (1-\sigma)} \left( R + (R \beta)^{\frac{1}{\sigma}} \right) c_{t-1}}{\sigma \beta H^2} < 0. \quad (6.3)
\]

where \( H \) is defined as in (3.15).

Note that we have evaluated the previous derivatives at \( \delta_c = 0 \) and \( \delta_x = 0 \), that is, the previous signs hold for low values of the intensity of aspirations. For higher values, the resulting expressions for the derivatives are extremely messy and cannot be sign explicitly. We see from the previous derivatives that the positive effect of warm-glow altruism on saving and bequest is dampened under aspirations associated with adult consumption. When this type of aspirations are introduced the utility the value of adjusted consumption, \( c_t - \delta_c c_{t-1} \), becomes smaller. Note that, since the isoelastic utility function displays decreasing absolute risk aversion, the degree of concavity of the utility becomes higher ceteris paribus. In our non-stochastic environment this translates into a lower willingness to change the level of adult consumption, and this results in turn in a lower impact of the introduction of bequest motive on saving and bequest. The positive effect of bequest motive on capital accumulation become smaller when a positive value of \( \delta_c \) is introduced (see (6.1) and (6.3)). Whether the introduction of aspirations exacerbate or dampen the effect of warm-glow altruism on wealth accumulation of an old individual depends on whether this accumulation is increasing or decreasing in aspirations (see (5.1) and (6.2)).

The impact of the intensity \( \delta_x \) of old aspirations associated with old consumption on the effects of the introduction of bequest motive are also summarized in the following partial cross-derivatives evaluated at \( \delta_c = 0 \) and \( \delta_x = 0 \):

\[
\frac{\partial s_t}{\partial p \delta_x} \bigg|_{\delta_c=0, \delta_x=0} = \frac{(R \beta)^{\frac{1}{\sigma}} \left( \frac{\rho}{n^2} \right)^{\frac{1}{\sigma} (1-\sigma)} x_t}{\sigma \beta H^2} < 0,
\]

\[
\frac{\partial b_{t+1}}{\partial p \delta_x} \bigg|_{\delta_c=0, \delta_x=0} = -\frac{(R \beta)^{\frac{1}{\sigma}} \left( R + (R \beta)^{\frac{1}{\sigma}} \right) \left( \frac{\rho}{n^2} \right)^{\frac{1}{\sigma} (1-\sigma)} x_t}{n \sigma \beta H^2} < 0,
\]

\[
\frac{\partial (s_t - b_t)}{\partial p \delta_x} \bigg|_{\delta_c=0, \delta_x=0} = \frac{n \left( \frac{\rho R}{n} \right)^{\frac{1}{\sigma}} x_t}{\sigma \rho H^2} < 0,
\]

\[
\frac{\partial (n b_{t+1} - s_t)}{\partial p \delta_x} \bigg|_{\delta_c=0, \delta_x=0} = -\frac{(R \beta)^{\frac{1}{\sigma}} \left( \frac{\rho}{n^2} \right)^{\frac{1}{\sigma} (1-\sigma)} \left( R - 1 + (R \beta)^{\frac{1}{\sigma}} \right) x_t}{\sigma \beta H^2} < 0,
\]

12
\[
\frac{\partial (nb_{t+1} - b_t)}{\partial \rho \partial \delta_x} \bigg|_{\delta_c=0,\delta_x=0} = \frac{- (R\beta)^{\frac{1}{2}} \left( \frac{\rho}{n\beta} \right)^{\frac{1}{2}(1-\sigma)} (R + (R\beta)^{\frac{1}{2}} x_t)}{\sigma \beta H^2} < 0.
\]

where \( H \) is defined as in (3.15). Here, the strength of old aspirations lowers the positive impact of the introduction of warm-glow altruism on saving, bequests and wealth accumulation of adult and old individuals. Like in the previous type of aspirations, parental consumption induce some inertia in the behavior of individuals, which results in a weaker response to the introduction of bequest motive. Table 2 summarizes the sign of the previous cross partial derivatives.

[Insert Table 2]

In order to analyze the robustness of the dampening effect of both types of aspirations when bequest motivations are introduced, we perform the following numerical analysis. We choose the value of the preference parameters \( \beta = 1/2 \) and, following Iacoviello (2008), the value \( w = 2/3 \) for the wage. We assume a rate of population growth of 1% per year and we consider that each period lasts for 30 years so that \( n = (1.01)^{30} = 1.3478 \). Finally, we choose an interest rate per year of 2% so that \( R = (1.02)^{30} = 1.8114 \). We maintain these parameter values for the remaining numerical exercises. As initial values of bequest \( b_t \), adult aspirations \( c_t - 1 \), and old aspirations \( x_t \), we choose the steady-state values corresponding to a benchmark with no warm-glow altruism, \( \rho = 0 \), which implies, \( b_t = 0 \), and no aspirations, \( \delta_c = \delta_x = 0 \). Thus, the initial values for parental consumption are \( c_t - 1 = 0.44444 \) and \( x_t = 0.40253 \) for \( \sigma = 1 \), \( c_t - 1 = 0.43705 \) and \( x_t = 0.41593 \) for \( \sigma = 2 \), and \( c_t - 1 = 0.45887 \) and \( x_t = 0.37641 \) for \( \sigma = 1/2 \). Here, we keep this initial values fixed since we are going to analyze how the optimal decisions of individuals are affected by bequest motive and aspirations and to do so we should also take as given the exogenous variables of the maximization problem faced by individuals. In the next section we will perform instead the comparative statics on the steady-state, that is, we will compare the pattern of wealth accumulation of individuals in stationary equilibria associated with different values of the parameters of the model. In this latter case, the initial values of bequest and aspirations should also be adjusted as a response to the change in parameter values.

In Figures 1 and 2 depict the effect of altruism on adult accumulation of wealth, \( s_t - b_t \), for different strictly positive values of the intensities \( \delta_c \) and \( \delta_x \) of aspirations and of the parameter \( \sigma \) referred to the inverse of elasticity of the utility functions. In particular, we compare the benchmark case where \( \delta_c = \delta_x = 0 \), with values of the aspirations intensities equal to 1/2. Figure 1 considers the values \( \sigma = 1 \) and \( \sigma = 2 \) while Figure 2 refers to the value \( \sigma = 1/2 \). We see that the slopes of all the wealth accumulations as a function of the intensity of warm-glow altruism are smaller for positive values of aspiration intensities. The same conclusion arises from Figures 3 and 4 concerning the accumulation of wealth for old individuals, \( nb_{t+1} - s_t \). Finally Figures 5 and 6 refer to the accumulation of wealth along the whole life of individuals, \( nb_{t+1} - b_t \). Therefore, the dampening effect presented above of aspirations extends beyond the simple introduction of bequest motive and aspirations.

[Insert Figures 1, 2, 3, 4, 5 and 6]
7. Steady-state effects

In this section we will analyze how the introduction of aspirations affects the change in the pattern of wealth accumulation brought about by bequest motivations in the steady state. This means that we are going to compare the individual optimal decision concerning saving and bequest of a generation in the steady state, where \( c_t = c \), \( x_{t+1} = x \), \( s_t = s \), and \( b_{t-1} = b \) for all \( t \) and the optimal decision under different values of the parameters characterizing warm-glow altruism and aspirations once the dynasty has reached the new steady-state associated with these new parameter values. The steady-state values of consumption and bequest are given in (4.1), (4.2), and (4.3), while the steady state of saving is, according to (2.1), given by

\[
s = w + b \delta_c.
\]

We start with the effects on the stationary value of saving and bequests of the intensity of bequest motive and aspirations \( c \) and \( x \). The following derivatives summarize the results:

\[
\frac{\partial s}{\partial \rho} = \frac{R (R\beta)^{\frac{1}{2}} (1 - \delta_c) (1 - \delta_x) \left( \frac{\rho}{\rho \beta} \right)^{\frac{1}{2} (1 - \sigma)} (n (1 - \delta_x) + (R\beta)^{\frac{1}{2}} (1 - \delta_c)) w}{n \sigma \beta J^2} > 0,
\]

\[
\frac{\partial b}{\partial \rho} = \frac{R (R\beta)^{\frac{1}{2}} (1 - \delta_c) (1 - \delta_x) \left( \frac{\rho}{\rho \beta} \right)^{\frac{1}{2} (1 - \sigma)} (R (1 - \delta_x) + (1 - \delta_c) (R\beta)^{\frac{1}{2}}) w}{n \sigma \beta J^2} > 0,
\]

\[
\frac{\partial s}{\partial \delta_c} = \frac{R (R\beta)^{\frac{1}{2}} \left( 1 + n \left( \frac{\rho}{\rho \beta} \right)^{\frac{1}{2} (1 - \delta_x)} \right) (1 - \delta_x) w}{J^2} < 0,
\]

\[
\frac{\partial b}{\partial \delta_c} = -\frac{R^2 (1 - \delta_x)^2 \left( \frac{\rho R}{\rho} \right)^{\frac{1}{2} w}}{J^2} < 0,
\]

\[
\frac{\partial s}{\partial \delta_x} = \frac{R (R\beta)^{\frac{1}{2}} \left( 1 - (1 - \delta_c) \left( \frac{\rho}{\rho \beta} \right)^{\frac{1}{2} w} \right) w}{J^2} \geq 0, \tag{7.1}
\]

\[
\frac{\partial s}{\partial \delta_x} \bigg|_{\rho = 0} = \frac{R (R\beta)^{\frac{1}{2}} (1 - \delta_c) w}{(R (1 - \delta_x) + (1 - \delta_c) (R\beta)^{\frac{1}{2}})^2} > 0, \tag{7.2}
\]

and

\[
\frac{\partial b}{\partial \delta_x} = -\frac{R (R\beta)^{\frac{1}{2}} (1 - \delta_c)^2 \left( \frac{\rho}{\rho \beta} \right)^{\frac{1}{2} w}}{J^2} < 0.
\]

where \( J \) is defined as in (4.4). We see that the sign of the partial derivatives are the same as the ones obtained in Section 3 for the non-stationary values of bequest and saving. Note however that here in order to sign the effect of the intensity \( \delta_x \) of old aspirations on saving we need to evaluate the partial derivative (7.1) at \( \rho = 0 \) (see (7.2)) so that the sign applies to low levels of warm-glow altruism. In general, the partial derivative (7.1) has an ambiguous sign.
Concerning the effects on the accumulation of wealth by adult individuals, \( s - b \), by old individuals, \( nb - s \), and total accumulation \( nb - b = (n - 1)b \) of changes in the intensity of bequest motive, we obtain the following partial derivatives:

\[
\frac{\partial (s - b)}{\partial \rho} = - \left( \frac{\pi_s}{\sigma} \right)^{\frac{1}{\sigma}} R \beta \frac{1}{\sigma} (1 - \delta_s) (R - n) \geq 0, \quad (7.3)
\]

\[
\frac{\partial (nb - s)}{\partial \rho} = \frac{Rw(R\beta)^{\frac{1}{\sigma}} \left( \frac{\pi_s}{\sigma} \right)^{\frac{1}{\sigma}} (1 - \delta_s) (1 - \delta_s) (n(1 - \delta_s)(R - 1 + (R\beta)^{\frac{1}{\sigma}} (1 - \delta_s)(n - 1))(n(1 - \delta_s)(R - 1 + (R\beta)^{\frac{1}{\sigma}} (1 - \delta_s)(n - 1)) \right) \geq 0, \quad (7.4)
\]

\[
\frac{\partial [(n - 1)b]}{\partial \rho} = (n - 1) \frac{\partial b}{\partial \rho} \geq 0 \quad \text{if and only if} \quad n \geq 1. \quad (7.5)
\]

where \( J \) is defined as in (4.4). We see in (7.3) that adult accumulation \( s - b \) increases (decreases) with the degree of warm-glow altruism in a steady state if \( R < (>)n \). If the gross return \( R \) from saving is small then saving must increase a lot relative to bequest per children \( b \) to generate an increase in the transfer to the next generation. Similarly, if the number \( n \) of children is high, saving should also increase a lot so as to raise the amount of bequest per child. Concerning the accumulation of wealth \( nb - s \) for old individuals individuals, we see that the sign of the partial derivative (7.4) is positive whenever the gross return \( R \) and the gross rate \( n \) of population growth are greater or equal than 1, and at least one of them is strictly grater than 1. Obviously, on the one hand, the higher is the return on saving the less is the need for increasing the amount of savings for bequest reasons. On the other hand, the higher is the number of children per parent, the larger should be the total amount bequeathed \( nb \). These two effects explain the positive sign of the partial derivative (7.4) for this empirically relevant case. Finally, the sign of the effect on total accumulation along the life cycle, 

\[
(n - 1)b, \quad \text{depends on whether the net rate of population growth is positive} \quad (n > 1) \quad \text{or negative} \quad (n < 1). \quad \text{Clearly, the higher is the rate of population growth the larger the amount of wealth individuals must accumulate to endow their children with the stationary amount of inheritance.}
\]

We next show the partial derivatives that characterize the effect of the intensity \( \delta_c \) of adult aspirations on the stationary pattern of wealth accumulation:

\[
\frac{\partial (s - b)}{\partial \delta_c} = - \frac{\left( R \beta \frac{1}{\sigma} (1 - \delta_s) \right) \left[ 1 - \left( \frac{\rho}{n\beta} \right)^{\frac{1}{\sigma}} (1 - \delta_s) (R - n) \right] w}{J^2} \geq 0,
\]

\[
\left. \frac{\partial (s - b)}{\partial \delta_c} \right|_{\rho=0} = - \frac{R \beta \frac{1}{\sigma} (1 - \delta_s) w}{\left( R(1 - \delta_s) + (1 - \delta_c) (R\beta)^{\frac{1}{\sigma}} \right)^2} \leq 0, \quad (7.6)
\]

\[
\frac{\partial (nb - s)}{\partial \delta_c} = \frac{Rw(R\beta)^{\frac{1}{\sigma}} \left( \frac{\pi_s}{\sigma} \right)^{\frac{1}{\sigma}} (1 - \delta_s) (1 - \delta_s) (n(1 - \delta_s)(R - 1 + (R\beta)^{\frac{1}{\sigma}} (1 - \delta_s)(n - 1))(n(1 - \delta_s)(R - 1 + (R\beta)^{\frac{1}{\sigma}} (1 - \delta_s)(n - 1))} \geq 0,
\]

\[
\left. \frac{\partial (nb - s)}{\partial \delta_c} \right|_{\rho=0} = \frac{R \beta \frac{1}{\sigma} (1 - \delta_s) w}{\left( R(1 - \delta_s) + (1 - \delta_c) (R\beta)^{\frac{1}{\sigma}} \right)^2} > 0, \quad (7.7)
\]
where $J$ is defined as in (4.4). Finally, the effect of the intensity $\delta_x$ of old aspirations on the stationary pattern of wealth accumulation is given by the following partial derivatives:

\[
\frac{\partial (n-1)b}{\partial \delta_c} = (n-1) \frac{\partial b}{\partial \delta_c} \gtrless 0 \text{ if and only if } n \gtrless 1.
\]

where $J$ is defined as in (4.4). Note that the signs of the comparative static exercises for adult and old accumulation of wealth are the same as the ones obtained in Section 3 when the dynasty was not in its steady state. We need however to assume in some cases that the intensity of bequest motive is close to zero (see (7.6), (7.7), and (7.8)).

Concerning the total accumulation $(n-1)b$ along the lifetime, the sign depends in an obvious way on the rate of population growth. Table 3 summarizes the signs of the comparative statics exercises on the stationary equilibrium.

To conclude our analysis we can evaluate whether the aspirations dampen or exacerbate the effect of warm-glow altruism when we make the comparison between steady states. To do so we need to compute the partial cross derivative of the steady state values $s, b, s-b, nb-s$, and $(n-1)b$ with respect to the bequest motive intensity $\rho$ and the aspirations intensities $\delta_c$ and $\delta_x$ when adult and old, respectively. The exact expressions of these cross derivatives are extremely messy and are available from the authors under request but their signs are shown in Table 4 under some parametric restrictions.

To assess the effect of aspirations when they are introduced non-marginally, we have the numerical exercises contained in Figures 7 to 12. These tables are the counterparts of Figures 1 to 6 but evaluated at the steady state values, i.e., we do not take the initial inheritance and aspirations as fixed since we use instead the stationary values $b$ of inheritance and $c$ and $x$ of aspirations, which vary endogenously with the parameter values $\rho, \delta_c$ and $\delta_x$. The main conclusion of these figures is that the presence of either adult or old aspirations dampens the effect of warm-glow altruism on the pattern of wealth accumulation. We see that the slopes of the functions displayed exhibit a lower slope in absolute value. In particular, observe in Figures 7 and 8 how the stationary adult accumulation of wealth $s-b$ is decreasing in the intensity of bequest motive $\rho$. 

\[
\frac{\partial [(n-1)b]}{\partial \delta_c} = (n-1) \frac{\partial b}{\partial \delta_c} \gtrless 0 \text{ if and only if } n \gtrless 1.
\]
However, this negative effect becomes less strong when aspirations are present. This numerical analysis allows to extend the conclusion at which we arrived when we considered the individual decision with exogenous initial values of bequest and aspirations to the stationary patterns of capital accumulation. In all these new graphs we have use the same parameter values as in the previous figures.

[Insert Figures 7, 8, 9, 10, 11 and 12]

8. Conclusion

We have developed a simple OLG model that enable us to study the effect of the introduction of warm-glow altruism and aspirations on the individual pattern of wealth accumulation of individuals. Our results show that the introduction of aspirations at different ages display different effects on the amount of saving for workers. However, both adult and old aspirations make the introduction of the bequest motive less effective in changing the pattern of accumulation. Therefore, under aspirations both bequest motivates and non-bequest motivated individuals will behave more similarly than when aspirational concerns are absent.

We have shown that the higher is the intensity of aspirations the faster a dynasty will converge to its steady state as intergenerational transfers are then allowed to adjust faster. Finally, the dampening effect of aspirations survive when we compare stationary allocations.
References

A. Appendix

Proof of Lemma 4.1. We can rewrite the system formed by the difference equations (3.11), (3.12), and (3.14) in matrix form as

$$\begin{bmatrix} c_{t+1} \\ x_{t+1} \\ b_{t+1} \end{bmatrix} = P \times \begin{bmatrix} c_{t-1} \\ x_{t} \\ b_{t} \end{bmatrix} + \frac{1}{H} \begin{bmatrix} R \\ R(R\beta)^{\frac{1}{\sigma}} \end{bmatrix} w,$$

where the coefficient matrix $P$ is given by

$$P = \frac{1}{H} \begin{bmatrix} R & [(\beta R)^{\frac{1}{\sigma}} + n \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}] \delta_c & -\delta_x \\ R(\beta R)^{\frac{1}{\sigma}} & -R(\beta R)^{\frac{1}{\sigma}} \delta_c & \left[R + n \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}\right] \delta_x \\ R\left(\frac{pR}{n}\right)^{\frac{1}{\sigma}} & -R\left(\frac{pR}{n}\right)^{\frac{1}{\sigma}} \delta_c & -\left(\frac{pR}{n}\right)^{\frac{1}{\sigma}} \delta_x \end{bmatrix},$$

where $H$ is given in (3.15). The characteristic polynomial of the matrix $P$ is

$$P(\lambda) = \lambda^3 - \mu_1 \lambda^2 + \mu_2 \lambda - \mu_3,$$

where

$$\mu_1 = \frac{R \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}} + \left(\beta R\right)^{\frac{1}{\sigma}} + n \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}} \delta_c + \left[R + n \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}\right] \delta_x}{H},$$

$$\mu_2 = \frac{\left(\frac{pR}{n}\right)^{\frac{1}{\sigma}} \delta_c \delta_x}{H} + \frac{R \delta_c + R \delta_x + n \delta_c \delta_x}{H},$$

$$\mu_3 = \frac{R \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}} \delta_c \delta_x}{H}.$$

Moreover, if $\delta_c = 0$ and $\delta_x = 0$ the characteristic polynomial becomes

$$P(\lambda) = \lambda^3 - \left(\frac{R \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}}{H}\right) \lambda^2 = \lambda^2 \left(\lambda - \left(\frac{R \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}}{H}\right)\right),$$

so that the eigenvalues become

$$\lambda_1 = \frac{R \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}}{R + (\beta R)^{\frac{1}{\sigma}} + n \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}} \in (0, 1),$$

where

$$1 = \frac{R(\beta R)^{\frac{1}{\sigma}}}{R + \left(\beta R\right)^{\frac{1}{\sigma}} + n \left(\frac{pR}{n}\right)^{\frac{1}{\sigma}}} \in (0, 1).$$
\( \lambda_2 = 0 \) and \( \lambda_3 = 0 \). Finally, since the eigenvalues are continuous functions of the parameters \( \delta_c \) and \( \delta_x \), the three eigenvalues will lie in the interior of the unit circle for a sufficiently small value of parameters measuring the intensity of aspirations. \textit{Q.E.D.}

**Proof of Lemma 4.2.** Since we are only interested in the effect of the introduction of aspirations, we will consider the marginal introduction of adult aspirations and old aspirations separately. Let us first consider the introduction of adult aspirations relative from a starting situation with no aspirations. To this end, let us take the characteristic polynomial in (A.2) and make \( x = 0 \) to get

\[
P(\lambda) = \lambda^3 - \frac{R \left( \frac{\rho R}{n} \right)^{\frac{1}{2}}}{H} + \left[ (\beta R)^{\frac{1}{2}} + n \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \right] \delta_c \lambda^2 + \frac{\left( \frac{\rho R}{n} \right)^{\frac{1}{2}} R \delta_c}{H} \lambda.
\]

Here one of the eigenvalues equals zero since the parental old consumption \( x_t \) is not a state variable for an individual of generation \( t \) and, hence the value of the initial condition \( x_0 \) is irrelevant. The other two eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are equal to the conjugate pair

\[
\delta_c (R\beta)^{\frac{1}{2}} + (n\delta_c + R) \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \pm \left[ \left( (R\beta)^{\frac{1}{2}} \left( (R - n\delta_c) \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} - \delta_c \right) \right)^2 - 4R^2 \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \delta_c \right]^{\frac{1}{2}}.
\]

If we take the largest of this two eigenvalues, \( \lambda_1 \) say, and perform the derivative with respect to the aspiration intensity \( \delta_c \) and then we evaluate the derivative when \( \delta_c = 0 \), we obtain

\[
\left. \frac{d\lambda_1}{d\delta_c} \right|_{\delta_c=0,\delta_x=0} = -\frac{R}{R + (\beta R)^{\frac{1}{2}} + n \left( \frac{\rho R}{n} \right)^{\frac{1}{2}}} < 0. \quad (A.3)
\]

Similarly, we can replicate the argument for the introduction of aspirations on old consumption. The characteristic polynomial in (A.2) with \( \delta_c = 0 \) becomes now

\[
P(\lambda) = \lambda^3 - \frac{R \left( \frac{\rho R}{n} \right)^{\frac{1}{2}}}{H} + \left( R + n \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \right) \delta_x \lambda^2 + \frac{R \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \delta_x}{H} \lambda.
\]

In this case, one of the eigenvalues is again equal to zero since the parental adult consumption \( c_t \) is not a state variable for an individual of generation \( t \) and, hence, the value of the initial condition \( c_{-1} \) does not affect his decision. The other two eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are equal to the conjugate pair

\[
\left( R\delta_x + (R + n\delta_x) \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \right) \pm \left( R\delta_x - (R - n\delta_x) \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \right)^2 - 4R \left( (R\beta)^{\frac{1}{2}} \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \delta_x \right) \right]^{\frac{1}{2}}
\]

\[
2 \left[ R + (\beta R)^{\frac{1}{2}} + n \left( \frac{\rho R}{n} \right)^{\frac{1}{2}} \delta_x \right]^{\frac{1}{2}}.
\]
If we take the largest of this two eigenvalues, $\lambda_1$ say, and perform the derivative with respect to the aspiration intensity $\delta_x$ and then we evaluate the derivative when $\delta_x = 0$, we obtain

$$\left. \frac{d\lambda_1}{d\delta_x} \right|_{\delta_c=0,\delta_x=0} = -\frac{(R\beta)^{\frac{1}{\sigma}}}{R + (\beta R)^{\frac{1}{\sigma}} + n \left( \frac{\rho R}{n} \right)^{\frac{1}{\sigma}}} < 0. \quad (A.4)$$

Therefore, we can conclude that the introduction of aspirations either on adult or in old consumption increases the speed of convergence around the steady state. Q.E.D.
Table 1. Comparative statics of saving, bequest and wealth accumulation.

<table>
<thead>
<tr>
<th></th>
<th>( s_t )</th>
<th>( b_{t+1} )</th>
<th>( s_t - b_t )</th>
<th>( n b_{t+1} - s_t )</th>
<th>( n b_{t+1} - b_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial}{\partial \rho} )</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial \delta c} )</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>( \geq 0 ) (if ( \frac{\beta}{\rho} \geq \frac{(R-1)^{\sigma}}{n^{1-\sigma}} ))</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial \delta \xi} )</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>

**Table 2.** The dampening effect of aspirations on bequest motive

<table>
<thead>
<tr>
<th></th>
<th>( s_t )</th>
<th>( b_{t+1} )</th>
<th>( s_t - b_t )</th>
<th>( n b_{t+1} - s_t )</th>
<th>( n b_{t+1} - b_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial}{\partial \rho} \bigg</td>
<td>_{\delta c=0, \delta \xi=0} )</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial \rho} \bigg</td>
<td>_{\delta c=0, \delta \xi=0} )</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>

23
\[
\begin{array}{|c|c|c|c|c|}
\hline
 & s & b & s - b & nb - s & (n - 1)b \\
\hline
\frac{\partial}{\partial p} & > 0 & > 0 & \geq 0 \ (\text{if } R \leq n) & > 0 \ (\text{if } n > 1) & \geq 0 \ (\text{if } n \geq 1) \\
\frac{\partial}{\partial \delta_c} & < 0 & < 0 & < 0 \ (\text{if } \rho = 0) & > 0 \ (\text{if } \rho = 0) & \geq 0 \ (\text{if } n \leq 1) \\
\frac{\partial}{\partial x} & > 0 & < 0 & > 0 & < 0 \ (\text{if } \rho = 0) & \geq 0 \ (\text{if } n \leq 1) \\
\hline
\end{array}
\]

**Table 3.** Comparative statics of stationary saving, bequest and wealth accumulation.

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & s_t & b_{t+1} & s_t - b_t & nb_{t+1} - s_t & nb_{t+1} - b_t \\
\hline
\frac{\partial}{\partial p} \bigg|_{\delta_c=0, \delta_x=0} & < 0 \ (\text{if } R \leq n/2) & < 0 & \geq 0 \ (\text{if } R \geq n) & < 0 & \geq 0 \ (\text{if } n \leq 1) \\
\frac{\partial}{\partial p} \bigg|_{\delta_c=0, \delta_x=0} & \geq 0 & < 0 & \geq 0 \ (\text{if } R \geq n) & < 0 & \geq 0 \ (\text{if } n \leq 1) \\
\hline
\end{array}
\]

**Table 4.** The dampening effect of aspirations on bequest motive.
Figure 1. The effects of aspirations on adult accumulation of wealth \((s_t - b_t)\).

Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1\).

Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\). Thick dash line: \(\delta_c = 0, \delta_x = 0, \sigma = 2\).

Medium dash line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 2\). Thin dash line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\).

Figure 2. The effects of aspirations on adult accumulation of wealth \((s_t - b_t)\).

Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1/2\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1/2\).

Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1/2\).
Figure 3. The effects of aspirations on old accumulation of wealth \((nbt_{t+1} - st)\).
Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1\).
Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\). Thick dash line: \(\delta_c = 0, \delta_x = 0, \sigma = 2\).
Medium dash line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 2\). Thin dash line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\).

Figure 4. The effects of aspirations on old accumulation of wealth \((nbt_{t+1} - st)\).
Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1/2\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1/2\).
Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1/2\).
Figure 5. The effects of aspirations on lifetime wealth accumulation \((n_{bt+1} - b_t)\).
Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1\).
Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\). Thick dash line: \(\delta_c = 0, \delta_x = 0, \sigma = 2\).
Medium dash line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 2\). Thin dash line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\).

Figure 6. The effects of aspirations on lifetime wealth accumulation \((n_{bt+1} - b_t)\).
Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1/2\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1/2\).
Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1/2\).
Figure 7. The effects of aspirations on stationary adult accumulation of wealth \((s - b)\).
Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1\).
Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\). Thick dash line: \(\delta_c = 0, \delta_x = 0, \sigma = 2\).
Medium dash line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 2\). Thin dash line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1\).

Figure 8. The effects of aspirations on stationary adult accumulation of wealth \((s - b)\).
Thick solid line: \(\delta_c = 0, \delta_x = 0, \sigma = 1/2\). Medium solid line: \(\delta_c = 1/2, \delta_x = 0, \sigma = 1/2\).
Thin solid line: \(\delta_c = 0, \delta_x = 1/2, \sigma = 1/2\).
Figure 9. The effects of aspirations on stationary adult accumulation of wealth \((nb - s)\).
Thick solid line: \(\delta_c= 0, \delta_x = 0, \sigma = 1\). Medium solid line: \(\delta_c=1/2, \delta_x = 0, \sigma = 1\).
Thin solid line: \(\delta_c=0, \delta_x = 1/2, \sigma = 1\). Thick dash line: \(\delta_c= 0, \delta_x = 0, \sigma = 2\).
Medium dash line: \(\delta_c= 1/2, \delta_x = 0, \sigma = 2\). Thin dash line: \(\delta_c=0, \delta_x = 1/2, \sigma = 1\).

Figure 10. The effects of aspirations on stationary adult accumulation of wealth \((nb - s)\).
Thick solid line: \(\delta_c= 0, \delta_x = 0, \sigma =1/2\). Medium solid line: \(\delta_c=1/2, \delta_x = 0, \sigma =1/2\).
Thin solid line: \(\delta_c=0, \delta_x = 1/2, \sigma =1/2\).
Figure 11. The effects of aspirations on stationary lifetime accumulation of wealth \((n-1)b\)
Thick solid line: \(\delta_c=0, \delta_x = 0, \sigma = 1\). Medium solid line: \(\delta_c=1/2, \delta_x = 0, \sigma = 1\).
Thin solid line: \(\delta_c=0, \delta_x = 1/2, \sigma = 1\). Thick dash line: \(\delta_c=0, \delta_x = 0, \sigma = 2\).
Medium dash line: \(\delta_c=1/2, \delta_x = 0, \sigma = 2\). Thin dash line: \(\delta_c=0, \delta_x = 1/2, \sigma = 1\).

Figure 12. The effects of aspirations on stationary lifetime accumulation of wealth \((n-1)b\)
Thick solid line: \(\delta_c=0, \delta_x = 0, \sigma =1/2\). Medium solid line: \(\delta_c=1/2, \delta_x = 0, \sigma =1/2\).
Thin solid line: \(\delta_c=0, \delta_x = 1/2, \sigma =1/2\).