A new index combining the absolute and relative aspects of income poverty: theory and application

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Abstract
I derive a new index combining the absolute and relative aspects of income poverty. Increasing the income of a poor individual reduces her absolute poverty but increasing the inequality she experiences worsens her relative poverty. Provided that individual poverty is not computed based on normalized income, the two aspects can be weighted such that absolutely poor individuals are always considered poorer than relatively poor individuals. Only the value of poverty aversion associated with the Poverty Gap Ratio is consistent with this approach. I identify which set of properties defines the new index. An application illustrates that the new index yields intuitive judgments about unequal growth experiences, for which all absolute (resp. relative) poverty indices systematically conclude that poverty has decreased (resp. increased).

Keywords: Income Poverty, Relative Poverty, Absolute Poverty, Income Inequality, Poverty Lines, Decomposable Index.

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1 Introduction

Income poverty reduction is a major political objective, both at national and international levels. Policy makers such as the EU Commission or the World Bank evaluate development programs on the basis of their impacts on poverty. To do so, they use income poverty measures, which are composed of two elements: a poverty line and an index (Sen, 1976). A poverty line specifies the income threshold below which individuals are considered to be poor. An index aggregates the poverty of all individuals in a society and, hence, allows us to compare poverty in different societies.

There exist two central approaches for measuring income poverty, absolute poverty and relative poverty. The difference between the two approaches lies in the type of poverty line used. An absolute line has its income threshold independent of the standard of living whereas a relative line has its income threshold evolves as a constant fraction of the standard of living. These two types of lines aim at capturing different deprivations. On the one hand, absolute poverty refers to the idea of subsistence. An individual is absolutely poor if her income is not sufficient to satisfy several of her basic needs, such as reaching a minimal level of nutrition. In a first approximation, the real cost of subsistence is absolute as it does not depend on standards of living. For example, 100 grams of rice contain the same amount of calories in New-York or in New-Delhi. On the other hand, relative poverty refers to the ideas of social participation or inclusion. An individual is relatively poor if her income is not sufficient to engage in the everyday life of her society (Townsend, 1979; Sen, 1983). The real cost of not being excluded from social participation is relative as it depends on standards of living. The archetypical example is that of the linen shirt (Smith, 1776). Adam Smith observed that in the England of his time, people would be too ashamed to appear in public without wearing a linen shirt, which, he argued, was not the case in the Roman Empire, which had a lower standard of living.\footnote{The normative foundations for taking a relativist approach in poverty measurement are reviewed in Ravallion (2008).}

Policy makers typically aim at reducing both absolute and relative poverties. These two objectives appear for example in the poverty reduction target of the EU Commission or in the new twin goals of the World Bank.\footnote{In its EU2020 strategy, the EU Commission targets to reduce by 20 millions the number of individuals that are at risk of poverty or social exclusion (AROPE). The AROPE individuals are inter alia those individuals that are at risk of poverty (relative poverty) or are severely materially deprived (absolute poverty) (see European Commission (2015)). In 2013, the World Bank committed itself to twin goals: eliminating extreme poverty (absolute poverty) and boosting shared prosperity (relative poverty). The second objective has a clear relative flavor since it is defined as raising the living standards of the bottom 40% of individuals in any given country (see World Bank (2015)).} One serious difficulty is that the two approaches make opposite extreme judgments on unequal growth. A country experiences unequal growth if its economic growth goes along with an increase in income inequality. That is, all individuals get more resources but the additional resources go disproportionately more to the middle class and the rich than to the poor. From the perspective of a poor individual, the additional income reduces her absolute poverty but the larger inequality she experiences worsens her relative poverty. On the one hand, absolute measures evaluate growth positively, regardless of its distributional aspects. On the other
hand, relative measures judge a reduction in the inequality experienced by the poor positively, regardless of the level of income earned by the poor.

In order to select among competing development programs, some of which create unequal growth, policy makers need to trade-off absolute and relative poverty. Clearly, neither absolute measures nor relative measures are able to make this trade-off. Measuring both forms of poverty in parallel does not solve the issue since, more often than not, the two approaches yield opposite conclusions.

This paper proposes a new way to measure poverty that combines the absolute and relative aspects of income poverty. Previous attempts to develop such a measure followed two different routes. One route measures both forms of poverty in parallel before looking for a way to aggregate them (Atkinson and Bourguignon, 2001; Anderson and Esposito, 2013). Unfortunately, this approach is confronted to several difficulties, including double counting issues. The other route aims at developing a single measure based on a poverty line making the trade-off between the absolute and relative aspects of income. So far, this second route has mostly focused on defining new poverty lines. The most influential proposals of such endogenous lines are the hybrid lines (Foster, 1998) and the weakly relative lines (Ravallion and Chen, 2011). Surprisingly, indices to use in combination with an endogenous line have not been rigorously studied. In empirical applications (Chen and Ravallion, 2013), the default practice is to use an endogenous line in combination with an index derived for absolute lines, such as the very popular Foster-Greer-Thorbecke (FGT) indices (Foster et al., 1984). As shown in this paper, there are two limitations associated with this practice. First, indices derived for absolute lines lose their desirable properties when combined with endogenous lines. Second, the endogenous measures obtained by this practice weigh the absolute and relative aspects of income poverty in a questionable way. They may consider that absolutely poor individuals are less poor than relatively poor individuals.

Why do measures combining an endogenous line with an index designed for an absolute line yield this debatable conclusion? Standard indices, such as the FGT indices, are additive. For a given poverty line, additive indices compute the average of the values of individual poverty that they attribute to each individual in the distribution. The individual poverty attributed by FGT indices to an individual depends only on her normalized income, i.e. her income divided by the income threshold in her society. For the weakly relative line used by Chen and Ravallion (2013), in 2010, an individual living on 1 $ a day in Ivory Coast has the same normalized income as an individual living on 3.6 $ a day in Brazil. As a result, FGT indices attribute to both the same individual poverty. This conclusion ignores that, unlike the latter, the individual in Ivory Coast is below the World Bank’s threshold for extreme poverty: 1.25 $ a day (Ravallion et al., 2009). Being above or below the threshold for extreme poverty is not reflected in normalized incomes. Hence, an extremely poor individual in Ivory Coast can be deemed less poor than a non-extremely poor individual in Brazil. The problem is so serious that these endogenous measures may conclude that there is more poverty in middle- and high-income countries than in low-income countries.

Footnotes:
3A common practice is to use absolute measures in low- and middle-income countries and relative measures in high-income countries. Official national poverty definitions mostly follow this practice (Ravallion, 2012), which leads to extreme judgments as explained above.
4See Zheng (1997) for a survey.
countries. As shown in the empirical illustration, such measures deem Brazil equally or more poor than Ivory Coast in 2010. Even if income inequality was larger in Brazil than in Ivory Coast, such judgment could be seriously questioned given that mean income in Brazil was more than four times larger than that of Ivory Coast. Moreover, 22.7% of individuals in Ivory Coast lived on less than 1.25 $ a day but only 5.4% in Brazil.

This paper proposes a new index combining the absolute and relative aspects of income poverty. In order to avoid the problem faced by standard indices, I depart from individual poverty comparisons based only on normalized incomes. To begin with, I define an absolute poverty threshold, which in the application is fixed at 1.25 $ a day. Below this subsistence threshold, an individual is deemed absolutely poor and her individual poverty does not depend on the standard of living in her society. For instance, two individuals living with 1.25 $ a day in Ivory Coast and Brazil contribute identically to poverty in their respective countries. Then, I define the endogenous poverty line above the absolute threshold. An individual above the absolute threshold but below the endogenous line is deemed relatively poor. Her individual poverty depends on the standard of living in her society. In the application, an individual living on 2 $ a day in Ivory Coast, where the mean is 3 $ a day, contributes identically to poverty as an individual living on 6.8 $ a day in Brazil, where the mean is 13.8 $ a day.

More generally, I formalize the comparison of individual poverties across societies having different standards of living by defining the concept of equivalence ordering. In a nutshell, two individuals that are attributed equal individual poverties are on the same equivalence curve. I constrain equivalence curves below the absolute threshold to be independent of standards of living. In contrast, the equivalence curves above the absolute threshold may evolve with standards of living. The constraints on equivalence curves imply that absolutely poor agents are always considered poorer than relatively poor agents. This judgment is in line with largely shared intuitions, as appeared from questionnaire studies run all over the world by Corazzini et al. (2011).

This paper has two main theoretical results. First, I characterize a family of additive indices based on mean-sensitive endogenous poverty lines. In other words, I identify the set of properties defining a family of indices based on poverty lines sensitive to mean income. This is the first characterization of indices based on non-absolute lines. This result extends the characterization of additive indices of Foster and Shorrocks (1991) to non-absolute lines. Then, I investigate which members of this additive family satisfy compelling properties. To do so, I define an extended family of FGT indices based on equivalence orderings meeting the constraints mentioned above. This family depends on two parameters, one of which is the poverty aversion parameter. The second result shows that a unique member of this extended FGT family satisfies two basic properties. One property is classical and requires that a progressive transfer between two poor individuals does not increase poverty. The other property is new and specific to indices based on endogenous lines. It requires that destroying part of the income of a poor individual does not reduce poverty. This property excludes all values of poverty aversion except the one associated to the Poverty Gap Ratio.

The index characterized is new and inherits the properties of its underlying equivalence ordering. That is, absolutely poor individuals are distinguished from
relatively poor individuals and the former are always considered poorer than the latter. Being additive, the new index is decomposable between the respective contributions of absolutely and relatively poor individuals. This last feature simplifies greatly the analysis of the evolution of poverty and its communication.

Finally, a poverty measure based on the new index is applied to World Bank data. The application illustrates that the judgments obtained from the new measure are more in line with general intuition than those obtained with standard measures. For instance, the new measure deems Brazil less poor than Ivory Coast. In a second step, the new measure is used to assess the evolution of poverty in several countries that experienced unequal growth. For example, income poverty in urban China was reduced by about 75% over the period 1990–2010. By decomposing the measure, one can see that this improvement almost entirely rests on the drastic reduction in absolute poverty. Absolute poverty accounted for about two-third of income poverty in 1990, but less than 10% in 2010. This shows that if the main issue in urban China was absolute poverty in 1990, it has become relative poverty in 2010. Studying different countries shows that the measure may yield different judgments on unequal growth. Over the period 1990–2010, income poverty did not change in Mexico as the reduction in absolute poverty was cancelled out by the increase in relative poverty. Conversely, unequal growth has lead to an increase in poverty in Hungary over the period 1996–2010, where the impact on relative poverty was dominant. In general, whether unequal growth reduces the poverty measure or not strongly depends on the initial importance of absolute poverty.

The paper is organized as follows. A literature review is presented in Section 2. The framework and the characterization of the additive family are presented in Section 3. The index proposed is derived and discussed in Section 4. The robustness of the results is investigated in Section 5. Other income standards than the mean are discussed in Section 6. The empirical illustration is presented in Section 7. I conclude in Section 8. All proofs are relegated in the Appendix.

2 Literature review

I review in this section the literature on income poverty measurement. More specifically, I present the poverty measures that are popular in empirical applications and I emphasize their limitations when comparing societies with different standards of living.

The objective of the literature on income poverty measurement is to rank income distributions with respect to the poverty they contain. I divide all existing measures between those based on absolute lines and those based on endogenous lines. Initially, most contributions were concerned with indices based on absolute lines. This early literature on absolute measures is nicely reviewed in Zheng (1997).

2.1 Absolute measures

Absolute measures are measures based on absolute lines. A poverty line is absolute if its income threshold does not evolve with standards of living. I make two remarks on absolute lines in order to avoid any confusion. First, the threshold of an absolute line is constant in real terms. To be sure, all incomes in
this paper are expressed in real terms. The threshold of an absolute line is often
defined by the cost of a particular bundle of goods. The line is then “anchored”
in that bundle. This does not prevent the \textit{nominal} threshold of the absolute
line to evolve over time with inflation or to vary from one country to another
as a function of purchasing power. Second, the bundle of goods “anchoring” an
absolute line can potentially capture \textit{both} subsistence and social participation
for a given society at a given time. Absolute measures can therefore account
for both functionings, but only when comparing two societies having the \textit{same}
standards of living.

I present here the notation necessary for exposing the relevant results in the
literature on absolute measures. Let an income distribution \( y := (y_1, \ldots, y_n) \)
be a list of non-negative incomes sorted in non-decreasing order \( y_1 \leq \cdots \leq y_n \).
Absolute poverty lines are defined by a constant threshold \( z^* \in \mathbb{R}_{++} \).
Agent \( i \) qualifies as poor if \( y_i < z^* \). The objective is to rank all distributions in a set \( Y \).

Let a poverty index be a real-valued function \( P : Y \times \mathbb{R}_{++} \to \mathbb{R} \)
representing the complete ranking on \( Y \). For any two \( y, y' \in Y \), there is strictly more poverty
in \( y \) than in \( y' \) if \( P(y, z^*) > P(y', z^*) \), and weakly more if \( P(y, z^*) \geq P(y', z^*) \).
The number of poor agents is denoted \( q(y) \) or simply \( q \) when no confusion is
possible.

A central result is the characterization of additive indices. Given an absolute
line, any index satisfying five basic properties must be ordinarily equivalent to
an additive index (Foster and Shorrocks, 1991):

\[
P(y, z^*) := \frac{1}{n} \sum_{i=1}^{n} d(y_i),
\]

where function \( d : \mathbb{R}_{+} \to [0, 1] \) is non-increasing in \( y_i \) and returns zero for all
incomes above \( z^* \). The value returned by the function \( d \) can be interpreted as
the individual poverty associated to earning income \( y_i \). This individual poverty
only depends on own income (and \( z^* \)). An additive index can be interpreted as
the average individual poverty in the distribution.

This family is very broad as very few restrictions are imposed on the function \( d \). The Foster-Greer-Thorbecke (FGT) subfamily proposes an exponential
expression for the function \( d \) (Foster et al., 1984):

\[
P_{\text{FGT}}(y, z^*) := \frac{1}{n} \sum_{i=1}^{n} \left( \frac{z^* - y_i}{z^*} \right)^{\alpha}.
\]

The FGT family has a unique parameter \( \alpha \in [0, \infty) \), which can be interpreted
as poverty aversion. The larger \( \alpha \), the higher is the priority given by the index
to agents at the bottom of the income distribution. This family allows for a
very wide variety of judgments and admits the Head-Count Ratio (HC) and the
Poverty Gap Ratio (PGR) as particular cases:

\[
\text{HC}(y, z^*) := \frac{q}{n} \quad \text{corresponds to } \alpha = 0,
\]

\[
\text{PGR}(y, z^*) := \frac{1}{n} \sum_{i=1}^{n} \left( \frac{z^* - y_i}{z^*} \right) \quad \text{corresponds to } \alpha = 1.
\]

\( \mathbb{R}_{+} \) denotes the set of non-negative reals and \( \mathbb{R}_{++} \) the set of strictly positive reals.

\( 6 \) Although it is not explicit in its notation, \( q \) depends on \( z^* \).
Virtually all empirical applications use a poverty measure based on an index in the FGT family. Many other absolute indices have been proposed, including those of Kakwani (1980), Chakravarty (1983) or Duclos and Gregoire (2002).

**Limitation of absolute measures**

Absolute measures are not well-suited for evaluating the impact on poverty of unequal growth. Growth increases the standard of living, which in turn raises the cost of social participation. The shortcoming of absolute measures is that they completely ignore these social participation effects. In a nutshell, they ignore the relative aspect of income poverty.

Table 1 presents an example illustrating the problem. This example compares the income distributions of two societies A and B, each populated by two agents, one poor and one non-poor. To fix ideas, assume incomes are given in $ a day. I assume that the subsistence threshold, denoted by $z^a$, is $1.25 a day, which corresponds to the threshold for extreme income poverty defined by the World Bank. This subsistence threshold is smaller than the income threshold $z^*$ defining income poverty in this example. Distribution B dominates distribution A but all extra resources go to the non-poor individual, except for an epsilon. Any absolute measure concludes there is less poverty in society B than in A. This conclusion is debatable for small epsilon. If the poor individual has more income in B than in A, she is worse off in A than in B in the two-dimensional space relevant for poverty evaluation: subsistence and social participation. Indeed, the poor individual is above the subsistence threshold in both societies but has more difficulties to participate in society B than in society A.

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$z^a$</th>
<th>$z^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Society A</td>
<td>3</td>
<td>15</td>
<td>1.25</td>
<td>5</td>
</tr>
<tr>
<td>Society B</td>
<td>$3 + \epsilon 100$</td>
<td>1.25</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

The problem illustrated in Table 1 results from the axiom of **Focus**. This axiom is satisfied by all indices derived for absolute lines. Focus encapsulates the key property distinguishing poverty indices from inequality indices, namely

7This assumption is in line with evidence provided by national poverty thresholds. In purchasing power parity, national income thresholds tend to increase with standards of living (Ravallion, 2012).

8Before Sen (1983) made the case that an absolute level in the space of capabilities translates into a relative level in the space of resources, other scholars recognized the importance of relative income poverty. Townsend (1979) discussed how individuals not having the resources for obtaining the living conditions that are widely encouraged in their society would be excluded from ordinary living patterns, customs and activities. Runciman (1966) pointed out that the comparison of own income with incomes of better-off individuals creates a feeling of deprivation. Relative deprivation and social exclusion are different concepts (Bossert et al., 2007), but both insist on relative income poverty. Ravallion (2008) critically reviews the normative foundations for taking a relativist approach in poverty measurement.

9Empirical Social Choice has shown from questionnaire experiments that resources dominance is far from being unanimously accepted by respondents as a sufficient normative criteria for concluding that one distribution is better than another one (Gaertner and Schokkaert, 2012). A tentative explanation put forward by this literature is that respondents consider other-regarding feelings and hence would agree with dominance in the space of utilities, but not in the space of resources.
that poverty indices are only concerned with the fate of poor agents. Formally, Focus requires the index not to be affected by the income of non-poor agents.

**Poverty axiom 1 (Focus).**

For all \( y, y' \in Y \) and \( z^* \in \mathbb{R}^{++} \), if \( n(y) = n(y') \), \( q(y) = q(y') \) and \( y_i = y'_i \) for all \( i \leq q(y) \), then \( P(y, z^*) = P(y', z^*) \).

A corollary of Focus is that a distribution has the same poverty as its associated censored distribution, obtained by replacing the income of all non-poor agents by the income threshold \( z^* \). This axiom, together with a monotonic property, implies that if a censored distribution first-order stochastically dominates another, then it has less poverty, excluding social participation effects.

Social participation effects can be accounted for in the identification of poverty by the use of endogenous lines. However, measures obtained by combining endogenous lines with indices derived for absolute lines fail to give a minimal priority to subsistence over social participation.

### 2.2 Endogenous measures

Endogenous measures are measures based on endogenous lines. The income threshold of an endogenous line may evolve with standards of living. In practice, the threshold is endogenously determined by the mediation of an income standard like mean or median income.

Relative lines are the most famous example of endogenous lines. The threshold of a relative line evolves as a constant fraction of the income standard. Relative lines are widely used in developed countries. For example, the “At Risk of Poverty” measure of the European Commission is based on a relative line. Nevertheless, relative lines have been heavily criticized, mainly on two grounds (Ravallion and Chen, 2011). First, their threshold goes to zero in low-income countries, making clear that subsistence is not accounted for. Second, relative lines are based on a rather extreme view on social participation. Indeed, any equi-proportionate growth does not get any individual out of poverty because the threshold is multiplied by the same factor as individual incomes. Poverty measures based on an FGT index together with a relative line are unaffected by equi-proportionate growth. In that sense, these relative measures ignore absolute gains and losses.

Given the shortcomings of relative measures, a literature emerged with the ambition to balance the absolute and relative aspects of income poverty, albeit most efforts concentrated on identification. Two main routes have been proposed for endogenous identification.

The first route consists in using two different lines for identification, one absolute and one relative (Atkinson and Bourguignon, 2001). The absolute line captures subsistence, referred to as “absolute poverty”, and the relative line captures social participation, referred to as “relative poverty”. As the relative line’s threshold is larger than that of the absolute line in high-income countries and smaller in low-income countries, the two lines cross. As a consequence, some individuals in low-income countries can be deemed absolutely poor but not relatively poor. If this route proposes a meaningful way of identifying the poor, the construction of a good index based on two lines has proved very difficult. The first solution is to construct two measures, one based on the absolute line and the other on the relative line. In order to judge unequal growth, the two
measures need to be aggregated. Atkinson and Bourguignon (2001) suggest to consider the two measures in lexicographic order, as they judge subsistence to be a more serious component of poverty than social participation. Lexicographic aggregation unfortunately makes the relative measure almost irrelevant in poverty judgments. Another possibility is to weight the two measures (Anderson and Esposito, 2013). The second solution is to derive a unique index by aggregating the income gaps with respect to each of the two lines (Atkinson and Bourguignon, 2001). Unfortunately, this raises a problem of double counting for individuals that are both absolutely and relatively poor. So far, a characterization of the properties of an index based on two lines remains missing. As a result, there is no guarantee that measures based on two lines give a minimal priority to subsistence over social participation.

The second route consists in using a unique endogenous line balancing the absolute and relative aspects of income poverty. Foster (1998) proposes hybrid lines that feature a constant income elasticity $\rho$. This income elasticity can be interpreted as the extent to which poor individuals should share the benefits of economic growth. Absolute lines have an income elasticity of zero and relative lines have an income elasticity of one, representing two extreme views on this parameter. A different proposal by Ravallion and Chen (2011) suggests using weakly relative lines, whose income elasticity is zero for low-income countries and then increases with standards of living, tending ultimately to a value of one. Both hybrid and weakly relative lines are interesting proposals for identifying the poor. Unfortunately, a characterization of the properties of an index based on a unique endogenous line remains missing. So far, all endogenous measures used in empirical applications are based on FGT indices, which are characterized for absolute lines (see for example Chen and Ravallion (2013)). This is problematic as those indices lose some of their desirable properties when combined with endogenous lines.

Limitations of endogenous measures

Current endogenous measures are not well-suited for evaluating the impact on poverty of unequal growth. They give no priority to subsistence over social participation. The judgment that subsistence should be given priority over social participation is not only the intuition of experts like Atkinson and Bourguignon (2001) but appears to be largely shared as shown in questionnaire studies conducted in different parts of the World (Corazzini et al., 2011).

Current endogenous measures consider that some individuals whose income is below the subsistence level are less poor than other individuals living in a richer society but whose income is above subsistence level. Notice that I make a pairwise comparison of individuals living in different societies. Such a comparison cannot be performed using a standard poverty axiom because axioms compare two distributions, not two individuals. The issue is nevertheless extremely serious as it often leads endogenous measures to conclude that there is more poverty in middle and high-income societies than in low-income societies. In other words, endogenous measures often conclude that growth increase

\[ z_h = \begin{cases} z_a & \text{if } z_r > 1 - \rho \\ z_r (1 - \rho) & \text{if } z_r < 1 - \rho \end{cases} \]

For a given income standard, letting $z_a$ be the threshold of an absolute line and $z_r$ be the threshold of a relative line, the hybrid threshold is given by $z_h = \begin{cases} z_a & \text{if } z_r > 1 - \rho \\ z_r (1 - \rho) & \text{if } z_r < 1 - \rho \end{cases}$.

Madden (2000) estimates empirically an upper-bound for the value of this parameter using Irish data.
Poverty even if many individuals were brought above the subsistence level.

Table 2 presents an example illustrating the problem. Assume for simplicity that the endogenous measure uses an FGT index $P$ based on a relative line $z$ whose threshold is defined as 50% of the mean income (denoted $\overline{y}$). For example, index $P$ could be the HC or the PGR. The normalized income gap of a poor individual is defined to be one minus her normalized income:

$$g(y_i, z(\overline{y})) := \frac{z(\overline{y}) - y_i}{z(\overline{y})} = 1 - \frac{y_i}{z(\overline{y})}.$$  (3)

FGT indices are then simply the average of normalized income gaps taken to the power $\alpha$. The example in Table 2 compares the income distributions of two societies C and D, each populated by two agents, on poor and one non-poor. The poor individual in society C has income below the subsistence threshold, i.e. below $1.25$ a day, whereas the poor individual in society D has income above the subsistence threshold. Nevertheless, the poor individual in society D has a larger normalized income gap, implying the endogenous measure concludes there is more poverty in D than in C.\textsuperscript{12} Observe that for any other monotonic endogenous line, another example featuring the same issue can be constructed.

Table 2: Endogenous measures give no priority to subsistence ($1.25$ a day).

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$z^a$</th>
<th>$z(\overline{y})$</th>
<th>$g(y_1, z(\overline{y}))$</th>
<th>$P(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Society C</td>
<td>1</td>
<td>5</td>
<td>1.25</td>
<td>1.5</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{2} \left( \frac{1}{3} \right)^a$</td>
</tr>
<tr>
<td>Society D</td>
<td>1.5</td>
<td>10.5</td>
<td>1.25</td>
<td>3</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2} \left( \frac{1}{3} \right)^a$</td>
</tr>
</tbody>
</table>

The problem illustrated in Table 2 results from the axiom of Scale Invariance. This axiom is satisfied by virtually all endogenous measures. This axiom is often defended on the grounds that it renders the currency units in which income is measured irrelevant. Formally, Scale Invariance requires the index not to be affected when the income of all agents are multiplied by the same factor as the income threshold.

**Poverty axiom 2 (Scale Invariance).**

For all $y \in Y$ and $\lambda > 0$, $P(\lambda y, \lambda z^*) = P(y, z^*)$.

A corollary of Scale Invariance is that a censored distribution has the same poverty as its associated normalized gaps censored distribution, obtained by replacing all censored incomes by their normalized gaps. Therefore, whether or not an individual has income below the subsistence level is irrelevant. The fact that Scale Invariance imposes more than just the irrelevance of currency units has already been emphasized by Zheng (2007). This author derives indices satisfying a weaker property than Scale Invariance. His approach is nevertheless only concerned with indices based on absolute lines (his indices satisfy Focus).

The empirical illustration provides examples of such debatable judgments by current endogenous indices, for example when comparing Brazil with Ivory Coast in 2010.\textsuperscript{13} Given the weakly relative line used, an individual with $1$ a

\textsuperscript{12}The HC concludes that there is equivalent poverty in both societies. A conclusion that is also questionable.

\textsuperscript{13}From a theoretical perspective, societies having different standards of living can be either the same country at different points in time or two different countries at the same point in time.
day in Ivory Coast has the same normalized gap as an individual with $3.6 a day in Brazil. Endogenous measures based on the PGR and the HC conclude that there is more (or equivalent) poverty in Brazil than in Ivory Coast. This judgment goes against standard intuitions. Indeed, in 2010 mean income in Ivory Coast is $3 a day, whereas it is $13.7 a day in Brazil. The percentage of individuals below the subsistence level is 22.7% in Ivory Coast and only 5.4% in Brazil. Although income inequality is higher in Brazil than in Ivory Coast - the percentage of individuals below a relative line whose threshold equals 50% of mean income is lower in Ivory Coast (30%) than in Brazil (43%) - a poverty measure based on the weakly relative line and my index concludes there is 32% less poverty in Brazil than in Ivory Coast.

The bottom line of this literature review is that standard measures provide very counter-intuitive judgments when comparing income poverty between societies with different standards of living. There is a need for a new measure that can provide sound judgments on unequal growth. Clearly, no absolute measure can account for social participation effects. Therefore, I study indices based on a unique endogenous line. As emphasized above, there exists no characterization of any such index. In the next section, I derive an additive family of indices based on endogenous lines.

3 Additive indices based on endogenous lines

I describe in this section the characterization of a family of additive indices based on endogenous lines. The family is based on the concept of an equivalence ordering. The presentation of this new object requires the introduction of additional notation.

3.1 Notations and basic restrictions

The notation is a slight modification of the notation used for absolute measures, presented in section 2. Mean income $\overline{y} := \frac{\sum y_i}{n}$ is the income standard capturing standards of living. This choice and the robustness of the results for other income standards are discussed in Section 6. I refer to $y_i$ as the absolute situation of agent $i$ and $\overline{y}$ as her relative situation.

An endogenous poverty line is defined by its associated threshold function $z : \mathbb{R}_+^+ \to \mathbb{R}_+^+$ specifying the income threshold $z(\overline{y})$ associated to $\overline{y}$. Agent $i$ qualifies as poor if $y_i < z(\overline{y})$. Letting $N := \{n \in \mathbb{N} | n \geq 3\}$, the set of income distributions considered is

$$Y := \{y \in \mathbb{R}_+^N | \overline{y} > 0 \text{ and } y_n \geq z(\overline{y})\}.$$  

For technical reasons, this set excludes lists of zeros and any distribution for which all agents are poor. These two restrictions are arguably rather mild.

In order to keep the notation minimal, the notation $P$ for poverty indices features the income distribution as its unique argument, suppressing its dependence on the line $z$. A poverty index is therefore a real valued function $P : Y \to \mathbb{R}$. For any two $y, y' \in Y$, there is strictly more poverty in $y$ than in $y'$ if $P(y) > P(y')$, and weakly more if $P(y) \geq P(y')$. 

11
Endogenous lines

This research does not provide a guide for the selection of an endogenous line. The endogenous line is assumed to be exogenously given. How can a practitioner select a good line? Ideally, for each value of mean income, the threshold function returns the minimal cost of a bundle of goods sufficient to secure subsistence and social participation. In practice, such a line could be regressed on the costs of a set of reference bundles, each bundle constructed for a different country (the sample should cover countries with different standards of living). A more pragmatic choice is to select an hybrid or a weakly relative line. Most important is that the line makes sense in the societies that are being compared.

The selection of an endogenous line is a normative choice. Therefore, this selection involves some arbitrariness. An important remark is that selecting an endogenous line does not involve more arbitrariness than selecting an absolute line. There are of course fewer parameters to fix when opting for an absolute line, but this is precisely because absolute lines assume the income threshold to be constant. This assumption is as arbitrary as selecting a positive slope for the line. What is more, this assumption implies that the associated poverty measure ignores completely social participation effects, as illustrated in Table 1.

For the results to hold, the endogenous line must meet two mild restrictions, besides being continuous. Possibility of Poverty Eradication requires the existence of an income level that, if earned by all agents, makes all agents non-poor.

**EL restriction 1** (Possibility of Poverty Eradication).
*There exists \( g > 0 \) such that \( g \geq z(g) \).*

The restriction Slope Less than One requires that the slope of \( z \) at mean income \( \overline{y} \), denoted \( s(\overline{y}) \), is never larger than one. This restriction implies that if an agent is not poor in an initial distribution and if her income and mean income increase by the same amount, this agent cannot be considered poor in the new distribution.

**EL restriction 2** (Slope Less than One).
*For all \( \overline{y} > 0 \) we have \( s(\overline{y}) \leq 1 \).*

Together, the two restrictions imply there exists a minimal value of mean income above which poverty-free income distributions always exist.

The presentation of the results is simplified by the introduction of specific subdomains of endogenous lines. The intercept of a line – the limit of the function \( z \) when \( \overline{y} \) tends to zero – is denoted by \( z^0 \).

- **Piecewise-linear lines:**
  There exists \( \overline{y}^d \geq 0 \) and \( \delta \geq 0 \) such that for all \( \overline{y} \leq \overline{y}^d \), we have \( z(\overline{y}) = z^0 \) and for all \( \overline{y} > \overline{y}^d \) we have \( z(\overline{y}) = z^0 + \delta(\overline{y} - \overline{y}^d) \).

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14This interpretation derives from Sen (1983): “absolute deprivation in terms of a person’s capabilities relates to relative deprivation in terms of commodities, incomes and resources”. This interpretation can also be found in Atkinson and Bourguignon (2001).

15Slope Less than One is necessary in order for the line to admit an EO meeting Translation Monotonicity, defined below.
Figure 1: (a) Equivalence curves of a generic EO in $\mathcal{R}$ and an income distribution $(y_1, y_2, y_3) \in Y$. (b) EO implied by an absolute measure. (c) EO implied by a relative measure.

- **Monotonic lines:**
  For all $\overline{y}, \overline{y}' > 0$ with $\overline{y} < \overline{y}'$, we have $z(\overline{y}) \leq z(\overline{y}')$ and there exists $g > 0$ with $g \geq z(g)$ such that $s(g) > 0$.

  Finally, **linear lines** are piecewise-linear lines such that $\overline{y}^k = 0$, **relative lines** are linear lines such that $z^0 = 0$ and $\overline{s} > 0$ and **absolute lines** are linear lines such that $z^0 > 0$ and $\overline{s} = 0$.

  Observe that weakly relative lines (Ravallion and Chen, 2011) are piecewise-linear lines. Both weakly relative lines and hybrid lines (Foster, 1998) are monotonic lines.

**Equivalence orderings**

I introduce here the concept of an equivalence ordering (EO). The set of accessible bundles is

$$X := \{(y_i, \overline{y}) \in \mathbb{R}^+ \times \mathbb{R}^+\}.$$  

These bundles are two-dimensional since the individual poverty of an agent depends on both her income and mean income in her society. Given an endogenous line $z$, the set of bundles at which an agent qualifies as poor is

$$X_p := \{(y_i, \overline{y}) \in X \mid y_i < z(\overline{y})\}.$$  

An EO is a preference relation for an ethical observer that compares individual bundles. See Figure 1 for examples of EOs.

**Definition 1** (Equivalence Ordering).

An equivalence ordering $\succeq$ is a continuous ordering on $X_p$.\(^\text{16}\)

The poverty line compares bundles in different societies. An EO extends this logic below the poverty line. The line is the frontier equivalence curve of the EO, defining the threshold below which an agent is deemed poor. Let $(y_i, \overline{y}) \succeq (y'_i, \overline{y}')$ denote the judgment that agent $i$ with income $y_i$ when mean income is $\overline{y}$ has a weakly smaller individual poverty than with income $y'_i$ when

\(^{16}\)An ordering is a reflexive, transitive and complete binary relation.
mean income is $\overline{y}$. The symmetric and asymmetric parts of $\succeq$ are denoted $\sim$ and $\succ$ respectively.

The selection of an EO is a normative choice. This choice is arbitrary to some extent. As a consequence, some might be afraid that this approach leads to a significant increase in the arbitrariness of poverty judgments. This is not the case for two reasons. First, all major poverty measures implicitly define such an EO. Their EO is constrained by axioms with little ethical content such as Scale Invariance.17 Second, the choice of an EO is guided by intuitive restrictions excluding exotic trade-offs between an agent’s absolute and relative situation.

Given an endogenous line, the basic domain $\mathcal{R}$ of continuous EO is defined by three restrictions. Strict Monotonicity in Income requires that, at any mean income, more income leads to strictly smaller individual poverty.

**EO restriction 1** (Strict Monotonicity in Income).
For all $(y_i, \overline{y}) \in X_p$ and $a > 0$, we have $(y_i + a, \overline{y}) \succ (y_i, \overline{y})$.

The other two restrictions limit the importance of a poor agent’s relative situation for her individual poverty. Translation Monotonicity requires that any poor agent is made weakly better-off by the equal distribution of an extra amount of income. It seems hard to conceive that the relative situation of a poor agent is made worse by such an equal distribution of income. As a result, the slopes of the EO’s equivalence curves are never larger than one.

**EO restriction 2** (Translation Monotonicity).
For all $(y_i, \overline{y}) \in X_p$ and $a > 0$, we have $(y_i + a, \overline{y} + a) \succeq (y_i, \overline{y})$.

Finally, Minimal Absolute Concern requires that an agent with zero income is strictly poorer than another agent with non-zero income, regardless of the mean incomes in their respective societies.

**EO restriction 3** (Minimal Absolute Concern).
For all $(y_i, \overline{y}) \in X_p$ with $y_i > 0$ and $\overline{y} > 0$, we have $(y_i, \overline{y}) \succ (0, \overline{y})$.

Domain $\mathcal{R}$ is very wide and admits an infinity of different EOs below each endogenous line. This domain is flexible as it admits the implied EOs of standard poverty measures as special cases. This is for example the case of absolute and relative measures, as illustrated in Figure 1.b and 1.c.18 As a result, additive indices presented in (1) are a special case of the family of additive indices based on endogenous lines that I derive below.

### 3.2 Characterization of an additive family

Any poverty index based on an absolute line and satisfying five basic axioms must have an additive mathematical expression (Foster and Shorrocks, 1991). I extend this standard result to indices based on endogenous lines. I present here the modified versions of the basic axioms used for this extension.

An EO captures the trade-offs at the individual level between the absolute and the relative situation. Domination among Poor requires poverty indices

17Another restriction implicitly constraining the EO is Translation Invariance, which requires the index not to be affected if the income of all agents is increased by the same amount as the poverty line.

18The EO below a relative line in Figure 1.c lies inside the domain $\mathcal{R}$ since bundle $(0, 0)$ is excluded from the set $X$ of bundles.
to respect the individual poverty comparisons encapsulated in an EO. It does so by imposing a monotonicity requirement in the space of individual poverty distributions, limited to poor agents. If the individual poverty of one poor agent decreases, while the individual poverties of all other agents do not increase, then poverty must decrease.

Poverty axiom 3 (Domination among Poor).
There exists $\succeq \in \mathbb{R}$ such that for all $y, y' \in Y$ with $n(y) = n(y')$, if $(y'_i, \mathbf{y'}) \succeq (y_i, \mathbf{y})$ for all $i \leq q(y')$, then $P(y) \geq P(y')$. If in addition there is $j \leq q(y)$ such that $(y'_j, \mathbf{y'}') \succ (y_j, \mathbf{y})$, then $P(y) > P(y')$.

Observe that Domination among Poor implies a weak version of Focus. Indeed, only the situation of poor agents is relevant for the index, but the incomes of non-poor agents can influence the index via the income standard.

Subgroup consistency is a standard axiom requiring that, if poverty decreases in a subgroup while it remains constant in the rest of the distribution, overall poverty must decline.\textsuperscript{19} Sen (1992) questioned the desirability of this axiom by arguing that incomes in one subgroup may affect poverty in another subgroup. Foster and Sen (1997) recommend not to use this axiom when the index aims at capturing relative aspects of income poverty. I subscribe to this point of view. The issue becomes transparent once the channel through which one subgroup affects the other is modeled. In this framework, incomes in a subgroup impact mean income which affects other poor agents’ individual poverty. If the line is absolute and the EO features flat equivalence curves (see Figure 1.b), relative income does not matter and subgroup consistency is compelling. If relative income does matter, then it is not always meaningful to extrapolate the judgments made on subgroups to the whole population. Weak Subgroup Consistency restricts such extrapolations to cases for which the incomes in a subgroup do not influence the individual poverty of agents in the other subgroup. These cases occur when the two subgroups of a population have the same mean income, implying that the subgroups have the same mean income as the total population. In such cases, poverty judgments made on subgroups are relevant for the total population.

Poverty axiom 4 (Weak Subgroup Consistency).
For all $y^1, y^2, y^3, y^4 \in Y$ such that $n(y^1) = n(y^3)$, $n(y^2) = n(y^4)$, $\mathbf{y}^1 = \mathbf{y}^2$ and $\mathbf{y}^3 = \mathbf{y}^4$, if $P(y^1) > P(y^3)$ and $P(y^2) = P(y^4)$, then $P(y^1, y^2) > P(y^3, y^4)$.

The remaining three auxiliary axioms need no specific modification. Symmetry requires that agents identities do not matter. Working with sorted distributions is therefore without loss of generality.

Poverty axiom 5 (Symmetry).
For all $y, y' \in Y$, if $y' = y \cdot \pi_{n(y) \times n(y)}$ for some permutation matrix $\pi_{n(y) \times n(y)}$, then $P(y) = P(y')$.

Continuity requires poverty indices to be continuous in incomes. This is important for empirical applications in order to avoid measurement errors having excessive impacts on poverty judgments.

Poverty axiom 6 (Continuity).
For all $y \in Y$, $P$ is continuous in $y$.

\textsuperscript{19}A formal definition of this axiom can be found in Foster and Shorrocks (1991).
Replication Invariance permits comparing poverty in distributions of different population sizes. If a distribution is obtained by replicating another one several times, then the latter’s poverty equals that of the original distribution.

Poverty axiom 7 (Replication Invariance).
For all \( y, y' \in Y \), if \( n(y') = k n(y) \) for some positive integer \( k \) and \( y' = (y, y, \ldots, y) \), then \( P(y) = P(y') \).

Those five axioms allow us to derive an extension of the additive separability result of Foster and Shorrocks (1991). Its formal statement needs two definitions. First, a numerical representation is a continuous function representing an EO.

Definition 2 (Numerical Representation \( d \)).
The continuous function \( d : X \rightarrow [0, 1] \) is a numerical representation of \( \succeq \in \mathcal{R} \) if
- for all \((y_i, \overline{y}), (y'_i, \overline{y}) \in X_p\) we have \((y_i, \overline{y}) \succeq (y'_i, \overline{y}) \Leftrightarrow d(y_i, \overline{y}) \leq d(y'_i, \overline{y})\),
- for all \((y, \overline{y}) \in X \setminus X_p\) we have \(d(y, \overline{y}) = 0\).

A numerical representation differs from a utility representation of equivalence levels in two ways. First, it is constant for all equivalence levels above the poverty threshold. Second, below the poverty threshold, its value decreases when individual poverty decreases. The values returned by this function can be interpreted as individual poverty, i.e. the opposite of utility.

Next, I define additive poverty indices which aggregate agents’ individual poverty by summing them.

Definition 3 (Additive Poverty Index).
\( P \) is an additive poverty index if it is ordinally equivalent to another index \( \hat{P} : Y \rightarrow [0, 1] \) such that for all \( y \in \mathbb{R}^N \)
\[
\hat{P}(y) := \frac{1}{n} \sum_{i=1}^{n} d(y_i, \overline{y}),
\]

where \( d \) is a numerical representation of an EO in \( \mathcal{R} \).

Theorem 1 characterizes the family of additive poverty indices based on endogenous lines. This is the first characterization of indices based on endogenous lines.

Theorem 1 (Characterization of additive poverty indices).
Let \( P \) be a poverty index based on an endogenous poverty line. The following two statements are equivalent.
1. \( P \) is an additive poverty index.
2. \( P \) satisfies Domination among Poor, Weak Subgroup Consistency, Symmetry, Continuity and Replication Invariance.

Proof. It is easy to check that additive poverty indices satisfy these five axioms, so the proof that statement 1 implies statement 2 is hence omitted. The proof of the reverse implication is in Appendix 9.1. In a nutshell, the proof shows that the result on additive separability of Gorman (1968) applies. The crucial assumption to verify is that the index satisfies a separability property. After applying Theorem 1 in Gorman (1968), the remaining part of the proof is a modification of Foster and Shorrocks (1991).
The difference with the result of Foster and Shorrocks (1991) is that numerical representations of individual poverty depend now on two-dimensional bundles, made of own income and mean income. This new dependence on mean income vanishes if the line is absolute and the EO has only flat equivalence curves.

The family of additive indices is very broad. Choosing an index in that family requires selecting both an EO below the line and a numerical representation for this EO. I show in the next section that both normative choices can be deduced from largely shared intuitions. A new index emerges then as the focal additive index with good properties.

4 A new index with good properties

In this section, I first describe how to select an EO and its numerical representation from largely shared intuitions. Then, I present the index defined by these choices and show that this index is workable, it distinguishes absolutely poor agents from relatively poor agents and it is decomposable between the contributions of these two kinds of poor agents.

4.1 Selection of an equivalence ordering

The example given in Table 2 showed that endogenous measures satisfying Scale Invariance give no priority to subsistence over social participation. The EO of an endogenous measure satisfying Scale Invariance has its equivalence curves evolve as constant fractions of the income threshold. Geometrically, these equivalence curves are homothetic.

An EO giving priority to subsistence can never consider that an agent whose income is below the subsistence threshold is less poor than another agent whose income is above, regardless of the standards of living in their respective societies. Accordingly, when comparing two agents with incomes below subsistence, the one with larger income can not be judged poorer than the other, independently of their respective relative situations. Only EOs having all their equivalence curves flat up to the subsistence threshold satisfy these intuitions.20

I define a subdomain of EOs based on a fourth restriction. Restriction Absolute-Homotheticity, illustrated in Figure 2(a), is defined from the absolute threshold $z^a$. This parameter can be interpreted as the subsistence threshold or alternatively as the threshold for absolute material deprivation. EOs satisfying Absolute-Homotheticity have all their equivalence curves flat up to $z^a$. This condition is formally expressed in part (i).

EO restriction 4 (Absolute-Homotheticity).
There exists $z^a \geq 0$ such that for all $(y_i, \bar{y}), (y'_i, \bar{y}') \in X_p$:

(i) Priority to subsistence over social participation.
   if $y_i = y'_i \leq z^a$ then $(y_i, \bar{y}) \sim (y'_i, \bar{y}')$.

(ii) Homothetic equivalence curves above the absolute threshold.
   if $y_i, y'_i \geq z^a$ and \[ \frac{y_i - z^a}{y_i - \bar{y}} = \frac{y'_i - z^a}{y'_i - \bar{y}} \]
   then $(y_i, \bar{y}) \sim (y'_i, \bar{y}')$.

20I assume that the cost of subsistence – and hence the subsistence threshold – does not evolve with the standard of living.
(iii) Cost of social participation is never zero.

If $z^a > 0$, then $z^a < z^0$.

Part (ii) of the restriction requires the equivalence curves above $z^a$ to evolve as constant fractions of the distance between $z^a$ and the income threshold. This simplifying assumption is a natural default option in the absence of reasons to deviate from it. In Section 5, I show that if equivalence curves above $z^a$ deviate too much from homotheticity, then there exists no numerical representation of the EO with good properties.

The last part is technical and needed for some results, even though it has some normative content. Part (iii) requires the cost of social participation to be strictly positive, even in low-income societies. Ravallion (2012) defends this point by giving several examples of expenditures playing a social role in low-income countries such as festivals and celebrations.

Given that restriction Absolute-Homotheticity incorporates important intuitions, the selected EO should meet this restriction. Before turning to the selection of a numerical representation for an absolute-homothetic EO, I make two important remarks on this subdomain of EOs.

Given an endogenous line, the only parameter of the absolute-homothetic subdomain is the absolute threshold $z^a$. A particularly interesting feature of absolute-homothetic EOs is that they allow categorizing poor agents between those that are absolutely poor and those that are “only” relatively poor. An equivalent categorization can be found in Foster et al. (2013). This categorization implies that relatively poor agents are never absolutely poor, contrary to the categorization obtained when identifying the poor with two lines that cross (Atkinson and Bourguignon, 2001).

Let the homothetic EO be the absolute-homothetic EO for which $z^a = 0$. As illustrated in Figure 2.b, the homothetic EO does not give priority to subsistence over social participation. Absolute measures are based on the homothetic EO below an absolute line. Standard endogenous measures are based on the homothetic EO below an endogenous line. This shows that the absolute-homothetic
domain of EOs generalizes the implied EOs of standard measures.

4.2 Selection of a numerical representation

Having selected an absolute-homothetic EO by fixing the absolute threshold, the only element of the additive index remaining unspecified is the EO’s numerical representation. Many numerical representations should be discarded for their counter-intuitive judgments. I consider two properties that strongly constrain the set of acceptable numerical representations.

The first property is specific to poverty indices considering both the absolute and relative aspects of income. In such a framework, increasing the income of an agent entails a worse relative situation for the others. Poverty indices must balance those gains and losses without giving excessive importance to relative losses. **Monotonicity in Income** requires that decreasing the income of some poor agent never leads to an unambiguous poverty reduction.

**Monotonicity in Income**

For all \( y, y' \in Y \), if \( y_i < y'_i < z(\overline{y}) \) and \( y'_j = y_j \) for all \( j \neq i \), then \( P(y) \geq P(y') \).

When a poor agent’s income increases, her individual poverty decreases as both her absolute and relative situation improve. On the other hand, mean income increases and this might increase the individual poverty of other poor agents, depending on the EO.\(^{21}\) Moreover, the income threshold might increase and some agents who were non-poor might therefore become considered as poor. **Monotonicity in Income** requires that the positive impact of such an income increase is dominant. Observe that the larger the number of agents, the lower is the impact of such an income increase on mean income and hence on the individual poverty of others.

It is worth emphasizing that the Head-Count Ratio, when combined with an endogenous line, can conclude that destroying part of the income of a poor agent reduces poverty. The problem is illustrated in Table 3. The relative line \( z \) has its threshold equal to 50% of mean income. The distribution in society F is obtained from the distribution in society E by decreasing the income of poor agent 1. Nevertheless, the HC concludes there is more poverty in society E than in F.

**Table 3**: Index HC violates **Monotonicity in Income**.

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( z(\overline{y}) )</th>
<th>( HC(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Society E</td>
<td>2.5</td>
<td>3</td>
<td>12.9</td>
<td>3.1</td>
</tr>
<tr>
<td>Society F</td>
<td>2</td>
<td>3</td>
<td>12.9</td>
<td>2.9</td>
</tr>
</tbody>
</table>

The second property is a standard requirement that most poverty indices satisfy. **Transfer among Poor** requires that a Pigou-Dalton transfer taking place between two poor agents never unambiguously increases poverty.\(^{22}\) This property is still very compelling when using mean income as an income standard since balanced transfers do not alter the mean. As a result, the individual poverty of agents not involved in the transfer is preserved.

\(^{21}\)The individual poverty of absolutely poor agents is not affected as their equivalence curve is flat for absolute-homothetic EOs.

\(^{22}\)A Pigou-Dalton transfer is a progressive balanced transfer preserving the relative ranks of the two agents involved in the transfer.
Poverty axiom 9 (Transfer among Poor).
For all \( y, y' \in Y \) and \( \lambda > 0 \), if \( y_j - \lambda = y_j' > y_k' = y_k + \lambda \), \( z(\overline{y}) > y_j \) and \( y_j' = y_i \) for all \( i \neq j, k \), then \( P(y) \geq P(y') \).

I investigate which additive indices respect both properties. It is well-known that poverty indices satisfying Transfer among Poor are based on convex numerical representations. Monotonicity in Income is a new axiom in this context and I show below that it has a strong discriminative power.

A central result of this paper is that – when selecting an absolute-homothetic EO below a monotonic line – there is a unique numerical representation belonging to the extended Foster-Greer-Thorbecke family that satisfies both Monotonicity in Income and Transfer among Poor. Before formally stating this result in Theorem 2, I define this family of numerical representations.

Beyond the subdomain of homothetic EOs, the graph of a numerical representation depends on the particular mean income at which it is drawn. As a result, the mathematical expression of a numerical representation depends on the reference mean income, denoted by \( \overline{y} \), at which it is expressed. Defining a particular family of numerical representations requires introducing a function that specifies for each bundle the income yielding the same individual poverty at the reference mean income.\(^{23}\)

Definition 4 (Equivalent Income Function at \( \overline{y} \)).
For any \( \geq \in \mathcal{R} \) and \( \overline{y} > 0 \), the equivalent income function \( e^* : X \to [0, z(\overline{y})] \) is the continuous function such that for all \((y, \overline{y}) \in X : (y, \overline{y}) \sim (e^*(y, \overline{y}), \overline{y})\).

Given the restrictions on the domain \( \mathcal{R} \) of EOs, the equivalent income function is well-defined.\(^{24}\) Using the concept of equivalent income function, I propose an extension of the Foster-Greer-Thorbecke (FGT) family of numerical representations.

Definition 5 (Extended FGT Family).
For any given EO in \( \mathcal{R} \), the numerical representation \( d \) belongs to the extended FGT family if there exist \( \overline{y} \geq 0 \) such that for all \((y, \overline{y}) \in X_p:\)

\[
d(y, \overline{y}) = \left( \frac{z(\overline{y}) - e^*(y, \overline{y})}{z(\overline{y})} \right)^\alpha \quad \text{with } \alpha \geq 0,
\]

where \( e^* \) is the equivalent income function at \( \overline{y} \).

The extended FGT family depends on two parameters: the reference mean income \( \overline{y} \) at which \( d \) takes an exponential expression and the exponent \( \alpha \), interpreted as poverty aversion. For homothetic EOs, this family coincides with the standard FGT family presented in (2) since the mathematical expression of their numerical representation does not depend on the reference mean income.

\(^{23}\)As the case \( \overline{y} = 0 \) is ruled out from my domain, the definition of this function must be modified if \( \overline{y} = 0 \). The function \( e^0 \), the equivalent income function at \( \overline{y} = 0 \), is defined from \( e^* \) with \( \overline{y} > 0 \). Take any \((y, \overline{y}) \in X \), \( e^0 : X \to [0, d] \) is the continuous function such that for all \( e > 0 \) there is \( \delta > 0 \) such that if \( \overline{y} < \delta \), then \(|e^0(y, \overline{y}) - e^*(y, \overline{y})| < c\).

\(^{24}\)The existence of an equivalent income function at any value of mean income is guaranteed for all EO’s in our domain by restriction Minimal Absolute Concern. Furthermore, it is a function – it returns a unique value – since EOs meet restriction Strict Monotonicity in Income and since its domain of images is bounded above by the income threshold.
In the extended FGT family, each value of poverty aversion defines a subfamily whose members are parameterized by the reference mean income. For example, the PGR at \( y_r \) is the numerical representation that is linear (\( \alpha = 1 \)) at mean income \( \overline{y} \). The PGR at the origin, defined by \( \overline{y} = 0 \) and illustrated in Figure 3, plays a key role in the remainder of this paper.

Theorem 2 formalizes the central result showing that in the extended FGT family, only the PGR at the origin satisfies \textit{Monotonicity in Income} and \textit{Transfer among Poor}.

**Theorem 2 (Characterization of PGR at the origin).**

Let \( z \) be a monotonic line. Let \( P \) be an additive poverty index based on an absolute-homothetic EO below \( z \) with a numerical representation in the extended FGT family.

1. \( P \) satisfies \textit{Monotonicity in Income} only if:
   \[
   \alpha = 1.
   \]

2. \( P \) satisfies \textit{Monotonicity in Income} and \textit{Transfer among Poor} if and only if:
   \[
   \alpha = 1 \quad \text{and} \quad \overline{y} = 0,
   \]
   that is, \( d \) is the PGR at the origin.

*Proof.* See in Appendix 9.3. The proof is based on Lemma 2, presented in Appendix 9.2, which gives a necessary condition and a sufficient condition for an index to satisfy \textit{Monotonicity in Income}. ■

Theorem 2 shows that a unique member of the very rich extended FGT family satisfies both properties.

Claim 1 shows that \textit{Monotonicity in Income} is responsible for the largest part of the result. First, among all values of poverty aversion, only the one associated to the PGR is acceptable. This characterization of the poverty aversion’s value is due to the exponential mathematical form of the extended FGT family. For the case \( \alpha < 1 \), as the income of a poor agent tends to the income threshold, the priority granted to her over – say – an absolutely poor agent tends to infinity. Therefore, when the income of an absolutely poor agent increases, the individual poverty of a relatively poor agent close to the income threshold is negatively affected and the index concludes poverty has increased. The case \( \alpha > 1 \) is plagued with the reverse problem. As the income of a poor agent tends to the income threshold, her priority over other poor agents tends to zero. An increase in her income can be negatively judged by the index. Second, not all members of the PGR at \( \overline{y} \) subfamily satisfy \textit{Monotonicity in Income}. If the monotonic line is piecewise-linear, then the PGR at \( \overline{y} \) satisfies the axiom if and only if \( \overline{y} \) is below an upper-bound whose value depends on the parameters of the line and the absolute threshold.\(^{25}\)

\(^{25}\)The proof for this claim can be found in Appendix 9.3.1. The intuition for the upper-bound goes as follows. The larger the reference mean income \( \overline{y} \), the lower is the individual poverty gain made when bringing an agent with zero income to the absolute threshold. In other words, the larger \( \overline{y} \), the lower the priority of absolutely poor agents over relatively poor agents. This priority tends to zero when \( \overline{y} \) tends to infinity. In this sense, the upper-bound requires the index to guarantee a minimal priority to absolutely poor agents.
Figure 3: (a) Poverty Gap Ratio at the origin representing an absolute-homothetic EO (its values $d$ are indicated at the end of three equivalence curves). (b) Graph of the PGR at the origin drawn at $\bar{y} = \bar{y} = 0$. (c) Graph of the PGR at the origin drawn at $\bar{y} = \bar{y}$. 

Claim 2 shows that Transfer among Poor further restricts the acceptable members of the PGR subfamily to a unique index. If the reference mean income is not $\bar{y} = 0$, then there exist mean incomes at which the numerical representation is concave, which violates Transfer among Poor. To see why, consider Figure 3. When drawn at the reference mean income, the graph of the PGR at $\bar{y}$ is linear, as shown in Figure 3.b for the PGR at the origin. When drawn at a larger mean income than the reference, its graph is piecewise-linear and convex because the income threshold is then larger than at the reference mean income, as shown in Figure 3.c for the PGR at the origin. If the reference value for mean income is not zero, then there exists values of mean income at which the income threshold is lower than at the reference mean income and the graph is piecewise-linear and concave.

The PGR at the origin representing an absolute-homothetic EO defines a new index of poverty. I present this new index in the coming subsection.

4.3 Presentation of the new index

Given an endogenous line $z$, how can the practitioner compute the index identified above?

The first step is to select an absolute-homothetic EO by fixing the absolute threshold $z^a$. The value for parameter $z^a$ is selected to be either the subsistence threshold or a meaningful threshold for absolute material deprivation for the empirical question tackled by the practitioner. The choice of $z^a$ defines an absolute-homothetic EO. The second step is mechanical and simply amounts to computing the mathematical expression of the index $P$. This mathematical expression, illustrated in Figure 3, is the PGR at the origin for the selected EO.

In practice, from distribution $y$ with mean income $\bar{y}$, compute the censored distribution $\hat{y}$ by setting the income of all non-poor agents equal to $z(\bar{y})$. Compute then the equivalent gap distribution $g^0$ from the censored distribution $\hat{y}$. The equivalent gap of agent $i$ is defined as:

$$g^0_i := \frac{z^0 - e^0(\hat{y}_i, \bar{y})}{z^0},$$ (5)
where $e^0(y, \overline{y})$ is the equivalent income at $\overline{y} = 0$ given the selected EO, and $z^0$ is the intercept of the endogenous line. See Section 7 for an empirical illustration.

The new index is then simply the average equivalent gap:

$$P(y) := \frac{1}{n} \sum_{i=1}^{n} g^0_i.$$

Notice that the equivalent gap is different from the normalized gap presented in (3). The conceptual difference is that the former gives priority to subsistence over social participation by comparing individual situations using an absolute-homothetic EO. The equivalent gap is only equal to the normalized gap in the special case in which the EO is homothetic ($z^a = 0$), thereby denying the existence of an absolute form of poverty.

**Interesting features of the index**

I argue in what follows that this new index is conceptually simple, it yields judgments in line with intuitions and it is decomposable between absolute and relative poverty.

The index is conceptually simple for two main reasons. First, the index makes a clear distinction between absolutely poor agents and relatively poor agents. The latter are never considered to be poorer than the former. Then, its expression is the average equivalent gap, interpretable as the average individual poverty in the population.

This additive index satisfies both *Monotonicity in Income* and *Transfer among Poor*. Furthermore, the index inherits the judgments of its absolute-homothetic EO. I emphasize that, when comparing poverty using this index:

- An extra dollar has the same impact on global poverty when it is given to an absolutely poor agent in a low-income country as when it is given to an absolutely poor agent in a high-income country.

- An extra dollar has more impact on global poverty when it is given to a relatively poor agent in a low-income country than when it is given to a relatively poor agent in a high-income country. Even if bringing an agent from the subsistence threshold to the poverty threshold has the same impact on her individual poverty in both countries, it is more costly to do so in the high-income country.

- Growth, however unequally distributed, decreases the individual poverty of absolutely poor agents.\(^{27}\)

- On the contrary, growth should not be too unequally distributed in order for the individual poverty of relatively poor agents to decrease.

A corollary of the last two bullet points is that this index concludes that growth, if strong enough, eventually eradicates absolute poverty but not necessarily relative poverty. Whether the latter form of poverty is eventually eradicated depends on the distributive aspects of growth.

\(^{26}\)Remember that from a theoretical perspective, comparing two different countries or the same country at different points in time is equivalent.

\(^{27}\)This judgment resonates with the ideas of Sen (1983): “If there is starvation and hunger, then - no matter what the relative picture looks like - there clearly is poverty. In this sense, the relative picture – if relevant – has to take a back seat…”
Finally, this additive index is decomposable between the absolute and relative aspects of income poverty. Absolutely poor agents have income below the absolute threshold $z_a$, whereas relatively poor have income between the absolute threshold and the poverty threshold in their society. The numerical representation attributes an individual poverty equal to zero for non-poor agents and equal to one for agents with zero income. The key parameter $z_a$ measures which fraction of the zero-one range is attributed to absolute poverty. This fraction corresponds to the evolution of the individual poverty of an agent, from 1 to $1 - z_a$, when bringing her income from zero to the absolute threshold. The complement of this fraction is attributed to relative poverty. Hence, the individual poverty $d(y_i, \overline{y})$ of an absolutely poor agent can be decomposed between its absolute contribution $d^{pa}(y_i, \overline{y})$ and its relative contribution $d^{pr}(y_i, \overline{y})$,

$$d(y_i, \overline{y}) = d^{pa}(y_i, \overline{y}) + d^{pr}(y_i, \overline{y}),$$

where

$$d^{pa}(y_i, \overline{y}) := \frac{z_a - y_i}{z_a},$$
$$d^{pr}(y_i, \overline{y}) := \frac{z_0 - z_a}{z_0}.$$

The individual poverty of a relatively poor agent is directly equal to its relative contribution $d^{r}(y_i, \overline{y})$. Let $q^a$ be the number of absolutely poor agents in $y$. By definition, the number of relatively poor agents equals $q - q^a$. The index can be decomposed in the following way:

$$P(y) = P^a(y) + P^r(y),$$

where

$$P^a(y) := \frac{1}{n} \left( \sum_{i=1}^{q^a} d^{pa}(y_i, \overline{y}) + \sum_{i=1}^{q^a} d^{pr}(y_i, \overline{y}) \right),$$
$$P^r(y) := \frac{1}{n} \sum_{i=q^a+1}^{q} d^{r}(y_i, \overline{y}).$$

Index $P$ is hence decomposable between the contribution $P^a$ of absolutely poor agents and the contribution $P^r$ of relatively poor agents. The contribution of absolutely poor agents can be further decomposed. Term 1 in (7) measures the absolute contribution due to their individual poverty, coming from earning less than the subsistence threshold. Term 1 is ordinally equivalent to the PGR based on an absolute line whose threshold is the absolute threshold. Term 2 measures the relative contribution due to the individual poverty of absolutely poor agents. Term 3 in (8) accounts for the individual poverty of relatively poor agents, and therefore measures the relative poverty of relatively poor agents. Terms 2 and 3 together measure the total relative poverty in the population.
A last remark relates to the key parameter \( \frac{z}{z_0} \). Given a particular poverty line, the domain of absolute-homothetic EOs has the absolute threshold as the unique parameter. Therefore, the index proposed is technically a family of indices, parameterized by \( \frac{z}{z_0} \). If this fraction tends to one, the index tends to consider absolute and relative poverty in lexicographic order. In this case, any two income distributions are first compared based on the absolute poverty of absolutely poor agents, using term 1 in \( (7) \). If the comparison is non-conclusive, then relative poverty enters the picture. On the other hand, if \( \frac{z}{z_0} \) tends to zero, there exist no absolutely poor agents and the index becomes the standard PGR based on the endogenous line. These two limit positions are rather extreme and the value of this fraction should hence not deviate too much from one half. Most importantly, the parameter \( z^a \) should be a meaningful absolute threshold for the question tackled.

Given its interesting features, this index is a good candidate for comparing poverty between societies having different standards of living. I conduct an empirical application using this index in Section 7. In the next two sections, I investigate the robustness of Theorems 1 and 2 to several assumptions.

5 Robustness with mean income as the income standard

I show in this section that, when mean income is the income standard, any other index satisfying Monotonicity in Income and Transfer among Poor should be "close" to the index presented above.

5.1 Outside the extended FGT family

The very sharp conclusions of Theorem 2 are valid for numerical representations in the extended FGT family. I investigate in this subsection the robustness of these conclusions outside that family. I show by means of an example that, for other families, the discriminating power of Monotonicity in Income is less strong but the PGR at the origin still emerges as the focal numerical representation. Any other numerical representation satisfying Monotonicity in Income and Transfer among Poor must be close to the PGR at the origin.

For simplicity, the poverty line is linear and the EO is homothetic. Given these assumptions, I define the quadratic family of numerical representations. This family has no particular ethical appeal but is useful to illustrate the trade-off emerging from Monotonicity in Income.

Definition 6 (Quadratic Family).
For any homothetic EO, the numerical representation \( d \) belongs to the quadratic family if for all \((y_i, \bar{y}) \in X_p\):

\[
d(y_i, \bar{y}) = \left(1 - \frac{y_i}{z(\bar{y})}\right) + \alpha \left(\frac{y_i}{z(\bar{y})}^2 - \frac{y_i}{z(\bar{y})}\right) \quad \text{with} \ \alpha \in [-1, 1].
\]

The quadratic family admits a unique parameter \( \alpha \) interpreted as poverty aversion. The case \( \alpha = 0 \) corresponds to the standard PGR. Quadratic poverty indices satisfy Domination among Poor only when \( \alpha \) belongs to \([-1, 1]\), a range
which allows for much less variety of judgments around the PGR than the extended FGT family. The restrictions on \( \alpha \) under which Monotonicity in Income is satisfied are stated in Theorem 3 and illustrated in Figure 4.b. The coefficient of poverty aversion is bounded above and below and those bounds depend monotonically on the poverty line’s slope.

**Theorem 3** (Bounds on poverty aversion around PGR).
Let \( z \) be a linear poverty line with slope \( \bar{s} \). Let \( P \) be an additive poverty index based on a homothetic EO below \( z \) with a numerical representation in the quadratic family.

\( P \) satisfies Monotonicity in Income if and only if:

\[
\frac{(\bar{s} - 1)(1 + \bar{s})}{(1 + \bar{s})} \leq \alpha \leq \frac{4 - \bar{s} + 4(1 - \bar{s})^{1/2}}{(\bar{s} + 8)}
\]  

(9)

**Proof.** See in Appendix 9.4. ■

The steeper the slope, the narrower is the range of acceptable values for poverty aversion around the case \( \alpha = 0 \), corresponding to the PGR. There is no collapse towards the PGR when the slope is equal to one. There exist indices exhibiting a – slightly – higher poverty aversion than the PGR that respect Monotonicity in Income and Transfer among Poor.

Theorem 3 is obtained for homothetic EOs and linear lines. Defining the extended quadratic family using equivalent income functions, a similar bound result can be derived for any absolute-homothetic EO below a monotonic line. The PGR at the origin is not the only index satisfying the two properties. Outside the extended FGT family, there are acceptable indices with larger poverty aversion. However, this bound result shows that the numerical representation of alternative indices should not be too far from the PGR. The steeper the poverty line, the closer these indices are to the PGR at the origin. In this sense, the PGR at the origin is focal.

So far, the EO has been assumed absolute-homothetic. As argued in sections 2 and 4, there are good ethical reasons for flat equivalence curves below the
Figure 5: Homothetic-homothetic EO based on a piecewise-linear line.

absolute threshold. Nevertheless, the homotheticity of equivalence curves above the absolute threshold has just been presented as a convenient assumption. A natural question to ask is whether choosing an EO from a different domain allows for a wider set of indices satisfying both Monotonicity in Income and Transfer among Poor. As shown in the next subsection, the mere existence of such indices is not guaranteed, even for EOs that “almost” belong to the absolute-homothetic domain. What is more, the PGR at the origin is still the focal index with good properties.

5.2 Beyond absolute-homothetic orderings

This section provides an additional reason to rely on absolute-homothetic EOs. Some additive indices based on absolute-homothetic EOs satisfy both Monotonicity in Income and Transfer among Poor.28 There are no such indices if the EO departs too much from being absolute-homothetic.

For simplicity, the poverty line is piecewise-linear. For such lines, I define a domain extending the absolute-homothetic domain. The homothetic-homothetic domain \( \mathcal{R}_{HH} \), illustrated in Figure 5, is defined from the general domain \( \mathcal{R} \) by the additional restriction Homothetic-Homothetic Piecewise-Linear.

EO restriction 5 (Homothetic-Homothetic Piecewise-Linear).

There exist two piecewise-linear curves \( x \) and \( z \) defined by:

\[
x(y) = \begin{cases} 
x_0 & \text{if } y \leq y_k^x, \\
 x_0 + s_x(y - y_k^x) & \text{else},
\end{cases} \quad
z(y) = \begin{cases} 
z_0 & \text{if } y \leq y_k^z, \\
 z_0 + s_z(y - y_k^z) & \text{else},
\end{cases}
\]

with \( 0 \leq s_x \leq s_z \), \( 0 < x_0 < z_0 \) and \( y_k^z \geq z_0 \) such that for all \((y_i, y)\), \((y_i', y)\) \( \in \mathcal{X}_p\):

(i) Homothetic equivalence curves below \( x \).

if \( y_i < x(y) \) and \( \frac{y_i}{x(y)} = \frac{y_i'}{x(y')} \), then \((y_i, y) \sim (y_i', y)\).

(ii) Homothetic equivalence curves between \( x \) and \( z \).

if \( y_i \geq x(y) \) and \( \frac{y_i - x(y)}{x(y) - y_k^x} = \frac{y_i' - x(y)}{x(y) - y_k^x} \), then \((y_i, y) \sim (y_i', y)\).

28I am grateful to Martin Ravallion for having pointed out that the existence of additive indices respecting Monotonicity in Income is not guaranteed for all EO in \( \mathcal{R} \).
For a given poverty line $z$ as defined in restriction 5 and $x^* < z^0$, the subdomain of EOs $\mathcal{R}^{HH}(z, x^*)$ is parameterized by the slope $s_x$ of the intermediate line $x$ defined in restriction Homothetic-Homothetic Piecewise-Linear:

$$\mathcal{R}^{HH}(z, x^*) := \{ \succeq \in \mathcal{R}^{HH} \mid \succeq \text{ is below line } z \text{ and } x^0 = x^* \}.$$  

Let $\succeq_{s_x}$ denote a generic element in $\mathcal{R}^{HH}(z, x^*)$. The case $s_x = 0$ corresponds to the absolute-homothetic EO at the origin of the subdomain $\mathcal{R}^{HH}(z, x^*)$. The case $s_x = s^h_x := \frac{z^0 - x^0}{x^0}$ corresponds to the homothetic EO below the poverty line. The case $s_x = s_z$ is the limit since larger intermediate slopes entail that both lines cross.

Theorem 4 consists of two claims. Claim 1 says there is a range of values for the slope parameter $s_x$, centered on the value $s_x = s^h_x$ making the EO homothetic, outside which no additive poverty index satisfies both properties. Claim 2 says that if the slope parameter $s_x$ is smaller than $s^h_x$ and if the numerical representation belongs to the extended FGT family, then the two properties force the numerical representation to be the PGR at the origin.

**Theorem 4** (Non absolute-homothetic EOs and PGR at the origin).

Let $z$ be a piecewise-linear poverty line with $y_k \geq z^0$ and $\bar{s} > 0$. Let $x^* > 0$ be such that $x^* < z^0$. Let $\succeq_{s_x}$ be an EO belonging to the subdomain $\mathcal{R}^{HH}(z, x^*)$.

1. There exists an additive index $P$ based on $\succeq_{s_x}$ satisfying both Monotonicity in Income and Transfer among Poor if and only if
   - either $s_x = 0$,
   - or for some $s_x$ and $\overline{s}_x$ with $s_x < \overline{s}_x < s_z$ we have:
     $$\underline{s}_x \leq s_x \leq \overline{s}_x.$$  

2. Assume $s_x \in [\underline{s}_x, \overline{s}_x]$ with $s_x \geq 0$. Let $P$ be an additive index based on $\succeq_{s_x}$ with a numerical representation belonging to the extended FGT family. The two following statements are equivalent:
   - $P$ satisfies both Monotonicity in Income and Transfer among Poor.
   - The numerical representation of $P$ is the PGR at the origin.

**Proof.** See in Appendix 9.5. $\blacksquare$

The domain $\mathcal{R}^{HH}$ of homothetic-homothetic EOs is defined without flat equivalence curves below an absolute threshold. Extending the definition of the domain $\mathcal{R}^{HH}$ to a domain $\mathcal{R}^{AHH}$ of absolute-homothetic-homothetic EOs with $z^a < x^0$ is straightforward. The last result can then be extended to EOs in $\mathcal{R}^{AHH}$. This extended result implies for the definition of absolute-homothetic EOs that have homothetic curves above the subsistence threshold is not just a

\[ s_x := \frac{z^0 - x^0}{x^0} \quad \text{and} \quad \overline{s}_x := \frac{1}{2} \left( \left( \frac{x^0}{z^0 - x^0} \right)^2 + 4\overline{s}_x \frac{x^0}{z^0 - x^0} \right)^{0.5} - \frac{x^0}{z^0 - x^0}. \]

29 The expressions for $\underline{s}_x$ and $\overline{s}_x$ are respectively:
convenient assumption but rather a precondition for the existence of indices satisfying both Monotonicity in Income and Transfer among Poor. Other EOs admitting poverty indices with these properties are not too far from being absolute-homothetic, as shown by the acceptable range around the homothetic value of the slope parameter given in Claim 1 of Theorem 4.

I have made the claim that Monotonicity in Income cannot be satisfied by an index based on an EO far from satisfying restriction Absolute-Homotheticity. This raises the question of the relationship that poverty axioms and EO restrictions have. What is their relative status in the case that an incompatibility arises? In my view, they have an equal status in the sense that they both constrain the set of acceptable indices. The difference is the channel through which they constrain them. Axioms constrain the comparison of distributions whereas EO restrictions constrain the comparison of individual bundles. When an incompatibility arise between a set of axioms and EO restrictions, one must arbitrate between them on the basis of their respective normative merits. I see no reason to systematically give priority to one type of “index constraint” over the other.

The message of this section is that the PGR at the origin is the focal numerical representation satisfying both Monotonicity in Income and Transfer among Poor. This conclusion is derived when considering that mean income is the relevant reference statistic for standards of living. In the next section, I argue why mean income is a good income standard for poverty measurement. Moreover, I show that the index proposed is still very relevant when using other income standards.

6 Income standards other than the mean

I discuss in this section the choice of the income standard to which the poverty line is sensitive. I argue that median income is not a good income standard for poverty measurement and that other income standards are preferable, such as the mean or a lower partial mean. Finally, I study the robustness of Theorems 1 and 2 when using poverty lines sensitive to income standards different than mean income.

An income standard is a reference statistic gauging the size of an income distribution. Let \( f : \mathbb{R}_+^N \to \mathbb{R}_+ \) denote an income standard. As for poverty indices, income standards can be derived from the properties defining its concept. The two properties more specific to income standards are Normalization and Linear homogeneity. Normalization requires that if all incomes in a distribution are equal, then the income standard is also equal to the common income. Linear homogeneity requires that if all incomes in a distribution are multiplied by a common factor, the value taken by the income standard is also multiplied by the same factor.

Besides the mean, there exist four types of income standards that are in common use (Foster et al., 2013). These four types are quantiles (e.g. the median), generalized means (e.g. the geometric mean), partial means (e.g. mean among the 99% least rich individuals) and the Sen mean. All four types of income standards are presented and discussed in this section.

The choice of income standard is important because it defines the channel through which the income of other agents affect individual poverty. More specif-
ically, it defines the distributional changes altering the income threshold and, hence, the individual poverty of poor agents. Poverty judgments hence depend on the income standard used.

I defend the use of mean-sensitive lines even if median-sensitive lines are often used in practice. For example, the AROP measure of the European Commission uses a median-sensitive line. Both statistics have different advantages and flaws.

The main advantage of the mean corresponds to a major flaw of the median. A poverty measure aims at evaluating the impact that economic policies have on the worse-off individuals. If some policies impact growth, many policies have only redistributive consequences. I find highly counter-intuitive that policies whose unique impact are regressive transfers from the middle class to the rich are deemed to be poverty reducing. Axiom Transfer among Non-Poor requires the index not to be affected by redistributions among non-poor individuals. An index satisfying this axiom is therefore immune to “redistributive manipulations”. Transfer among Non-Poor is a weakening of Focus.

Poverty axiom 10 (Transfer among Non-Poor).
For all $\delta > 0$ and all $y, y' \in Y$ with $n(y') = n(y)$, if $y_j - \delta = y'_j > y'_k = y_k + \delta$, $k > q(y)$ and $y'_i = y_i$ for all $i \neq j, k$, then $P(y) = P(y')$.

For mean-sensitive indices, Transfer among Non-Poor is implied by Domination among Poor. In contrast, de Mesnard (2007) has shown that median-sensitive indices behave very counter-intuitively when income distributions experience an increase in inequality. The issue does not only show up in theory, it is particularly problematic in a World in which intra-country inequalities are on the rise (Bourguignon, 2013). An illustration of such behavior took place in New-Zealand between 1981 and 1992. According to Easton (2002), the implementation of policies inducing regressive transfers led to a decrease in the income of the bottom 80% of households. Nevertheless, the median-sensitive HC dropped due to the large decline in median income and some institutions used these figures to argue the regressive policies were a success.

The main drawback of mean-sensitive indices is that the mean is affected by “outliers”. What if a policy incentivize a very rich individual – say Bill Gates – to immigrate to the country, or simply allows an individual to flourish and become very rich? This could be good news but some fear that a mean-sensitive index systematically concludes otherwise. In theory, this need not necessarily be the case. Indeed, the conclusion depends on the redistributive system of the country. If the presence of very rich benefits to the poor – say via the country’s tax-and-transfers system – then the value returned by a mean-sensitive index can decrease. In this sense, a mean-sensitive index can be used to judge a country’s institutions. In practice, however, the income standard is not always computed from administrative data but often from random samples. The median is known to be more robust than the mean in random samples (Cowell and Victoria-Feser, 1994). Median-sensitive lines have hence a less volatile income threshold.

For those who judge that the lower robustness of the mean is a more serious issue in practice than the “manipulability” of the median, mean income among the 99% least rich individuals can offer a good compromise. This partial mean

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30 The At Risk of Poverty measure is the Head-Count Ratio based on a relative line whose threshold is 60% of the median income.
is much less affected by outliers than the mean. The downside of such a partial
mean is that regressive redistributions among non-poor individuals benefiting
the 1% richest individuals affect it.

In the remainder of this section, I study the robustness of the results to
the use of different income standards. The median is first investigated before
turning to other income standards.

6.1 Median income

The median is a particular quantile. Let \( x \in [0, 100] \) be a percentile. The
quantile income at the \( x^{th} \) percentile in distribution \( y \) is the income level \( y_x \)
such that \( x \) percent of individuals earn more than \( y_x \) and \( 1 - x \) individuals
earn less. Quantile incomes are crude as they only provide information about a
specific point of the distribution.

Median income, corresponding to the case \( x = 50 \), is the income standard
such that half of the population earn more and half of the population earn less.
Formally, the definition of median income is slightly different for distributions
with even or odd number of dimensions. Median income \( y_m \) is defined to be the
income of agent \( m \) where \( m : \mathbb{N}_+ \to \mathbb{N}_+ \) is defined by:

\[
m := \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd}, \\ \frac{n}{2} & \text{otherwise}. \end{cases}
\]

Changing the income standard requires modifying several definitions. I
present here only the major non-straightforward modifications. For a given
median-sensitive poverty line \( z \), the results depend on the domain of income
distributions considered. Let \( Y^r \) be the domain of distributions containing a
strict minority of poor agents and let \( Y^p \) be its complement:

\[
Y^r := \{ y \in \mathbb{R}_+^n \mid z(y_m) \leq y_m \}, \\
Y^p := \{ y \in \mathbb{R}_+^n \mid 0 < y_m < z(y_m) \}.
\]

Given the median-sensitive line \( z \), let \( y_m^* \) be the lowest value of median income
for which \( y \in Y^r \), implicitly defined by \( z(y_m^*) = y_m^* \). This is the limit value for
median income above which a distribution belongs to \( Y^r \). Let \( Y := Y^p \cup Y^r = \{ y \in \mathbb{R}_+^n \mid y_m > 0 \} \) be the general domain of distributions. Poverty indices are
based on an equivalence ordering \( \succeq^m \) ranking the set of poor bundles

\[
X_p := \{ (y_i, y_m) \in X \mid z(y_m) > y_i \},
\]

where \( X := \mathbb{R}_+ \times \mathbb{R}_+^+ \). Poverty axioms as well as restrictions to endogenous
lines and EOs in the general domain \( \mathcal{R}_m \) are easily modified. Such modifications
allow characterizing additive poverty indices with median-sensitive lines for the
domain of income distributions containing a strict minority of poor agents.

**Theorem 5** (Characterization of median-sensitive additive poverty indices).

Let \( P : Y \to \mathbb{R} \) be a poverty index based on a median-sensitive poverty line.
Statement 2 implies statement 1.

1. On \( Y^r \), \( P \) is ordinally equivalent to an index \( P' : Y \to [0, 1] \) defined by

\[
P'(y) = \frac{1}{n} \sum_{i=1}^{n} d(y_i, y_m), \tag{10}
\]
where $d$ is a numerical representation of an EO in $\mathbb{R}_m$.

2. $P$ satisfies the modified versions of Domination among Poor, Weak Subgroup Consistency, Symmetry, Continuity and Replication Invariance.

Proof. See in Appendix 9.6.

With median-sensitive lines, the characterization of additive poverty indices is only valid on $Y^r$. On the general domain $Y$, additive indices satisfy the five axioms, but there might be other indices to do so.

The consequences of the modified version of Monotonicity in Income are different than for mean-sensitive poverty lines. This axiom constrains the domain of median-sensitive lines for which there exists additive indices respecting it. Additive indices respect Monotonicity in Income when their poverty line is flat for all values of median income below $y_m^*$.

Theorem 6 (Flat median-sensitive lines for low median incomes).

Let $z$ be a monotonic median-sensitive poverty line with $z^0 > 0$. Let $P: Y \rightarrow [0, 1]$ be an additive poverty index based on an absolute-homothetic EO below $z$. The following two statements are equivalent.

1. $P$ satisfies Monotonicity in Income.
2. $z$ is flat for all $y_m < y_m^*$.

Proof. See in Appendix 9.7.

For all distributions in $Y^r$, median income is above the poverty threshold and hence the incomes of poor agents do not affect the reference statistic. The modified version of Monotonicity in Income puts no extra constraint on additive indices as this axiom is implied by Domination among Poor. For distributions in $Y^p$, the median income is below the threshold. If the median income increases by an amount not sufficient for the median agent to change, the reference statistic changes by the same amount, irrespective of the number of agents. This drastic impact drives the result. In contrast, mean income changes only by a fraction $\frac{1}{n}$ of the amount gained by a poor agent.

I argued above that median-sensitive lines lead to counter-intuitive judgments about the impacts of regressive redistributive policies. If their flaws are not judged serious enough for switching to mean-sensitive lines, this research still provides good reasons for adopting the index proposed in Section 4. Indeed, contrary to the HC or the PGR, this index gives priority to subsistence over social participation.

6.2 Other income standards

Besides the mean and the median, other income standards can be used as reference statistic. I discuss the robustness of the results in each case.

Partial means

Partial means return mean income for a subset of the distribution. Two types are in common use: lower partial means and upper partial means. As for quantiles, they are attached to a percentile $x \in [0, 100]$. In distribution $y$, the
lower partial mean below $x$, denoted $f_{lpm}(y, x)$, is the mean income among the bottom $x$ percent of income earners. On the contrary, the upper partial mean above $x$ is the mean income among the top $100 - x$ percent of income earners.

I only consider lower partial means because they better capture the evolution of the cost of social participation for poor individuals. Furthermore, since lower partial means are not affected by outliers, lines sensitive to these income standards offer a good compromise between the issues attached to mean and median-sensitive lines, respectively.

The additive representation result holds for the lower partial mean below $x$ if the set of distributions considered only contains distributions for which the percentage of poor individuals is less than $x$:

$$Y_{lpm} := \{ y \in \mathbb{R}_+^N | z(f_{lpm}(y)) \leq \frac{y}{x} \frac{x}{n} \},$$

where $y_{\frac{x}{n}}$ denotes the income of the agent whose index $i$ is the largest natural number less than or equal to $\frac{x}{100}n$.

The characterization of the PGR at the origin as the only numerical representation inside the FGT family satisfying modified versions of Monotonicity in Income and Transfer among Poor holds when using lower partial means. Formal statements and proofs of these two claims may be found in Appendix 9.8.

**Generalized means**

Generalized means form a class of income standards putting more emphasis on the bottom or on the top of the income distribution, depending on the value taken by its unique parameter $\beta \in (-\infty, +\infty)$. Generalized means, denoted $f_{gm}(y, \beta)$, are defined in the following way (Atkinson, 1970):

$$f_{gm}(y, \beta) := \begin{cases} \left( \frac{y_1 + \cdots + y_n}{n} \right)^{\frac{1}{\beta}} & \text{if } \beta \neq 0, \\ (y_1 \times \cdots \times y_n)^{\frac{1}{\beta}} & \text{if } \beta = 0. \end{cases}$$

If $\beta < 1$, then the bottom of the distribution is emphasized, if $\beta > 1$, then the top of the distribution is emphasized. The most popular members of this class are the arithmetic mean ($\beta = 1$), the geometric mean ($\beta = 0$) and the harmonic mean ($\beta = -1$).

I only consider generalized means with $\beta < 1$ as they better capture the evolution of the cost of social participation for poor individuals. These generalized means are not well-suited income standards for my purpose. Any EO respecting modified versions of the basic restrictions defining $\mathcal{R}$ must have all its equivalence curve flat. Such an EO cannot account for the impact that the relative situation has on individual poverty.

**Theorem 7** (Non-flat EO violates Translation Monotonicity).

Let $f_{gm}$ be an income standard in the generalized mean family with $\beta < 1$. If $\succeq$...
is an EO respecting the modified version of **Strict Monotonicity in Income** and \( \succeq \) is non-flat, then \( \succeq \) violates the modified version of **Translation Monotonicity**.\(^{32}\)

**Proof.** See in Appendix 9.9. \( \blacksquare \)

This result is a consequence of the exponential expression of this income standard. A small increment given to an agent whose income is close to zero has a disproportionate impact on the value taken by the generalized mean. Then, the small increment received by another poor agent whose bundle is on a non-flat indifference curve cannot compensate for this disproportionate increase in income standard. The non-flat EO considers that the equal distribution of the additional resource made this other agent poorer, which violates **Translation Monotonicity**.

Notice that **Translation Monotonicity** is imposed as a restriction on the EO rather than as a poverty axiom. Imposing that equal increments reduces poverty as an axiom would be less strong. When **Domination among Poor** is imposed, the EO restriction **Translation Monotonicity** implies this associated axiom. Nevertheless, this EO restriction appears as a minimal limitation at the individual level to the importance of relative aspects of income. Therefore, I conclude from Theorem 7 that generalized means should not serve as income standards for poverty measurement.

**Sen mean**

The Sen mean, denoted \( f_{sm} \), is interpreted as the expected minimum value among two income draws with replacement in the distribution.

\[
\begin{align*}
    f_{sm}(y) &:= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min\{y_i, y_j\} \\
    &= \frac{1}{n^2} \left( (2n-1)y_1 + (2n-3)y_2 + \cdots + 3y_{n-1} + y_n \right).
\end{align*}
\]

By definition, this income standard cannot be larger than the mean income. The Sen mean places more emphasis on incomes at the bottom of the distribution.

Since the Sen mean does not satisfy a separability property, the equivalent of the additive representation theorem does not hold. In particular, additive indices do not satisfy the modified version of **Weak Subgroup Consistency**.

Unlike for generalized means, the modified version of **Translation Monotonicity** only requires the equivalence curves of the EO to have slopes no larger than one.\(^{33}\) Using the Sen mean as the income standard therefore allows us to account for the impact that relative income has on individual poverty. Nevertheless, the exact implications of the modified version of **Monotonicity in Income** for Sen-mean-sensitive indices is still an open question.

---

\(^{32}\) An EO is **non-flat** if there exists \((y_i, f_{sm}(y)) \in X_p\) such that \(s(y_i, f_{sm}(y)) > 0\), i.e. the slope in \((y_i, f_{sm}(y))\) is strictly positive.

\(^{33}\) For all \(y \in \mathbb{R}_+^n\), given the mathematical expression of the Sen mean, we have \((\nabla f_{sm}(y) \cdot \mathbf{1}_n) = 1\).
Moving average of an income standard

I discuss in this subsection an important point valid for any choice of income standard. Endogenous measures are regularly criticized for the counter-intuitive judgments they sometimes provide when a distribution is affected by a negative shock. Think of a transient economic crisis. Assume that even if all incomes decrease, the crisis has a smaller effect on the incomes at the bottom of the distribution than at the top. Endogenous measures can conclude that poverty has decreased, a highly debatable judgment. This problem is of course coming from the endogeneity of the line.

Based on such examples, some argue against the use of endogenous measures. Instead, they suggest using absolute measures whose absolute line is unchanged over many years and then updated to account for changes in the standard of living. After having changed the line, comparisons across the two periods – the periods before and after the update – are typically made using the new absolute line. It should be clear that this approach does not account for social participation effects (illustrated in Table 1). As a result, growth is deemed poverty reducing, regardless of how unequally distributed its gains are.

Another point of view on the counter-intuitive judgments made by endogenous measures is that the income standard is not appropriate. In practice, endogenous lines have their income threshold updated each time the poverty measure is recomputed, typically every year. The cost of social participation has some inertia and does not react as quickly. At the beginning of a crisis, poor individuals have a lower income and face almost the same costs of social participation. It takes some time before people adapt their social standards and expectations. A solution would then be to introduce some inertia in the income standard. This can be done by letting the poverty line evolve with a moving average of the values taken by the income standard over several years.\[34\]

This section has discussed the choice of an income standard. I emphasized that median-sensitive lines lead to counter-intuitive judgments when intra-country inequality increases. Therefore, median-sensitive lines are not well-suited for the evaluation of unequal growth. If the lack of robustness of the mean in random samples is judged too serious, a good compromise is to use a partial mean, such as mean income among the 99\% least rich agents. For any choice of income standard, the index proposed in Section 4 is a strong candidate for replacing standard indices that give no priority to subsistence over social participation, such as the HC and the PGR.

7 Empirical illustration

In this section, I apply the new index using World Bank data. The objective is to verify that the index proposed is well-suited for evaluating unequal growth. First, using different poverty measures, I compare poverty between several low-income low-inequality countries and middle-income high-inequality countries. I show that the judgments obtained by a poverty measure based on my index are more in line with intuition than those obtained by standard measures. Second, I use the poverty measure based on my index in order to evaluate whether

\[34\] I am grateful to Karel Van den Bosch and Tim Goedemé for having pointed out to me the usefulness of moving average income standards.
the economic growth taking place over the last 20 years in low- and middle-income countries was poverty reducing in spite of the increase in intra-country inequality. I discuss the variables influencing the answer.

The data is taken from PovcalNet, a website built by the World Bank that provides income and consumption data. This data is gathered from more than 850 surveys of randomly sampled households in 127 low- and middle-income countries between 1981 and 2010. The frequency and precision of the surveys vary from one country to another. In some countries, the surveys focus on income, whereas in others on the value of total consumption. In order to permit cross-country comparisons, the Bank translates the survey data by making use of the Purchasing Power Parity (PPP) exchange rates for household consumption from the 2005 International Comparison Program. The national income distributions presented in PovcalNet are estimated from the survey data. More information about the data can be found in Chen and Ravallion (2013).

7.1 A poverty measure based on my index

This section demonstrates how to apply the new index. I assume that the selected endogenous line has the following weakly relative definition, illustrated in Figure 6:

\[ z(\bar{y}) = \max(2, 0.625 + 0.5\bar{y}). \]

Its income threshold equals $2 a day in countries whose mean income is lower than $2.75 a day. The World Bank considers that $2 a day is the threshold for income poverty in developing countries. For mean incomes higher than $2.75 a day, this line has a constant slope of one half. Observe that the intercept $0.625 of this second part is positive. As a result, the line does not evolve as a constant fraction of the mean.

This line is very close to that used by Chen and Ravallion (2013). The only difference is that the income threshold for low-income countries used by these authors is $1.25 a day, considered by the World Bank as the threshold for


\[ \text{PovcalNet is the database used in Chen and Ravallion (2013).} \]
extreme poverty.\textsuperscript{37} For richer countries, these authors fit their line on national thresholds. Their premise is that thresholds adopted at a country level reflect a balance made between absolute and relative aspects of income. The endogenous line selected is of course debatable but the objective pursued here is not to argue in favor of its use but rather to pick one that seems reasonable and that serves for purposes of illustration.

The new index is based on an absolute-homothetic EO below the endogenous line. The only parameter of this family of EOs is the subsistence threshold $z^a$. I take $z^a$ to be the threshold for extreme poverty: $\$1.25$ a day. This threshold was computed as an average of income thresholds in the fifteen poorest countries of the World (Ravallion et al., 2009). Many among these countries establish their national thresholds based on the cost of a bundle of goods whose consumption guarantees to reach a minimal level of physical survival (including a minimal nutrition level). Therefore this choice seems natural for $z^a$. Individuals earning less than $\$1.25$ a day are deemed absolutely poor and those earning more than $\$1.25$ a day but less than the endogenous threshold are deemed relatively poor.

The absolute-homothetic EO defined is illustrated in Figure 6a. The poverty measure based on my index is denoted $P^{EL}$, where the superscript is meant to indicate that it is based on the endogenous line. Given the endogenous line selected and the choice of $z^a$, this poverty measure has the following mathematical expression.\textsuperscript{38}

\begin{equation}
P^{EL}(y) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{2 - e^k(y_i, \overline{y})}{2} \right),
\end{equation}

where $e^k(y_i, \overline{y}) = \begin{cases} y_i & \text{if } y_i \leq 1.25, \\ 1.25 + (2 - 1.25) \frac{y_i - 1.25}{\overline{y} - 1.25} & \text{otherwise}. \end{cases}$

Judgements based on $P^{EL}$ are compared with those obtained by four other measures. Among the four alternative measures, three are based on the Head-Count Ratio while the last is based on the Poverty Gap Ratio. The first measure, $HC^{AL}$, is an absolute measure corresponding to the fraction of individuals whose income is below the absolute line defined by the subsistence threshold $\$1.25$ a day. The second, $HC^{RL}$, is a relative measure corresponding to the fraction of individuals whose income is below the relative line whose threshold is half the mean income. This measure provides some information about the inequality in the distribution. The third measure, $HC^{EL}$, is an endogenous measure corresponding to the fraction of individuals whose income is below the endogenous line defined above. The last measure, $PGR^{EL}$, is the Poverty Gap Ratio below the endogenous line, defined by:

\begin{equation}
PGR^{EL}(y) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{z(\overline{y}) - y_i}{z(\overline{y})} \right).
\end{equation}

\textsuperscript{37}It makes little sense for my purpose to consider that agents whose income is $\$1.25$ a day have the same individual poverty than agents at the poverty line in richer countries since I consider $\$1.25$ a day to be the subsistence threshold.

\textsuperscript{38}Given that the endogenous line is flat for mean incomes below $y^k = 2.75$, the PGR at the origin is equivalent to the PGR at $y^k$. 

37
This last measure satisfies both Monotonicity in Income and Transfer among Poor but gives no priority to subsistence over social participation.

I now consider the relations existing between $P^{EL}$ and $P^{GREL}$. For mean incomes below $2.75$ a day, the endogenous line is flat. The respective EOs of $P^{EL}$ and $P^{GREL}$, illustrated respectively in Figure 6.a and 6.b, are hence equivalent for these low values of mean incomes. As a result, $P^{EL}$ and $P^{GREL}$ return equal values for very poor countries. Above $2.75$ a day, $P^{EL}$ systematically returns lower figures than $P^{GREL}$ because the absolute-homothetic EO of $P^{EL}$ associates to any bundle an equivalent income at $2.75$ a day larger than the one associated to the same bundle by the homothetic EO of $P^{GREL}$. Therefore, if distribution A has a larger mean income than distribution B and $P^{EL}$ concludes that there is more poverty in A than in B, then $P^{GREL}$ draws the same conclusion. Index $P^{GREL}$ places more emphasis on poverty in richer countries as its homothetic EO weighs more the relative aspect of individual poverty.

7.2 Empirical results

The data extracted from PovcalNet is used for computing the five poverty measures. I first show that, when dealing with unequal growth, $P^{EL}$ makes poverty judgments that are more in line with intuition than those of the other four measures.

Table 4 provides figures for six countries in 2010. The countries are sorted in increasing order of mean income. Three low-income low-inequality countries are considered, namely Ethiopia, Nepal and Ivory Coast. Their mean incomes amount to $2$, $2.2$ and $3$ a day respectively and their Gini coefficients in 2010 amount to $34\%$, $33\%$ and $43\%$. Three middle-income high-inequality countries are considered, namely Bolivia, South Africa and Brazil. Their mean incomes amount to $8.3$, $8.4$ and $13.8$ a day respectively and their Gini coefficients in 2010 amount to $50\%$, $63\%$ and $54\%$. Remember that for my purpose, the distributions of two countries can equally be interpreted as two distributions corresponding to the same country but at different points in time.

<table>
<thead>
<tr>
<th>Countries</th>
<th>Mean</th>
<th>Gini</th>
<th>$HC^{AL}$</th>
<th>$HC^{RL}$</th>
<th>$HC^{EL}$</th>
<th>$P^{GREL}$</th>
<th>$P^{EL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ethiopia</td>
<td>2.0</td>
<td>34</td>
<td>30.6</td>
<td>17.7</td>
<td>65.0</td>
<td>23.1</td>
<td>23.1</td>
</tr>
<tr>
<td>Nepal</td>
<td>2.2</td>
<td>33</td>
<td>24.8</td>
<td>18.5</td>
<td>56.3</td>
<td>18.7</td>
<td>18.7</td>
</tr>
<tr>
<td>Ivory Coast</td>
<td>3.0</td>
<td>43</td>
<td>22.7</td>
<td>30.0</td>
<td>47.6</td>
<td>18.3</td>
<td>17.4</td>
</tr>
<tr>
<td>Bolivia</td>
<td>8.3</td>
<td>50</td>
<td>13.4</td>
<td>43.3</td>
<td>48.3</td>
<td>25.3</td>
<td>16.5</td>
</tr>
<tr>
<td>South Africa</td>
<td>8.4</td>
<td>63</td>
<td>13.8</td>
<td>57.1</td>
<td>61.3</td>
<td>32.8</td>
<td>17.6</td>
</tr>
<tr>
<td>Brazil</td>
<td>13.8</td>
<td>54</td>
<td>5.4</td>
<td>43.1</td>
<td>46.5</td>
<td>22.1</td>
<td>11.7</td>
</tr>
</tbody>
</table>

All poverty measures and the Gini coefficients are expressed in %. Mean incomes are expressed in $a$ a day (2005 PPP). Source: PovcalNet.

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39The Gini coefficient is a popular measure of inequality. The larger the Gini coefficient, the larger is inequality. The figures were obtained online from the World Bank Poverty and Equity Database on the 24th of August 2015, www.povertydata.worldbank.org. The Gini coefficient is measured in 2010 for Ethiopia and Nepal; in 2009 for Bolivia, South Africa and Brazil and in 2008 for Ivory Coast.
$HC^{AL}$ is strongly negatively correlated with mean income and $HC^{RL}$ is strongly positively correlated with inequality, as measured by the Gini coefficient.$^{40}$ $HC^{RL}$ concludes that middle-income countries, having a larger income inequality, have by far the largest poverty. $HC^{AL}$ reaches opposite conclusion. On the sole basis of these two measures, it is hence difficult to balance the absolute and relative aspects of growth. The three measures based on the endogenous line are more nuanced. $PGR^{EL}$ places more emphasis on poverty in richer countries and concludes that the two poorest countries are Bolivia and South Africa. In contrast, the two poorest countries according to $P^{EL}$ are low-income countries, namely Ethiopia and Nepal.

Pairwise comparisons of countries having different mean incomes illustrate the different judgments made by $P^{EL}$, $PGR^{EL}$ and $HC^{EL}$. $PGR^{EL}$ and $HC^{EL}$ conclude that there is less – or approximately equal – poverty in Ivory Coast than in Brazil, even if the fraction of absolutely poor individuals is much higher in the former (22.7 %) than in the latter (5.4%). In contrast, $P^{EL}$ places more emphasis on the absolute aspects of income poverty and concludes that there is more poverty in Ivory Coast than in Brazil. Oppositions of the same type can be found when comparing South Africa with Nepal or Ivory Coast, or when comparing Brazil with Bolivia. Observe that $P^{EL}$ does not always follow the judgments of $HC^{AL}$. Unlike $P^{EL}$, $HC^{AL}$ concludes that there is much more poverty in Nepal and Ivory Coast than in South Africa. If Nepal and Ivory Coast underwent a very unequal growth transforming their distributions into that of South Africa, whose distribution is very polarized, $P^{EL}$ would not lead to conclusions as enthusiastic as those obtained from $HC^{AL}$. Observe that the difference in judgments described above are based on large differences in the respective figures.

Table 4 demonstrates that the poverty judgments drawn from $P^{EL}$ can be radically different from those obtained with the other four measures. Moreover, the judgments drawn from $P^{EL}$ seem to be in line with basic intuitions. Next, $P^{EL}$ is used in order to evaluate the impact of the unequal growth taking place over the period 1990-2010 in different geographic entities. The decomposability of $P^{EL}$ allows us to decompose the fraction of poor individuals ($HC^{EL}$) between those that are absolutely poor ($HC^{AL}$) and those that are “only” relatively poor. Furthermore, the figure for $P^{EL}$ can be decomposed between the contribution of absolutely poor agents ($P^a$) and that of relatively poor agents ($P^r$). These decompositions are illustrated in Figure 7 for the World, urban China and Mexico. In Figure 7, the contribution of absolutely poor agents ($P^a$) is further decomposed between its absolute ($P^aa$) and relative components ($P^ar$).$^{42}$

$^{40}$In the sample, the coefficients of correlations are -0.97 and 0.99 respectively.
$^{41}$The figures for the World are an aggregate of the figures for the low- and middle-income countries, weighted by their population. The figures for urban China are obtained by computing the endogenous threshold for the mean income in urban China.
$^{42}$ $P^aa$ and $P^ar$ correspond respectively to term 1 and term 2 in (7).
Table 5: Evaluation of several unequal growths.

<table>
<thead>
<tr>
<th>Geo Entity</th>
<th>Year</th>
<th>Mean</th>
<th>$HC^{RL}$</th>
<th>$HC^{AL}$</th>
<th>$HC^{EL}$</th>
<th>$P^{EL}$</th>
<th>$P^a/P^{EL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>World</td>
<td>1990</td>
<td>3.0</td>
<td>21.2</td>
<td>43.0</td>
<td>70.7</td>
<td>30.7</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>2010</td>
<td>4.9</td>
<td>26.4</td>
<td>20.8</td>
<td>52.7</td>
<td>17.7</td>
<td>0.66</td>
</tr>
<tr>
<td>Urban China</td>
<td>1990</td>
<td>1.9</td>
<td>9.1</td>
<td>23.4</td>
<td>61.2</td>
<td>18.9</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>2010</td>
<td>7.1</td>
<td>21.7</td>
<td>0.6</td>
<td>30.6</td>
<td>4.7</td>
<td>0.08</td>
</tr>
<tr>
<td>Costa Rica</td>
<td>1990</td>
<td>7.0</td>
<td>31.5</td>
<td>8.4</td>
<td>40.0</td>
<td>11.4</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>2010</td>
<td>15.3</td>
<td>40.3</td>
<td>2.6</td>
<td>43.7</td>
<td>8.9</td>
<td>0.22</td>
</tr>
<tr>
<td>Mexico</td>
<td>1990</td>
<td>7.8</td>
<td>24.1</td>
<td>4.5</td>
<td>29.2</td>
<td>7.4</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>2010</td>
<td>10.6</td>
<td>35.8</td>
<td>0.7</td>
<td>41.2</td>
<td>7.5</td>
<td>0.05</td>
</tr>
<tr>
<td>Hungary</td>
<td>1996</td>
<td>8.8</td>
<td>9.8</td>
<td>0.2</td>
<td>16.0</td>
<td>1.7</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>2010</td>
<td>12.5</td>
<td>15.2</td>
<td>0.2</td>
<td>20.1</td>
<td>2.2</td>
<td>0.06</td>
</tr>
</tbody>
</table>

All poverty measures are expressed in %. Mean income is expressed in $ a day (2005 PPP). $P^a$ corresponds to the contribution to $P^{EL}$ of absolutely poor agents, defined by (7). Source: PovcalNet.

The World and urban China experienced a large decline in income poverty over the period: $P^{EL}$ dropped by 42% and 75%, respectively. In other words, in spite of the increase in income inequality, particularly important in urban China as indicated by $HC^{RL}$, $P^{EL}$ concludes unambiguously that growth has been poverty reducing.\footnote{It is the intra-country inequality that is accounted for when discussing the evolution of inequality in the World. Intra-country inequality influences poverty in the World via its impact on the country-specific endogenous thresholds.} These reductions reflect primarily the changes in absolute poverty. Absolute poverty was a main concern in both entities in 1990. In the World for example, 43% of individuals were absolutely poor in 1990 and these individuals contributed to 82% of $P^{EL}$. In 2010, only 20.8% of individuals remained absolutely poor in the World, contributing then to 66% of $P^{EL}$. For urban China, absolute poverty has been almost eradicated over the period. These evolutions and trade-offs appear clearly when studying the graphs decomposing $P^{EL}$ in Figure 7. The decrease in $P^{EL}$ in both entities is clearly driven by changes in $P^a$ whereas at the same time $P^r$ does not change much.

Costa Rica and Mexico experienced a lower reduction in poverty than the World and urban China over this period. $P^{EL}$ dropped by 22% in Costa Rica whereas it returned to its initial value in Mexico. The increase in relative poverty mitigated the significant reduction in absolute poverty achieved by the two countries. Absolute poverty was an important concern in 1990 – 53% of $P^{EL}$ for Costa Rica and 39% of $P^{EL}$ for Mexico – although not as dominant as for the World and urban China. The fraction of absolutely poor individuals fell from 8.4% to 2.6% in Costa Rica and from 4.5% to 0.7% in Mexico. At the same time however, the large increase in inequality in these two countries implied that more individuals were poor in 2010 than in 1990, as shown by $HC^{EL}$. Again, the trade-offs for Mexico appear clearly when studying the graphs decomposing $P^{EL}$ in Figure 7. In Mexico, the large increase in inequality taking place between 1990 and 1994 increased significantly $P^r$. The later reduction in $P^a$ only compensated for the increase in $P^r$. Appendix 9.10 contains a further analysis of the Mexican case based on cumulative distributions of income and individual poverty.
Figure 7: Evolution of income poverty between 1990 and 2010 in urban China, Mexico and the World as measured by $P^{EL}$. The left graphs show the decomposition of poor agents ($HC^{EL}$) between absolutely poor ($HC^{AL}$) and relatively poor, together with the endogenous threshold. The right graphs show the decomposition of $P^{EL}$ between the contribution of absolutely poor agents ($P^a$) and that of relatively poor agents ($P^r$), together with mean income. $P^a$ is further decomposed between its absolute ($P^{aa}$) and relative contributions ($P^{ar}$). Source: PovcalNet.
Hungary experienced an increase in poverty over the period 1996 – 2010, in spite of an increase of 43% of its mean income. $P^{EL}$ increased by 30% in Hungary over the period. Absolute poverty was not an important concern in 1996 – $P^a$ was less than 10% of $P^{EL}$ in 1996 – and did not change significantly over the period. On the contrary, income inequality increased and 20% of individuals were poor in 2010 whereas only 16% of individuals were poor in 1996. The increase in $P^{EL}$ is directly driven by the increase in $P^r$.

The distinction between absolutely and relatively poor agents and the decomposability of the index make it possible to separately track these two forms of poverty and aggregate them in a coherent way. I illustrate this possibility based on Figure 7. In urban China in 1990, 23.4% of individuals were absolutely poor and 37.8% were relatively poor, adding up to 61.2% of poor individuals. Overall, the poverty index for the income distribution in 1990 takes a value of 18.9%. This value of income poverty can be decomposed into the contribution of absolutely poor agents (11.8%) and that of relatively poor agents (7.1%). Hence, absolutely poor agents contributed to 62% of income poverty, which shows that absolute poverty was the main issue in urban China in 1990. In 2010, 0.6% of individuals were absolutely poor and 30% were relatively poor, adding up to 30.6% of poor individuals. Overall, the poverty index for the income distribution in 2010 takes a value of 4.7%, a figure 75% lower than that of 1990. This lower value of income poverty can be decomposed into the contribution of absolutely poor agents (0.4%) and that of relatively poor agents (4.3%). Hence, absolutely poor agents contributed to 8% of income poverty. This demonstrates that the reduction in absolute poverty is responsible for most of this three-quarters reduction in income poverty. Moreover, it shows that relative poverty became the main issue in urban China in 2010.

Analyzing with $P^{EL}$ several unequal growths has shown that very different conclusions can be drawn by this measure. Different factors influence the conclusions of $P^{EL}$. A key factor is the importance for $P^{EL}$ of absolute poverty at the beginning of the period. If absolute poverty is not the main concern, like in Hungary, the increase in inequality entails an increase in $P^{EL}$.

Altogether, $P^{EL}$ confirms that poverty reduction has been impressive over the last decades in low- and middle-income countries (“the World” in Table 5). In fact, poverty decreased even more than Head-Count based measures suggest. Over the period 1990-2010, even if the fraction of poor individuals decreased only by 25%, $P^{EL}$ concludes that income poverty was reduced by 42%.

8 Concluding remarks

Comparing income poverty between societies with different standards of living has always been done with extreme caution. This caution follows from the inability of standard poverty measures to consider simultaneously the absolute and relative aspects of income poverty. Bringing together the concepts of endogenous lines (Foster, 1998; Ravallion and Chen, 2011) and other-regarding preferences, I show how these aspects can be combined by endogenous poverty measures based on a new index; therefore providing a firmer foundation for these comparisons.

The distrust of standard poverty measures has complicated the evaluation of unequal growth. A literature proposing several definitions for pro-poor growth
emerged in order to fill the gap. Araar and Duclos (2009) classified the different proposals in two categories, the absolute and the relative pro-poorness measures. The existence of these two categories shows that the pro-poor growth literature is confronted to the difficulty of considering simultaneously the absolute and relative aspects of income. My index constitutes a possible answer to this difficulty. In the spirit of Ravallion and Chen (2003), growth could be deemed pro-poor if it leads to a decrease in an endogenous poverty measure based on my index. The endogenous line and the subsistence threshold become then the key parameters for the evaluation of the pro-poorness of growth.

There are several direct applications for this research. A prominent example is the measurement of income poverty by the World Bank. This institution recently established a commission aimed at advising it on the best way to monitor the realization of its twin goals. The decomposition of the new index between absolute and relative poverty should simplify the analysis and the communication on the progress achieved towards its twin goals. In the same vein, the EU Commission could replace the AROPE measure by a measure based on the new index. Countries whose official income poverty definition is judged nonsatisfactory could find interest in the new index. The United States constitute a prominent example as several observers like Ruggles (1990) and Citro and Michael (1995) questioned its absolute line. See Blank (2008) for a review of the political initiatives that have attempted to modify it.

Switching the poverty measure changes the evaluation of policies aimed at reducing poverty. Up to now, policy makers used absolute measures for policy evaluation in low- and middle-income countries and relative measures in high-income countries. This practice ensures that the most relevant aspect of income poverty was captured in each case, at the cost of ignoring the other aspect. The limitation of this practice is that it yields extreme judgments on growth. On the one hand, absolute measures evaluate policies creating economic growth positively, regardless of their distributional aspects. On the other hand, relative measures judge redistributive policies positively, regardless of their impact on growth, as long as the inequality experienced by the poor decreases. The evaluation of policies with a measure based on the new index solves these limitations. This index combines both aspects and weighs more the most relevant aspect in each case. Indeed, its judgments depend on the importance of absolute poverty in the initial distribution. As a consequence, the policies recommended by this index should be in line with what the specific situation requires without being extreme.

The index proposed has applications outside income poverty measurement. If the emphasis has been put on income, the index can measure the poverty in any other resource for which both the absolute and relative aspects matter, like

\footnote{44A basic definition of pro-poorness is to require that growth reduces a poverty measure based on the Watts index (Ravallion and Chen, 2003). This definition has been called “weak” as it does not specify a minimal extent of poverty reduction for a given growth in mean income (Kakwani, 2008). Alternatively, growth can be deemed pro-poor if the average growth among the poor is higher than the growth in mean income (Duclos, 2003). Another contribution from Foster and Szekely (2008) aims at getting around the arbitrariness inherent in a poverty line. These authors suggest comparing the growth rate in mean income with that of different generalized means. The lower the parameter $\beta$ defining a particular generalized mean, the more emphasize is put on incomes at the bottom of the distribution.}

education or health.

More generally, this research contributes to attempts at introducing relative considerations into the normative evaluation of economic outcomes. Adapting utilitarian indicators to other-regarding preferences is of course straightforward. Nevertheless, utilitarian indicators often provide judgments that are at odds with equality of opportunity principles. Social ordering functions, i.e. indicators of well-being derived from efficiency and fairness principles, offer in that respect a good alternative to utilitarian indicators (Fleurbaey and Maniquet, 2011). Recently, a nascent literature has started investigating how to derive social ordering functions for economies populated with other-regarding agents. See Treibich (2014) for the single-good case and Decerf and Van der Linden (2014) for the multi-good case.
9 Appendix

9.1 Proof of Theorem 1

I show that statement 2 implies statement 1. Take any endogenous line \( z \) and any poverty index \( P \) satisfying the five axioms.

**STEP 1:** From a poverty ordering on income distributions to a poverty ordering on distributions of individual poverty.

I define a continuous mapping \( m : Y \to \mathbb{R}^N \), where \( N := \{ n \in \mathbb{N} | n \geq 2 \} \). Let \( \succeq \) be a EO in \( \mathcal{R} \) whose unanimous judgments among the poor are respected by \( P \). By *Domination among Poor*, such \( \succeq \) exists. Consider any numerical representation \( d \) of \( \succeq \). For each \(( y, \overline{y}) \in X \), let \( \nu := d(y, \overline{y}) \). Mapping \( m \) is defined for all \( y \in Y \) such that

\[
m(y) = (\nu_1, \cdots, \nu_{n-1}) \equiv \nu.
\]

Observe that if distribution \( y \) has \( n \) components, then \( m(y) \) has \( n - 1 \) components. The size of distribution \( \nu \) is taken to be \( n - 1 \) as for all \( y \in Y \) we have \( d(y_n, \overline{y}) = 0 \) since \( y_n \geq z(\overline{y}) \) and is hence omitted. Mapping \( m \) is continuous since \( d \) is continuous in both its arguments and the mean is a continuous function of its arguments. Given the numerical representation \( d \), mapping \( m \) returns the distribution of individual poverties corresponding to any income distribution.

I show for the mapping defined that \( m(Y) = V_d := [0, 1]^N \). The domain of images of \( Y \) through mapping \( m \) is hence a product space: \( V_d = \times_{i=1}^{N'} [0, 1] \). This means that (i) \( m(Y) \subseteq V_d \) and (ii) \( V_d \subseteq m(Y) \), that is for all \( \nu \in V_d \) there exists \( y \in Y \) such that \( m(y) = \nu \). If (i) follows directly from the definition of mapping \( m \), (ii) remains to be proven. Lemma 1 proves that \( V_d \subseteq m(Y) \).

**Lemma 1.** For all endogenous lines \( z \), \( \succeq \in \mathcal{R} \) and \( \nu \in V_d \), there exists \( y \in Y \) such that \( \nu = m(y) \).

**Proof.** Take any endogenous line \( z \), \( \succeq \in \mathcal{R} \) and \( \nu \in V_d \). Let \( g > 0 \) be such that \( g \geq z(g) \). Such \( g \) always exists by restriction Possibility of Poverty Eradication.

We construct \( y \) such that \( \overline{y} = g \) and \( m(y) = \nu \). For all \( i \leq q \), \( y \) is such that \( y_i := a_i \) defined implicitly by \( \nu_i := d(a_i, g) \). By restriction Minimal Absolute Concern and the continuity of \( d \), we have that \( a_i \in [0, z(g)) \) for all \( i \leq q \). Let \( y' \) be such that \( y'_j := y_i \) for all \( i \leq q \) and \( y'_j := g \) for all \( j \) with \( q + 1 \leq j \leq n \). We have \( \overline{y} \leq g \leq z(g) \leq g \). There exists hence \( l \geq g \) such that, if \( y_j := l \) for all \( j \) with \( q + 1 \leq j \leq n \), then we have \( \overline{y} = g \). As \( l \geq g \geq z(g) \), all agents \( j \) with \( q + 1 \leq j \leq n \) are non-poor. By construction we have \( m(y) = \nu \).

\( P \) is by definition the representation of a complete poverty ordering \( \succeq_Y \) on \( Y \). By *Domination among Poor*, for any two \( y, y' \in Y \) such that \( m(y) = m(y') \), we have \( P(y) = P(y') \). Therefore, the complete ordering \( \succeq_Y \) implies a complete ordering \( \succeq_{V_d} \) on \( V_d \) since \( V_d = m(Y) \). Ordering \( \succeq_{V_d} \) is defined such that for all \( y, y' \in Y \) we have \( y \succeq_Y y' \Leftrightarrow m(y) \succeq_{V_d} m(y') \). Ordering \( \succeq_{V_d} \) is continuous since the ordering on \( Y \) is continuous by Continuity and mapping \( m \) is continuous. Being continuous, ordering \( \succeq_{V_d} \) can be represented by a continuous index \( P^* : V_d \to \mathbb{R} \). In particular, ordering \( \succeq_{V_d} \) is represented by \( P^* \) defined such that for all \( \nu \in V_d \) and \( y \in Y \) with \( m(y) = \nu \), we have \( P^*(\nu) = P^*(m(y)) = P(y) \).
**STEP 2:** Index \( P^\nu \) representing ordering \( \succeq_{V_d} \) on distributions of individual poverty is additively separable.

If the assumptions of Theorem 1 in Gorman (1968) are all met, then for any \( n \in \mathbb{N} \) and any \( \nu \) of size \( n - 1 \), index \( P^\nu \) has the following functional form:

\[
P^\nu(u) = \tilde{F} \left( \sum_{i=1}^{n-1} \tilde{\phi}(\nu_i) \right)
\]

where \( \tilde{F} \) and \( \tilde{\phi} \) are strictly increasing functions.

Take any \( n \in \mathbb{N} \). For the remaining part of Step 2, I abuse slightly notation by denoting \( V_d \) the subset of \( V_d \) containing elements of size \( n - 1 \). The three assumptions required for this theorem are the following:

**Assumption 1:** There exists a complete and continuous ordering on a product space.

I proved in Step 1 that the ordering \( \succeq_{V_d} \) is complete and continuous on \( V_d \), which is a product space \( V_d = \times_{i=1}^{n-1} [0,1] \).

**Assumption 2:** Each sector \([0,1]_i\) of \( V_d \) has a countably dense subset, is arc-connected and is strictly essential. Strict essentiality means that for any subdistribution \( (\nu_1, \cdots, \nu_{i-1}, \nu_{i+1}, \cdots, \nu_{n-1}) \in \times_{j=i}^{n-1} [0,1] \), not all elements of \([0,1]_i\) are indifferent for the ordering \( \succeq_{V_d} \).

As all sectors are real intervals. Any sector therefore has a countably dense subset and is arc-connected. Strict essentiality follows directly from Domination among Poor together with the fact that for any \( i \leq n - 1 \) and any subdistribution \( (\nu_1, \cdots, \nu_{i-1}, \nu_{i+1}, \cdots, \nu_{n-1}) \), the individual poverty \( \nu_i \) is not constrained as the individual poverty of the agent \( n \) with highest income is discarded.\(^{46}\)

**Assumption 3:** Let \( S := \{[0,1]_1, \cdots, [0,1]_{n-1}\} \) be the set of sectors in \( V_d \) and \( A \subseteq S \) be any subset of sectors, we have that each \( A \) is separable. Separability means that for all \((u,w),(v,w),(u,t),(v,t) \in V_d \), we have \( P^\nu(u,w) \geq P^\nu(v,w) \iff P^\nu(u,t) \geq P^\nu(v,t) \). Separability is proven in two substeps.

**Substep 1:** Construct for each of the four distributions of individual poverty \((u,w),(v,w),(u,t),(v,t)\) a particular income distribution associated to it.

Construct \( y^1, y^2, y^3, y^4 \in Y \) such that \( m(y^1) = (u,w), m(y^2) = (v,w), m(y^3) = (u,t), m(y^4) = (v,t) \) and \( \overline{y}^1 = \overline{y}^2 = \overline{y}^3 = \overline{y}^4 = g \) with \( g \geq z(g) \). Such distributions exist and are constructed following the procedure given in Lemma 1.

Decompose in subgroups \( y^1 = (y^1_A, y^1_B, y^1_n) \), such that subdistributions \( y^1_A \) and \( y^1_B \) are associated – via the numerical representation \( d \) – to the subdistributions \( u \) and \( w \) respectively.\(^{47}\) Typically, \( \overline{y}^1_A \neq \overline{y}^1_B \neq g \) but the next operations aims at obtaining such equality.

\(^{46}\)In the definition and the proof of strict essentiality, the indices are not sorted by income level but refer to the identities.

\(^{47}\)For each element \( u_i \in u \) there exists \( y^1_i \in y^1_A \) such that \( u_i = d(y^1_i, \overline{y}^1) \). The same holds for \( w \) and \( y^1_B \).
Triplicate $y^1$ and re-organize the subgroups to obtain at least one non-poor agent per subgroup. Let $y_i^1 := (y_1^1, y_2^1, y_3^1) = (y_{A_1}^1, y_{B_1}^1, y_{C_1}^1)$. This triplication does not affect the mean: $\overline{y}^1 = \overline{y}^2$. Reorganize subgroups: $y_i^1 = (y_{A_i}, y_{B_i}, y_{C_i})$ with $y_{A_i} := (y_{A_1}^1, y_{A_2}^1, y_{A_3}^1)$ and $y_{B_i} := (y_{B_1}^1, y_{B_2}^1, y_{B_3}^1)$.

Letting $u := (u, u, w, w, 0)$ and $u' := (w, w, w)$, we have that

$$m(y_i^1) = (u, u, u, 0, w, w, 0) = (u', 0, w', 0),$$

as $d(y_i, g) = 0$ for any $y_i \geq z(g)$.

Construct $y_i^3$ such that $m(y_i^3) = u'$ with $\overline{y}_A^3 = g$ and $y_{B_i}^3$ such that $m(y_{B_i}^3) = w'$ with $\overline{y}_B^3 = g$. Those income distributions exist as proven in Lemma 1, as both subgroups $A'$ and $B'$ contain at least one non-poor agent.

The income distribution $y_i^* := (y_{A_i}^*, y_{B_i}^*, g)$ is such that $m(y_i^*) = (u', 0, w', 0)$. This distribution is such that $\overline{y}_i^* = g$ as its three subgroups have mean $g$.

Using the same procedure (decomposition, triplication, reorganization), construct successively $y^2, y^3, y^4$ and $y^{i*}, y^{i*}, y^{i*}$ such that:

- $y_i^* = (y_{A_i}^*, y_{B_i}^*, g)$ with $m(y_i^*) = (u', 0, w', 0) = (u, u, u, 0, w, w, 0)$,
- $y_i^{3*} = (y_{A_i}^{3*}, y_{B_i}^{3*}, g)$ with $m(y_i^{3*}) = (u', 0, w', 0) = (v, v, v, 0, w, w, 0)$,
- $y_i^{4*} = (y_{A_i}^{4*}, y_{B_i}^{4*}, g)$ with $m(y_i^{4*}) = (u', 0, w', 0) = (u, u, 0, t, t, t, 0)$,
- $y_i^{5*} = (y_{A_i}^{5*}, y_{B_i}^{5*}, g)$ with $m(y_i^{5*}) = (u', 0, t', 0) = (v, v, v, 0, t, t, 0)$.

For all $m \in \{1, 2, 3, 4\}$, we have $P(y_{i+m}) = P(y_{i})$ by Replication Invariance. As $(y_{i+m}, g) \sim (y_{i+m}, g)$ for all $i \leq q(y_{i+m})$, we have $P(y_{i+m}) = P(y_{i})$ by Domination among Poor. Therefore, proving $P(y_{i+m}) \geq P(y_{i+2}) \iff P(y_{i+3}) \geq P(y_{i+4})$ is equivalent to proving $P(u, w) \geq P'(v, w) \iff P''(v, t) \geq P''(v, t)$ for notational simplicity, drop the symbols * and ′ to name the new distributions and subgroups as the old ones.

Substep 2: Prove separability from judgments on the associated income distributions: $P(y_{A_i}, y_{B_i}, g) \geq P(y_{A_i}^*, y_{B_i}^*, g) \iff P(y_{A_i}^*, y_{B_i}^*, g) \geq P(y_{A_i}^*, y_{B_i}^*, g)$.

These income distributions are constructed such that $P(y_{A_i}^*) = P(y_{A_i}^*) = P(y_{A_i}^*) = P(y_{A_i}^*) = P(y_{A_i}^*)$. By assumption, we have $P(y_1^*) = P(y_2^*)$ and $P(y_3^*) = P(y_4^*)$ by Domination among Poor. By assumption, we have that $P(y_{A_i}^*) \geq P(y_{B_i}^*)$ by Weak Subgroup Consistency (remember all our subgroups have their mean equal to $g$). By Weak Subgroup Consistency again, this implies $P(y_{A_i}^*) \geq P(y_{A_i}^*)$.

Then, $P(y_{A_i}^*) \geq P(y_{A_i})$ together with $P(y_{A_i}^*) = P(y_{A_i}^*)$ and $P(y_{A_i}^*) = P(y_{A_i}^*)$ imply $P(y_{A_i}^*) \geq P(y_{A_i}^*)$. Two cases can arise.

- Case 1: $P(y_{A_i}^*) > P(y_{A_i}^*)$.
  As $P(y_{A_i}^*) = P(y_{A_1}^*)$, we have $P(y_{A_i}^*) = P(y_{A_i}^*)$ by Domination among Poor. Together we obtain $P(y_{A_i}^*, y_{B_i}^*, g) \geq P(y_{A_i}^*, y_{B_i}^*, g)$ by Weak Subgroup Consistency. This case is hence such that $P(y_{A_i}) \geq P(y_{A_i})$, as desired.

- Case 2: $P(y_{A_i}^*) = P(y_{A_i}^*)$.
  I show by contradiction this case is such that $P(y_{A_i}) \geq P(y_{A_i})$. Assume

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48 Strictly speaking Weak Subgroup Consistency cannot be applied again as subgroup $g$ contains a unique agent and hence does not belong to $Y$. Nevertheless, further replications of the income distributions solve the issue.
we have \( P(y_A^3, y_B^4, g) < P(y_A^3, y_B^4, g) \). As \( P(y_A^3) = P(y_A^4) \), Weak Subgroup Consistency implies that \( P(y_A^3, y_B^4, y_A^4, g) < P(y_A^3, y_B^4, y_A^4, g) \). Again, as \( P(y_B^4) = P(y_B^4) \), we obtain \( P(y_A^3, y_B^4, y_A^4, y_B^4, g) < P(y_A^3, y_B^4, y_A^4, y_B^4, g) \). This is a contradiction as the two distributions have equal poverty by Symmetry.

The two cases lead to \( P(y^3) \geq P(y^4) \), which proves separability.

As all three assumptions hold, we can use Theorem 1 in Gorman (1968) and obtain, for all \( \nu \in V_d \):

\[
P^\nu (\nu) = \tilde{F}' \left( \sum_{i=1}^{n-1} \tilde{\phi}_i (\nu_i) \right)
\]

where \( \tilde{F}' \) and \( \tilde{\phi}_i \) are strictly increasing functions. Functions \( \tilde{\phi}_i \) might still depend on the rank \( i \) of the considered agent. Nevertheless, since \( \leq_{V_d} \) is separable, we must have \( \phi_i = \tilde{\phi} + f(i) \). Defining \( \tilde{F}(x) := \tilde{F}(x + \sum f(i)) \), a translation of \( \tilde{F}' \), we can use (33) with function \( \tilde{\phi} \) independent of rank \( i \).

**STEP 3:** Show functions \( \tilde{F} \) and \( \tilde{\phi} \) do not depend on the number \( n \) of agents.

Theorem 1 in Gorman (1968) is valid for a fixed number \( n \) of agents. Therefore, when \( n \) is allowed to vary, equation (33) must be written:

\[
P^\nu (\nu) = \tilde{F}_n \left( \sum_{i=1}^{n-1} \phi_n (\nu_i) \right)
\]

I modify the proof of Foster and Shorrocks (1991) in order to show that these functions are independent of \( n \).

**Step 3.1:** Define transformations of \( \tilde{F}_n \) and \( \phi_n \) for normalization purposes.

Let \( F_n \) and \( \phi_n \) be the following transformations of \( \tilde{F}_n \) and \( \phi_n \):

\[
\phi_n (\nu_i) = n \left[ \phi_n (\nu_i) - \phi_n (0) \right],
\]

\[
F_n (x) = \tilde{F}_n \left[ x + (n-1) \phi_n (0) \right].
\]

These transformations allows rewriting last equation in the following way

\[
P^\nu (\nu) = F_n \left( \frac{1}{n} \sum_{i=1}^{n-1} \phi_n (\nu_i) \right),
\]

where \( \phi_n (0) = 0 \).

Since agent \( n \) is non-poor by definition, we have \( d(y_n, \mathbf{y}) = 0 \). Therefore, we obtain – slightly abusing notation (by introducing agent \( n \)'s zero individual poverty at the end of distribution \( \nu \)) – that for all \( n \geq 3 \):

\[
P^\nu (\nu) = F_n \left( \frac{1}{n} \left( \phi_n (0) + \sum_{i=1}^{n-1} \phi_n (\nu_i) \right) \right) = F_n \left( \frac{1}{n} \sum_{i=1}^{n} \phi_n (\nu_i) \right),
\]

where \( F_n \) and \( \phi_n \) are continuous, strictly increasing and \( \phi_n (0) = 0 \).
Step 3.2: Use Replication Invariance to prove functions $F_n$ and $\phi_n$ do not depend on $n$.

From the previous step, we have $\phi_n : [0, 1] \rightarrow [0, b_n]$ with $\phi_n(0) = 0$ for all $n \in \mathbb{N}_{++}$. Take any $y \in Y$ with dimension $n = 5$ such that a single agent is poor in $y$. Consider $x := (y, \ldots, y)$ a k-replication of $y$. Let $\nu := m(y) = (t, 0, 0, 0)$ be the individual poverty distribution associated to $y$ where $t$ can be any element in $[0, 1]$. Let $\nu' := m(x) = (t, \ldots, t, 0, \ldots, 0)$ be the individual poverty distribution associated to $x$ which contains $4k - 1$ zeros and $k \ t$'s. The dimension of $\nu$ is $r = 4$ and the dimension of $\nu'$ is $s = 5k - 1$. Therefore we have $s = k(r + 1) - 1 = kr + k - 1$.

Denoting $F := F_4$ and $\phi := \phi_4$, the relationship between $\phi$, $\phi_s$, $F$ and $F_s$ for all $t \in [0, 1]$ is computed using (13) and Replication Invariance:

\[ P^\nu(\nu) = F \left[ \frac{1}{5} \phi(t) \right] = F_s \left[ \frac{k}{5k} \phi_s(t) \right] = P^{\nu'}(\nu'), \]

\[ \phi_s(t) = 5F_s^{-1} \left[ F \left( \frac{1}{5} \phi(t) \right) \right]. \quad (14) \]

Replacing $\phi_s(t)$ in (13) by its value obtained in (14), we get:

\[ F^{-1}[P^\nu(\nu')] = F^{-1} \left[ F_s \left( \frac{1}{5} \sum_{i=1}^{5k} 5F_s^{-1} \left[ F \left( \frac{1}{5} \phi(\nu_i') \right) \right] \right) \right] \]

\[ = G_s^{-1} \left( \frac{1}{5} \sum_{i=1}^{5k} 5G_s \left( \frac{1}{5} \phi(\nu_i') \right) \right), \quad (15) \]

(16)

where $G_s(w) := F_s^{-1}(F(w))$ and $G_4(w) = F^{-1}(F(w)) = w$.

By Replication Invariance, we have that $F^{-1}[P^\nu(\nu')] = F^{-1}[P^{\nu'}(\nu')]$, which by (16) yields:

\[ G_s \left( \frac{1}{5} \phi(t) \right) = \left( \frac{1}{5} \sum_{i=1}^{5k} 5G_s \left( \frac{1}{5} \phi(\nu_i') \right) \right) \]

\[ = G_s \left( \frac{1}{5} \phi(t) \right) + \frac{4k - 1}{k} G_s(0), \]

which shows that $G_s(0) = 0$.

Consider now any $y' \in Y$ with dimension $n = 5$ such that two agents are poor in $y'$. Consider $x' := (y', \ldots, y')$ a k-replication of $y'$. Let $\nu := m(y') = (t, u, 0, 0)$ be the individual poverty distribution associated to $y'$ where $t$ and $u$ can be any element in $[0, 1]$. Let $\nu' := m(x') = (t, \ldots, t, u, \ldots, u, 0, \ldots, 0)$ be the individual poverty distribution associated to $x$ which contains $3k - 1$ zeros, $k \ t$'s and $k \ u$’s.

By Replication Invariance, we have that $F^{-1}[P^\nu(\nu)] = F^{-1}[P^{\nu'}(\nu')]$, which by (16) yields:

\[ \frac{1}{5} \phi(t) + \frac{1}{5} \phi(u) = G_s^{-1} \left( G_s \left( \frac{1}{5} \phi(t) \right) + G_s \left( \frac{1}{5} \phi(u) \right) \right), \]

which can be rewritten as the Jensen equation:

\[ G_s(x + x') = G_s(x) + G_s(x'), \]
that admits as general solution $G_s(x) = rsx + qs$. As $G_s(0) = 0$ we have $q_s = 0$.

Replacing $G_s$ by its expression in (16), we obtain

$$F^{-1}[P^\nu(\mu')] = \frac{1}{5k} \sum_{i=1}^{5k} \phi(\nu'_i).$$

Therefore, for any $y \in Y$ with dimension $5k$ and its associated $\nu = m(y)$:

$$P^\nu(\nu) = F\left(1\sum_{i=1}^{5k} \phi(\nu_i)\right)$$  \hspace{1cm} (17)

The same expression is valid for all $y \in Y$ with dimension $n$ as the same reasoning can be applied between $n(y)$ and the least common multiple between $n(y)$ and $5$.

Finally, transformations $d'$ and $G$ of respectively functions $\phi$ and $F$ guarantee that the domain of image of $d'$ is $[0,1]$. Letting $d'(y_i, \overline{y}) = \frac{\phi(d(y_i, \overline{y}))}{\phi(1)}$ and $G(x) = F(x\phi(1))$, we have for all $y \in Y$:

$$P^\nu(\nu) = G\left(1\sum_{i=1}^{n} d'(y_i, \overline{y})\right) = P(y)$$  \hspace{1cm} (18)

where $G$ is a continuous and strictly increasing function and $d'$ is a numerical representation of $\succeq$. As function $G$ is strictly increasing, $P$ is ordinally equivalent to $P' : Y \to [0,1]$ with $P'(y) = \frac{1}{n} \sum_{i=1}^{n} d'(y_i, \overline{y})$. This proves $P$ is an additive poverty index.

### 9.2 Proof of Lemma 2

The proof of Theorem 2 relies on Lemma 2, which gives a necessary condition and a sufficient condition for satisfying Monotonicity in Income. These conditions hold for the general domains of poverty lines, absolute-homothetic EOs and numerical representations.

The presentation of Lemma 2 requires introducing two definitions. For a given additive poverty index, the degree of priority of an income level over another at a certain mean income measures the ratio of the increase in the index if a marginal increase takes place at one income level rather than at the other.

\textbf{Definition 7} (Degree of Priority of $y_i$ over $y_j$ at $\overline{y}$). \hspace{1cm} $DP_{ij}(\overline{y}) := \frac{\partial_1 d(y_i, \overline{y})}{\partial_1 d(y_j, \overline{y})}$

$DP_{ij}(\overline{y})$ can be interpreted as the priority given by the index to an income level $y_i$ over another income level $y_j$ when mean income is $\overline{y}$.

Monotonicity in Income sets a lower and an upper bound on the degrees of priority granted by additive indices. These bounds depend on the slopes of the
equivalence curves at the bundles of the concerned agents. These slopes can be
defined using the numerical representation.\footnote{Again, the modification of this definition
for points at which $d$ is not differentiable is in Appendix 9.2.1. This definition allows attributing
a unique value of the slope even at points for which the equivalence curves of the EO exhibit a kink.}

**Definition 8** (Slope at $(y, \overline{y})$).

$$s(y, \overline{y}) := -\frac{\partial_2 d(y, \overline{y})}{\partial_1 d(y, \overline{y})}$$

The two general conditions are the following.

**Lemma 2** (Bounds on degrees of priority).

An additive poverty index based on an absolute-homothetic EO below an endogenous line satisfies **Monotonicity in Income**:

1. (sufficient condition) if for all $\overline{y} > 0$ and $y_i, y_j < z(\overline{y})$, we have:

   $$s(y_j, \overline{y}) \leq DP_{ij}(\overline{y})$$

   \hspace{1cm} (19)

2. (necessary condition) only if for all $\overline{y} > 0$ with $z(\overline{y}) \leq \overline{y}$ and all $y_i, y_j < z(\overline{y})$, (19) holds.

**Proof.** Consider any additive index $P$ based on an absolute-homothetic EO below an endogenous line. The index $P$ satisfies **Monotonicity in Income** if and only if for all $y \in Y$ and $i \leq q$ we have $\partial_i P(y_1, \ldots, y_n) \leq 0$. By the additively separable form of $P$, this inequality becomes by chain derivation:\footnote{The case of points at which $d$ is not differentiable is treated in Appendix 9.2.1.}

$$\partial_1 d(y_i, \overline{y}) + \sum_{j=1}^{n} \partial_2 d(y_j, \overline{y}) \partial_i \overline{y} \leq 0. \hspace{1cm} (20)$$

From the definition of the mean, we have $\partial_\overline{y} = \frac{1}{n}$. From the definition of $s(y_j, \overline{y})$, we get $\partial_2 d(y_j, \overline{y}) = -\partial_1 d(y_j, \overline{y}) s(y_j, \overline{y})$ for all $(y_j, \overline{y}) \in X$. Inequality (20) becomes:

$$\partial_1 d(y_i, \overline{y}) - \frac{1}{n} \sum_{j=1}^{n} \partial_1 d(y_j, \overline{y}) s(y_j, \overline{y}) \leq 0. \hspace{1cm} (21)$$

In the remainder of the proof, (21) is shown to imply the necessary and sufficient conditions linked to (19). Inequality (19) can be rewritten:

$$\underbrace{\partial_1 d(y_i, \overline{y}) - \partial_1 d(y_j, \overline{y}) s(y_j, \overline{y})}_{L_{21}} \leq 0. \hspace{1cm} (22)$$

**Necessity** of condition 2 is proved by contradiction. Assume (22) does not hold for some $y^1 \in Y$ with $z(\overline{y}^1) \leq \overline{y}^1$ and $y^1_i, y^1_j$ are such that $0 \leq y^1_i < y^1_j < z(\overline{y}^1)$. Therefore, at $(y^1_i, \overline{y}^1)$, $(y^1_j, \overline{y}^1) \in X_p$, we have for some $l > 0$ that $L_{22} = l$. I prove that for all $\epsilon > 0$, there exists $y^2 \in Y$ with $\overline{y}^2 = \overline{y}^1$ such that $|l - L_{21}(y^2)| < \epsilon$ and hence, for $\epsilon < l$, there exists an $y^2$ such that $L_{21}(y^2) > 0$, violating **Monotonicity in Income**. Construct $y^2$ such that
\begin{itemize}
\item \(y_1^2 := y_1^1\),
\item \(y_k^2 := y_k^1\) for all \(k\) with \(2 \leq k \leq n(y^2) - 1\) and
\item \(y_n^2 := n(y^2)\mathbf{y} - \sum_{h=1}^{n(y^2)-1} y_h^2\).
\end{itemize}
Notice \(y_n^2 \geq z(\mathbf{y}^1)\) since \(\mathbf{y} \geq z(\mathbf{y}^1)\), which implies \(y^2 \in Y\). For distribution \(y^2\), remembering that \(\partial_1 d(y_n^2, \mathbf{y}^1) = 0\), we have:
\begin{align*}
l - L_{21}(y^2) &= L_{22} - L_{21}(y^2) \\
&= -\frac{1}{n(y^2)} (2\partial_1 d(y_j^1, \mathbf{y}^1)s(y_j^1, \mathbf{y}^1) - \partial_1 d(y_l^1, \mathbf{y}^1)s(y_l^1, \mathbf{y}^1)).
\end{align*}
In order to show that \(|l - L_{21}(y^2)| < \epsilon\), two cases must be considered:
\begin{itemize}
\item Case 1: \(\partial_1 d(y_j^1, \mathbf{y}^1)\) and \(\partial_1 d(y_l^1, \mathbf{y}^1)\) are finite.
The distance \(|l - L_{21}(y^2)|\) can be made arbitrarily small by taking \(n(y^2)\) sufficiently large, implying \(L_{21}(y^2) > 0\), which violates (21) and hence Monotonicity in Income.
\item Case 2: \(\partial_1 d(y_j^1, \mathbf{y}^1)\) or \(\partial_1 d(y_l^1, \mathbf{y}^1)\) are not finite.
Observe first that if \(\partial_1 d(y_j^1, \mathbf{y}^1) = -\infty\) and \(\partial_1 d(y_l^1, \mathbf{y}^1)\) is finite, then (22) must hold.
Assume \(\partial_1 d(y_j^1, \mathbf{y}^1) = -\infty\). If (22) does not hold, then we have \(s(y_j^1, \mathbf{y}^1) > 0\) as \(d\) is strictly decreasing in \(y_j\). If \(\partial_1 d(y_j^1, \mathbf{y}^1)\) is finite, then \(L_{21}(y^2) > 0\) and Monotonicity in Income does not hold. If \(\partial_1 d(y_j^1, \mathbf{y}^1) = -\infty\), by the continuity of \(d\), there exists \(y_k^1\) close to \(y_1^1\) for which the equivalent of (22) does not hold and \(\partial_1 d(y_k^1, \mathbf{y}^1)\) is finite, leading again to a violation of the axiom.
\end{itemize}

The case for which \(0 \leq y_j^1 < y_k^1 < z(\mathbf{y}^1)\) leads to the same contradiction. The only difference lies in the construction of \(y^2\): \(y_{n(y^2)-1}^2 := y_j^1\), \(y_k^2 := y_j^1\) for all \(k\) with \(1 \leq k \leq n(y^2) - 2\). The condition is therefore necessary.

Sufficiency of condition 1 follows from the fact that, if there exists an \(y \in Y\) violating (21), inequality (22) is violated as well for a particular value of \(y_j\). For all \(y \in Y\) there exists \(y_j^2 \in [0, z(\mathbf{y})]\) such that, taking \(y_j := y_j^2\) in \(L_{22}\), we have
\[
L_{21}(y) < L_{22},
\]
which is:
\[
-\frac{1}{n} \sum_{j=1}^{n} \partial_1 d(y_j, \mathbf{y})s(y_j, \mathbf{y}) < -\partial_1 d(y_j^*, \mathbf{y})s(y_j^*, \mathbf{y}),
\]
where the strict inequality comes from the presence of the non-poor agent \(n\) for whom \(\partial_1 d(y_n, \mathbf{y}) = 0\). The key property for last inequality to hold is that \(\partial_1 d(y_j, \mathbf{y})\) and \(s(y_j, \mathbf{y})\) depend on the income of other agents only through their impact on mean income \(\mathbf{y}\). At mean income \(\mathbf{y}\), \(y_j^*\) is obtained by solving the following problem:
\[
y_j^* := \arg \max_{y_j \in [0, z(\mathbf{y})]} -\partial_1 d(y_j, \mathbf{y})s(y_j, \mathbf{y}).
\]
The symmetry of degrees of priority implies that the lower bound given in (19) is associated with an upper bound. Lemma 2 shows that the steeper the equivalence curves, the narrower is the range of acceptable degrees of priority. These rather obscure constraints have strong implications that are best illustrated on specific domains of poverty lines, EO’s and families of numerical representations.

9.2.1 Non-differentiability of numerical representation \( d \)

I extend in this subsection the definitions of degrees of priorities and slopes for bundles at which the numerical representation is not differentiable. I show how these extended definitions allows Lemma 2 to hold even at those bundles and hence everywhere for absolute-homothetic EOs.

Function \( d \) is differentiable almost everywhere as the function \( d \) is continuous. Consider any \((y_j^1, \overrightarrow{y})\), \((y_j^1, \overrightarrow{y}) \in X_p \) at which \( d \) is not differentiable. The definition of \( DP_{ij} \) at these bundles is given by

\[
DP_{ij}(\overrightarrow{y}) := \lim_{y_j \to y_j^1} \frac{\partial_1 d(y, \overrightarrow{y})}{\partial_1 d(y_j, \overrightarrow{y})}
\]

Either these limits take non-negative finite values or they tend to infinity, showing that \( DP_{ij} \in [0, \infty) \). For any \((y_j^1, \overrightarrow{y}) \in X_p \) at which \( d \) is not differentiable, the definition of the slope becomes

\[
s(y_j, \overrightarrow{y}) := \frac{\lim_{\overrightarrow{y} \to \overrightarrow{y}^+} \partial_2 d(y, \overrightarrow{y}) - \lim_{\overrightarrow{y} \to \overrightarrow{y}^-} \partial_2 d(y, \overrightarrow{y})}{\lim_{\overrightarrow{y} \to \overrightarrow{y}^-} \partial_2 d(y, \overrightarrow{y}) - \lim_{\overrightarrow{y} \to \overrightarrow{y}^+} \partial_2 d(y, \overrightarrow{y})} = \begin{cases} 0 & \text{if } y_j \leq z^a, \\ \frac{y_j - z^a}{z(\overrightarrow{y}) - z^a} \partial_z(\overrightarrow{y}) & \text{otherwise.} \end{cases}
\]

The definition of the slope at bundle \((y_j^1, \overrightarrow{y})\) where \( d \) is not differentiable implies

\[
\lim_{\overrightarrow{y} \to \overrightarrow{y}^+} \partial_2 d(y, \overrightarrow{y}) = - \lim_{\overrightarrow{y} \to \overrightarrow{y}^-} \partial_1 d(y, \overrightarrow{y}) s(y_j^1, \overrightarrow{y}).
\]

From the previous equation, we can extend (22) in Lemma 2, which must now be compared with an extended version of (20), that is obtained by chain derivation of \( P \) at \( y \):

\[
\lim_{y_j \to y_j^1} \partial_1 d(y, \overrightarrow{y}) + \sum_{j=1}^n \lim_{y_j \to y_j^1} \partial_2 d(y_j, \overrightarrow{y}) \partial_\overrightarrow{y} \leq 0.
\]

The reasoning given in the proof of Lemma 2 is then valid even at those points. This extension of the validity of Lemma 2 is only necessary for the
proof of Theorem 4. Indeed, other theorems relies on families of numerical representations that are differentiable everywhere.

Observe that non-smooth equivalence curves are not ruled out from absolute-homothetic EO\'s. Indeed, poverty lines can exhibit kinks, as it is the case at \( y^k \) for piecewise-linear lines. This non-smoothness is not problematic as the extended definition of slope given above guarantees there is a unique value of slope at these bundles. The evolution of slopes with \( \overline{y} \) is not continuous at \( \overline{y}^* \), but this does not affect Lemma 2, which provides conditions to be checked independently at each particular value of mean income \( \overline{y} \).

9.3 Proof of Theorem 2

Take any monotonic endogenous line \( z \) and any absolute-homothetic EO \( \succeq \) below \( z \). Take additive index \( P \) whose numerical representation \( d \) of \( \succeq \) belongs to the extended FGT family. I prove Theorem 2 claim by claim.

Claim 1: \( P \) satisfies Monotonicity in Income only if \( \alpha = 1 \).

The numerical representation of \( P \) belongs to the extended FGT family which means there exists \( \overline{z} \geq 0 \) such that for all \((y_i, \overline{y}) \in X_p\) we have

\[
d(y_i, \overline{y}) = \left( \frac{z(\overline{y}) - e^\alpha(y_i, \overline{y})}{z(\overline{y})} \right)^\alpha.
\] (23)

I show that if \( \alpha \neq 1 \) then the necessary condition given in Lemma 2 (see Appendix 9.2) for the associated additive poverty index \( P \) to satisfy Monotonicity in Income is violated.

Since line \( z \) is monotonic, there exists \( g > 0 \) with \( g \geq z(g) \) such that \( s(g) > 0 \). Consider any \((y^i_1, \overline{y}^i) \in X_p\) with \( \overline{y}^i = g \) and \( y^i_1 > z^\alpha \). By the monotonicity of \( z \), we have hence that \( \overline{y}^i \geq z(\overline{y}) \). From the necessary condition in Lemma 2, if there exists \( y^i_1 \) with \( y^i_1 < z(\overline{y}) \) and \( s(y^i_1, \overline{y}^i) > DP_{y^i}(\overline{y}^i) \) then Monotonicity in Income does not hold.\textsuperscript{55} I show below there exists \( y^i_j \) with \( y^i_1 < y^i_j < z(\overline{y}) \) leading to a violation of the necessary condition when \( \alpha \neq 1 \).

The degree of priority given by \( P \) to agent \( i \) over \( j \), when \( y^i_1 \leq y^i_j \) is obtained by chain derivation of (23):

\[
DP_{y^i}(\overline{y}^i) = \frac{\partial_i d(y^i_1, \overline{y}^i)}{\partial_1 d(y^i_1, \overline{y}^i)} = \left( \frac{z(\overline{y}) - e^\alpha(y^i_1, \overline{y})}{z(\overline{y}) - e^\alpha(y^i_j, \overline{y})} \right)^{\alpha-1} \frac{\partial_1 e^\alpha(y^i_1, \overline{y})}{\partial_1 e^\alpha(y^i_j, \overline{y})}, \tag{24}
\]

where \( \overline{y}^\alpha \) denotes the value of mean income at which \( d \) takes the exponential mathematical expression. Factor F1 in (24) is equal to one because the EO is absolute-homothetic. Indeed, absolute-homothetic implies that for all \( \overline{y} > 0 \) and \( y_i, y_j \in [z^\alpha, z(\overline{y})] \) we have

\[
\frac{e^\alpha(y_i, \overline{y}) - z^\alpha}{e^\alpha(y_j, \overline{y}) - z^\alpha} = \frac{y_j - z^\alpha}{y_i - z^\alpha}.
\] (25)

\textsuperscript{55}See Appendix 9.2 for the definitions of slope \( s(y^i_1, \overline{y}^i) \) and degree of priority \( DP_{y^i}(\overline{y}^i) \).
Therefore (24) can be simplified to

\[ DP_{ij}(\underline{y}) = \left( \frac{z(\underline{y}) - e^r(y_i, \underline{y})}{z(\underline{y}) - e^r(y_j, \underline{y})} \right)^{\alpha - 1}. \tag{26} \]

I now prove that the necessary condition is violated for \( y^*_j \) sufficiently close to \( z(\underline{y}) \). Three cases must be considered depending on the value taken by \( \alpha \).

- Case 1: \( 0 < \alpha < 1 \):
  When \( y^*_j \) tends to \( z(\underline{y}) \), we have that \( e^r(y_j, \underline{y}) \) tends to \( z(\underline{y}) \). From the exponential functional form of \( DP_{ij}(\underline{y}) \), for all \( \epsilon > 0 \), there exists hence \( y^*_j \in [y^*_j, z(\underline{y})] \) such that

\[ DP_{ij}(\underline{y}) = \left( \frac{z(\underline{y}) - e^r(y_i, \underline{y})}{z(\underline{y}) - e^r(y_j, \underline{y})} \right)^{\alpha - 1} < \epsilon. \]

As the poverty line is monotonic and \( \underline{y} = g \), we have \( s(z(\underline{y})) > 0 \). As the EO is absolute-homothetic and \( y^*_j > z^a \), we have \( s(y^*_j, \underline{y}) > 0 \) and for all \( y^*_j > z^a \) we have \( s(y^*_j, \underline{y}) > s(y^*_j, \underline{y}) \). As a result, for any \( \epsilon < s(y^*_j, \underline{y}) \) we have \( s(y^*_j, \underline{y}) > DP_{ij}(\underline{y}) \) and the necessary condition is violated.

- Case 2: \( \alpha > 1 \):
  As the numerical representation is differentiable for all \( y^*_j, y^*_j \) with \( z^a < y^*_j < y^*_j \), we have \( DP_{ij}(\underline{y}) = \frac{1}{DP_{ij}(\underline{y})} \). From the reasoning given for the case \( 0 < \alpha < 1 \), we have that for all \( \epsilon > 0 \), there exists \( y^*_j \in [y^*_j, z(\underline{y})] \) such that \( DP_{ij}(\underline{y}) < \epsilon. \) This leads to a violation of Monotonicity in Income for identic reasons.

- Case 3: \( \alpha = 0 \):
  Index \( P \) is an increasing transformation of the Head-Count Ratio. Monotonicity in Income is violated for any \( y \in Y \) with \( \underline{y} = g \) and one non-poor agent \( i \) has income \( y_i = z(\underline{y}) \).

**Claim 2:** \( P \) satisfies Monotonicity in Income and Transfer among Poor if and only if \( \alpha = 1 \) and \( \underline{y} = 0 \).

By Claim 1, \( P \) satisfies Monotonicity in Income only if \( \alpha = 1 \). Claim 2 is therefore proven by the combination of steps 1 and 2.

**Step 1:** If \( \alpha = 1 \), then \( P \) satisfies Transfer among Poor if and only if \( \underline{y} = 0 \).

\( P \) satisfies Transfer among Poor if and only if the numerical representation \( d \) is convex at all values of mean income. Formally, \( P \) satisfies Transfer among Poor if and only if for all \( \underline{y} > 0 \) and all \( y_i, y_j \in [0, z(\underline{y})] \) with \( y_i < y_j \) we have \( DP_{ij}(\underline{y}) \geq 1 \). Lemma 3 shows that we have \( DP_{ij}(\underline{y}) \neq 1 \) only in the case \( y_i < z^a < y_j \).

**Lemma 3.** Let \( \succeq \) be an absolute-homothetic EO below an endogenous line. Let \( d \) be a numerical representation of \( \succeq \) in the extended FGT family with \( \alpha = 1 \).
For all \((y_i, \overline{y}), (y_j, \overline{y}) \in X_p\) with \(y_i \leq y_j\), if \(DP_{ij}(\overline{y}) \neq 1\), then \(y_i < z^a < y_j\) and
\[
DP_{ij}(\overline{y}) = \frac{z(\overline{y}) - z^a}{z(\overline{y}) - z^a}. \tag{27}
\]

Proof. Consider any \((y_i, \overline{y}), (y_j, \overline{y}) \in X_p\) with \(y_i \leq y_j\). Given \(\alpha = 1\), the value taken by \(DP_{ij}(\overline{y})\) depends only on \(\overline{y}\) and on the relative positions of \(y_i, y_j\) and \(z^a\). Four cases must be considered.

• Case 1: \(y_i = y_j\)
  \(DP_{ij}(\overline{y}) = 1\) by the definition of \(DP_{ij}(\overline{y})\).

• Case 2: \(y_i < y_j \leq z^a\)
  Equation (24) holds as it does not depend on the particular value of mean income \(\overline{y}\). By absolute-homotheticity we have for all \(\overline{y} > 0\) and \(y_i \leq z^a\) that \(e^r(y_i, \overline{y}) = y_i\). As a result (26) holds as well. Replacing \(\alpha = 1\) leads to \(DP_{ij}(\overline{y}) = 1\).

• Case 3: \(z^a \leq y_i < y_j\)
  Equation (26) holds. Replacing \(\alpha = 1\) leads to \(DP_{ij}(\overline{y}) = 1\).

• Case 4: \(y_i < z^a < y_j\)
  As \(\alpha = 1\), the numerical representation at any point \((y_i, \overline{y}) \in X_p\)
  \[
d(y_i, \overline{y}) = \left(\frac{z(\overline{y}) - e^r(y_i, \overline{y})}{z(\overline{y})}\right).
\]
  As the EO is absolute-homothetic, for any \(y_i \leq z^a\) we have \(e^r(y_i, \overline{y}) = y_i\). As a result we have for any \(y_i \leq z^a\) that
  \[
  \partial_1 d(y_i, \overline{y}) = \frac{-1}{z(\overline{y})}.
\]
  As the EO is absolute-homothetic, for any \(y_j \geq z^a\) we have that
  \[
  e^r(y_j, \overline{y}) - z^a = (y_j - z^a)\frac{z(\overline{y})}{z(\overline{y})} - z^a,
\]
  implying for any \(y_j \geq z^a\) that
  \[
  \partial_1 d(y_j, \overline{y}) = \frac{-1}{z(\overline{y})} \left(\frac{z(\overline{y}) - z^a}{z(\overline{y}) - z^a}\right).
\]
  By the definition of \(DP_{ij}(\overline{y})\), we find for any \(y_i < z^a < y_j\) that (27) holds.

I show that \(\overline{y}^r = 0\) is sufficient and necessary for \(P\) to satisfy Transfer among Poor.

• Case \(\overline{y}^r = 0\) (sufficiency)
  This case is such that \(\overline{y}^r < \overline{y}\) for all \(\overline{y} > 0\). As \(z\) is assumed monotonic, we have that \(z(\overline{y}) \leq z(\overline{y})\). Then for all \(y_i, y_j \in [0, z(\overline{y})]\) with \(y_i < z^a < y_j\) we have by (27) and \(z(\overline{y}) \leq z(\overline{y})\) that \(DP_{ij}(\overline{y}) \geq 1\). This implies by Lemma 3 that, when \(y_i \leq y_j\), we have \(DP_{ij}(\overline{y}) \geq 1\) and hence the sufficient condition for Transfer among Poor holds.
• Case $\overline{y} > 0$ (necessity)
  This case is such that there exists $y < \overline{y}$ such that $z(y) < z(\overline{y})$. Indeed, if the poverty line is flat for all $y < \overline{y}$, then the numerical representation is linear in $y = 0$ as the EO is absolute-homothetic. Therefore the numerical representation is equivalent to $\overline{y} = 0$ and we have $y = 0$.

At $y < \overline{y}$ such that $z(y) < z(\overline{y})$ we have for any $y_t < z^a < y$ that $DP_{ij}(\overline{y}) < 1$ from (27), violating the necessary condition for Transfer among Poor.

Step 2: If $\alpha = 1$ and $\overline{y} = 0$, then $P$ satisfies Monotonicity in Income.

The sufficient condition for Monotonicity in Income given in Lemma 2 requires that for all $\overline{y} > 0$ and $y_t, y < z(\overline{y})$, we have $s(y_t, \overline{y}) \leq DP_{ij}(\overline{y})$.

As the EO satisfies Translation Monotonicity, we have for all $(y_t, \overline{y}) \in X_p$ that $s(y_t, \overline{y}) \leq 1$. The sufficient condition can therefore only be violated if $DP_{ij}(\overline{y}) < 1$.

From Lemma 3, the case $DP_{ij}(\overline{y}) \neq 1$ can only happen if one agent is absolutely poor (income below $z^a$) and the other relatively poor (income above $z^a$). As $\overline{y} = 0$, $\alpha = 1$ and since the line is monotonic, (27) shows that the relatively poor agent cannot have a priority over the absolutely poor agent strictly larger than 1. Therefore, the case $DP_{ij}(\overline{y}) < 1$ only happens if $y_t < z^a < y$.

As the EO is absolute-homothetic, if $y_t < z^a$ then $s(y_t, \overline{y}) = 0$ and the sufficient condition holds since $DP_{ij}(\overline{y})$ is non-negative by definition.

9.3.1 Piecewise-linear poverty line

This subsection shows that if the endogenous line is piecewise-linear, then there exists an upper-bound for the value of reference mean income below which the PGR at $\overline{y}$ satisfies Monotonicity in Income.

Theorem 8 (Upper-bound for reference mean income).

Let $z$ be a piecewise-linear poverty line with $\overline{y} \geq z^0$ and slope $\bar{s} > 0$. Let $P$ be an additive poverty index based on an absolute-homothetic EO below $z$ with a numerical representation in the extended FGT family with $\alpha = 1$.

1. $P$ satisfies Monotonicity in Income if and only if:

$$\overline{y} \leq \overline{y}^k + \left(\frac{1 - \bar{s}}{\bar{s}^2}\right)(z^0 - z^a).$$

Proof.

Step 1: $P$ satisfies Monotonicity in Income if and only if for all $\overline{y} \geq \overline{y}^k$ and all $y_t, y < z(\overline{y})$, we have $s(y_t, \overline{y}) \leq DP_{ij}(\overline{y})$.

Given $z$ is piecewise-linear, for all $\overline{y} \leq \overline{y}^k$ and all $y_t < z(\overline{y})$ we have $s(y_t, \overline{y}) = 0$. As a result, for all $\overline{y} \leq \overline{y}^k$ and all $y_t, y < z(\overline{y})$ inequality $s(y_t, \overline{y}) \leq DP_{ij}(\overline{y})$ holds. Therefore, the necessary condition for Monotonicity in Income given in Lemma 2 is also sufficient.
Step 2: $P$ satisfies *Monotonicity in Income* if and only if for all $\overrightarrow{y} \geq \overrightarrow{y}^k$ and $y_j < z(\overrightarrow{y})$ we have $s(y_j, \overrightarrow{y}) \leq \frac{z(\overrightarrow{y}) - z^a}{z(\overrightarrow{y}) - z^a}$.

As the EO satisfies Translation Monotonicity we have $s(y_j, \overrightarrow{y}) \leq 1$ for all $(y_j, \overrightarrow{y}) \in X_p$. Assuming without loss of generality that $y_i \leq y_j < z(\overrightarrow{y})$, by Lemma 3 we have that inequality $s(y_j, \overrightarrow{y}) \leq DP_{ij}(\overrightarrow{y})$ is violated only if $y_i < z^a < y_j$. In that case, by (27) we get

$$DP_{ij}(\overrightarrow{y}) = \frac{z(\overrightarrow{y}) - z^a}{z(\overrightarrow{y}) - z^a}.$$ 

Therefore, condition of Step 2 is a simplified version of the necessary and sufficient condition of Step 1.

Step 3: $P$ satisfies *Monotonicity in Income* if and only if

$$\overrightarrow{y} \leq \overrightarrow{y}^k + \left(\frac{1 - \overrightarrow{s}}{\overrightarrow{s}^2}\right)(z^0 - z^a).$$

For all $\overrightarrow{y} \geq \overrightarrow{y}^k$, slope $s(y_j, \overrightarrow{y})$ is maximal and tends to $\overrightarrow{s}$ when $y_j$ tends to $z(\overrightarrow{y})$. When $y_i < z^a < y_j$, considering any $y_j \geq z^a$ does not affect the value of $DP_{ij}(\overrightarrow{y})$ found in Step 2. Therefore replacing $s(y_j, \overrightarrow{y})$ by $\overrightarrow{s}$ in the condition of Step 2 is without loss of generality.

Given $\overrightarrow{y}$, $DP_{ij}(\overrightarrow{y})$ is weakly decreasing in $\overrightarrow{y}$ (constant on $\overrightarrow{y} \geq \overrightarrow{y}^k$) and reach a minimal value for $\overrightarrow{y} = \overrightarrow{y}^k$. Therefore, if the inequality given in Step 2 holds for $\overrightarrow{y} = \overrightarrow{y}^k$, then it holds for all $\overrightarrow{y} > 0$. Therefore *Monotonicity in Income* holds if and only if:

$$\overrightarrow{s} \leq \frac{z(\overrightarrow{y}) - z^a}{z(\overrightarrow{y}) - z^a} = \frac{z^0 - z^a}{z(\overrightarrow{y}) - z^a},$$

which yields the desired threshold for $\overrightarrow{y}$ as for all $\overrightarrow{y} \geq \overrightarrow{y}^k$ we have $z(\overrightarrow{y}) = z^0 + \overrightarrow{s}(\overrightarrow{y} - \overrightarrow{y}^k)$.

9.4 Proof of Theorem 3

Take any linear line $z$ with $\overrightarrow{s} > 0$ and any additive index $P$ with a numerical representation $d$ of the homothetic EO below $z$ belonging to the quadratic family.\footnote{If $\overrightarrow{s} = 0$, then the line is absolute and $P$ satisfies *Monotonicity in Income* since in that case *Monotonicity in Income* is implied by Domestic among Poor.} This proof is made in two steps, which together constitute the proof.

**STEP 1:** $P$ satisfies *Monotonicity in Income* if and only if for some arbitrary $\overrightarrow{y}^l > 0$ and all $y_i, y_j \in [0, z(\overrightarrow{y}^l))$ we have $s(y_j, \overrightarrow{y}^l) \leq DP_{ij}(\overrightarrow{y}^l)$.

Take any $\overrightarrow{y}^l$ with $z(\overrightarrow{y}^l) \leq \overrightarrow{y}^l$. As the poverty line is linear and hence $\overrightarrow{s} > 0$, such $\overrightarrow{y}^l$ exists. As shown in the necessary condition of Lemma 2, $P$ satisfies *Monotonicity in Income* only if for all $y_i, y_j \in [0, z(\overrightarrow{y}^l))$ we have $s(y_j, \overrightarrow{y}^l) \leq DP_{ij}(\overrightarrow{y}^l)$.

By assumption the EO is homothetic. I show that homotheticity implies that the degree of priority of one equivalence level over another does not depend on
mean income. For all \( \overline{y}^1, \overline{y}^2 > 0 \), \((y_i, \overline{y}^1), (y_j, \overline{y}^1) \in X_p \) if \( y_k := e^2(y_j, \overline{y}^1) \) and \( y_l := e^2(y_j, \overline{y}^1) \) then \( DP_{ij}(\overline{y}^1) = DP_{kl}(\overline{y}^2) \). Homotheticity implies that
\[
e^2(y_j, \overline{y}^1) = \frac{y_j}{y_j} e^2(y_i, \overline{y}^1) \cdot \frac{y_i}{y_i} = e^2(y_i, \overline{y}^1).
\]
By chain derivation, the degree of priority of \( y_i \) over \( y_j \) at \( \overline{y}^1 \) is hence:
\[
DP_{ij}(\overline{y}^1) = \partial_i d(e^2(y_i, \overline{y}^1), \overline{y}^2) \cdot \partial_j e^2(y_j, \overline{y}^1) = DP_{kl}(\overline{y}^2) \frac{e^2(y_j, \overline{y}^1)}{y_j} = DP_{kl}(\overline{y}^2).
\]
By assumption, the poverty line \( z \) is linear which, together with a monotonic EO implies that all bundles yielding the same equivalence level have a constant slope, for all values of mean income. For all \((y_i, \overline{y}^1) \in X_p \) and all \((y_k, \overline{y}^2) \in X_p \) with \((y_k, \overline{y}^2) \sim (y_i, \overline{y}^1)\), we have:
\[
s(y_i, \overline{y}^1) = \overline{y} \frac{y_i}{z(\overline{y}^1)} = \overline{y} \frac{y_k}{z(\overline{y}^2)} = s(y_k, \overline{y}^2). \tag{28}
\]
Therefore, if for all \( y_i, y_j \in [0, z(\overline{y}^1)] \) we have \( s(y_i, \overline{y}^1) \leq DP_{ij}(\overline{y}^1) \), then for all \( \overline{y}^2 > 0 \) and all \( y_i, y_j \in [0, z(\overline{y}^1)] \) we have \( s(y_j, \overline{y}^1) \leq DP_{ij}(\overline{y}^1) \). Therefore the sufficient condition in Lemma 2 holds as well in that case.

**STEP 2:** For any \( \overline{y} > 0 \), we have \( s(y_j, \overline{y}^1) \leq DP_{ij}(\overline{y}^1) \) for all \( y_i, y_j \in [0, z(\overline{y}^1)] \) if and only if inequalities (9) hold.

Take any \( \overline{y} > 0 \). For all \( y_i \in [0, z(\overline{y}^1)] \), since \( d \) belongs to the quadratic family:
\[
\partial_i d(y_i, \overline{y}) = -\frac{1}{z(\overline{y})} \left( 1 + \frac{\alpha}{1 - 2\frac{y_i}{z(\overline{y})}} \right).
\]
Therefore for all \( y_i, y_j \in [0, z(\overline{y}^1)] \), using the expression of \( s(y_i, \overline{y}) \) given in 28, inequality \( s(y_j, \overline{y}) \leq DP_{ij}(\overline{y}) \) is rewritten:
\[
\frac{y_j}{z(\overline{y})} \frac{1 + \alpha}{1 + \frac{\alpha}{1 - 2\frac{y_i}{z(\overline{y})}}} \leq \frac{1}{\overline{s}}. \tag{29}
\]
Two cases must be considered for this inequality:

- **Case 1:** \( \alpha < 0 \).
  \( L_{29} \) is maximal when (i) \( y_i = 0 \) and (ii) \( y_j \) tends to \( z(\overline{y}) \), implying that \( s(y_j, \overline{y}) \) tends to \( \overline{s} \). Replacing those values yields the lower bound on \( \alpha \).

- **Case 2:** \( \alpha \geq 0 \).
  \( L_{29} \) is maximal when (i) \( y_i \) tends to \( z(\overline{y}) \) and (ii) \( \frac{y_j}{z(\overline{y})} = \frac{1 + \alpha}{4\alpha} \) for all \( \alpha \) with \( \frac{1}{3} \leq \alpha \leq 1 \). Replacing those values yields the upper bound on \( \alpha \). For all \( \alpha \in \left[ 0, \frac{1}{4} \right] \), inequality (29) is respected for all \( y_i, y_j \in [0, z(\overline{y}^1)] \) as \( \overline{s} \leq 1 \).
9.5 Proof of Theorem 4

Take any piecewise-linear line $z$ with $y^0 \geq z^0$ and $s > 0$. Take any $x^* > 0$ with $x^* < z^0$. Let $P$ be an additive index based on an EO in $R^{HH}(z, x^*)$. Let $s_{s_x}$ be an EO belonging to the subdomain $R^{HH}(z, x^*)$ and hence $x^0 = x^*$.

First, I prove Claim 1. If $s_x = 0$, then $s_{s_x}$ is absolute-homothetic and Theorem 2 shows that if the numerical representation of $P$ is the PGR at the origin, then it satisfies both properties. I focus hence on proving the claim for EOs with $s_x > 0$. The proof relies on Lemma 4 giving a necessary and sufficient condition for an index to satisfy both properties.

**Lemma 4.** $P$ satisfies Monotonicity in Income and Transfer among Poor if and only if for all $\mathbf{y} \geq \mathbf{y}^0$ and all $y_i, y_j \in [0, z(\mathbf{y})]$ with $y_i < y_j$ we have $1 \leq DP_{ij}(\mathbf{y}) \leq \frac{1}{s(y_i, y_j)}$.

**Proof.**

$P$ satisfies Transfer among Poor if and only if for all $\mathbf{y} > 0$ and all $y_i, y_j \in [0, z(\mathbf{y})]$ with $y_i \leq y_j$, we have $1 \leq DP_{ij}(\mathbf{y})$. Given the poverty line is piecewise-linear and the EO is homothetic-homothetic, for all $\mathbf{y}, \mathbf{y}'$ with $\mathbf{y} < \mathbf{y}' \leq \mathbf{y}^0$ and all $y_i, y_j \in [0, z(\mathbf{y})]$ we have $DP_{ij}(\mathbf{y}) = DP_{ij}(\mathbf{y}')$. This implies that if the condition for Transfer among Poor holds for all $\mathbf{y} \geq \mathbf{y}^0$, then it holds for all $\mathbf{y} > 0$.

Given the poverty line is piecewise-linear and the EO is homothetic-homothetic, for all $\mathbf{y} < \mathbf{y}^0$ and all $y_i, y_j \in [0, z(\mathbf{y})]$, inequality $s(y_i, \mathbf{y}) \leq DP_{ij}(\mathbf{y})$ is trivially satisfied as $s(y_j, \mathbf{y}) = 0$. By assumption we have $\mathbf{y}^0 \geq z^0$, which implies $z(\mathbf{y}^0) \leq \mathbf{y}^0$ and hence $z(\mathbf{y}) \leq \mathbf{y}$ for all $\mathbf{y} \geq \mathbf{y}^0$. The necessary condition for Monotonicity in Income stated in Lemma 2 is therefore also sufficient: $P$ satisfies Monotonicity in Income if and only if for all $\mathbf{y} \geq \mathbf{y}^0$ and all $y_i, y_j \in [0, z(\mathbf{y})]$ we have $s(y_j, \mathbf{y}) \leq DP_{ij}(\mathbf{y})$.\footnote{Although Lemma 2 is only proven for absolute-homothetic EOs, extending its validity to homothetic-homothetic EOs is straightforward.}

If $P$ satisfies Transfer among Poor, this condition is met for all $y_i \leq y_j$ as $s(y_j, \mathbf{y}) \leq 1$ by Translation Monotonicity. For all $y_j < y_i$, the condition for Monotonicity in Income is based on inequality $s(y_i, \mathbf{y}) \leq DP_{ij}(\mathbf{y})$. As $DP_{ij}(\mathbf{y}) = \frac{1}{DP_{ij}(\mathbf{y})}$, we have that $P$ satisfies both Monotonicity in Income and Transfer among Poor if and only if for all $\mathbf{y} \geq \mathbf{y}^0$ and all $y_i, y_j \in [0, z(\mathbf{y})]$ with $y_i < y_j$ we have $1 \leq DP_{ij}(\mathbf{y}) \leq \frac{1}{s(y_i, y_j)}$.

Given a particular EO, choosing an additive index $P$ is equivalent to choosing its numerical representation $d$. At the reference mean income $\mathbf{y}$, at which $d$ is expressed, selecting $d$ is equivalent to selecting for all $y_i, y_j \in [0, z(\mathbf{y})]$ with $y_i < y_j$ the degrees of priority $DP_{ij}(\mathbf{y})$.\footnote{This selection is under the constraint that for any $y_i \in [0, z(\mathbf{y})]$ with $y_i < y_i < y_j$ we have $DP_{ij}(\mathbf{y}) = DP_{ij}(\mathbf{y})DP_{ij}(\mathbf{y})$, at least if $d$ is differentiable at $(y_i, \mathbf{y})$.} Each income level $y_i$ at $\mathbf{y}$ is associated to an equivalence level corresponding to the equivalence curve of the EO passing through the bundle $(y_i, \mathbf{y})$. Being selected at $\mathbf{y}$, the degrees of priority between two equivalence levels evolve with mean income $\mathbf{y}$, according to the evolution of the equivalence curves of the EO. I compute below how these degrees of priority between two equivalence levels evolve with mean income $\mathbf{y}$. Then, I derive the conditions on the slope $s_x$ of the homothetic-homothetic EO under which it is possible that, for the whole range of mean incomes $[\mathbf{y}, \infty)$, the degrees of priority stay inside $[1, \frac{1}{s(y_i, y_j)}]$. The proof of Claim 1 is in three steps.
**STEP 1:** Evolution of $DP_{ij}$ with mean income depends on $s_x$.

Take $\overline{y}$ as reference mean income. Any reference mean income is taken without loss of generality as there is no constraint on the mathematical expression of $d$.

Consider any two $y_i, y_j \in [0, z(\overline{y})^k]$ with $y_i < y_j$. Let $e^k : X_p \rightarrow [0, z(\overline{y})^k]$ be the equivalent income function at $\overline{y}^k$. For any $\overline{y} > \overline{y}^k$, consider the bundles $(y_i, \overline{y})$ and $(y_j, \overline{y})$ yielding equivalent individual poverty, that is $y_i = e^k(y_i, \overline{y})$ and $y_j = e^k(y_j, \overline{y})$. As $EO \succeq_{s_x}$ is homothetic-homothetic, we have for all $\overline{y} \geq \overline{y}^k$ and all $y_i \in [0, x(\overline{y})]$ that

$$e^k(y_i, \overline{y}) = y_i \frac{x^0}{x(\overline{y})},$$

and for all $y_i \in [x(\overline{y}), z(\overline{y})]$ that

$$e^k(y_i, \overline{y}) = x^0 + \frac{z^0 - x^0}{z(\overline{y}) - x(\overline{y})} (y_i - x(\overline{y})).$$

The evolution of $DP_{ij}$ as a function of $\overline{y}$ depends on the relative positions of $y_i, y_j$ and $x(\overline{y}^k) = x^0$. Three cases must be considered.

- **Case 1:** $y_i < y_j \leq x^0$.
  A direct extension of the reasoning on homotheticity in Step 1 of Theorem 3’s proof shows that for all $\overline{y} > \overline{y}^k$ and all $y_i, y_j < x(\overline{y}^k)$ we have $DP_{ij}(\overline{y}) = DP_{ij}(\overline{y}^k)$. This implies that if the necessary and sufficient condition of Lemma 4 is met at $\overline{y}^k$, it is met for all $\overline{y} \geq \overline{y}^k$.

- **Case 2:** $x^0 \leq y_i < y_j$.
  We have $DP_{ij}(\overline{y}) = DP_{ij}(\overline{y}^k)$. Again, checking the condition at $\overline{y}^k$ is necessary and sufficient.

- **Case 3:** $y_i < x^0 < y_j$.
  By chain derivation, we obtain successively:

  $$DP_{ij}(\overline{y}) = \frac{\partial d(e^k(y_i, \overline{y}, \overline{y}^k))}{\partial d(e^k(y_j, \overline{y}, \overline{y}^k))} \frac{\partial e^k(y_i, \overline{y})}{\partial e^k(y_j, \overline{y})} = DP_{ij}(\overline{y}^k) \left( \frac{x^0}{x(\overline{y})} \frac{z(\overline{y}) - x(\overline{y})}{z^0 - x^0} \right),$$

  and finally

  $$DP_{ij}(\overline{y}) = DP_{ij}(\overline{y}^k) \frac{x^0}{z^0 - x^0} \left( \frac{z^0 - x^0}{x^0 + s_x(\overline{y} - \overline{y}^k)} \right). \quad (30)$$

Taking the partial derivative of $DP_{ij}(\overline{y})$ with respect to mean income yields

$$\frac{\partial DP_{ij}(\overline{y})}{\partial y} = DP_{ij}(\overline{y}^k) \frac{x^0}{z^0 - x^0} \frac{s_xz^0 - s_xx^0}{(x^0 + s_x(\overline{y} - \overline{y}^k))^2}.$$

There are three subcases to consider, depending on the value of $s_x$.  

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Subcase 3.1: \( s_x \in \left( 0, \frac{x^0}{z^0} s_z \right) \).

The partial derivative of \( DP_{i|j}(\overline{y}) \) is strictly increasing for all \( \overline{y} > \overline{y}^k \).

Subcase 3.2: \( s_x \in \left( \frac{x^0}{z^0} s_z, 0 \right) \).

The partial derivative of \( DP_{i|j}(\overline{y}) \) is strictly decreasing for all \( \overline{y} > \overline{y}^k \).

Subcase 3.3: \( s_x = \frac{x^0}{z^0} s_z \).

The partial derivative of \( DP_{i|j}(\overline{y}) \) is zero for all \( \overline{y} > \overline{y}^k \). For \( s_x = s_x^h \), the EO \( \succeq_{s_x} \) corresponds to an homothetic EO. If the numerical representation is the PGR at the origin, then \( P \) respects both properties as shown in Theorem 2.

The degree of priority between two equivalence levels evolve with \( \overline{y} \) only if \( y_i < x^0 < y_j \). As shown by the three subcases, this evolution depends on the value taken by \( s_x \). I study in Step 2 the conditions under which an EO in subcase 3.1 admits a numerical representation such that \( P \) satisfies both properties. In Step 3, I study those conditions in subcase 3.2.

**STEP 2:** For subcase 3.1, derive the lower bound \( s_x \) for \( s_x \) above which an additive index satisfies the necessary and sufficient conditions of Lemma 4.

Subcase 3.1 is such that \( 0 < s_x < \frac{x^0}{z^0} s_z \). Step 1 showed for these values of \( s_x \) that \( DP_{i|j}(\overline{y}) \) is strictly increasing in \( \overline{y} \). The necessary and sufficient condition for both properties given in Lemma 4 requires that we have \( 1 \leq DP_{i|j}(\overline{y}) \leq \frac{1}{s(y_i, \overline{y})} \) for all \( \overline{y} \geq \overline{y}^k \). As \( DP_{i|j}(\overline{y}) \) is strictly increasing with \( \overline{y} \), it is sufficient to check these inequalities at the boundaries of the domain for mean income, that is at \( \overline{y} = \overline{y}^k \) and when \( \overline{y} \) tends to \( \infty \). From (30), the condition in Lemma 4 is satisfied only if for all \( y_i, y_j \in [0, z(\overline{y}^k)] \) with \( y_i < x(\overline{y}^k) < y_j \) we have:

\[
1 \leq DP_{i|j}(\overline{y}^k) \quad \text{and} \quad DP_{i|j}(\overline{y}^k) \frac{x^0 - x^0}{\beta} \frac{s_z - s_x}{s_x} \leq \frac{1}{s(y_i, \overline{y})} \quad (31)
\]

As this subcase is such that \( s_z > s_x \frac{x^0}{z^0} \), we have \( \beta > 1 \). Observe that the slope of a given equivalence curve is constant for all \( \overline{y} \geq \overline{y}^k \), which implies the second inequality can be bounded above by the slope at \( \overline{y}^k \).

If inequalities (31) are not met when taking \( DP_{i|j}(\overline{y}^k) = 1 \) for all \( y_i, y_j \in [0, z(\overline{y}^k)] \) with \( y_i < x(\overline{y}^k) < y_j \), then any other value for \( DP_{i|j}(\overline{y}^k) \) also implies their violation.\(^{59}\) In other words, if the PGR at \( \overline{y}^k \) cannot respect these conditions, no other numerical representation can. On the contrary, if the PGR at \( \overline{y}^k \) does respect inequalities (31), then the index based on this numerical representation satisfies both *Monotonicity in Income* and *Transfer among Poor*. Indeed, I showed in Step 1 that respecting the condition of Lemma 4 for all \( y_i, y_j \in [0, z(\overline{y}^k)] \) with \( y_i < y_j < x^0 \) or with \( x^0 \leq y_i < y_j \) at mean income \( \overline{y}^k \) was sufficient to respect it for all \( \overline{y} \geq \overline{y}^k \).

I show that the PGR at \( \overline{y}^k \) respect inequalities (31) for all \( y_i, y_j \in [0, z(\overline{y}^k)] \) with \( y_i < x^0 < y_j \) if and only if \( s_x \geq s_x^h \). The first of these inequality holds as

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\(^{59}\)The reason is that this definition of \( DP_{i|j} \) considers its minimal value such that inequality \( 1 \leq DP_{i|j}(\overline{y}^k) \) holds.
$DP_{ij}(\overrightarrow{y}) = 1$. I show that provided $s_x \geq \underline{s}_x$, the second holds as well for the subcase 3.1. The left-hand side of this second inequality does not depend on the specific value taken by $y_i$ and $y_j$, given they meet $y_i < x^0 < y_j$. The tightest upper bound is obtained when $y_i$ tends to $x^0$ and hence $s(y_i, \overrightarrow{y})$ tends to $s_x$. Replacing $DP_{ij}(\overrightarrow{y})$ by 1 and $s(y_i, \overrightarrow{y})$ by $s_x$ yields successively:

$$\frac{x^0}{(z^0 - x^0)} \frac{(s_z - s_x)}{s_z} \leq \frac{1}{x^0},$$

$$\underline{s}_x := s_z - \frac{(z^0 - x^0)}{x^0} \leq s_x.$$ 

This $\underline{s}_x$ corresponds to the threshold for $s_x$ below which the PGR at the origin leads to an index violating either Transfer among Poor or Monotonicity in Income (and hence any other numerical representation as well) and above which the PGR at the origin leads to an index satisfying both properties. This bound is such that $\underline{s}_x < s_x^*$:

$$s_z - \frac{(z^0 - x^0)}{x^0} < \frac{x^0}{z^0} s_z,$$

$$s_z < \frac{x^0}{z^0},$$

which holds as $s_z \leq 1$ and we assumed $x^* < z^0$.

**STEP 3:** For subcase 3.2, derive the upper bound $\overline{s}_x$ for $s_x$ below which an additive index satisfies the necessary and sufficient conditions of Lemma 4.

Subcase 3.2 is such that $\overline{s}_x^0 s_z < s_x < s_z$. Step 1 showed for this case that $DP_{ij}(\overrightarrow{y})$ is strictly decreasing in $\overrightarrow{y}$. As for Step 2, it is sufficient to check the condition in Lemma 4 at the boundaries:

$$DP_{ij}(\overrightarrow{y}) \leq \frac{1}{s(y_i, \overrightarrow{y})}$$

and

$$1 \leq DP_{ij}(\overrightarrow{y}) \frac{x^0}{z^0 - x^0} \frac{(s_z - s_x)}{s_x}.$$ (32)

As this subcase is such that $s_z < s_x \frac{x^0}{z^0}$, we have $\beta < 1$. If inequalities (32) are not met when taking $DP_{ij}(\overrightarrow{y})/\beta = 1$ for all $y_i, y_j \in [0,z(\overrightarrow{y})]$ with $y_i < x^0 < y_j$ when $\overrightarrow{y}$ tends to $\infty$, then any other value for $DP_{ij}(\overrightarrow{y})$ also implies their violation. In other words, if the PGR at $\overrightarrow{y}_\infty$ cannot respect these conditions, no other numerical representation of $\geq s_x$ can.\footnote{The PGR at $\overrightarrow{y}_\infty$ is defined as the numerical representation in the extended FGT family with $\alpha = 1$ granting a degree of priority at $\overrightarrow{y}_\infty$ to absolute over relatively poor agents of $\overrightarrow{y}$.} On the contrary, if the PGR at $\overrightarrow{y}_\infty$ does respect inequalities (31), then the index based on this numerical representation satisfies both Monotonicity in Income and Transfer among Poor, as explained in Step 2.

I show that the PGR at $\overrightarrow{y}_\infty$ respect inequalities (32) for all $y_i, y_j \in [0,z(\overrightarrow{y})]$ with $y_i < x^0 < y_j$ if and only if $s_x \leq \overline{s}_x$. The second of these inequality holds as $DP_{ij}(\overrightarrow{y})/\beta = 1$. I show that provided $s_x \leq \overline{s}_x$, the first holds as well for subcase 3.2. The tightest upper bound is obtained when $y_i$ tends to $x^0$ and
hence \( s(y, \overline{y}^k) \) tends to \( s_x \). Replacing \( DP_{ij}(\overline{y}^k) \) by \( \frac{1}{y} \) and \( s(y, \overline{y}^k) \) by \( s_x \) yields successively:

\[
\frac{(z^0 - x^0)}{x^0}(s_x - s_x) \leq \frac{1}{s_x},
\]

\[
s_x^2 + s_x \frac{x^0}{z^0 - x^0} - s_x \frac{x^0}{z^0 - x^0} \leq 0.
\]

This second order equation in \( s_x \) has two roots \( r_- \) and \( r_+ \), one negative and one positive. The images of this parabola are negative between the two roots. The positive root constitutes the threshold \( s_x := r_+ \), given by:

\[
s_x = \frac{\left(\frac{x^0}{z^0 - x^0}\right)^2 + 4s_x \frac{x^0}{z^0 - x^0}}{2}.
\]

This \( s_x \) corresponds to the threshold for \( s_x \) above which the PGR at \( \overline{y}^\infty \) leads to an index violating either \textit{Transfer among Poor} or \textit{Monotonicity in Income} (and hence any other numerical representation as well) and above which the PGR at \( \overline{y}^\infty \) leads to an index satisfying both properties. This bound is such that \( s_x^0 := \frac{x^0}{z^0}s_x < s_x \) as successively we have:

\[
2\frac{x^0}{z^0}s_x + \frac{x^0}{z^0 - x^0} < \left(\frac{x^0}{z^0 - x^0}\right)^2 + 4s_x \frac{x^0}{z^0 - x^0} \right)^{0.5},
\]

\[
\left(\frac{x^0}{z^0}s_x\right)^2 < \frac{x^0}{z^0}s_x,
\]

which is guaranteed as \( s_x \leq 1 \) and \( x^0 < z^0 \). This bound is also such that \( s_x < s_z \) as successively we have:

\[
\left(\frac{x^0}{z^0 - x^0}\right)^2 + 4s_z \frac{x^0}{z^0 - x^0} \right)^{0.5} < 2s_z + \frac{x^0}{z^0 - x^0},
\]

\[
\frac{x^0}{z^0} < \frac{x^0}{z^0} + s_z,
\]

which is guaranteed as \( s_z > 0 \) as \( \bar{s} > 0 \). This concludes the proof of Claim 1.

I prove Claim 2, based on arguments exposed in the proof of Claim 1. A direct application of the reasoning proving Claim 1 of Theorem 2 shows that any \( P \) having its numerical representation in the extended FGT family satisfies \textit{Monotonicity in Income} only if \( \alpha = 1 \). Then, as by assumption \( s_x \leq \frac{x^0}{z^0}s_z \), the expression of \( DP_{ij}(\overline{y}) \) for any \( y_i < x(\overline{y}) < y_j \) given in (30) is strictly increasing in \( \overline{y} \) as shown in Step 1 of Claim 1. Therefore, if the numerical representation \( d \) of \( P \) is the PGR at \( \overline{y} \) then \( P \) satisfies \textit{Transfer among Poor} only if \( \overline{y} = \overline{y}^k \), that is \( d \) is the PGR at the origin. Finally, Step 2 of Claim 1 showed that if \( s_x \geq 0 \) is such that \( s_x \in [s_x, s_x^0] \) and if \( d \) is the PGR at the origin, then \( P \) satisfies both \textit{Monotonicity in Income} and \textit{Transfer among Poor} when \( s_x \leq \frac{x^0}{z^0}s_x \). This shows the equivalence of the two statements in Claim 2.
9.6 Proof of Theorem 5

The proof is very close to that of Theorem 1. I therefore omit parts that are straightforward modifications in order to emphasize the differences.

Take any median-sensitive endogenous line \( z \) and any poverty index \( P \) satisfying the five modified axioms. Let \( Y_{even}^r := \{ y \in Y^r | n(y) \in 2N \} \) be the subdomain of \( Y^r \) containing only income distributions with an even number of dimensions. I prove in Steps 1 to 3 that the additive representation is implied for all \( y \in Y_{even}^r \), then in Step 4 I use Replication Invariance to extend this result to the whole \( Y^r \).

**STEP 1:** From a poverty ordering on income distributions to a poverty ordering on distributions of individual poverty.

The definition of the mapping is different. Let the continuous mapping be \( M : Y_{even}^r \rightarrow \mathbb{R}^{N'} \), where \( N' := \{ n \in 2N + 1 | n \geq 1 \} \). Let \( \succeq^m \) be an EO in \( \mathcal{R}_m \), whose minimal judgments among the poor are respected by \( P \). By modified Domination among Poor, such \( \succeq^m \) exists. Consider any numerical representation \( d \) of \( \succeq^m \). For each \( (y, r) \in X \), let \( \nu_i := d(y, y_m) \). Mapping \( M \) is defined for all \( y \in Y_{even}^r \) such that \( M(y) = (\nu_1, \cdots, \nu_m) := \nu \).

Observe that if distribution \( y \in Y_{even}^r \) has \( m \) components, then \( M(y) \) has \( m - 1 \) components. The size of distribution \( \nu \) is taken to be \( n \) as for all \( y \in Y^r \) and all \( i \geq m \) we have \( d(y, y_m) = 0 \) since \( y_i \geq z(y_m) \) and are hence omitted.

I show for the continuous mapping defined that \( M(Y_{even}^r) = V_d := [0, 1]^{N'} \). The domain of images of \( Y_{even}^r \) through mapping \( M \) is hence a product space: \( V_d = \times_{i=1}^{N'} [0, 1] \). This means that (i) \( M(Y_{even}^r) \subseteq V_d \) and (ii) \( V_d \) is a subdomain of \( Y_{even}^r \), that is for all \( \nu \in V_d \) there exists \( y \in Y_{even}^r \) such that \( M(y) = \nu \). If (i) follows directly from the definition of mapping \( M \), (ii) remains to be proven. Lemma 5 proves that \( V_d \subseteq M(Y_{even}^r) \).

**Lemma 5.** For all \( \succeq^m \in \mathcal{R}_m \) and \( \nu \in V_d \), there exists \( y \in Y_{even}^r \) such that \( M(y) = \nu \).

**Proof.** Consider any \( \nu \in V_d \) and any \( g \in \mathbb{R}_+ \) such that \( g \geq z(g) \). For any endogenous line, such a \( g \) exists by modified Possibility of Poverty Eradication. I construct \( y \in \mathbb{R}^{n(\nu)+1} \) such that \( y_m = g \) and \( M(y) = \nu \). Let \( y \) be such that, for all \( i \leq n(\nu) \), \( y_i := a_i \) defined implicitly by \( \nu_i = d(a_i, g) \). If \( \nu_i = 0 \) take \( a_i := g \). By modified Minimal Absolute Concern and the continuity of \( d \), we have that \( a_i \in [0, z(g)] \) for all \( i \leq g \). For all \( z \in \{ m(y), \ldots, n(y) \} \), take \( y_j := g \). This construction implies \( y_m = g \). Therefore we have \( y \in Y_{even}^r \) since (i) \( n(y) = 2(n(\nu) + 1) \) is even, (ii) a majority of agents are non-poor as \( n(\nu) + 2 \) agents earn \( g \geq z(g) \) and we have \( n(\nu) + 2 > n(y) / 2 \). We have by construction \( M(y) = \nu \), the desired result. \[ \blacksquare \]

Again, the poverty ordering \( \succeq_{Y_{even}^r} \) on the set of income distributions is associated to an ordering \( \succeq_{V_d} \) on distributions of individual poverty, by modified


**Domination among Poor.** This ordering $\succeq_{V_d}$ can be represented by a continuous poverty index $P^\nu : V_d \to \mathbb{R}$. In particular, ordering $\succeq_{V_d}$ is represented by $P^\nu$ defined such that for all $\nu \in V_d$ and $y \in Y_{even}$, with $M(y) = \nu$, we have $P^\nu(\nu) = P^\nu(M(y)) = P(y)$.

**STEP 2:** Index $P^\nu$ representing ordering $\succeq_{V_d}$ is additively separable.

We verify that the assumptions of Theorem 1 in Gorman (1968) are all met. This allows deriving the following functional form for the index $P^\nu$, for a given $y(y) \in 2\mathbb{N}$:

$$P^\nu(\nu) = \tilde{F} \left( \sum_{i=1}^{m-1} \tilde{\phi}(\nu_i) \right) \quad (33)$$

where $\tilde{F}$ and $\tilde{\phi}$ are strictly increasing functions and $m - 1 = n(\nu)$.

The assumptions required for this Theorem are the following:

**Assumption 1:** As before.

**Assumption 2:** As before.

**Assumption 3:** Let $S = \{[0,1]_1, \ldots, [0,1]_{m-1}\}$ be the set of sectors in $V_d$ and $A \subseteq S$ be any subset of sectors, we have that each $A$ is separable. Separability means that for all $(u,w), (v,w), (u,t), (v,t) \in V_d$, we have $P^\nu(u,w) \geq P^\nu(v,w) \iff P^\nu(u,t) \geq P^\nu(v,t)$. Separability is proven in two substeps.

**Substep 1:** Construct an income distributions associated to each distribution of individual poverty.

Construct $y^1, y^2, y^3, y^4 \in Y_{even}$ such that $M(y^1) = (u,w), M(y^2) = (v,w), M(y^3) = (u,t), M(y^4) = (v,t), y^1, y^2, y^3, y^4 \in \mathbb{R}_+^{m(u,v)+1}$ and $y^1_m = y^2_m = y^3_m = y^4_m = g$ with $g \geq z(g)$. Such distributions exist and are constructed following the procedure presented in Lemma 5.

Decompose in subgroups $y^1 = (y^1_A, y^1_B, y^1_C)$, such that subdistributions $y^1_A$ and $y^1_B$ are associated –via the numerical representation $d$ – to the individual poverty subdistributions $u$ and $w$ respectively and $y^1_C$ is the subdistributions containing the income for all $j \in \{m(y), \ldots, n(y)\}$, for whom by construction we have $y_j = g$. By construction, $y^1 \in \mathbb{R}_+^{2m(u,v)+1}$ and has hence an even number of dimensions. We can hence decompose $y^1_C = (y^1_{C1}, y^1_{C2}) = (g, \ldots, g)$ such that $n(y^1_{C1}) = n(y^1_A) + 1$ and $n(y^1_{C2}) = n(y^1_B) + 1$. Typically, the median in $y^1_A$ is different from the median in $y^1_B$, which is different from $g$ but our next operations will aim at equalizing median income in subgroups by distributing the agents in subgroup $C$ between the subgroups $A$ and $B$.

Duplicate $y^1$ and re-organize the subgroups in a way that equalizes median income in each subgroup with $g$. Let $x^i := (y^i_1, y^i_2) = (x^i_A, x^i_B)$ with $x^i_A := (y^i_1, y^i_A, y^i_{C1}, y^i_{C1})$ and $x^i_B := (y^i_1, y^i_B, y^i_{C2}, y^i_{C2})$, implying both $x^i_A$ and $x^i_B$ have an even number of dimensions. This duplication does not affect the median:

\footnote{Mapping $M$ cannot be used to obtain the image of income distributions with odd number of dimensions, otherwise the same $\nu$ is obtained for two income distributions with different poverty. For example $\nu = 1(1)$, corresponds to both $y_{even} := (0, g, g, g)$ and $y_{odd} := (0, g, g)$; and the ranking on $V_d$ cannot discriminate these two income distributions having different poverty.}
Substep 2: Prove $P(x^1_A, y^1_B) \geq P(x^2_A, y^2_B) \Leftrightarrow P(x^3_A, y^3_B) \geq P(x^4_A, y^4_B)$.

Our income distributions are constructed such that $P(x^1_A) = P(x^2_A), P(x^3_A) = P(x^4_A), P(x^1_B) = P(x^2_B)$ and $P(x^3_B) = P(x^4_B)$ by Domination among Poor. By assumption we have $P(x^1) \geq P(x^2)$. As $P(x^1_B) = P(x^2_B)$, we have that $P(x^1_A) \geq P(x^3_A)$ by Weak Subgroup Consistency (remember all our subgroups have their median equal to $g$).

Then, $P(x^1_A) \geq P(x^3_A)$ together with $P(x^1_A) = P(x^2_A)$ and $P(x^3_A) = P(x^4_A)$ imply $P(x^1_A) \geq P(x^1_A)$. Since $P(x^1_B) = P(x^2_B)$, this implies $P(x^1) \geq P(x^2)$. Two cases can arise.

- Case 1: $P(x^1_A) > P(x^3_A)$.
  As $P(x^1_B) = P(x^2_B)$, we obtain $P(x^1_A, y^1_B) > P(x^3_A, y^3_B)$, by Weak Subgroup Consistency. This case is hence such that $P(x^1) \geq P(x^2)$ as desired.

- Case 2: $P(x^1_A) = P(x^3_A)$.
  I show by contradiction that $P(x^1) \geq P(x^2)$. Assume we have $P(x^1_A, x^1_B) < P(x^3_A, x^3_B)$. As $P(x^1_A) = P(x^3_A)$, Weak Subgroup Consistency implies that $P(x^1_A, x^1_B) < P(x^3_A, x^3_B)$. Again, as $P(x^1_B) = P(x^2_B)$, we obtain $P(x^1_A, x^1_B) < P(x^3_A, x^3_B)$. This is a contradiction as the two distributions have identical poverty by Symmetry.

The two cases lead to $P(x^1) \geq P(x^2)$, which proves separability. All assumptions of Theorem 1 in Gorman (1968) are met.

**STEP 3:** Show functions $\tilde{F}$ and $\tilde{\phi}$ do not depend on the number of agents.

Theorem 1 in Gorman (1968) is valid for a fixed number of potentially poor agents $n(\nu)$. I modify the proof of Foster and Shorrocks (1991) in order to prove these functions are independent of $n$. When $n(\nu)$ is allowed to vary -- $n(\nu)$ will be denoted $n$ below -- (33) must be written:

$$P^*(\nu) = \tilde{F}_n \left( \sum_{i=1}^n \tilde{\phi}_n(v_i) \right)$$
Step 3.1: Define transformations of \( \tilde{F}_n \) and \( \tilde{\phi}_n \) for normalization purposes.

Let \( F_n \) and \( \phi_n \) be the following transformations of \( \tilde{F}_n \) and \( \tilde{\phi}_n \):

\[
\phi_n(x) = 2(n + 1)\left[\tilde{\phi}_n(x) - \phi_n(0)\right],
\]

\[
F_n(x) = \tilde{F}_n[x + n\tilde{\phi}_n(0)].
\]

These transformations imply successively:

\[
F_n\left(\frac{1}{2(n + 1)} \sum_{i=1}^{n} \phi_n(x_i)\right) = F_n\left(\frac{2(n + 1)}{2(n + 1)} \sum_{i=1}^{n} \tilde{\phi}_n(x_i) - \phi_n(0)\right),
\]

\[
= F_n\left(\sum_{i=1}^{n} \tilde{\phi}_n(x_i) - \tilde{\phi}_n(0)\right) + n\tilde{\phi}_n(0).
\]

This yields

\[
P^r(\nu) = F_n\left(\frac{1}{2(n + 1)} \sum_{i=1}^{n} \phi_n(x_i)\right),
\]

where \( \phi_n(0) = 0 \) and by the definition of mapping \( M \), we have \( 2(n + 1) = n(y) \). As any agent \( j \in \{m(y), \ldots, n(y)\} \) is non-poor in \( Y^r \), we have \( d(y_j, y_m) = 0 \). By slightly abusing notation (by introducing the zero individual poverty of those non-poor agents at the end of the distribution \( \nu \)), we obtain for all even \( n(y) \):

\[
P^r(\nu) = F_n\left(\frac{1}{n(y)} \left(\sum_{i=1}^{n(y)} \phi_n(x_i) + \sum_{i=m(y)}^{n(y)} \phi_n(0)\right)\right) \quad (34)
\]

\[
= F_n\left(\frac{1}{n(y)} \sum_{i=1}^{n(y)} \phi_n(x_i)\right) \quad (35)
\]

with \( F_n \) and \( \phi_n \) continuous, strictly increasing and \( \phi_n(0) = 0 \).

Step 3.2: Use Replication Invariance to prove functions \( F_n \) and \( \phi_n \) do not depend on \( n \).

Same as before.

**STEP 4**: The additively separable expression obtained for all \( y \in Y^r_{even} \) is valid for all \( y \in Y^r_{odd} \).

Consider any \( y \in Y^r_{odd} \) and its duplication \( x := (y, y) \in Y^r_{even} \). By Replication Invariance, we have \( P(y) = P(x) \), which means that the mathematical expression of any \( y \in Y^r_{odd} \) also take the additively separable form of (35) as:

\[
P(x) = F\left(\frac{1}{n(x)} \sum_{i=1}^{n(x)} d(x_i, x_m)\right) = F\left(\frac{1}{2n(y)} \sum_{i=1}^{n(y)} 2d(y_i, y_m)\right) = P(y),
\]

where \( d(y_i, y_m) = d(x_{2i}, x_m) \) for all \( i \leq n(y) \) as \( y_m = x_m \) and \( y_i = x_{2i} \). This completes the proof as \( Y^r = Y^r_{odd} \cup Y^r_{even} \).
9.7 Proof of Theorem 6

Let \( z \) be any median-sensitive line with \( z^0 > 0 \). Let \( P \) be any additive poverty index based on an absolute-homothetic EO below \( z \).

I don’t provide a complete proof showing that the second statement implies the first. The intuition is the following. For any additive index based on an absolute-homothetic EO below a line \( z \) meeting the second statement, \textit{Monotonicity in Income} is implied by \textit{Domination among Poor}. As additive indices respect \textit{Domination among Poor}, such \( P \) satisfies \textit{Monotonicity in Income}.

I prove by contraposition that the first statement implies the second. As by assumption \( z^0 > 0 \), we have \( y^*_m > 0 \). Assume there exists \( y^*_m < y^*_m \) with \( s(y^*_m) > 0 \). I construct an \( y \in \mathcal{Y} \) at which a violation of \textit{Monotonicity in Income} arise for a particular increment. Let \( y_m := y^*_m \), and let \( y_{m+1} \) be constructed such that \( y_{m+1} \leq z(y^*_m) \) and \( y_{m+1} - y_m = \epsilon > 0 \). The numerical representation of \( P \) is \( d \). Let \( \Delta := d(y_m, y^*_m) - d(y_m + \epsilon, y^*_m + \epsilon) \) be the individual poverty gain obtained by the median agent when her income is increased by the increment \( \epsilon \). Observe that the increase in median income with \( \epsilon \) does not depend on the number of agents, contrary to the increase in mean income. By modified Strict Monotonicity in Income and Translation Monotonicity, we have that \( \Delta \geq 0 \).

Consider income level \( a < z(y^*_m) \) with \( s(a, y^*_m) > 0 \). Such an income \( a \) exists as the EO is absolute-homothetic and \( s(y^*_m) > 0 \). Let \( \delta := d(a, y^*_m + \epsilon) - d(a, y^*_m) \) be the individual poverty loss obtained by an agent earning \( a \) when the income of the median agent is increased by \( \epsilon \). We have \( \delta > 0 \) since \( s(a, y^*_m) > 0 \) and numerical representation \( d \) is strictly decreasing in equivalence levels by modified Strict Monotonicity in Income.

Let \( n^a \) be the number of agents earning income \( a \) in the income distribution \( y \). If \( n^a > \frac{\Delta}{\delta} \), giving an additional \( \epsilon \) to the median agent in distribution \( y \) strictly increases poverty. Let \( y' \) be obtained from \( y \) when median agent earns an extra \( \epsilon \). As \( P \) is an additive index, we have:

\[
P(y) - P(y') = \frac{1}{n} \sum_{i=1}^{n} (d(y_i, y_m) - d(y'_i, y^*_m)),
\]

\[
= \frac{1}{n} \left( \Delta - \left( \sum_{j \neq m} d(y_j, y^*_m + \epsilon) - d(y_j, y^*_m) \right) \right),
\]

\[
= \frac{1}{n} (\Delta - n^a \delta - A),
\]

where term \( A \) stands for the sum of individual poverty losses obtained by agents different than the median agent and not earning \( a \). We have \( A \geq 0 \) as the EO is absolute-homothetic and \( \epsilon > 0 \), implying that the individual poverty of all poor agents except the median agent cannot decrease when passing from \( y \) to \( y' \). As a result, if \( n^a > \frac{\Delta}{\delta} \), then we have \( P(y) - P(y') < 0 \), which violates \textit{Monotonicity in Income}. There exists such \( y \in \mathcal{Y} \) since the number of agents in distributions belonging to \( \mathcal{Y} \) is not bounded above.

9.8 Partial means

I extend here Theorem 1 and Theorem 2 for the class of lower partial means. The case \( x = 100 \) corresponds to the mean.
Monotonicity in Income has different implications when imposed on mean-sensitive or median-sensitive poverty lines. For mean-sensitive lines, Theorems 2 and 3 show this axiom forces the index to be close to the PGR at the origin. Unlike the median, the mean as well as lower partial means are influenced by the income of all poor agents.62

The definition of a lower partial mean given in (36) is in two parts since I consider finite income distributions and there is hence no guarantee that \( \frac{x}{100n} \) be a natural number. For the sake of notational simplicity, let \( x \) denote a fraction rather than a percentage, that is \( x \in (0, 1] \).

**Definition 9** (Lower partial mean).

The income standard \( f_{lpm}^n : \mathbb{R}_+^n \to \mathbb{R}_+ \) is a lower partial mean if

\[
\begin{align*}
  f_{lpm}^n(y) := \\
  \frac{1}{xn} \sum_{i=1}^{xn} y_i & \quad \text{if } xn \in \mathbb{N}, \\
  \frac{1}{xn} \left( \sum_{i=1}^{r} y_i + (xn - r)y_{r+1} \right) & \quad \text{otherwise},
\end{align*}
\]

(36)

where \( x \in \{ a \in \mathbb{Q} | 0 < a \leq 1 \} \) and \( r := \max_{a \in \mathbb{N}} a \leq xn \).

Again, changing the income standard requires modifying several definitions. I present here only the main non-straightforward modifications. The domain of poverty indices are based on an equivalence ordering \( \succeq \) restriction 6 (modified Translation monotonicity).

**EO restriction 6** (modified Translation monotonicity).

For all \( (y, f_{lpm}^n(y)) \in X_p \) and \( a > 0 \), we have \( (y_i + a, f_{lpm}^n(y + a1^n)) \succeq (y_i, f_{lpm}^n(y)) \).

**Poverty axiom 11** (modified Weak Subgroup Consistency).

For all \( y^1, y^2, y^3, y^4 \in Y^f \) such that \( n(y^1) = n(y^3) \), \( n(y^2) = n(y^4) \), \( f_{lpm}^n(y^1) = f_{lpm}^n(y^2) = f_{lpm}^n(y^3) = f_{lpm}^n(y^4) \) and \( f_{lpm}^n(y^1) = f_{lpm}^n(y^3, y^4) \), if \( P(y^1) \geq P(y^2) \) and \( P(y^3) = P(y^4) \), then \( P(y^1, y^2) > P(y^3, y^4) \).

---

62The incomes of all poor agents influence the lower partial mean if the percentage of poor agents is lower than \( x \).
63Where \( 1^n \) denotes a \( n \)-dimensional distribution of ones. Giving an equal increment to all agents cannot increase the individual poverty of a poor agent.
Such modifications allow characterizing additive poverty indices with partial-mean-sensitive lines for the domain of income distributions containing at least $100(1-x)\%$ of non-poor agents. For notational simplicity, the lower partial mean $f_{pm}^l$ is denoted $f$.

**Theorem 9** (Characterization of additive partial-mean-sensitive indices).

Let $f$ be a lower partial mean. Let $P : Y^f \to \mathbb{R}$ be a poverty index based on a $f$-sensitive poverty line. The following two statements are equivalent.

1. $P$ is ordinally equivalent to an index $P' : Y^{f} \to [0, 1]$ with
   \[
   P'(y) = \frac{1}{n} \sum_{i=1}^{n} d(y_i, f(y)),
   \]  
   where $d$ is a numerical representation of an EO in $\mathbb{R}$.

2. $P$ satisfies the modified versions of **Domination among Poor**, **Weak Subgroup Consistency**, **Symmetry**, **Continuity** and **Replication Invariance**.

**Proof.** The proof is very close to the proof of Theorem 1. I therefore omit parts that are straightforward modifications in order to emphasize differences. Again, I just prove that statement two implies statement one.

Take any partial-mean-sensitive endogenous line $z$ and any poverty index $P$ satisfying the five modified axioms. By assumption we have $x \in Q$ so $x$ can be expressed as $x = \frac{a}{b}$ with $a, b \in \mathbb{N}$ and their greatest common divider $\gcd(a, b) = 1$. Let $Y_{bb}^{f}$ be the subset of income distributions in $Y^f$ for which $xn$ belongs to the natural:

\[
Y_{bb}^{f} := \{ y \in Y^f | n(y) \in b\mathbb{N} \}.
\]

I prove in Steps 1 to 3 that the additive representation is implied for all $y \in Y_{bb}^{f}$, then in Step 4 I use **Replication Invariance** to extend it to the whole $Y^f$.

**STEP 1:** From a poverty ordering on income distributions to a poverty ordering on distributions of individual poverty.

The definition of the mapping is different. Let the continuous mapping be $M : Y_{bb}^{f} \to \mathbb{R}^{N'}$, where $N' := \{ n \in \mathbb{N} | n + 1 \in a\mathbb{N} \text{ and } n \geq 1 \}$. Let $\succeq^f$ be an EO in $\mathcal{R}_f$ whose unanimous judgments among the poor are respected by $P$. By modified **Domination among Poor**, such $\succeq^f$ exists. Consider any numerical representation $d$ of $\succeq^f$. For each $(y_i, f(y)) \in X$, let $\nu_i := d(y_i, f(y))$. Mapping $M$ is defined for all $y \in Y_{bb}^{f}$ such that

\[
M(y) = (\nu_1, \ldots, \nu_{xn-1}) := \nu.
\]

Observe that if distribution $y \in Y_{bb}^{f}$ has $n$ components, then $M(y)$ has $xn - 1$ components. The size of distribution $\nu$ is taken to be $xn - 1$ as for all $y \in Y$ and all $i \geq xn$ we have $d(y_i, f(y)) = 0$ since $y_i \geq z(f(y))$ and are hence omitted.

I show for the continuous mapping defined that $M(Y_{bb}^{f}) = V_d := [0, 1]^{N'}$. The
Possibility of Poverty Eradication and the continuity of

\( \text{Minimal Gorman} \)

where

As before.

Assumption 1

Assumption 2

Assumption 3

\( \text{As before.} \)

\( \text{\textbf{Proof.}} \)

\( \text{Consider any} \ y \ \text{such that} \ g(y) = r \) and

\( \nu \in \mathbb{R}^n \) defined implicitly by \( \nu \in d(c_i, g) \). If \( \nu = 0 \) take \( c_i := g \). By modified Minimal Absolute Concern and the continuity of \( d \), we have that \( c_i \in \{0, z(g)\} \) for all \( i \leq q \). Take then \( y_{zn} \) such that \( g(y) = \frac{1}{zn} \sum_{i=1}^{zn} y_i = g \). I prove now that such a \( y_{zn} \) exists and is such that \( y_{zn} \geq z(g) \). Consider \( y' \) with \( n(y') = n \) and for which \( y'_i := y_i \) for all \( i \leq zn \), and \( y'_i := g \). By construction, we have \( y'_i \leq g \) for all \( i \in \{1, \ldots, zn(y')\} \) as \( g \geq z(g) \). Therefore we have \( g(y') \leq g \). Now, there exists \( c \geq g \) such that if \( y_n := c \) then \( f(y) = g \). Since \( c \geq g \) and \( g \geq z(g) \) by assumption, we have that agent \( zn \) is non-poor. Take finally \( y_i := y_{zn} \) for all \( i \in \{zn + 1, \ldots, n\} \). This ensures these agents are non-poor and have a weakly higher income than agent \( zn \) and hence \( y \in \mathbb{F}^i \). By construction of \( y \), we have \( f(y) = g \) and \( M(y) = \nu \).

\( \square \)

Again, the poverty ordering \( \succeq_{V_d} \) on the set of income distributions is associated to an ordering \( \succeq_{V_d} \) on distributions of individual poverty, by modified Domination among Poor. This ordering \( \succeq_{V_d} \) can be represented by a continuous poverty index \( P^\nu : V_d \to \mathbb{R} \). In particular, ordering \( \succeq_{V_d} \) is represented by \( P^\nu \) defined such that for all \( \nu \in V_d \) and \( y \in \mathbb{F}^i \) with \( M(y) = \nu \), we have

\( P^\nu(\nu) = P^\nu(M(\nu)) = P(y) \).

\( \text{STEP 2: Index} \ P^\nu \text{representing ordering} \succeq_{V_d} \text{is additively separable.} \)

We verify that the assumptions of Theorem 1 in Gorman (1968) are all met. This allows deriving the following functional form for the index \( P^\nu \), for a given \( n(y) \in \mathbb{R} \):

\( P^\nu(\nu) = \hat{F} \left( \sum_{i=1}^{zn-1} \phi(\nu_i) \right) \) \hspace{1cm} (38)

where \( \hat{F} \) and \( \phi \) are strictly increasing functions.

The assumptions required for this Theorem are the following:

\( \text{Assumption 1: As before.} \)

\( \text{Assumption 2: As before.} \)

\( \text{Assumption 3: Let} \ S := \{[0,1], \ldots,[0,1]\} \text{be the set of sectors in} \ V_d \) and \( A \subseteq S \) be any subset of sectors, we have that each \( A \) is separable. Separability means that for all \( (u,w), (v,w), (u,t), (v,t) \in \mathbb{F} \), we have \( P^\nu(u,w) \geq P^\nu(v,w) \Leftrightarrow P^\nu(u,t) \geq P^\nu(v,t) \). Separability is proven in two substeps.
Substep 1:  Construct the income distributions associated to these distributions of individual poverty.

Construct \( y_1^1, y_2^1, y_3^1, y_4^1 \in \mathbb{R}_+^+ \) such that \( M(y^1) = (u,w) \), \( M(y^2) = (v,w) \), \( M(y^3) = (u,t) \), \( M(y^4) = (v,t) \), \( y_1^1, y_2^1, y_3^1, y_4^1 \in Y_{bn}^f \) and \( f(y^1) = f(y^2) = f(y^3) = f(y^4) = g \) with \( g \geq z(g) \). Such distributions exist in \( Y_{bn}^f \) and are constructed following the procedure presented in Lemma 6.

Decompose in subgroups \( y^1 = (y_A^1, y_B^1, y_C^1) \), such that subdistributions \( y_A^1 \) and \( y_B^1 \) are associated to the individual poverty subdistributions \( u \) and \( w \) respectively and \( y_C^1 \) is the subdistributions containing the income for all \( j \in \{xn, \ldots, n\} \), for whom by construction we have \( y_j \geq g(j) \).

By construction we have \( n(y^1) = \frac{n(u,w) + 1}{2} \) and \( n(v,w) + 1 \in \mathbb{N} \), \( xn(y^1) \in \mathbb{N} \).

Typically, we have \( f(y_A^1) \neq f(y_B^1) \neq g \), but next operations equalize the reference statistic in new subgroups \( A' \) and \( B' \) of a k-replication of \( y^1 \) by distributing the agents in the k-replication of subgroup \( C \) between the k-replications of subgroups \( A \) and \( B \) and modifying income among some non-poor agents.

Let \( s^1 := (y^1, \ldots, y^4) \) be the k-replication of \( y^1 \) with \( k := b(n(u) + n(w)) \).

Re-organize the k-subgroups \( A, B \) and \( C \) in a way to obtain two subgroups \( A' \) and \( B' \): \( s^1 = (s_{A'}^1, s_{B'}^1) \) such that \( \frac{n(s_{A'}^1)}{n(s_{A'}^1) + n(s_{B'}^1)} = \frac{n(u)}{n(u) + n(w)} \) and all agents associated to the k-replication of subgroup \( A \) are in subgroup \( A' \). Given \( k = b(n(u) + n(w)) \) and \( x = \frac{a}{b} \) with \( a, b \in \mathbb{N} \), this equality can be obtained with

\[
\begin{align*}
n(s_{A'}^1) &\in \mathbb{N}, \\
xn(s_{A'}^1) &\in \mathbb{N}.
\end{align*}
\]

since \( bx = a \in \mathbb{N} \) and \( xn(y^1) \in \mathbb{N} \). The numbers \( n(s_{A'}^1) \) and \( xn(s_{A'}^1) \) belong to the naturals as we have

\[
\begin{align*}
n(s_{A'}^1) &= k \left( n(u) + \frac{n(u)}{n(u) + n(w)} [n(y^1) - n(u) - n(w)] \right) = bn(u)n(y^1), \\
xn(s_{A'}^1) &= k \left( n(w) + \frac{n(w)}{n(u) + n(w)} [n(y^1) - n(u) - n(w)] \right) = bn(w)n(y^1),
\end{align*}
\]

which are such that \( n(s_{A'}^1) + n(s_{B'}^1) = n(s^1) = kn(y^1) \). Income distribution \( s_{A'}^1 \) contains at least \( bn(u) \) non-poor agents and whose income is taken into account by the lower partial mean \( f \) when computing \( f(s_{A'}^1) \).\(^{64}\) Accordingly, there are at least \( bn(w) \) non-poor agents and whose income is taken into account by \( f \) when computing \( f(s_{B'}^1) \).

Construct \( s_{A'}^1 = (s_{A'}^{1*}, s_{B'}^{1*}) \) from \( s^1 \) in such a way that \( f(s_{A'}^{1*}) = f(s_{B'}^{1*}) = f(s_{A'}^1) = g \) and \( \{s_{A'}^{1*}, g\} \sim \{s_{A'}^1, g\} \) for all \( i \in \{1, \ldots, k(n(u) + n(v))\} \). In order to construct \( s_{A'}^{1*} \), take

- \( s_{A'}^{1*} := s_{A'}^i \) for all \( i \leq k(n(u) + n(v)) \),
- for all agents \( l \in \{kn(u) + 1, \ldots, kn(u) + bn(u)\} \) in \( s_{A'}^{1*} \):

\[
s_{A'}^{1*} := \frac{xn(s_{A'}^{1*})}{bn(u)} - \sum_{i=1}^{kn(u)} s_{A'}^{1*}.
\]

\(^{64}\)The number \( bn(u) \) is obtained from \( xn(s_{A'}^1) - kn(u) \) which is equal to \( bn(u)xn(y^1) - b(n(u) + n(w))n(u) \).
• for all $j \in \{kn(w) + 1, \ldots, kn(w) + bn(w)\}$ in $s_{B'}^1$

$$s_j^1 := \frac{\text{xn}(s_{B'}^1)g - \sum_{i=1}^{kn(w)} s_i^{1*}}{bn(w)},$$

• $s_i^{1*} := \max(s_i^1, s_j^1)$ for all $i \in \{\text{xn}(s^1*) + 1, \ldots, n(s^1*)\}$,

where both $s_i^{1*} \geq g$ and $s_j^1 \geq g$, which implies those agents are non-poor. We have by construction that $f(s_i^{1*}) = g$ and $f(s_j^1) = g$.

Observe now that we still have $f(s^{1*}) = g$ as

$$f(s^{1*}) = \frac{1}{\text{xn}(s^{1*})} \sum_{i=1}^{\text{xn}(s^{1*})} s_i^{1*} = \frac{1}{\text{xn}(s_A^{1*}) + \text{xn}(s_{B'}^{1*})} \left( \sum_{i=1}^{\text{xn}(s_A^{1*})} s_i^{1*} + \sum_{i=1}^{\text{xn}(s_{B'}^{1*})} s_i^{1*} \right)$$

$$= \frac{\text{xn}(s_A^{1*})g + \text{xn}(s_{B'}^{1*})g}{\text{xn}(s_A^{1*}) + \text{xn}(s_{B'}^{1*})} = g.$$

By construction we have

$$M(s_A^{1*}) = (u, \ldots, u, 0, \ldots, 0),$$

$$M(s_{B'}^{1*}) = (w, \ldots, w, 0, \ldots, 0),$$

where $M(s_A^{1*})$ contains $k$ subdistributions $u$ and $bn(u) = 1$ zeros; while $M(s_{B'}^{1*})$ contains $k$ subdistributions $w$ and $bn(w) = 1$ zeros.

Using the same procedure (decomposition, k-replication, reorganization), construct $s^{2*}, s^{3*}, s^{4*}$ such that:

$$s^{1*} = (s_A^{1*}, s_{B'}^{1*}) \quad \text{with} \quad M(s^{1*}) = (u, \ldots, u, w, \ldots, w, 0, \ldots, 0),$$

$$s^{2*} = (s_A^{2*}, s_{B'}^{2*}) \quad \text{with} \quad M(s^{2*}) = (v, \ldots, v, w, \ldots, w, 0, \ldots, 0),$$

$$s^{3*} = (s_A^{3*}, s_{B'}^{3*}) \quad \text{with} \quad M(s^{3*}) = (u, \ldots, u, t, \ldots, t, 0, \ldots, 0),$$

$$s^{4*} = (s_A^{4*}, s_{B'}^{4*}) \quad \text{with} \quad M(s^{4*}) = (v, \ldots, v, t, \ldots, t, 0, \ldots, 0),$$

where the number of zeros in $M(s^{1*})$ is equal to $b(n(u) + n(w)) - 1$. For all $m \in \{1, 2, 3, 4\}$, we have $P(s^{m*}) = P(g^m)$ by Replication Invariance. As by construction $(s_A^{m*}, g) \sim (s_A^{m*}, g)$ for all $i \leq g(s^{m*}) = g(s^m)$, we have $P(s^{m*}) = P(s^m)$ by Domination among Poor. Therefore, proving $P(s^{1*}) \geq P(s^{2*}) \geq P(s^{3*})$ is equivalent to proving $P(y^1) \geq P(y^2) \geq P(y^3)$, which is equivalent to proving $P(y^1(u, w)) \geq P(y^1(u, t)) \geq P(y^1(v, t))$. For notational simplicity, drop the symbols '*' and ' ′ to name the new distributions $s^{m*}$ and subgroups $A'$ and $B'$ as the old ones.

Substep 2: Prove $P(s_A^{1*}, s_{B'}^{1*}) \geq P(s_A^{2*}, s_{B'}^{2*}) \iff P(s_A^{3*}, s_{B'}^{3*}) \geq P(s_A^{4*}, s_{B'}^{4*})$.

Income distributions are constructed such that $s_A^{1*}, s_{A'}^{2*}, s_{B'}^{2*}, s_{A'}^{3*}, s_{B'}^{3*}, s_A^{4*}, s_{B'}^{4*} \in Y_{bn}^f$, $P(s_A^{1*}) = P(s_A^{3*}) = P(s_A^{4*}) = P(s_B^{1*})$ and $P(s_B^{2*}) = P(s_B^{3*})$ by Domination among Poor. The proof is the same as before.

All assumptions of Theorem 1 in Gorman (1968) are met.
STEP 3: Show functions $\tilde{F}$ and $\tilde{\phi}$ do not depend on the number $n$ of agents.

Theorem 1 in Gorman (1968) is valid for a fixed number of potentially poor agents $n(\nu)$. I modify the proof of Foster and Shorrocks (1991) in order to prove these functions are independent of $n$. When $n$ is allowed to vary – but still respecting $\frac{n+1}{x} \in bN$ – (38) must be written:

$$P^\nu(\nu) = \tilde{F}_n \left( \sum_{i=1}^{n} \tilde{\phi}_n(\nu_i) \right)$$

**Step 3.1:** Define transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$ for normalization purposes.

Let $F_n$ and $\phi_n$ be the following transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$:

$$\phi_n(\nu_i) = \frac{n+1}{x} \left[ \tilde{\phi}_n(\nu_i) - \tilde{\phi}_n(0) \right],$$
$$F_n(X) = \tilde{F}_n \left[ X + n\tilde{\phi}_n(0) \right].$$

These transformations imply successively:

$$F_n \left( \frac{x}{n+1} \sum_{i=1}^{n} \phi_n(\nu_i) \right) = F_n \left( \frac{x}{n+1} \frac{n+1}{x} \sum_{i=1}^{n} [\tilde{\phi}_n(\nu_i) - \tilde{\phi}_n(0)] \right),$$
$$= \tilde{F}_n \left( \sum_{i=1}^{n} \tilde{\phi}_n(\nu_i) - \tilde{\phi}_n(0) \right) + n\tilde{\phi}_n(0).$$

This yields

$$P^\nu(\nu) = F_n \left( \frac{x}{n+1} \sum_{i=1}^{n} \phi_n(\nu_i) \right),$$

where $\phi_n(0) = 0$ and by the definition of $N'$, we have $\frac{n+1}{x} \in N$.

As any $j \in \{xn(y), \ldots, n(y)\}$ is non-poor in $Y^f$, we have $d(y_j, y) = 0$. Therefore, we obtain that for all $n(y)$ for which $xn(y) \in bN$, by slightly abusing notation (by introducing the zero individual poverty of those non-poor agents at the end of the distribution $\nu$):

$$P^\nu(\nu) = F_n \left( \frac{1}{n(y)} \left( \sum_{i=1}^{xn(y)-1} \phi_n(\nu_i) + \sum_{i=xn(y)}^{n(y)} \phi_n(0) \right) \right)$$
$$= F_n \left( \frac{1}{n(y)} \sum_{i=1}^{n(y)} \phi_n(\nu_i) \right),$$

with $F_n$ and $\phi_n$ continuous, strictly increasing and $\phi_n(0) = 0$.

**Step 3.2:** Use Replication Invariance to prove functions $F_n$ and $\phi_n$ do not depend on $n$.

Same as before.
STEP 4: The additively separable expression obtained for all \(y \in Y_f\) is valid for all \(y \in Y_f\setminus Y_{bn}\).

Consider any \(y \in Y_f\) and \(s := (y, \ldots, y)\) a k-replication of \(y\) with \(s \in Y_{bn}\). We have \(f(y) = f(s)\) as

\[
f(s) = \frac{1}{xn(s)} \sum_{i=1}^{xn(s)} s_i, = \frac{1}{kxn} \left( k \sum_{i=1}^{r} y_i + k(xn - r)y_{r+1} \right), = \frac{1}{xn} \left( \sum_{i=1}^{r} y_i + (xn - r)y_{r+1} \right) = f(y),
\]

where \(r := \max_{c \in \mathbb{N}} c \leq xn\) and \(k(xn - r) \in \mathbb{N}\).

The mathematical expression of \(P(s)\) takes the additive separable form. By Replication Invariance, we have \(P(y) = P(s)\), which means that the mathematical expression of any \(y \in Y_f\) also take the same additive separable form as:

\[
P(s) = \frac{1}{n(s)} \sum_{i=1}^{n(s)} d(s_i, f(s)) = \frac{1}{kn} \sum_{i=1}^{n} k \cdot d(y_i, f(y)) = P(y),
\]

where \(d(y_i, f(y)) = d(s_{ki}, f(s))\) for all \(i \leq n\) as \(f(y) = f(s)\) and \(y_i = s_{ki}\).

More interestingly, next result shows that the modified version of Monotonicity in Income has equivalent implications to those derived when the mean is used as income standard. Observe that on the domain of income distribution \(Y_f\), balanced transfers among poor agents never affect lower partial means.

Theorem 10 (Poverty Gap Ratio for partial-mean-sensitive lines). Let \(f\) be a lower partial mean. Let \(z\) be a monotonic \(f\)-sensitive poverty line. Let \(P\) be an additive index based on an absolute-homothetic EO below \(z\) with a numerical representation in the extended FGT family.

1. \(P\) satisfies modified Monotonicity in Income only if \(\alpha = 1\).

2. \(P\) satisfies modified Monotonicity in Income and modified Transfer among Poor if and only if \(\alpha = 1\) and \(f^r = 0\), that is \(d\) is the PGR at the origin.

Proof. The proof of both claims relies on a modification of Lemma 2 for partial means. This modification provides a necessary condition for Claim 1 and a sufficient condition for Claim 2 under which modified Monotonicity in Income is satisfied by additive indices. The modified definitions for the notions of degree of priority and slope are straightforward:

**Definition 10 (Degree of Priority partial means).**

\[
DP_{ij}(f(y)) := \frac{\partial^2 d(y_i, f(y))}{\partial d(y_j, f(y))}
\]

**Definition 11 (Slope at \((y_i, f(y))\)).**

\[
s(y_i, f(y)) := -\frac{\partial^2 d(y_i, f(y))}{\partial d(y_j, f(y))}
\]
Lemma 7. An additive poverty index based on an absolute-homothetic EO satisfies modified Monotonicity in Income:

1. (sufficient condition) if for all \( y \in Y^f \) and \( y_i, y_j < z(f(y)) \), we have:

\[
s(y_j, f(y)) \leq DP_{ij}(f(y))
\]

(39)

2. (necessary condition) only if for all \( y \in Y^f \) such that there exists \( g > 0 \)

with \( f(y) = g \) and \( z(g) \leq g \), and all \( y_i, y_j < z(f(y)) \), inequality (39) holds.

Proof. Let \( f \) be a lower partial mean. Consider any additive index \( P \) based on any \( f \)-sensitive line, EO in \( \mathcal{R} \) and numerical representation \( d \). Monotonicity in Income requires that for all \( y \in Y \) and \( i \leq q \) we have \( \partial_i P(y) \leq 0 \). By the additively separable form of \( P \), we obtain by chain derivation:

\[
\partial_1 d(y_i, f(y)) + \sum_{j=1}^{n} \partial_2 d(y_j, f(y)) \partial_i f(y) \leq 0
\]

(40)

From the definition of lower partial means, we have \( \partial_i f(y) = \frac{1}{x} \).\(^{65} \) From the definition of \( s(y_j, f(y)) \), we get \( \partial_2 d(y_j, f(y)) = -\partial_1 d(y_j, f(y)) \) \( s(y_j, f(y)) \) for all \((y_j, f(y)) \in X \). Inequality (40) becomes:

\[
\partial_1 d(y_i, f(y)) - \frac{1}{x} \sum_{j=1}^{n} \partial_1 d(y_j, f(y)) \ s(y_j, f(y)) \leq 0
\]

\( \text{L}_{41} \)

(41)

In the remainder of the proof, inequality (41) is shown to imply the necessary and sufficient conditions linked to (39). Inequality (39) can be rewritten:

\[
\partial_1 d(y_i, f(y)) - \partial_1 d(y_j, f(y)) \ s(y_j, f(y)) \leq 0.
\]

\( \text{L}_{42} \)

(42)

Necessity of condition 2 is proved by contradiction. Assume (42) does not hold for some \( y \in Y^f \) with \( f(y) = g \), \( z(g) \leq g \), \( y_i := a \), \( y_j := b \) with \( 0 \leq a < b < z(g) \).\(^ {66} \) Therefore, at \((a, g), (b, g) \in X \), we have for some \( l > 0 \) that \( L_{42} = l \). I prove that for all \( \epsilon > 0 \), there exists \( y' \in Y^f \) with \( f(y') = g \) such that \( |l - L_{41}(y')| < \epsilon \) and hence there exists an \( y' \in Y^f \) such that \( L_{41}(y') > 0 \). Construct \( y' \) such that

- \( y'_1 := a \),
- \( y'_k := b \) for all \( 2 \leq k \leq xn - 1 \),
- \( y'_{xn} := xn \cdot a - (xn - 2)b \) and
- \( y'_j = y'_{xn} \) for all \( xn + 1 \leq j \leq n \).

\(^{65} \) We assume for simplicity that \( xn \in \mathbb{N} \).

\(^{66} \) I take \( a < b \) without loss of generality as the same reasoning can be held for the other assumption.
Monotonicity in Income

Translation Monotonicity does not hold. For this $y'$, as $y'_{xn} > z(g)$ we have for all $i \in \{xn, \ldots, n\}$ that $\partial_1 d(y'_i, g) = 0$. Therefore

$$l - L_{41}(y') = L_{42} - L_{41}(y') = -\frac{1}{xn} \left(2\partial_1 d(b, g) s(b, g) - \partial_1 d(a, g) s(a, g)\right).$$

Taking $n(y')$ sufficiently large, we can make $|l - L_{41}(y')| < \epsilon$, implying $L_{41}(y') > 0$, which violates (41) and hence modified Monotonicity in Income does not hold. The case for which $\partial_1 d(b, g)$ and $\partial_1 d(a, g)$ are not finite is treated as in the proof of Lemma 2.

 Sufficiency of condition 2 follows from the fact that if there exists an $y \in Y^f$ violating (41), inequality (42) is violated as well for a particular value of $y_j$. For all $y \in Y^f$ there exists $y^*_j \in [0, z(f(y))]$ such that, taking $y_j := y^*_j$ in $L_{42}$, we have $L_{41}(y) < L_{42}$:

$$-\frac{1}{xn} \sum_{j=1}^{n} \partial_1 d(y_j, f(y)) s(y_j, f(y)) < -\partial_1 d(y^*_j, f(y)) s(y^*_j, f(y)),$$

where the strict inequality comes from the presence of the non-poor agent $xn$. Observe we can consider only the $xn$ first agents since for all agents $k \in \{xn, \ldots, n\}$, we have $y_k > z(f(y))$ and hence $\partial_1 d(y_k, f(y)) = 0$. At the value of reference statistic $f(y)$, $y^*_j$ is obtained by solving the following problem:

$$y^*_j := \arg \max_{y_j \in [0, z(f(y))]} -\partial_1 d(y_j, f(y)) s(y_j, f(y)).$$

9.9 Proof of Theorem 7

The proof is by contradiction. Assume that EO $\succeq$ is such that for some $(y^*_i, f^{gm}(y^*)) \in X_p$ we have $k := s(y^*_i, f^{gm}(y^*)) > 0$.

The modified version of Translation Monotonicity imposes that for all $(y_i, f^{gm}(y)) \in X_p$ and all $\delta > 0$ we have

$$(y_i + \delta, f^{gm}(y + \delta 1_n)) \succeq (y_i, f^{gm}(y)).$$

$\text{If } y_{en} = z(f(y)) \text{ and } s(f(y)) > 0, \text{ then the increment } \epsilon \text{ given to a poor agent implies } y' \notin Y^f \text{ as } y'_{en} < z(f(y')).$
Take any continuous and differentiable numerical representation \( d \) of \( \succeq \) (such \( d \) exists since \( \succeq \) is a continuous ordering). Translation Monotonicity can be equivalently restated using the numerical representation \( d \): for all \( y \in Y \), \( i \leq q \) and \( \delta > 0 \) we have:

\[
d(y_i + \delta, f^{gm}(y + \delta \mathbb{1}_n)) \leq d(y_i, f^{gm}(y)).
\]

A necessary condition for the previous condition to hold is that we have for all \( y \in Y \) and \( i \leq q \)

\[
\partial_i d(y_i, f^{gm}(y)) + \partial_j d(y_j, f^{gm}(y)) \ (\nabla f^{gm}(y) \cdot \mathbb{1}_n) \leq 0.
\]

Given the slope is defined as

\[
s(y, f^{gm}(y)) := -\frac{\partial d(y_i, f^{gm}(y))}{\partial d(y_j, f^{gm}(y))},
\]

at \((y^*_i, f^{gm}(y^*))\), the previous inequality amounts successively to:

\[
\partial_i d(y^*_i, f^{gm}(y^*)) - k\partial_j d(y^*_i, f^{gm}(y^*)) \ (\nabla f^{gm}(y^*) \cdot \mathbb{1}_n) \leq 0,
\]

\[
\partial_i d(y^*_i, f^{gm}(y^*)) (1 - k(\nabla f^{gm}(y^*) \cdot \mathbb{1}_n)) \leq 0.
\]

Since the first factor is strictly negative by modified Strict Monotonicity in Income, a necessary condition for Translation Monotonicity is that:

\[
k(\nabla f^{gm}(y^*) \cdot \mathbb{1}_n) \leq 1. \tag{43}
\]

I construct \( y^1 \in Y \subset \mathbb{R}^N_+ \) such that \((y^1_{n-1}, f^{gm}(y^1)) = (y^*_i, f^{gm}(y^*))\) and (43) is violated at \( y^1 \), leading to a violation of Translation Monotonicity. Let \( y^1 \) be constructed such that

- \( y^1_j := 0 \) for all \( j \leq n - 2 \),
- \( y^1_{n-1} := y^*_i \), and
- \( y^1_n \) is such that \( y^1_n \geq z(f^{gm}(y^1)) \) and \( f^{gm}(y^1) = f^{gm}(y^*) \).

For \( n \) sufficiently large, there exists such an \( y_n \) (unshown). By the definition of the generalized mean, we have:

\[
\partial_i f^{gm}(y) = \frac{1}{n} \left( \frac{y^1_i + \cdots + y^1_n}{n} \right)^{\frac{1}{\beta}} - 1 y^1_i^{\frac{1}{\beta} - 1},
\]

\[
= \frac{1}{n} (f^{gm}(y))^{1-\beta} y^1_i^{-1}.
\]

Therefore, we have:

\[
\nabla f^{gm}(y^1) \cdot \mathbb{1}_n \geq \frac{n-2}{n} (f^{gm}(y^1))^{1-\beta} 0^{\beta-1}.
\]

As I assumed \( \beta < 1 \), the factor \( 0^{\beta-1} = +\infty \) and (43) is violated. As a result, Translation Monotonicity cannot hold.

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68Where \( \cdot \) is the notation for a dot product and \( \nabla \) is the notation for the gradient.
69The inequality sign comes from the fact I ignored the positive terms coming from the increments given to agents \( n - 1 \) and \( n \).


9.10 Mexican poverty: further analysis

This section aims at presenting two extra graphical tools. These tools are modifications of well-known tools introduced in the poverty measurement literature, which are both intuitive and helpful in analyzing the evolution of income poverty. I illustrate the changes that occurred in Mexico using these graphical tools.

The economic growth of Mexico between 1990 and 2010 has lead to an almost complete eradication of absolute poverty. Nevertheless, the increase in income inequality over that period, as measured by the relative measure $HC_{RL}$, increased the fraction $HC_{EL}$ of poor individuals. $P_{EL}$ concludes that income poverty has not changed, even if its nature became more relative than absolute.

Figure 8 shows for several points in time cumulative distributions limited to individuals whose income is below the endogenous threshold. The upper figure shows standard cumulative income distributions. The lower figure shows cumulative well-being distributions. Well-being is defined as $1 - d(y_i, y_j)$, where the numerical representation $d$ is based on the absolute-homothetic EO illustrated in Figure 6.a. Any individual with a well-being of one or above is non-poor.

The two graphs are such that the cumulative distributions for 2010 first-order stochastically dominate those of 1999. There is hence an unambiguous improvement over that period of both incomes and well-beings. The cumulative income distributions inform on the evolution of several variables. As it is limited to poor individuals, it shows the evolution of the endogenous threshold, which was $4.5$ a day in 1990, $4.2$ a day in 1999 before increasing up to $5.9$ a day in 2010. This evolution translates the changes in mean income, from $7.8$ a day in 1990 to $7.1$ a day in 1999 and up to $10.6$ a day in 2010. The evolution of $HC_{EL}$, from $29\%$ in 1990 and $47\%$ in 1999 to $41\%$ in 2010, can be read from the graphs by considering the end points’ ordinate of these cumulative distributions. Similarly, the same graph presents the evolution of $HC_{AL}$, from $4.5\%$ in 1990 and $7.5\%$ in 1999 to $0.7\%$ in 2010.

Income cumulative distributions can be drawn without making any normative choice on how to balance absolute and relative income in individual well-being (except those already made by the endogenous line). The cumulative well-being distributions in the lower graph make such choices with its absolute-homothetic EO. These graphs provide again the evolution of $HC_{EL}$. More interestingly, they show the evolution of $P_{EL}$. The values of $P_{EL}$ is equal to the areas below the cumulative well-being distributions.

Comparing the graphs related to 1990 and 2010 shows that less individuals have very low well-being in 2010 than in 1990 (the well-being threshold for absolute poverty is $\frac{1}{2.25} = 0.625$) but more individuals have well-being levels between 0.75 and 1, indicating that more individuals are in relative poverty. The index balances these two aspects and concludes income poverty has not changed (the area below the two curves is the same): $P_{EL}$ equals $7.4\%$ in 1990, $11.6\%$ in 1999 and $7.5\%$ in 2010.

If the EO used for assessing the well-being of individuals is homothetic, then the area below the cumulative distributions of “homothetic” well-being equals

\[ P_{EL} = \frac{1}{0} \int F^{-1}(x) dx, \]

where $1 - F^{-1}(x)$ is the individual poverty level such that a fraction $x$ of the population has lower well-being.

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\textsuperscript{70}Letting $F : [0, 1] \rightarrow [0, 1] : 1 - d(y_i, y_j) \rightarrow F(1 - d(y_i, y_j))$ be the cumulative well-being distribution function associated with $\equiv_{AH}$, we have that $P_{EL} = \int_0^{HC_{EL}} \int F^{-1}(x) dx$, where $1 - F^{-1}(x)$ is the individual poverty level such that a fraction $x$ of the population has lower well-being.
**Figure 8:** Evolution of Mexico’s cumulative distributions below the endogenous line. The well-being levels presented above are defined as $1 - d(y, \tau)$. Source: PovcalNet.
This cumulative distribution of “homothetic” well-being, which gives no priority to individuals below the subsistence level $z^a$, concludes that income poverty has changed from 12.4% in 1990, 20.1% in 1999 to 15.6% in 2010.
References


