DURABLE GOODS IN THE RESOLUTION OF LONG-RUN RISK*

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October 2017

Abstract

The Long Run Risk model (LRR) presented in Bansal and Yaron (2004) has become the benchmark in the macro-finance literature for its capability to explain several financial markets stylised facts. However, LRR implies unrealistic high levels of time and risk premia, see Epstein, Farhi, and Strzalecki (2014). Our study introduces a LRR model where durable and non-durable consumption goods are non-separable and gross complements, thus generating households’ concern with short and long run composition risk, that is, fluctuations in the relative share of durables in their consumption basket. We estimate the endowment economy with particle MCMC method to exploit time-varying properties of the data. We then solve the model in a fully non-linear fashion in order to recollect important higher order effects in asset prices. Our model explains financial data well with parameters values that are consistent with the macroeconomic literature and, at the same time, it reduces time and risk premia by more than 50 percent when compared with its single consumption good counterpart.

JEL classification: C11, E21, G11, G12

Keywords: Durable Goods, Composition Risk, Long-Run Risk Model, Equity Premium Puzzle, Elasticity of Substitution.

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*We thank Cahal Moran, Ákos Valentinyi and the participants at various seminars and conferences where the paper has been presented for useful comments. Myroslav Pidkuyko acknowledges the financial support from the Economic and Social Research Council UK [grant number ES/J500094/1].

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1 Introduction

The long-run risks model (LRRM) of Bansal and Yaron (2004) emphasizes that variations in the long-run growth rate, the variance of shocks to the growth rate (stochastic volatility) and Epstein and Zin (1989)’s preferences can deliver a unified explanation of several otherwise puzzling aspects of asset markets. However, a possible nuisance of LRRM is that in its standard parametrization, it implies unrealistic levels of timing and risk premium. With a parameter of risk aversion of 7.5 and an elasticity of intertemporal substitution of 1.5, agents populating the LRRM are willing to give up 24 percent of their lifetime consumption in order to resolve their risk one period ahead and 48 percent of their lifetime consumption to live in a world without risk, see Epstein, Farhi, and Strzalecki (2014).

Our main contribution consists in presenting a modification of the LRRM that can explain several facts of asset markets and, at the same time, can deliver realistic levels of timing and risk premium. At its core, we extend Bansal and Yaron (2004)’s benchmark by considering an endowment economy where non-durable and durable goods are gross complements in the agents’ utility function and both durable and non-durable consumption and dividend growth rates contain small long-run (possibly correlated) predictable components. As for the rest, our model is isomorphic to the original LRRM. In our benchmark estimation, our LRRM matches financial data well with a risk aversion of 1.86, an elasticity of intertemporal substitution of 1.18 and an elasticity of substitution between durable and non-durable goods of 0.78. With this parametrization, the timing and risk premia are 11 and 16 percent, respectively. Compared to the its single consumption good counterpart, our model reduces the timing premium by around 54 percent and risk premia by roughly 66 percent.

In the first part of our empirical analysis we provide strong evidence for a persistent component of durable and non-durable consumption growth as well as its time-varying volatility, which contradicts the commonly held view that consumption follows a random walk. Our basic empirical finding is robust across a wide range of model specifications and consists of a tri-variate model with dividend growth as third observable. The tri-variate model features two distinct persistent factors in cash-flow
growth rates, one coming from durable consumption and one from non-durable consumption. An important conclusion from our analysis is that both the predictable components of durables and non-durables play an important role in the dividend growth equation. Given we do not want to restrict our empirical specification of the returns, we do not include the measurement equations for those in our state-space model, nevertheless, the volatility process still affects the conditional means and volatilities of the asset prices. Due to the high-dimensionality of our state space model, the non-linear filtering is a seemingly daunting task. As in Schorfheide, Song, and Yaron (2014) we exploit the partially linear structure of the state-space model to derive a very efficient sequential Monte Carlo (particle) filter.

To carry out asset pricing, we embed the estimated time-series process for dividend, non-durable and durable consumption into an endowment economy with a representative agent that has Epstein-Zin preferences over a CES consumption bundle of durable and non-durable goods. We then estimate the model by combining the semi-parametric estimation technique presented in Chen, Favilukis, and Ludvigson (2013) with a fully fledged non-linear solver. This analysis generates predictions for the average equity premium, the volatility of equity returns, price-dividend ratio and so on. Then we compare these predictions with averages found in the quarterly data for the US over the period 1952-2016 and we carefully disentangle how each ingredient of the model as well as the complex non-linear dynamics matters for the results.

The rest of the paper is organized as follows. Section 2 describes the economic environment, Section 3 presents the empirical analysis and the quantitative assessment of the model. Finally, Section 4 concludes.

2 The Model

Consider an economy in which in every period $t$ a representative agent derives utility from a consumption bundle $u(C_t, D_t)$ represented by a CES function

$$u(C_t, D_t) = \left((1 - \alpha)C_t^{\frac{\beta-1}{\beta}} + \alpha D_t^{\frac{\beta-1}{\beta}}\right)^{\frac{\beta}{\beta-1}}. \quad (2.1)$$

$C_t$ is the non-durable consumption good that is non-storable and is entirely consumed in period $t$, and $D_t$ is the service flow from durable consumption good, $\alpha$ is the share
of durable consumption in the utility whereas $\rho$ is the elasticity of substitution between non-durable and durable consumption. Note that when $\rho = 1$ the expression (2.1) collapses to the familiar Cobb-Douglas case, while for $\rho < 1$ ($\rho > 1$) durable and non-durable consumption goods are gross complements (substitutes). Without loss of generality we assume that the service flow from durable consumption good is proportional to the stock of durable goods, which evolves according to the law of motion

$$D_t = (1 - \delta_t)D_{t-1} + E_t,$$

where $\delta_t \in (0, 1)$ is the (possibly) time varying depreciation rate and $E_t$ is the investment (or expenditure) in durable consumption goods. The utility function of the agent is recursive as in Epstein and Zin (1989, 1991) and Weil (1989) and is given by

$$U_t = \left\{ (1 - \beta)u(C_t, D_t)^{1-\gamma} + \beta \left( E_t [U_{t+1}]^{1-\gamma} \right) \right\}^{\frac{1}{1-\gamma}}. \quad (2.2)$$

The parameters of the agent’s utility function are the subjective discount factor $\beta \in (0, 1)$, the relative risk aversion coefficient $\gamma > 0$, and the elasticity of intertemporal substitution $\psi \geq 0$ with $\theta \equiv (1 - \gamma)/(1 - \frac{1}{\psi})$.

Consider a Lucas (1978) type of endowment economy with four assets: a non-durable consumption good, a durable consumption good, a stock, and a risk-free bond. The risk-free bond is in zero net supply and acts purely as a discount bond. The other three trees are in positive net supply. In each period $t$ the agent chooses the level of consumption (both non-durable and durable) and asset holdings to satisfy her budget constraint

$$C_t + P_tE_t + B_{b,t} + B_{s,t} = B_{b,t-1}R_{b,t} + B_{s,t-1}R_{s,t}, \quad (2.3)$$

where $P_t$ is the relative price of durable goods in terms of non-durable goods, $B_{b,t}$ is the $t$–period risk-free bond holdings, $B_{s,t}$ is the $t$–period stock holdings, $R_{b,t}$ is the return on risk-free bond, and $R_{s,t}$ is the return on stock.

### 2.1 Endowments

Now we describe the stochastic endowment process. In each period, a non-durable good $C_t$, a durable good $D_t$, and a dividend from stock $S_t$ arrive. We follow Bansal
and Yaron (2004) and model the growth rate of non-durable consumption \( \Delta C_{t+1} = \log(C_{t+1}/C_t) \) as containing a small persistent predictable component \( x_t \):

\[
\Delta C_{t+1} = \mu_c + x_t + \sigma_t \epsilon^c_{t+1},
\]

\( x_{t+1} = \rho_x x_t + \psi_x \sigma_t \epsilon_x^{x_t+1} \). \hfill (2.4)

We assume that the growth rate of durable consumption, \( \Delta D_{t+1} = \log(D_{t+1}/D_t) \), also contains a small persistent predictable component \( y_t \), that is different from \( x_t \):

\[
\Delta D_{t+1} = \mu_d + y_t + \psi_y \sigma_t \epsilon_y^{d_t+1},
\]

\( y_{t+1} = \rho_y y_t + \psi_y \sigma_t \epsilon_y^{y_t+1} \). \hfill (2.5)

Finally, we assume that dividend growth, \( \Delta S_{t+1} = \log(S_{t+1}/S_t) \), is exposed to both \( x_t \) and \( y_t \), as well as to shocks from \( \Delta C_{t+1} \) and \( \Delta D_{t+1} \):

\[
\Delta S_{t+1} = \mu_s + \phi_x x_t + \phi_y y_t + \tau_c \sigma_t \epsilon_c^{c_t+1} + \tau_d \sigma_t \epsilon_d^{d_t+1} + \psi_s \sigma_t \epsilon_s^{s_t+1},
\]

\( \epsilon_c^{c_t+1}, \epsilon_d^{d_t+1}, \epsilon_x^{x_t+1}, \epsilon_y^{y_t+1}, \epsilon_s^{s_t+1} \) and \( w_{t+1} \) are all i.i.d. \( N(0,1) \) and mutually independent.

\subsection*{2.2 Euler Equations}

Let \( W_t \) denote the period \( t \) wealth of the agent given by

\[
W_t = C_t + P_t E_t + B_{b,t} + B_{s,t},
\]

while \( W_{t+1} \) is given by

\[
W_{t+1} = B_{b,t} R_{b,t+1} + B_{s,t} R_{s,t+1}.
\]

We define the total wealth of the agent \( \tilde{W}_t \) as the sum of his current wealth and the value of the stock of durable goods

\[
\tilde{W}_t = W_t + (1 - \delta)P_t D_{t-1}.
\]
If we treat the durable consumption good as an asset, we can define

\[ B_{d,t} = P_tD_t \]

as the share of total wealth invested into durable goods and let

\[ R_{d,t+1} = \frac{(1 - \delta)P_{t+1}}{P_t} \]

denote the return on durable consumption good. Let define \( \omega_{i,t} = \frac{B_{i,t}}{(W_t-C_t)} \), i.e. the share of wealth net of non-durable consumption invested in activity \( i \). Then we can rewrite the agent’s budget constraint in a recursive form as

\[ \tilde{W}_{t+1} = \tilde{W}_t - C_t (\omega_{b,t}R_{b,t+1} + \omega_{s,t}R_{s,t+1} + \omega_{d,t}R_{d,t+1}) \] (2.10)

with

\[ \sum_{i \in \{b,s,d\}} \omega_{i,t} = 1. \] (2.11)

The consumption-portfolio choice problem of the agent is the following. Given current total wealth level \( \tilde{W}_t \), the household chooses consumption \( C_t \) and investment shares \( \omega_{b,t}, \omega_{s,t} \) and \( \omega_{d,t} \) to maximise utility (2.2) subject to (2.10) and (2.11). Therefore, we can write the Bellman equation for the problem as,

\[ J_t = \max_{\{C_t,\omega_{b,t},\omega_{s,t},\omega_{d,t}\}} \left\{ (1 - \beta)u(C_t, D_t)^{\frac{1-\gamma}{\sigma}} + \beta \left( \mathbb{E}_t[J_{t+1}]^{1-\gamma} \right)^{\frac{\theta}{1-\gamma}} \right\}^{\frac{\theta}{\gamma}}. \] (2.12)

where \( J_t \) is the period \( t \) value function.

The solution to this maximisation problem yields to two margins. First, in any given period, the marginal rate of substitution between the durable and non-durable consumption good needs to equal their relative prices, i.e.

\[ \frac{u_{D,t}}{u_{C,t}} = P_t - (1 - \delta)\mathbb{E}_t[M_{t+1}P_{t+1}] = Q_t, \] (2.13)

where \( M_{t+1} \) is the stochastic discount factor between period \( t \) and \( t + 1 \), and \( Q_t \) is the user cost of the service flow for the durable good. The second margin defines the intertemporal marginal rate of substitution (IMRS) between any two adjacent periods as,

\[ M_{t+1} = \beta^\theta \left( \frac{C_{t+1}}{C_t} \right)^{-\theta/\psi} \left( \frac{v(D_{t+1}/C_{t+1})}{v(D_t/C_t)} \right)^{\theta(1/\rho - 1/\psi)} R_{W,t+1}^{\theta-1}. \] (2.14)
where we define the function $v\left(\frac{D_t}{C_t}\right)$ as
\[
v\left(\frac{D_t}{C_t}\right) = \left[1 - \alpha + \alpha \left(\frac{D_t}{C_t}\right)^{1-1/\rho}\right]^{1/(1-1/\rho)},\]
and $R_{W,t+1} = \frac{\tilde{W}_{t+1}}{W_t - C_t - Q_t D_t}$ is the return on total consumption.

As in Epstein and Zin (1989, 1991) we can show that the first-order condition with respect to non-durable consumption and portfolio choice (risk-free bond and stock) implies that return on any tradable asset in the economy satisfies the Euler equation
\[
E_t [M_{t+1}R_{i,t+1}] = 1, \quad (2.15)
\]
where $i = b$ implies that the equation above holds for risk-free bond, and $i = s$ implies that the equation above holds for return on stock. Finally, the first-order condition with respect to durable consumption choice implies another Euler equation
\[
E_t [M_{t+1}(R_{b,t+1} - R_{d,t+1})] = \frac{u_{D,t}}{P_t u_{C,t}}. \quad (2.16)
\]

3 Empirical Analysis

3.1 Data

Personal consumption data is from the US National Income and Product Accounts and is available from Bureau of Economic Analysis (BEA). We measure non-durable consumption as the sum of personal consumption expenditures on non-durable goods and services. This measure includes food, clothing items, housing and utilities, health care services, transportation. Sample period is 1947:Q1 - 2014:Q4.

Durable consumption includes motor vehicles and parts, furnishings and durable household equipment, recreational goods and services, jewellery and watches. Since the BEA reports only annual series for consumers stock of durable goods, we interpolate the quarterly series by assuming that the depreciation rate is constant within year, such that the implied value of the depreciation rate is consistent with annual stocks of durable goods both at the beginning and at the end of the year, and with quarterly series of personal consumption expenditure (PCE) on durable goods. Sample period is 1947:Q1 - 2014:Q4.
Figure 1 plots the durable consumption as a ratio of non-durable consumption (black solid line) from 1952:I to 2014:IV. The time series exhibits an upward trend during the sample period, with the value of durable consumption relative to non-durable consumption in 2014:IV being about 3.5 larger than corresponding value in 1952:I. The upward trend in the series is also consistent with the downward trend in price of durable goods relative to non-durable goods (red dashed line in Figure 1).

We retrieve the US Population data for the sample period 1947:Q1-2014:Q4 from Federal Reserve Bank of St. Louis to obtain the per-capita quantities.

The returns on the stock market and the short-term interest rate for the sample period 1947:Q1 - 2014:Q4 are from the Center for Research in Security Prices (CRSP) available through Wharton Research Data Services. All asset returns are deflated with the PCE price index for non-durable consumption. The real dividend series are from Robert Shiller’s website. Sample period is 1947:Q1-2014:Q4. We construct the ex-ante real risk-free as a fitted value from a projection of ex post real rate on the current nominal yield and inflation over the previous year (nominal yield is the CRSP Fama Risk Free Rate, inflation is CPI rate available from CRSP).
3.2 Quantitative Assessment

3.2.1 State-Space Representation and Bayesian Inference

In order to conduct empirical analysis we rewrite the model in the state-space form. The full set of equations for the endowment process is

\[ \Delta C_{t+1} = \mu_c + x_t + \sigma_t \epsilon^c_{t+1}, \]
\[ x_{t+1} = \rho_x x_t + \psi_x \sigma_t \epsilon^x_{t+1}. \]
\[ \Delta D_{t+1} = \mu_d + y_t + \psi_d \sigma_t \epsilon^d_{t+1}, \]
\[ y_{t+1} = \rho_y y_t + \psi_y \sigma_t \epsilon^y_{t+1}. \]
\[ \Delta S_{t+1} = \mu_s + \phi_x x_t + \phi_y y_t + \pi_c \sigma_t \epsilon^c_{t+1} + \pi_d \sigma_t \epsilon^d_{t+1} + \psi_x \sigma_t \epsilon^x_{t+1} + \psi_y \sigma_t \epsilon^y_{t+1}, \]
\[ \sigma_t = \bar{\sigma} \exp(h_t) \]
\[ h_{t+1} = \rho_h h_t + \sigma_h \sqrt{1 - \rho_h^2} w_{t+1}, \]
The state-space system consists of two parts: the measurement equation and the transition equation. The measurement equation is

\[ y_{t+1} = D + Z s_{t+1}, \]  

(3.2)

where \( y_{t+1} \) includes the consumption and dividends growth

\[ y_{t+1} = \begin{pmatrix} \Delta C_{t+1} \\ \Delta D_{t+1} \\ \Delta S_{t+1} \end{pmatrix}, \]

and \( s_{t+1} \) includes the state variables of the model

\[ s_{t+1} = \begin{pmatrix} x_t \\ y_t \\ \sigma_t e_{t+1}^x \\ \sigma_t e_{t+1}^d \\ \sigma_t e_{t+1}^s \end{pmatrix}, \]

with

\[ D = \begin{pmatrix} \mu_c \\
\mu_d \\
\mu_s \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & \psi_d & 0 \\
\phi_x & \phi_y & \pi_c & \pi_d & \psi_s \end{pmatrix}. \]

The transition equations take the form

\[ s_{t+1} = \Phi s_t + v_{t+1}(h_t), \]

(3.3)

\[ h_{t+1} = \Psi h_t + \Sigma h_{t+1}, \]

(3.4)

where

\[ v_{t+1}(h_t) = \begin{pmatrix} \psi_x \sigma_t e_{t+1}^x \\
\psi_y \sigma_t e_{t+1}^y \\
\sigma_t e_{t+1}^d \\
\sigma_t e_{t+1}^s \\
\sigma_t e_{t+1}^h \end{pmatrix} = \begin{pmatrix} \psi_x \sigma \exp(h_t) e_{t+1}^x \\
\psi_y \sigma \exp(h_t) e_{t+1}^y \\
\sigma \exp(h_t) e_{t+1}^d \\
\sigma \exp(h_t) e_{t+1}^s \\
\sigma \exp(h_t) e_{t+1}^h \end{pmatrix}, \quad \Phi = \begin{pmatrix} \rho_x & 0 & 0 & 0 & 0 \\
0 & \rho_y & 0 & 0 & 0 \\
0 & 0 & \rho_x & 0 & 0 \\
0 & 0 & 0 & \rho_y & 0 \\
0 & 0 & 0 & 0 & \rho_h \end{pmatrix} \]

and (3.4) is the same as (2.9).

The system (3.2)-(3.4) defines a non-linear state-space system in which we use a Bayesian inference to estimate the parameter vector

\[ \Theta = (\rho_x, \psi_x, \psi_d, \rho_y, \psi_y, \phi_y, \pi_c, \pi_d, \psi_s, \rho_h, \sigma_h). \]  

(3.5)

The posterior distribution of \( \Theta \) given the data \( Y \) can be expressed as

\[ P(\Theta|Y) = \frac{P(Y|\Theta)P(\Theta)}{P(Y)}. \]  

(3.6)
To generate draws from the posterior distribution we specify the prior distribution $P(\Theta)$ and numerically evaluate the likelihood function $P(Y|\Theta)$. Because our model is non-linear, we use particle filtering to approximate the likelihood function by $\hat{P}(Y|\Theta)$. We follow the approach of Schorfheide, Song, and Yaron (2014) and estimate the model in the following way. Since the model is linear and Gaussian conditional on the volatility state $h_t$, we approximate the likelihood function $P(Y|\Theta)$ by a computationally efficient particle filter approximation $\hat{P}(Y|\Theta)$ in which we replace the particle values of $s_t$ by the mean and covariance matrix of the conditional distribution $s_t|(h_t,Y_{1:t})$ which we obtain by linear Kalman filter. We then insert the approximation $\hat{P}(Y|\Theta)$ into a fairly standard Metropolis-Hastings algorithm (Andrieu, Doucet, and Holenstein, 2010, show that use of the approximation in the MCMC algorithms still delivers draws from the true posterior distribution).

3.2.2 Estimation Results

Table 1 summarizes the estimation results for the parameter vector $\Theta$. We fix the values of the parameters $\mu_c$, $\mu_d$ and $\mu_s$ to their sample averages and $\tilde{\sigma}$ to 0.0096. Moreover, we fix the loading of the dividends on the non-durable long-run component $x_t$, $\phi_x$, to 4 for the identification purposes.

Parameter Estimates. We choose fairly uninformative priors for the estimation and in most cases we use Uniform distribution as a prior (the exception is the volatility of the volatility parameter $\sigma^2_h$ for which we choose the Inverse-Gamma distribution). For the loading parameters we choose highly dispersed priors between 0 and 10 or between 0 and 20. The 90% credible intervals for the priors are reported in third column of Table 1. For the persistence coefficients we also choose a fairly dispersed priors, the 90% credible interval for $\rho_x$ and $\rho_y$ ranges from 0.7 to 0.99 and covers the values reported in Bansal and Yaron (2004); Bansal et al. (2009); Yang (2011); Schorfheide et al. (2014); Eraker et al. (2016) and others.

The last column of Table 1 reports the percentiles of the posterior distribution for the estimated parameters. The estimates of the persistence of the long-run components, $\rho_x$ and $\rho_y$, are 0.85 and 0.91, accordingly, with the durable long-run component being
more persistent. Durable long-run component is also more volatile, with $\psi_y = 0.69$, compared to that of non-durable, with $\psi_x = 0.31$. The dividends depend on both non-durable and durable long-run components, with larger effect coming from the non-durable long-run component (we fixed $\phi_x = 4$, and $\phi_y$ is estimated at $\phi_y = 0.69$, similar to Yang, 2011). We also find a non-zero effect on dividend growth coming from the noise to non-durable and durable consumption, with $\pi_c$ and $\pi_d$ both being strictly positive. Finally, the volatility process is highly persistent, with $\rho_h = 0.96$.

**Table 1: Estimated Coefficients of the Endowment Process.**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Prior 5%</th>
<th>Prior 95%</th>
<th>Posterior 50%</th>
<th>Posterior 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_x$</td>
<td>U 0.71</td>
<td>0.99</td>
<td>0.79</td>
<td>0.85</td>
</tr>
<tr>
<td>$\psi_x$</td>
<td>U 0.05</td>
<td>9.95</td>
<td>0.25</td>
<td>0.31</td>
</tr>
<tr>
<td>$\psi_d$</td>
<td>U 0.05</td>
<td>9.95</td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td>$\rho_y$</td>
<td>U 0.71</td>
<td>0.99</td>
<td>0.88</td>
<td>0.91</td>
</tr>
<tr>
<td>$\psi_y$</td>
<td>U 0.05</td>
<td>9.95</td>
<td>0.60</td>
<td>0.69</td>
</tr>
<tr>
<td>$\phi_y$</td>
<td>U 1</td>
<td>19</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>$\pi_c$</td>
<td>U 1</td>
<td>19</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>$\pi_d$</td>
<td>U 1</td>
<td>19</td>
<td>0.08</td>
<td>0.60</td>
</tr>
<tr>
<td>$\psi_s$</td>
<td>U 0.05</td>
<td>9.95</td>
<td>0.13</td>
<td>0.83</td>
</tr>
<tr>
<td>$\rho_h$</td>
<td>U 0.90</td>
<td>0.99</td>
<td>0.93</td>
<td>0.96</td>
</tr>
<tr>
<td>$\sigma^2_h$</td>
<td>IG 0.06</td>
<td>0.37</td>
<td>0.24</td>
<td>0.40</td>
</tr>
</tbody>
</table>

*Notes: The estimation results are based on quarterly consumption and dividend data from 1952 to 2014. We fix $\mu_c = 0.0049$, $\mu_d = 0.0083$ and $\mu_s = 0.0016$ at their sample averages and we fix $\hat{\sigma} = 0.0096$ and $\phi_x = 4$. U and IG are the Uniform and Inverse-Gamma distributions, respectively.*

**Filtered Mean and Volatility States.** We now provide some further evidence that the economy is driven by two different long-run components, one for the non-durable and one for the durable consumption. Top and middle panel in Figure 2 depicts filtered estimates of the predictable long-run components $x_t$ and $y_t$. Both $x_t$ and $y_t$ tend to fall sharply during the recessions and tend to recover immediately after the recession. The bottom panel of Figure 2 also depicts the implied volatility state $h_t$. The sudden increase in volatility $h_t$ is often associated with a recession and tends to peak with the periods of NBER recessions.
3.2.3 Estimating the Elasticity of Substitution

Equation (2.13) provides a way to estimate the elasticity of substitution $\rho$ directly from the data. Taking logs of equation (2.13) we get

$$\log \left( \frac{\alpha}{1 - \alpha} \right) + \frac{1}{\rho} (c_t - d_t) - p_t = q_t - p_t,$$

where lowercase variables denote the logs of the uppercase variables. Assuming that the user cost and the spot price of durable goods are cointegrated (so that $q_t - p_t$ is stationary) implies that $c_t - d_t$ and $p_t$ are cointegrated with the cointegrating vector equal to $(1, -\rho)$. Hence, we can estimate the elasticity of substitution without observing the user cost of durable goods (see Ogaki and Reinhart, 1998, where $\rho$ is estimated by regressing $c_t - d_t$ on $p_t$). We estimate the elasticity of substitution by a dynamic ordinary least square regression of $c_t - d_t$ on $p_t$ with four leads and lags as proposed.
by Stock and Watson (1993):

\[ c_t - d_t = \text{const} + \rho p_t + \sum_{s=-4}^{4} b_{p,s} \Delta p_{t-s} + \varepsilon_t. \]

For the full sample 1952:I - 2014:IV our estimate of \( \rho = 0.78 \) with standard error of 0.03. We test the null hypothesis of no composition \( H_0 : \rho = 1. \) The t-statistics is \( t = -6.85 \) and thus we reject the hypothesis of no composition on 1% significance level.

3.2.4 Estimating the Linear Model

To accommodate an analytical solution, we utilize a linear approximation to the conditional volatility process (2.9) and express volatility as a process that follows a Gaussian distribution:

\[
\sigma_{t+1}^2 \approx \bar{\sigma}^2 (1 - \rho_h) + \rho_h \sigma_t^2 + 2\bar{\sigma}^2 \sigma_h \sqrt{1 - \rho_h^2} w_{t+1} \\
= \hat{\sigma} + \rho_h \sigma_t^2 + \sigma_w w_{t+1}.
\]

We derive the asset prices using the standard asset pricing condition

\[ \mathbb{E}_t [\exp(m_{t+1} + r_{i,t+1})] = 1 \]

for any asset \( r_{i,t+1} = \log(R_{i,t+1}) \), where the log-pricing kernel of the economy is

\[ m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) \Delta f_{t+1} + (\theta - 1) r_{w,t+1}. \]

\( r_{w,t+1} \) is the log return on the consumption claim, and \( r_{m,t+1} \) is log market return. We use the approximation of Campbell and Shiller (1988) for the returns:

\[ r_{w,t+1} = z_{w,t+1} - \kappa_0 - \kappa_1 z_{w,t} - z_t + \Delta c_{t+1}, \]
\[ r_{m,t+1} = \kappa^m_0 + \kappa^m_1 z_{m,t+1} - z_{m,t} + \Delta s_{t+1}, \]

where \( z_t = \log(D_t/C_t) \), \( z_{w,t} \) is the log-wealth-consumption ratio and \( z_{m,t} \) is the log-price-dividend ratio. The solution to the log-wealth-consumption ratio and to the log-price-dividend ratio are linear in states and are given by

\[ z_{w,t} = A_0 + A_1 x_t + A_2 y_t + A_3 z_t + A_4 \sigma_t^2, \]
\[ z_{m,t} = B_0 + B_1 x_t + B_2 y_t + B_3 z_t + B_4 \sigma_t^2, \]

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with $A$’s and $B$’s derived in Appendix C and are functions of the preference parameters. Given the solution (3.7) we can derive analytical expressions for both the market return and for the risk-free rate. The detailed derivations are in Appendix C.

We use the analytical solution of the linear model estimate the set of preference parameters

$$\Lambda = (\gamma, \psi, \beta, \alpha),$$

where $\gamma$ is the risk aversion coefficient, $\psi$ is the elasticity of the intertemporal substitution, $\beta$ is the subjective discount factor, and $\alpha$ is the share of durable consumption in the intraperiod utility function. We estimate $\Lambda$ by solving a sample minimum distance problem with the identity weighting matrix. We simulate 100,000 samples of length equal to our sample size and use those to calculate the market and the risk-free returns and estimate $\Lambda$ to reflect values that are required to match first two unconditional moments of the market and risk-free returns.

Panel A of Table 2 (“Linear Model”) reports the values of the estimated parameters. The estimate of risk aversion coefficient $\gamma$ is around 3 and the estimate of the elasticity of intertemporal substitution $\psi$ is around 1.3. We also find that the subjective discount factor is estimated at $\beta = 0.9985$ and the share of durable consumption in the intraperiod utility function $\alpha$ is about 30%.

Panel B of Table 2 reports the model implied simulated moments. The simulated mean of the risk-free rate and of the risky return is about 1% and 6%, close to the values observed in the data. The linear model also generates high volatility of risky return (about 20% compared to 19% observed in the data) and high value of price-dividend ratio. The linear model fails short of reproducing the standard deviation of risk-free rate and the standard deviation of price-dividend ratio.

### 3.2.5 Semi-parametric Estimation of the Preference Parameters

Pohl, Schmedders, and Wilms (2016) argue that numerical errors that are introduced in the LRR models using the Cambell-Shiller linearization are economically and statistically significant and could lead to wrong model predictions. We re-estimate the model taking into account all possible nonlinear effects. We employ a semiparametric estimation methodology similar to that of Chen, Favilukis, and Ludvigson (2013) and...
we proceed in two steps. On the first step, for a fixed value of preference parameters, we approximate the unknown wealth-consumption and price-dividend ratios as a series of Chebyshev polynomials and we nonparametrically estimate these functions using the wealth-Euler and the Euler equation. On the second step, given the estimate of these functions, we estimate the preference parameters by a sample minimum distance estimator (analog of GMM). The further details are in Appendices A and B.

Panel A of Table 2 (“Full Model”) reports the values of the estimated parameters for a full model. The estimate of risk aversion coefficient $\gamma$ is 1.86 and the estimate of the elasticity of intertemporal substitution $\psi$ is around 1.1865. We also find that the subjective discount factor is estimated at $\beta = 0.9914$ and the share of durable consumption in the intraperiod utility function $\alpha$ is about 15.5%.

Panel B of Table 2 (“Full Model”) reports the model implied simulated moments. The simulated mean of the risk-free rate and of the risky return is 1.01% and 5.78%, close to the values observed in the data. The model implied volatility of risky return is about 17% (compared to 19% observed in the data) and the volatility of the risk-free rate is about 1.61% (compared to 1.61% observed in the data). The model also matches the mean and the volatility of the price-dividend ratio. All the data moments lie comfortably inside the corresponding model implied 95% confidence intervals.


Table 2: Estimated Preference Parameters and Unconditional Moments of Returns.

<table>
<thead>
<tr>
<th>A. Estimated Preference Parameters</th>
<th>Linear Model</th>
<th>Full Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk aversion</td>
<td>( \gamma = 2.78 )</td>
<td>( \gamma = 1.86 )</td>
</tr>
<tr>
<td>EIS</td>
<td>( \psi = 1.29 )</td>
<td>( \psi = 1.18 )</td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>( \beta = 0.998 )</td>
<td>( \beta = 0.991 )</td>
</tr>
<tr>
<td>Share of durable consumption</td>
<td>( \alpha = 0.30 )</td>
<td>( \alpha = 0.15 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. Unconditional Moments of Returns</th>
<th>Data</th>
<th>Linear Model</th>
<th>Full Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean ((r_f))</td>
<td>0.95</td>
<td>0.69 1.08 1.62</td>
<td>-0.59 1.01 8.16</td>
</tr>
<tr>
<td>StdDev ((r_f))</td>
<td>1.61</td>
<td>0.26 0.36 0.49</td>
<td>1.08 1.61 3.40</td>
</tr>
<tr>
<td>Mean ((r_m))</td>
<td>5.57</td>
<td>2.50 6.21 9.84</td>
<td>1.66 5.78 10.89</td>
</tr>
<tr>
<td>StdDev ((r_m))</td>
<td>18.94</td>
<td>16.16 20.52 25.44</td>
<td>13.03 16.85 20.97</td>
</tr>
<tr>
<td>Mean ((p - d))</td>
<td>4.93</td>
<td>4.20 4.27 4.35</td>
<td>3.83 4.60 5.20</td>
</tr>
<tr>
<td>StdDev ((p - d))</td>
<td>0.38</td>
<td>0.12 0.17 0.23</td>
<td>0.18 0.31 0.58</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C. Timing and Risk Premia</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Timing Premium</td>
<td>( \pi^* = 11% )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk Premium</td>
<td>( \bar{\pi} = 16% )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.2.6 Timing and Risk Premia

In this section we describe the model’s implications for timing and risk premia.

Definitions of the timing and risk premia. Suppose you are facing an endowment process described in (3.1), with \( t = 0, 1, 2, \ldots \), for which consumption and dividends risk is resolved gradually over time (\( C_t, D_t, S_t, x_t \) and \( y_t \) are realized at time \( t \) only). Consider the alternative process in which all the risk is resolved in period 1. You are given an option to choose the alternative endowment process over the original one. The cost for you to choose the alternative process is to give up a fraction \( \pi \) of consumption today and in subsequent periods. We call the maximum value \( \pi^* \) for which you are willing to accept this offer a timing premium. Formally, we define it in a following way. Let \( U_0 \) be the utility with the original endowment process and \( U_0^* \) is the utility of the alternative endowment process in which all risk is resolved at time 1.
Then, $\pi^*$ is then defined as

$$\pi^* = 1 - \frac{U_0}{U^*_0}.$$

Now, consider another alternative endowment process, in which the risk is resolved entirely, and the consumption and dividend processes are deterministic. The maximum fraction of current and future consumption $\bar{\pi}$ which you are willing to give up in favor of this deterministic process is the risk premium and is formally defined as

$$\bar{\pi} = 1 - \frac{U_0}{\bar{U}^*_0},$$

where $\bar{U}_0$ is the utility associated with the deterministic endowment process.

**Calculating the Timing and Risk Premia.** We rely on numerical methods to calculate the value of $U_0$. The value function $U_0(C, D, x, y, \sigma^2)$ is the solution for the recursive functional equation

$$U_t = \left\{ (1 - \beta)u(C_t, D_t)^{\frac{1-\gamma}{\sigma}} + \beta \left( E_t[U_{t+1}]^{1-\gamma}\right) \right\}^{\frac{\gamma}{1-\gamma}}.$$

Noting that value function $U$ can be rewritten as $U(C, D, x, y, \sigma^2) = C \mathcal{H}(z, x, y, \sigma^2)$, where $z = D/C$, the function equation above is

$$\mathcal{H}(z_t, x_t, y_t, \sigma_t^2) =$$

$$\left\{ (1 - \beta)\tilde{u}(z_t)^{\frac{1-\gamma}{\sigma}} + \beta e^{\left(1 - \frac{1}{2}\right)(\mu_c + x_t + \frac{\sigma^2}{2})}\left( E_t[\mathcal{H}_{t+1}^{1-\gamma}(z_{t+1}, x_{t+1}, y_{t+1}, \sigma_{t+1}^2)]^{\frac{\gamma}{1-\gamma}}\right) \right\}^{\frac{\gamma}{1-\gamma}},$$

where $E_t$ is the expectation conditional on state variables $z_t, x_t, y_t$ and $\sigma_t^2$, and $\tilde{u}(z_t) = \tilde{u}(D_t/C_t) = u(C_t, D_t)/C_t$. We approximate $\mathcal{H}$ by Chebyshev polynomials and solve the functional equation using orthogonal collocation method; the expectation is approximated by Gauss-Hermite quadrature. We then run Monte-Carlo simulations with a fixed time horizon $T$ and pass $U_0$ as the continuation value at time $T$ to obtain both $U^*_0$ and $\bar{U}_0$.

The numerical results are reported in Panel C in Table 2. For the fixed time horizon of 1000 years (4000 periods) model generates a timing premium of 11% and a risk premium of 16%. We also simulate the model for different time horizons: starting with $T = 30$ years (corresponding to the duration of US Treasury bonds), $T = 63$ years
Figure 3 displays the simulation results. We see that the timing premium increases from 5% for $T = 30$ to 11% for $T = 300$ and remains stable thereafter. The risk premium increases from 11% for $T = 30$ to 16% for $T = 100$ and remains stable thereafter.

**Figure 3: Timing and Risk Premium.** The figure displays timing (dashed line) and risk (solid line) premium as a function of time horizon.

### 3.2.7 Discussion

To further understand the mechanics of the model, we also analyze several different scenarios. We start by re-estimating both the linear and full models when the composition effect is missing (by setting $\rho = 1$). Table 3 analyzes the results. Comparing with the benchmark case (Table 2) shutting down the composition effect has substantial results on both the values of the estimated parameters as well as on the estimated moments. The values of risk aversion and EIS are much higher in the no-composition scenario. Moreover, the model fails to match some of the asset pricing moments. There are also substantial difference between the linear and full model, with linear model failing to match the mean values of the risk-free and equity returns, among others. While the no-composition model generates a low value of the timing premium, it generates unreasonably high value of the risk premium, about twice as high as in Bansal and Yaron (2004). This suggests that composition effect plays an important role for the
estimated preference parameters and matching the asset markets moments.

### Table 3: No Composition (\(\rho = 1\))

#### A. Preference Parameters

<table>
<thead>
<tr>
<th></th>
<th>Linear Model</th>
<th>Full Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk aversion (\gamma)</td>
<td>15.35</td>
<td>4.83</td>
</tr>
<tr>
<td>EIS (\psi)</td>
<td>1.18</td>
<td>1.79</td>
</tr>
<tr>
<td>Subjective discount factor (\beta)</td>
<td>0.997</td>
<td>0.998</td>
</tr>
<tr>
<td>Share of durable consumption (\alpha)</td>
<td>0.42</td>
<td>0.47</td>
</tr>
</tbody>
</table>

#### B. Unconditional Moments of Returns

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Linear Model</th>
<th>Full Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (r_f)</td>
<td>StdDev (r_f)</td>
<td>Mean (r_m)</td>
</tr>
<tr>
<td>--------------------</td>
<td>------</td>
<td>---------------</td>
<td>------------</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>-0.87 0.34 1.39</td>
<td>-0.99 0.26 1.31</td>
</tr>
<tr>
<td></td>
<td>1.61</td>
<td>0.90 1.27 1.70</td>
<td>0.91 1.27 1.71</td>
</tr>
<tr>
<td></td>
<td>5.57</td>
<td>-3.09 0.72 4.78</td>
<td>2.67 6.20 10.04</td>
</tr>
<tr>
<td></td>
<td>18.94</td>
<td>14.61 18.88 23.41</td>
<td>13.46 17.43 21.61</td>
</tr>
<tr>
<td></td>
<td>4.93</td>
<td>8.43 8.49 8.55</td>
<td>4.22 4.28 4.33</td>
</tr>
<tr>
<td></td>
<td>0.38</td>
<td>0.11 0.15 0.21</td>
<td>0.10 0.14 0.19</td>
</tr>
</tbody>
</table>

#### C. Timing and Risk Premia

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Timing Premium (\pi^*)</td>
<td>4%</td>
</tr>
<tr>
<td>Risk Premium (\bar{\pi})</td>
<td>52%</td>
</tr>
</tbody>
</table>

We also analyze what happens to the asset pricing moments when we use the original preference parameter calibration by Bansal and Yaron (2004) (see table 4). There are several important things to note from the results. Firstly, the model generates very high values of risk-free rate (about 5% per year) and equity return (more than 100% per year). Secondly, the model generates a timing and risk premia of 69% and 80%, respectively. This suggests that the presence of durable consumption indeed dampens the values of the risk aversion and the EIS require to match the key financial data moments as well as to generate reasonable values of timing and risk premia.
Table 4: Bansal & Yaron (2004) Calibration with Durable Goods

A. Preference Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk aversion</td>
<td>$\gamma = 7.5$</td>
</tr>
<tr>
<td>EIS</td>
<td>$\psi = 1.5$</td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>$\beta = 0.998$</td>
</tr>
<tr>
<td>Share of durable consumption</td>
<td>$\alpha = 0.30$</td>
</tr>
</tbody>
</table>

B. Unconditional Moments of Returns

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean($r_f$)</td>
<td>0.95</td>
<td>5.07</td>
</tr>
<tr>
<td>StdDev($r_f$)</td>
<td>1.61</td>
<td>1.66</td>
</tr>
<tr>
<td>Mean($r_m$)</td>
<td>5.57</td>
<td>103.75</td>
</tr>
<tr>
<td>StdDev($r_m$)</td>
<td>18.94</td>
<td>9.43</td>
</tr>
<tr>
<td>Mean($p - d$)</td>
<td>4.93</td>
<td>1.22</td>
</tr>
<tr>
<td>StdDev($p - d$)</td>
<td>0.38</td>
<td>0.07</td>
</tr>
</tbody>
</table>

C. Timing and Risk Premia

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Timing Premium</td>
<td>$\pi^* = 69%$</td>
</tr>
<tr>
<td>Risk Premium</td>
<td>$\pi = 80%$</td>
</tr>
</tbody>
</table>

4 Conclusions

We introduce a LRRM where durable and non-durable consumption goods are non-separable and gross complements, thus generating households’ concern with short and long run composition risk, that is, fluctuations in the relative share of durables in their consumption basket. We show that our model matches the key stylized facts of financial markets and at the same time generates levels of timing and risk premia that are consistent with the conventional macroeconomic wisdom. In our benchmark calibration, our model matches financial data well with a risk aversion of 1.86, an elasticity of intertemporal substitution of 1.18 and an elasticity of substitution between durable and non-durable goods of 0.78. With this parametrization the timing and risk premia are 11 and 16 percent, respectively. Compared to its single consumption good counterpart, our model reduces the timing and risk premia by more than 50 percent.
Appendices

A Pricing Kernel

Define the return on total consumption as

\[ R_{W,t+1} = \frac{\tilde{W}_{t+1}}{\tilde{W}_t - C_t - Q_tD_t} \]

where total consumption \( G_t \) is given by

\[ G_t = C_t + Q_tD_t. \]

Here, \( Q_t \) denotes the user cost of the service flow for the durable good. Following Yogo (2006), it is given as a marginal rate of substitution between non-durable and durable consumption good:

\[ Q_t = \frac{\partial C_t}{\partial D_t}. \]

Given the functional form for \( C_t \), we get

\[ Q_t = \frac{\alpha}{1 - \alpha} \left( \frac{D_t}{C_t} \right)^{-\frac{1}{\rho}}. \]

Let

\[ F_t = \left( 1 - \alpha + \alpha \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\rho}} \right)^{\frac{1}{1 - \rho}}. \]

The IMRS is given by

\[ M_{t+1} = \beta^\theta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\varphi}} \left( \frac{F_{t+1}}{F_t} \right)^{\frac{\theta(1 - \frac{1}{\rho})}{\varphi}} R_{W,t+1}^{\theta - 1}. \]

Furthermore,

\[ R_{W,t+1} = \frac{\tilde{W}_{t+1}}{\tilde{W}_t - G_t} = \frac{\tilde{W}_{t+1}}{\tilde{W}_t - G_t} \frac{G_{t+1}}{G_t}, \]

where we can further rewrite \( G_t \) as

\[ G_t = C_t + Q_tD_t = C_t + \frac{\alpha}{1 - \alpha} \left( \frac{D_t}{C_t} \right)^{\frac{1}{\rho}} D_t = C_t \left( 1 + \frac{\alpha}{1 - \alpha} \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\rho}} \right). \]

Finally, we can also rewrite the expression for \( F_t \) as

\[ F_t = (1 - \alpha)^{\frac{1}{1 - \rho}} \left( 1 + \frac{\alpha}{1 - \alpha} \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\rho}} \right)^{\frac{1}{1 - \rho}}. \]
Substituting all the equations above into the one for \( M_{t+1} \) we get

\[
M_{t+1} = \beta^\theta \left( \frac{C_{t+1}}{C_t} \right)^{\theta \left( 1 - \frac{1}{n} \right) - 1} \left( \frac{A_{t+1}}{A_t} \right)^{\theta \left( 1 - \frac{1}{n} \right) - 1} \left( \frac{\bar{W}_{t+1}}{\bar{W}_t \psi} \right)^{\theta - 1},
\]

where

\[
A_t = 1 + \frac{\alpha}{1 - \alpha} \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\rho}}.
\]

We are also interested in the evolution of \( \frac{D_{t+1}}{C_{t+1}} \) (which enters \( A_{t+1} \)) in terms of other variables. We have

\[
\frac{D_{t+1}}{C_{t+1}} = \frac{D_{t+1}/D_t \cdot D_t}{C_{t+1}/C_t \cdot C_t} = \frac{D_{t+1}}{D_t} \left( \frac{C_{t+1}}{C_t} \right)^{-1} \left( \frac{C_{t+1}}{C_t} \right) = \frac{D_{t+1}}{C_t}.
\]

Let \( z_t = \log \frac{D_t}{C_t} \). Then,

\[
z_{t+1} = \Delta D_{t+1} - \Delta C_{t+1} + z_t = \mu_d + y_t + \sigma_d \epsilon_{t+1}^{d} - \mu_c - x_t - \sigma_x \epsilon_{t+1}^{x} + z_t.
\]
B Application of Projection Method

We apply the projection method twice: first, we use the wealth-Euler equation to solve for return on wealth \( R_{W,t+1} \); second, we use the obtained return on wealth to solve for the return on any asset \( i \) using the Euler equation.

We start with wealth-Euler equation

\[
\mathbb{E}_t [M_{t+1} R_{W,t+1}] = 1,
\]

where

\[
M_{t+1} = \beta^\theta \left( \frac{C_{t+1}}{C_t} \right)^{-\theta/\psi} \left( \frac{v(D_{t+1}/C_{t+1})}{v(D_t/C_t)} \right)^{\theta(1/\rho-1/\psi)} \theta^{\frac{1}{\rho}} R_{W,t+1}^{\theta-1}
\]

and

\[
v \left( \frac{D_t}{C_t} \right) = \left[ 1 - \alpha + \alpha \left( \frac{D_t}{C_t} \right)^{-1/(1-1/\rho)} \right]^{1/(1-1/\rho)}.
\]

For simplicity, let \( F_t = v \left( \frac{D_t}{C_t} \right) \).

In logs, the wealth-Euler equation becomes

\[
\mathbb{E}_t \left[ \exp \left( \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) \Delta f_{t+1} + (\theta - 1) r_{w,t+1} \right) \right] = 1,
\]

where lowercase variables denote the logs of the corresponding uppercase variables, \( \Delta c_{t+1} = c_{t+1} - c_t \) and \( \Delta f_{t+1} = f_{t+1} - f_t \). Log-return on wealth \( r_{w,t+1} \) can be further written as

\[
r_{w,t+1} = \log \left( \frac{\tilde{W}_{t+1}}{\tilde{W}_t - C_t - Q_t D_t} \right) = \log \left( \frac{\tilde{W}_{t+1}}{\tilde{W}_t} \times \frac{C_{t+1}}{C_t} \right)
\]

\[
= wc_{t+1} - \log \left( wc_t - 1 - Q_t D_t \right) + \Delta c_{t+1},
\]

where \( wc = \log \left( \frac{\tilde{W}_t}{C_t} \right) \) is the log-wealth-consumption ratio.
C Solving Linear Model

To accommodate an analytical solution, we utilize a linear approximation to the conditional volatility process (2.9) and express volatility as a process that follows a Gaussian distribution:

\[ \sigma_{t+1}^2 \approx \bar{\sigma}^2 (1 - \rho_h) + \rho_h \sigma_t^2 + 2 \bar{\sigma}^2 \sigma_h \sqrt{1 - \rho_h^2} w_{t+1} \]
\[ = \hat{\sigma} + \rho_h \sigma_t^2 + \sigma_w w_{t+1}. \]

The endowment process for the economy is then given by

\[
\begin{align*}
\Delta C_{t+1} &= \mu_c + x_t + \sigma_t^c e_t^{c,+1}, \\
\Delta D_{t+1} &= \mu_d + y_t + \psi_d \sigma_t^{d} e_t^{d,+1}, \\
\Delta S_{t+1} &= \mu_s + \phi_x x_t + \phi_y y_t + \pi_c \sigma_t e_t^{c,+1} + \pi_d \sigma_t e_t^{d,+1} + \psi_s \sigma_t e_t^{s,+1}, \\
x_{t+1} &= \rho_c x_t + \psi_c \sigma_t e_t^{c,+1}, \\
y_{t+1} &= \rho_y y_t + \psi_y \sigma_t e_t^{y,+1}, \\
\sigma_{t+1}^2 &= \hat{\sigma} + \rho_h \sigma_t^2 + \sigma_w w_{t+1}, \\
e_{t+1}, e^{c,+1}, e^{d,+1}, e^{s,+1}, w_{t+1} &\sim \mathcal{N}(0, 1).
\end{align*}
\]

We derive the asset prices using the standard asset pricing condition

\[ \mathbb{E}_t e^{m_{t+1} + r_{w,t+1}} = 1 \]

for any asset \( r_{i,t+1} = \log \left( R_{i,t+1} \right) \), where the log-pricing kernel of the economy is

\[
m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) \Delta f_{t+1} + \left( \theta - 1 \right) r_{w,t+1}.
\]

\( r_{w,t+1} \) is the log return on the consumption claim, and \( r_{m,t+1} \) is log market return. We use the approximation of Campbell and Shiller (1988) for the returns:

\[
r_{w,t+1} = z_{w,t+1} - \kappa_0 - \kappa_1 z_{w,t} - z_t + \Delta c_{t+1}, \\
r_{m,t+1} = \kappa^m_0 + \kappa^m_1 z_{m,t+1} - z_{m,t} + \Delta s_{t+1},
\]

where \( z_t = \log \left( D_t / C_t \right) \), \( z_{w,t} \) is the log-wealth-consumption ratio and \( z_{m,t} \) is the log-price-dividend ratio. The approximating constants are given by

\[
\kappa_0 = \log \left( e^{z_w} - 1 - q(z) \right) + \frac{1}{e^{z_w} - 1 - q(z)} \left[ -e^{z_w} z_w - \frac{\alpha}{1 - \alpha} \left( 1 - \frac{1}{\rho} \right) e^{(1-\frac{1}{\rho}) z_w} \right],
\]
and

\[ \kappa^m_0 = \log \left( 1 + e^{z_m^e} \right) - \frac{e^{z_m^e} z_m}{1 + e^{z_m^e}}, \quad \kappa^m_1 = \frac{e^{z_m^e}}{1 + e^{z_m^e}}. \]

### C.1 Consumption Claim

We conjecture that the log-wealth-consumption ratio \( z_{w,t} \) is a linear function of state variables

\[ z_{w,t} = A_0 + A_1 x_t + A_2 y_t + A_3 z_t + A_4 \sigma_t^2. \] (C.1)

Then,

\[
\begin{align*}
r_{w,t+1} &= z_{w,t+1} - \kappa_0 - \kappa_1 z_{w,t} - \kappa_2 z_t + \Delta c_{t+1} \\
&= A_0 + A_1 x_{t+1} + A_2 y_{t+1} + A_3 z_{t+1} + A_4 \sigma_{t+1}^2 - \kappa_0 - \kappa_1 A_0 - \kappa_1 A_1 x_t - \kappa_1 A_2 y_t - \kappa_1 A_3 z_t - \kappa_1 A_4 \sigma_t^2 - \kappa_2 z_t + \\
&\quad \{ A_0 (1 - \kappa_1) - \kappa_0 + A_3 (\mu_d - \mu_c) + A_4 \sigma_t^2 \} + \\
&\quad \{ A_1 (1 - \kappa_1) - \kappa_1 A_1 + 1 \} x_t + \\
&\quad \{ A_2 (1 - \kappa_1) - \kappa_1 A_2 \} y_t + \\
&\quad \{ A_3 (1 - \kappa_1) - \kappa_2 \} z_t + \\
&\quad \{ A_4 (1 - \kappa_1) A_4 \} \sigma_t^2 + \\
&= A_1 \psi_x \sigma_t \epsilon_{x,t+1}^c + A_2 \psi_y \sigma_t \epsilon_{y,t+1}^c + (1 - A_3) \sigma_t \epsilon_{x,t+1}^c + A_3 \psi_d \sigma_t \epsilon_{d,t+1}^c + A_4 \sigma_w \sigma_{w,t+1}. \] (C.2)
\]

Using

\[
\Delta f_{t+1} = \frac{\rho}{\rho - 1} \theta \exp \left( \left( 1 - \frac{1}{\rho} \right) \bar{z} \left( 1 - \frac{1}{\rho} \right) (z_{t+1} - z_t) \right) \\
= K \left( \mu_d + y_t + \psi_d \sigma_t \epsilon_{x,t+1}^d - \mu_c - x_t - \sigma_t \epsilon_{x,t+1} \right) \] (C.3)

and

\[
m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) \Delta f_{t+1} + (\theta - 1) r_{w,t+1} \] (C.4)

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we get

\[
mt_{t+1} = \left\{ \begin{array}{l}
\theta \log \beta - \frac{\theta}{\psi} \mu_c + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K (\mu_c - \mu_d) + \\
(\theta - 1)(A_0 (1 - \kappa_1) - \kappa_0 + A_3 (\mu_d - \mu_c) + A_4 \delta + \mu_c) + \\
- \frac{\theta}{\psi} - \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K + (\theta - 1) (A_1 \rho_x - A_3 - \kappa_1 A_1 + 1) \end{array} \right\} x_t + \\
\left\{ \begin{array}{l}
\theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K + (\theta - 1) (A_2 \rho_y + A_3 - \kappa_1 A_2) \end{array} \right\} y_t + \\
\left\{ (\theta - 1) (A_3 - \kappa_1 A_3 - \kappa_2) \right\} z_t + \\
\left\{ (\theta - 1) (A_4 \rho h - \kappa_1 A_4) \right\} \sigma_t^2 + \\
(\theta - 1) A_1 \psi_x \sigma_t \varepsilon_{t+1}^{x} + \\
(\theta - 1) A_2 \psi_y \sigma_t \varepsilon_{t+1}^{y} + \\
\left\{ (\theta - 1) (1 - A_3) - \frac{\theta}{\psi} - \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K \right\} \sigma_t \varepsilon_t^{c} + \\
\left\{ (\theta - 1) A_3 + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K \right\} \psi \sigma_t \varepsilon_t^{d} + \\
(\theta - 1) A_4 \sigma_t \omega w_{t+1}.
\]

(C.5)

Since both \( m_{t+1} \) and \( r_{w,t+1} \) are conditionally normal, the wealth-Euler equation can be written as

\[
\mathbb{E}_t [m_{t+1} + r_{w,t+1}] + \frac{1}{2} \operatorname{Var}_t [m_{t+1} + r_{w,t+1}] \approx 0.
\]

We use this equation to solve for \( A \)’s. They are

\[
A_0 = - \frac{(\theta - 1) (\kappa_0 - \mu_c - A_4 \delta + A_3 (\mu_c - \mu_d)) - \theta \log \beta - \frac{\mu_c^2 (\rho - 1)^2}{\theta} + \psi \rho + K \theta \left( \frac{1}{\psi} - \frac{1}{\rho} \right) (\mu_c - \mu_d)}{(\kappa_1 - 1) (\theta - 1)} ,
\]

\[
A_1 = \frac{(\kappa_2 \tau_1 + 1) (\theta - 1) + K \theta \left( \frac{1}{\psi} - \frac{1}{\rho} \right) - \psi \theta}{(\kappa_1 - \rho_c) (\theta - 1)} ,
\]

\[
A_2 = - \frac{K \theta \left( \frac{1}{\psi} - \frac{1}{\rho} \right) + \kappa_2 (\theta - 1)}{(\kappa_1 - \rho_c) (\theta - 1)} ,
\]

\[
A_3 = - \frac{K \theta \left( \frac{1}{\psi} - \frac{1}{\rho} \right) + \kappa_2 (\theta - 1)}{(\kappa_1 - \rho_c) (\theta - 1)} ,
\]

\[
A_4 = \frac{\mu_c^2 A_3 (\theta - 1) - K \theta \left( \frac{1}{\psi} - \frac{1}{\rho} \right)}{2 (\kappa_1 - \rho_c) (\theta - 1)} + \left( (A_3 - 1) (\theta - 1) - K \theta \left( \frac{1}{\psi} - \frac{1}{\rho} \right) + \psi \theta \right)^2 + A_2^2 \psi \mu_c^2 (\theta - 1)^2 + A_2^2 \psi \mu_c^2 (\theta - 1)^2
\]

(C.7)

The innovation to \( m_{t+1} \) is given by

\[
m_{t+1} - \mathbb{E}_t [m_{t+1}] = \lambda_x \sigma_t \varepsilon_{t+1}^{x} + \lambda_y \sigma_t \varepsilon_{t+1}^{y} + \lambda_c \sigma_t \varepsilon_{t+1}^{c} + \lambda_d \sigma_t \varepsilon_{t+1}^{d} + \lambda_w \sigma_t \omega w_{t+1},
\]

(C.8)
where \( \lambda \)s represent the market price of risk for each source of risk and are given by

\[
\lambda_x = (\theta - 1) A_1 \psi_x, \quad \lambda_y = (\theta - 1) A_2 \psi_y, \quad \lambda_c = (\theta - 1)(1 - A_3) - \frac{\theta}{\psi} - \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K,
\]

\[
\lambda_d = \left( (\theta - 1) A_3 + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K \right) \psi_d, \quad \lambda_w = (\theta - 1) A_4. \quad (C.9)
\]

Similarly, the innovation to \( r_{w,t+1} \) is given by

\[
r_{w,t+1} - \mathbb{E}_t[r_{w,t+1}] = -\beta_x \sigma_t \epsilon_{x,t+1} - \beta_y \sigma_t \epsilon_{y,t+1} - \beta_c \sigma_t \epsilon_{c,t+1} - \beta_d \sigma_t \epsilon_{d,t+1} - \beta_w \sigma_t \epsilon_{w,t+1}, \quad (C.10)
\]

where

\[
\beta_x = -A_1 \psi_x, \quad \beta_y = -A_2 \psi_y, \quad \beta_c = -(1 - A_3), \quad \beta_d = -A_3 \psi_d, \quad \beta_w = -A_4. \quad (C.11)
\]

The risk premium for the consumption claim is

\[
\mathbb{E}_t[r_{w,t+1} - r_f] + \frac{1}{2} \text{Var}_t[r_{w,t+1}] = -\text{Cov}_t[m_{t+1}, r_{w,t+1}]
\]

\[
= (\beta_x \lambda_x + \beta_y \lambda_y + \beta_c \lambda_c + \beta_d \lambda_d) \sigma_t^2 + \beta_w \lambda_w \sigma_w^2.
\]

### C.2 Market Return

Conjecture that the log-price-dividend ratio for the claim on dividends is

\[
z_{m,t} = B_0 + B_1 x_t + B_2 y_t + B_3 z_t + B_4 \nu_t^2.
\]

Then,

\[
r_{m,t+1} = \kappa_0^m + \kappa_1^m z_{m,t+1} - z_{m,t} + \Delta s_{t+1}
\]

\[
= \kappa_0^m + \kappa_1^m \left( B_0 + B_1 x_{t+1} + B_2 y_{t+1} + B_3 z_{t+1} + B_4 \sigma_{t+1}^2 \right)
\]

\[
- B_0 - B_1 x_t - B_2 y_t - B_3 z_t - B_4 \sigma_t^2
\]

\[
+ \mu_s + \phi_s x_t + \phi_y y_t + \pi_c \sigma_t \epsilon_{c,t+1} + \pi_d \sigma_t \epsilon_{d,t+1} + \psi_s \sigma_t \epsilon_{s,t+1}
\]

\[
= \{ \kappa_0^m + B_0(\kappa_1^m - 1) + \kappa_1^m (B_3(\mu_d - \mu_c) + B_4 \sigma) + \mu_s \}
\]

\[
+ \{ \kappa_1^m B_1 \rho x - \kappa_1^m B_3 - B_1 + \phi x \} x_t
\]

\[
+ \{ \kappa_1^m B_2 \rho y + \kappa_1^m B_3 - B_2 + \phi y \} y_t
\]

\[
+ \{ \kappa_1^m B_3 - B_3 \} z_t
\]

\[
+ \{ \kappa_1^m B_4 \rho h - B_4 \} \sigma_t^2
\]

\[
+ \{ \kappa_1^m B_1 \psi_x \} \sigma_t \epsilon_{x,t+1}
\]

\[
+ \{ \kappa_1^m B_2 \psi_y \} \sigma_t \epsilon_{y,t+1}
\]

\[
+ \{ -\kappa_1^m B_3 + \pi_c \} \sigma_t \epsilon_{c,t+1}
\]

\[
+ \{ \kappa_1^m B_3 \psi_d + \pi_d \} \sigma_t \epsilon_{d,t+1}
\]

\[
+ \{ \psi_s \} \sigma_t \epsilon_{s,t+1}
\]

\[
+ \{ \kappa_1^m \} \sigma_{w,t+1} \sigma_{w,t+1} \quad (C.14)
\]

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Again, since both \( m_{t+1} \) and \( r_{m,t+1} \) are conditionally normal, the Euler equation can be written as

\[
E_t [m_{t+1} + r_{m,t+1}] + \frac{1}{2} \text{Var}_t [m_{t+1} + r_{m,t+1}] \approx 0. \tag{C.15}
\]

We use this equation to solve for \( B \)'s. They are

\[
\begin{align*}
\beta_0 &= -\frac{\left( \frac{\kappa_1^m}{2} + \frac{M_0^2}{2} \right) \psi + \kappa_0 + \kappa_1 + \kappa_2^m (B_4 \psi - B_3 (\mu_c - \mu_d))}{\kappa_1^m - 1}, \\
\beta_1 &= -\frac{\psi (\kappa_1^m + 1)}{\kappa_1^m \rho - 1}, \\
\beta_2 &= -\frac{\psi (\kappa_1^m + 1)}{\kappa_1^m \rho - 1}, \\
\beta_3 &= -\frac{\psi (\kappa_1^m + 1)}{\kappa_1^m \rho - 1}, \\
\beta_4 &= -\frac{2M_0 + (\pi_c - B_3 \psi)^2 + (\pi_y - B_3 \psi)^2}{\kappa_1^m \rho - 2}, \tag{C.16}
\end{align*}
\]

where

\[
\begin{align*}
M_0 &= \left\{ \theta \log \beta - \frac{\theta}{\psi} \mu_c + \theta \left( \frac{1}{\rho - 1} \psi \right) K (\mu_c - \mu_d) + \right. \\
&\left. (\theta - 1) (A_0 (1 - \kappa_1) - \kappa_0 + A_3 (\mu_d - \mu_c) + A_4 \psi + \mu_c) \right\}, \\
M_x &= -\frac{\theta}{\psi} - \theta \left( \frac{1}{\rho - 1} \psi \right) K + (\theta - 1) (A_2 \rho_y + A_3 - \kappa_1 A_2), \\
M_y &= \theta \left( \frac{1}{\rho - 1} \psi \right) K + (\theta - 1) (A_2 \rho_y + A_3 - \kappa_1 A_2), \\
M_z &= (\theta - 1) (A_3 - \kappa_1 A_3 - \kappa_2), \\
M_{cx} &= (\theta - 1) A_1 \psi_x, \\
M_{cy} &= (\theta - 1) A_2 \psi_y, \\
M_{ce} &= (\theta - 1) (1 - A_3) - \theta \psi - \theta \left( \frac{1}{\rho - 1} \psi \right) K, \\
M_{cd} &= \left\{ (\theta - 1) A_3 + \theta \left( \frac{1}{\rho - 1} \psi \right) K \right\} \psi_d, \\
M_{cw} &= (\theta - 1) A_4. \tag{C.17}
\end{align*}
\]

The innovation to \( r_{m,t+1} \) is given by

\[
\begin{align*}
r_{m,t+1} - E_t [r_{m,t+1}] &= \beta_{m,x} \sigma_t \varepsilon_{t+1}^x + \beta_{m,y} \sigma_t \varepsilon_{t+1}^y - \beta_{m,c} \sigma_t \varepsilon_{t+1}^c - \beta_{m,d} \sigma_t \varepsilon_{t+1}^d - \beta_{m,s} \sigma_t \varepsilon_{t+1}^s - \beta_{m,u} \sigma_t \varepsilon_{t+1}^u, \tag{C.18}
\end{align*}
\]

where

\[
\begin{align*}
\beta_{m,x} &= -\kappa_1^m B_1 \psi_x, \beta_{m,y} = -\kappa_1^m B_2 \psi_y, \beta_{m,c} = \kappa_1^m B_3 - \pi_c, \\
\beta_{m,d} &= -\kappa_1^m B_3 \psi_d - \pi_d, \beta_{m,s} = -\psi_s, \beta_{m,u} = -\kappa_1^m B_4. \tag{C.19}
\end{align*}
\]
The risk premium for the dividend claim is
\[
\mathbb{E}_t[r_{m,t+1} - r_{f,t}] + \frac{1}{2} \text{Var}_t[r_{m,t+1}] = -\text{Cov}_t[m_{t+1}, r_{m,t+1}]
\]
\[
= (\beta_{m,x}\lambda_x + \beta_{m,y}\lambda_y + \beta_{m,c}\lambda_c + \beta_{m,d}\lambda_d) \sigma_t^2 + \beta_{m,w}\lambda_w\sigma_w^2.
\] (C.20)

C.3 Risk-Free Rate

Using the Euler equation the model-implied risk-free rate is given by
\[
r_{f,t} = -\mathbb{E}_t[m_{t+1}] - \frac{1}{2} \text{Var}_t[m_{t+1}].
\] (C.21)

Using the derived expression for \(m_{t+1}\), the risk-free rate will be given by
\[
r_{f,t} = C_0 + C_1 x_t + C_2 y_t + C_3 z_t + C_4 \sigma_t^2,
\] (C.22)

where
\[
C_0 = -\left\{ \theta \log \beta - \frac{\theta}{\psi} \mu_c + \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K(\mu_c - \mu_d)
\]
\[
+ (\theta - 1)(A_0(1 - \kappa_1) - \kappa_0 + A_3(\mu_d - \mu_c) + A_4\tilde{\sigma} + \mu_c) + \frac{\lambda_w^2\sigma_w^2}{2} \right\},
\]
\[
C_1 = -\left\{ \frac{\theta}{\psi} - \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K + (\theta - 1)(A_1\rho_x - A_3 - \kappa_1 A_1 + 1) \right\},
\]
\[
C_2 = -\left\{ \theta \left( \frac{1}{\rho} - \frac{1}{\psi} \right) K + (\theta - 1)(A_2\rho_y + A_3 - \kappa_1 A_2) \right\},
\]
\[
C_3 = -\left\{ (\theta - 1)(A_3 - \kappa_1 A_3 - \kappa_2) \right\},
\]
\[
C_4 = -\left\{ (\theta - 1)(A_4\rho_h - \kappa_1 A_4) + \frac{\lambda_x^2 + \lambda_y^2 + \lambda_c^2 + \lambda_d^2}{2} \right\}. \tag{C.23}
\]

C.4 Unconditional Moments

\[
\mathbb{E}(\sigma_t^2) = \frac{\sigma}{1 - \rho_h^2}, \quad \text{Var}(\sigma_t^2) = \frac{\sigma_w^2}{1 - \rho_h^2},
\]
\[
\mathbb{E}(x_t) = 0, \quad \text{Var}(x_t) = \frac{\psi_x^2 \left( \text{Var}(\sigma_t^2) + (\mathbb{E}(\sigma_t^2))^2 \right)}{1 - \rho_x^2},
\]
\[
\mathbb{E}(y_t) = 0, \quad \text{Var}(y_t) = \frac{\psi_y^2 \left( \text{Var}(\sigma_t^2) + (\mathbb{E}(\sigma_t^2))^2 \right)}{1 - \rho_y^2},
\]
\[
\mathbb{E}(z_t) = -0.5391, \quad \text{Var}(z_t) = 0.1877. \tag{C.24}
\]
References


