Adverse Selection and Financial Crises

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Abstract

We propose a macroeconomic model in which adverse selection in investment drives the amplification of macroeconomic fluctuations, in line with the prominent roles played by the credit crunch and collapse of the asset backed security market in the financial crisis. This can explain the deep fall in economic activity during the Great Recession without resorting to implausibly large productivity shocks. Whereas existing macroeconomic models of financial frictions are incapable of creating any endogenous movements in total factor productivity (TFP), by generating capital misallocation, our friction generates large endogenous movements in TFP following financial disturbances, explaining the much documented productivity puzzle. These results are supported by the Business Cycle Accounting approach of Chari, Kehoe & McGrattan (2007a), who find that economic fluctuations are driven by movements in both TFP and the marginal efficiency of investment, both of which our model generates endogenously.

JEL: E22, E32, E44, G01

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1 Introduction

This paper proposes a model of financial frictions that can map to both movements in total factor productivity (TFP) caused by the inefficient allocation of productive resources, and movements in the marginal efficiency of investment. Mapping to the former is required to explain the deep decline in economic activity during the Great Recession. Recent research has largely focused on financial frictions that cause fluctuations in the marginal efficiency of investment, and whilst able to explain business cycle amplifications, are unable to explain the large fall in output without resorting to implausibly large productivity or capital quality shocks. In the proposed model, adverse selection in investment drives endogenous movements in TFP by generating misallocation of capital on both the intensive margin when there is heterogeneity in productivity, and the extensive margin due to the restriction of capital to productive projects. This is in line with the prominent role played by the collapse of the asset backed security market in the financial crisis, and the ‘credit crunch’ proceeding the Great Recession.

The empirical evidence offers support for financial frictions that generate large declines in TFP; Chari, Kehoe & McGrattan (2007a) find that the movements in TFP, which they describe as the efficiency wedge, have historically played a more important role in driving economic fluctuations than the marginal efficiency of investment, or the so-called investment wedge. The authors show that many macroeconomic models can be generalised to a prototype RBC model with ‘wedges’ replacing exogenous shocks and frictions in the model. For example, the external finance premium in the Bernanke, Gertler & Gilchrist (1999) financial accelerator model will appear as a time-varying tax on investment. The authors estimate four wedges including government consumption and a labour tax in addition to the efficiency and investment wedges, and propose an accountancy procedure to assess each of the shocks relative importance. Following this approach, we estimate the history of these wedges on U.S. data 1980Q1-2014Q1 using maximum likelihood, and simulate as if the efficiency wedge were constant. This allows us to quantify the role of the efficiency wedge during the Great Recession. Figure 1 plots the empirical time-series of output and investment over the period including the Great Recession.

\footnote{A ‘wedge’ between the intertemporal marginal rate of substitution in consumption and the return to capital.} \footnote{This approach is discussed further in Brinca, Chari, Kehoe & McGrattan (2016) and extended to cover the Great Recession.}
Recession against the simulated time-series with the efficiency wedge held constant. This simulates the model over the same period using the investment, labour and government consumption wedges with the same probability distributions as those estimated with all wedges in order to formulate expectations, and highlights the crucial role played by the efficiency wedge during this episode.

We find that a model of adverse selection in investment generates time-varying investment and efficiency wedges that can cause significant asymmetries in the response to exogenous shocks and lead to endogenous financial crises. This introduces a negative skewness to the simulated time-series of investment observed in the data that standard models of financial frictions are unable to match. Consider one such model, the Gertler & Kiyotaki (2010) banking model, in which an agency problem results in banks facing borrowing constraints that tighten in their leverage ratio. Shocks that reduce the value of bank assets can generate an increase in the investment wedge due to higher borrowing costs; this causes deeper falls in investment relative to the efficient financial intermediation case. As a positive shock would have the reverse effect, the asymm-
try observed in the data is not an inherent feature of the model.\textsuperscript{3} Secondly, whilst it is possible to introduce shocks that generate large declines in output and investment in the Gertler & Kiyotaki (2010) model, across many advanced economies, the fall in investment was insufficient to produce the observed decline in output unless there was also a fall in TFP, or a large-scale destruction of physical capital (see Brinca, Chari, Kehoe & McGrattan 2016). An estimated time-series for TFP using the Chari, Kehoe & McGrattan (2007\textsuperscript{a}) procedure with U.S. data between 1980 and 2014 is shown in figure 2.

![Figure 2: HP filtered estimated efficiency wedge in the U.S. using methodology proposed in Chari et al. (2007\textsuperscript{a}). Grey bars correspond to NBER recessions.](image)

The financial friction in the proposed model is caused by an information problem; borrowers seeking funds to finance a productive project observe private information about the project risk. We show how this feature can lead to an adverse selection problem that can introduce both an investment wedge due to information rents, and an efficiency wedge caused by the misallocation of capital. The efficiency wedge drives episodes where falls in investments are significantly deepened, which we describe as financial

\textsuperscript{3}In Swarbrick, Holden & Levine (2016), myself and co-authors analyse a model in which the borrowing constraint is only occasionally binding. This does introduce asymmetries, but is shown to significantly reduce the impact of the accelerator mechanism.
crises. These introduce negative skewness in simulated time series that match observed macroeconomic data.

1.1 Related Literature

We build on a large literature examining the role of credit frictions in generating and propagating business cycle fluctuation, and whilst many models of financial frictions in this literature can be characterised by a time-varying investment wedge, there is a growing literature drawing focus on the efficiency wedge. The context of this literature is usually development economics, attempting to explain the differences in TFP growth rates across emerging economies, and although much of the research analyses differences in steady-state TFP growth rates (see e.g. Hsieh & Klenow 2009, Midrigan & Xu 2014), there have been several papers looking at transition dynamics, including Jeong & Townsend (2007), Banerjee & Moll (2010), Buera & Shin (2013), Moll (2014). These papers concentrate on the interaction between collateral constraints credit frictions and some form of heterogeneity in productivity. For example, as in this paper, Banerjee & Moll (2010) look at both the intensive and extensive margins of capital misallocation where the collateral constraints prevent efficient allocation; there is misallocation on the intensive margin when the marginal product of capital is unequal across entrepreneurs, and on the extensive margin when there are entrepreneurs with no capital at all. The latter might occur due to entry costs for example, and is likely to lead to much greater persistence in the efficiency wedge than misallocation on the intensive margin. Other related work includes Caggese & Cuñat (2013) which studies the interaction between financial constraints and trade with firm heterogeneity in productivity; Pratap & Urrutia (2012) and Oberfield (2013) who both analyse endogenous TFP during crisis episodes with financial constraints and productivity heterogeneity across sectors, but applied to Mexico and Chile respectively; and Gilchrist, Sim & Zakrajšek (2013) in which firms face heterogeneous borrowing costs and productivity, finding that the observed distribution in borrowing costs is an order of magnitude too small to explain observed changes in TFP. More generally, Buera & Moll (2015) examines the importance of heterogeneity, finding that when there is heterogeneity in productivity, a credit friction will emerge as an efficiency wedge, whereas it will emerge as an investment or labour wedge when the heterogeneity is in the cost of investment or cost of recruitment respectively. The key
novelties of the model proposed in this paper include the nature of the credit friction which abstracts from the more usual balance sheet channel, and that changes in risk can generate endogenous movements in TFP without requiring heterogeneity of movements in the productivity of projects.

As well as the macroeconomic literature, the model proposed in this paper builds on a literature examining optimal financial contracts with hidden information. The structure of the hidden information problem closely follows Stiglitz & Weiss’ (1981, 1992) model of credit rationing. This was extended in Bester (1985) who allows for collateral constraints to be used as a signalling device, finding that credit rationing need not occur (see also Besanko & Thakor 1987). More recently Martin (2009) analyses the relationship between entrepreneur wealth and investment under adverse selection. Martin finds that a pooling (separating) equilibrium will occur when net wealth is low (high) and consequently, an increase in net wealth can generate a drop in investment. Other recent works include Guerrieri, Shimer & Wright (2010) who examine search equilibria with adverse selection; Scheuer (2013) who analyses business tax policy with adverse selection in credit markets and occupational choice, finding that a less progressive tax regime on profits can be justified as a corrective measure mitigating occupational misallocation; and Clementi & Hopenhayn (2006) who study the impact on firm behaviour of borrowing constraints that emerge from an asymmetric information problem. Recent research in this area has largely focused on the optimal financial contract in a partial equilibrium context, whereas this paper attempts to bridge this to a macroeconomic literature that has typically focused more on the costly-state verification model rather than hidden information (cf. Bernanke et al. 1999, Christiano, Motto & Rostagno 2010).

The empirical literature assessing the importance of adverse selection in credit markets is relatively limited. An econometric test to evaluate for the presence of asymmetric information formulated in Chiappori & Salanié (2000) is developed in Crawford, Pavanini & Schivardi (2015) to test for adverse selection in loan markets. It is proposed that the correlation between the probability of taking a loan and probability of default is computed; a statistically significant correlation would indicate the presence of adverse selection. Using this test, the authors find evidence for the presence of adverse selection in Italian lending markets. They then use a structural model of the lending market to analyse the interaction between adverse selection and market imperfectness, finding the impact of adverse selection on outcomes varies depending on the market structure.
Cressy & Toivanen (2001) define a structural model with symmetric and asymmetric information, using the results to state propositions about borrower behaviour, similar to those of Chiappori & Salanié (2000), which are used to test for adverse selection using 1987-1990 U.K. bank lending data. The authors conclude that information is symmetric. Tang (2009) provides evidence of asymmetric information in U.S. credit markets using a Moody’s credit rating refinement in 1982, and finds that it has significant impact on economic outcomes. Away from firm lending markets, Ausubel (1999) and Dobbie & Skiba (2013) find evidence of adverse selection in credit card markets and the market for pay-day loans respectively. The former using a randomised field experiment, and the latter using discontinuities in the relationship between borrower pay and loan eligibility to estimate a regression discontinuity design.

1.2 Critiques of the Business Cycle Accounting Approach

The empirical evidence supporting the proposition that the efficiency wedge has greater importance in economic fluctuations than the investment wedge is not without controversy. The business cycle accounting approach proposed in Chari, Kehoe & McGrattan (2007a) was initially challenged in Christiano & Davis (2006) on two fronts; firstly, that the conclusions of the accounting procedure are sensitive to model assumptions and, secondly, there are likely to be important spillover effects between wedges that are ignored. Christiano & Davis (2006) propose an alternative VAR-based decomposition. Chari, Kehoe & McGrattan (2007b) argue that the different methods of decomposing the business cycle attempt to answer different questions; namely, the business cycle accounting approach estimates the effect on aggregate outcomes of a specific wedge that summarizes a number of underlying fundamental shocks, whereas Christiano & Davis (2006) examine the aggregate impact of specific fundamental shocks.

More generally, other research (such as Justiniano, Primiceri & Tambalotti, 2010, 2011) find greater support for changes in the marginal efficiency of investment in generating aggregate economic fluctuations. The ultimate drivers of business cycle fluctuations remains an open question, but we would argue that this offers support for papers such as this. We do not make a strong stand on the methodological approaches used to decompose business cycle fluctuations to a set of wedges in a general sense, except to acknowledge different approaches seek answers to different questions, and that each
approach has their relative merit. Rather, we examine how one credit friction might endogenously emerge as both efficiency and investment wedges.

1.3 Summary of Model and Results

To analyse the macroeconomic effects of the credit friction, we propose a model of occupational allocation in which entrepreneurs seek external debt finance to fund a project. Every period, a household head allocates members to act as entrepreneurs, workers or unemployed; and allocates capital to either debt or equity finance where, by assumption, every project requires a fixed amount of internal and external finance. When allocated, entrepreneurs privately draw either a risky or safe project, where the project type is independent and identically distributed across time and entrepreneurs. The lenders observe the aggregate state of the economy, including the proportion of risky and safe entrepreneurs, and propose incentive compatible one period contracts. In equilibrium, it is possible for the number of loans to differ from the number of borrowers when the value of debt and equity diverge. For instance, if the relative value of equity increases, there will a higher number of household members allocated to entrepreneurship seeking outside finance, but the relative availability of loans will fall as households allocate a smaller proportion of capital to debt finance.

In section 2, we discuss the contract in a partial equilibrium economy to gain insight into the contract structure and possible outcomes. We highlight firstly that the contract optimal to the lender can be modelled as a static one-period contract. Both separating and pooling equilibriums are available depending on the distribution of risk and expected values of safe and risky projects. Within a large range of parameter values, the optimal contract implies a separating equilibrium with full funding for risky borrowers, except when funding is limited and either the value of risky projects is sufficiently less than the value of safe projects, or risky projects are particularly risky. With either of these characteristics, the lender will choose a pooling equilibrium to reduce the information rents paid to risky borrowers. Under a separating equilibrium, the optimal contract is first-best for a range of parameters if the expected value of risky projects is weakly greater than safe projects unless: the expected return to risky projects is very high; risky projects are particularly risky; or risky projects particularly plentiful. Under these conditions, the information rents increase to the point where the lender restricts credit to
safe borrowers so to reduce the information rents received by risky borrowers. If funding is limited due to a feasibility constraint that implies some borrowers seeking funds will be unable to secure loans, the safe borrowers will never be fully funded.

In the general equilibrium model, we focus on shocks to aggregate total factor productivity, the productivity of risky projects which allow differing values of safe and risky projects, and shocks to the risk of risky projects. For the last, to isolate the role of risk, the productivity is also adjusted to keep the project values equal. As a point of comparison, we solve the first-best economy; this is analogous to a standard real business cycle model. The adverse selection problem introduces significant non-linearities. The specific impact of the non-linearities differ for each type of shock but the general effects can be discussed in the context of the outcomes implied by the contract. Shocks that increase the information surplus received by risky borrowers will cause the marginal value of safe loans to fall; if this reaches zero, the lender will begin to restrict credit to safe projects. As the lender leaves capital unallocated despite profitable opportunities, an efficiency wedge is introduced. This generates sharp declines in output, investment and employment that could well be described as a financial crisis.

Full analysis is given in section 5 although we highlight several key results here. Shocks to total factor productivity within a typical range imply higher volatility in investment relative to the friction-less real business cycle model. This ‘financial accelerator’ occurs because of fluctuations in information rents that produce movements in the investment wedge. Positive shocks to the risk of risky projects manifest as both investment and efficiency wedges, the former again coming from increased information rents. If the shock is sufficiently large, a 2 percentage point fall in the probability of project success under the model calibrations, then the lender begins to ration credit to safe borrowers. As this implies misallocation of capital on the extensive margin, an efficiency wedge emerges that produces much larger drops in output, investment and employment, and leads to increased negative skewness in the simulated time series of these variables. The third shock is to the productive of risky projects; specifically. For large enough positive shocks to the value of risky projects, a 2% increase from the ergodic mean under model calibrations, there are sharp adverse effects. Whilst the first-best economy exhibits the expected positive effects from higher productivity, with adverse selection, the lender begins to restrict lending to safe projects in order to receive additional surplus from risky projects. This, again, causes the efficiency wedge to emerge.
We next turn to the derivation and discussion of the optimal contract in a partial equilibrium setting in section 2. In section 3, we propose a stylized general equilibrium model with adverse selection. This is followed with a discussion of the numerical strategy employed in section 4 before analysing the results in section 5. We offer some concluding remarks to end in section 6.

2 Adverse Selection in a Market for Loans

We begin by characterising the adverse selection problem in partial equilibrium. There are a large number of borrowers and a single lender, each borrower has investment projects that arrive exogenously each period with a type $\theta \in \{s, r\}$ representing safe and risky respectively. $\lambda$ is the probability of receiving a safe project, and $1 - \lambda$ that of receiving a risky project. The projects yield return $X(\theta) \in \{R(\theta), 0\}$ where $p(\theta) \in (0, 1]$ is the probability of return $R(\theta)$. Each project requires a single unit of investment. The lender only has funds for a proportion $\gamma$ of borrowers, where we assume $\max\{\lambda, 1 - \lambda\} < \gamma \leq 1$. Denoting variables of type $\theta$ with superscripts, we make the following assumptions:

**Assumption 1.** $p^s > p^r$ and $R^s < R^r$

This defines the concept of risky and safe; the safe project has a higher probability of success but yields lower returns when it is successful.

**Assumption 2.** The bank cannot observe the risk-type of the project but whether the project is successful is public knowledge.

The first point, following Stiglitz & Weiss (1981, 1992), is the borrower heterogeneity at the heart of the credit friction and we make the second to simplify the contract. A number papers look the optimal long-term contract when the success of the project is private information (see e.g. Clementi & Hopenhayn 2006); making the assumption that the information is public allows us to characterise the optimal contract as a sequence of static contracts and to ensure tractability.\(^4\)

Borrowers must source external debt finance to fund the project and we allow lender and borrower to agree a contract over the provision of finance, where the optimal contract $C$  

\(^4\)If the lender cannot observe the success of the project then there cannot be short-term contracts, as the borrower will always declare a bad state.
will specify financing probabilities $x$ and transfers to the lender $\tau$. Following Prescott & Townsend (1984), the lottery $x$ is used to allow the lender to offer incentive compatible contracts to all borrowers. Consider a single period contract $C^i = \{x^s, x^r, \tau^s, \tau^r\}$; the ex ante value of the contract to the borrower is given by

$$V = \lambda x^s p^s (R^s - \tau^s) + (1 - \lambda) x^r p^r (R^r - \tau^r)$$  \hspace{1cm} (2.1)$$

and the value to the lender by

$$U = \lambda x^s (p^s \tau^s - 1) + (1 - \lambda) x^r (p^r \tau^r - 1)$$  \hspace{1cm} (2.2)$$

The objective of the lender is to propose a contract $C(x, \tau)$ to maximise $U$. We denote the total expected value of a project $W = V + U$.

2.1 First-Best

As a point of comparison to quantify the inefficiency introduced by asymmetric information, we solve the optimal contract under full information. The contract is solved subject to individual rationality (IR) constraints

$$R(\theta) - \tau(\theta) \geq 0$$  \hspace{1cm} (2.3)$$

which means there must be a weakly positive surplus to the borrower. We can verify that the IR constraint will bind, and the resulting solution implies that any projects with positive net present value will receive funding whilst those with negative value will not. Whichever project has the highest value will receive funding with probability one and the other type will receive any remaining funds. As all borrowers will earn their reservation value and so are indifferent about receiving funding, there is no restriction of credit even with $\gamma < 1$. Total social value $W^*$ will be maximised as all resources are utilised in the most productive projects.

2.2 Optimal Contract with Asymmetric Information

We assume that all projects require equal funding, and that matching occurs in an anonymous market which implies that the lender cannot observe the history of the bor-
rowers. It follows that the optimal contract must be a single period, static contract. The lender would like the risky borrower to pay a higher rate but of course the asymmetric information implies the risky borrower could just pretend to be safe. Introducing the lottery highlighted in Bolton & Dewatripont (2005, pp.59-60), the lender can design incentive compatible contracts that ensure the risky borrower does pay a higher rate by offering a higher probability of receiving finance. The problem of the lender is to post contract offers that maximise their value subject to the individual rationality constraints in equation (2.3) and subject to two incentive compatibility (IC) constraints given by

\[ x^i p^i (R^i - \tau^i) \geq x^i p^i (R^i - \tau^i), \quad i = r, s \]  

(2.4)

That is, the value to each borrower of declaring their type truthfully must be weakly greater than lying. As is standard in these mechanism design problems, the problem can be simplified by dropping two constraints; the IR constraint of the risky type and the IC constraint of the safe type. This is straightforward to show: the IC constraint of the risky type is given as

\[ x^r p^r (R^r - \tau^r) \geq x^s p^r (R^r - \tau^s) > x^s p^r (R^s - \tau^s) \geq 0 \]  

(2.5)

where the relationship between the second and third argument follows from \( R^r > R^s \), and the last because of the safe IR constraint. This implies that \( R^r > \tau^r \) and so the safe IR constraint must be the relevant one, indeed, we find this constraint will be always binding as the objective function is increasing in the repayment rate. It follows from this that the risky IC constraint is the relevant one, and again is found to be binding. Using these binding constraints, the repayment terms can be written

\[ \tau^s = R^s \]  

(2.6)

\[ \tau^r = R^r - \frac{x^s}{x^r} (R^r - R^s) \]  

(2.7)

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6 As the borrower type is i.i.d. across time and space, the contract optimal to the lender will revert to an optimal static contract even without the anonymity assumption. In online appendix B we look at a long-term case, and whilst there is a socially optimal long-term contract that produces a pooling equilibrium, it is not optimal to the lender who will only offer one-period terms for funding.

7 We assume a two-stage game following, for example, Rothschild & Stiglitz (1976) and Wilson (1977), whereby lenders post contract offers that borrowers can choose to accept. There are some consequences of the choice of assumptions; choosing a three-stage game, for instance, could lead to pooling or separating equilibria depending on the starting agent. A discussion of these issues is given in Hellwig (1987).
The safe IC constraint which, using \( R^s = \tau^s \), becomes \( 0 \geq x^r p^s (R^s - \tau^r) \), implies \( \tau^r \geq R^s \) which from the binding risky IC constraint, implies \( x^r \geq x^s \). Using this restriction and substituting in the expressions for \( \tau^s \) and \( \tau^r \), the lender’s problem is written
\[
\max_{x^s, x^r} \{ \lambda x^s (p^s R^s - 1) + (1 - \lambda) x^r (p^r R^r - 1) - (1 - \lambda) p^r x^s (R^r - R^s) \}
\]
\[
\text{s.t. } 0 \leq x_s \leq x_r \leq 1
\]
\[
\lambda x^s + (1 - \lambda) x^r \leq \gamma
\]
leading to
\[
(\lambda p^s + (1 - \lambda) p^r) R^s - 1 = \varrho + \varphi^r - \varphi^s
\]
\[
p^r R^r = 1 = \varrho + \frac{1}{1 - \lambda} \varphi^r - \frac{1}{1 - \lambda} \psi
\]
where \( \varrho \) is the Lagrange multiplier on the overall finance limit, \( \varphi^s \) and \( \varphi^r \) are lower bound on \( x^s \) and the upper bound on \( x^r \) respectively, and \( \psi \) is the Lagrange multiplier on \( (x^r - x^s) \). These first order conditions are also subject to Kuhn-Tucker conditions
\[
\varphi^s \geq 0
\]
\[
\varphi^r \geq 0
\]
\[
\varrho \geq 0
\]
\[
\psi \geq 0
\]
\[
\varphi^s x^s = 0
\]
\[
\varphi^r (1 - x^r) = 0
\]
\[
\psi (x^r - x^s) = 0
\]
\[
\varrho (\gamma - \lambda x^s - (1 - \lambda) x^r) = 0.
\]
Using these it is possible to identify four clear outcomes from the two first order conditions. These are (i) for the lender to lend to all risky borrowers, using the remaining funds to finance safe borrowers, offering a lottery for funding; (ii) the lender to lend to all risky borrowers but no safe borrowers; (iii) the lender to offer a single contract with a lottery for funding so that an equal proportion of risky and safe borrowers receive finance; and (iv) to extend no credit at all. The last only occurs if both
\[
1 > (\lambda p^s + (1 - \lambda) p^r) R^s
\]
\[
1 > p^r R^r.
\]
The first condition is that the net present value of lending without differentiating between risky and safe projects is negative, and the second that the net present value of risky projects is negative. The implication from the first condition, given that \( p^r < p^s \) by definition, is that it is possible that positive net present value safe projects will not receive funding at any price. Whilst it is possible for all positive value safe projects to go unfunded, and certainly no negative value safe project will ever be funded, it turns out that negative net present value risky projects can get financing provided that the surplus from safe projects is sufficiently high. This occurs when

\[
(\lambda p^s + (1 - \lambda) p^r) R^s > 1
\]  
(2.14)

in which case, the lender finances each project type with equal probability \( \gamma \), and the surplus from the safe projects subsides for the risky projects. The equal financing will occur providing (2.14) holds and if \( p^r R^r < (\lambda p^s + (1 - \lambda) p^r) R^s \). Once \( p^r R^r > (\lambda p^s + (1 - \lambda) p^r) R^s \), the lender will finance risky projects with probability 1, with any remaining funds going to safe projects which are financed with probability \( (\gamma - 1 + \lambda) \frac{1}{\lambda} \) unless the expected return to risky projects is particularly high. In this case, it is in the interest of the lender to cease funding risky projects; by excluding safe projects, the lender can charge risky borrowers a higher price for funds, this allows the lender to receive all surplus from the risky projects and occurs when

\[
(\lambda p^s + (1 - \lambda) p^r) R^s < \lambda + (1 - \lambda) p^r R^r.
\]  
(2.15)

It is clear that the higher (lower) the proportion of risky projects, the more (less) likely the safe projects will be excluded from the credit market.  

\(^8\)Further to the four outcomes given, if any of the conditions hold with equality there opens additional possible outcomes. If \( (\lambda p^s + (1 - \lambda) p^r) R^s = p^r R^r \) then the only binding constraint is the feasibility constraint; all funds will be lent. The lender and borrowers are indifferent as the inverse relationship between \( \tau^r \) and \( x^r \) implies the expected distribution of surplus is unchanged. When \( (\lambda p^s + (1 - \lambda) p^r) R^s = \lambda + (1 - \lambda) p^r R^r \), the lender is indifferent about whether to finance safe projects or not; as the safe borrowers receive no surplus anyway, they are already indifferent. In partial equilibrium, the conditions are insufficient to determine the outcomes; this will not usually be the case in general equilibrium.
2.2.1 Misallocation of Funds

Let the safe project be perfectly safe, so $p^s = 1$, and let the risk level of risky projects and expected return of safe projects be fixed at $p^r < 1$ and $R^s$ respectively. If we evaluate the contract for different values of $R^r$ we can assess the inefficiency of the misallocation of funds introduced by the information problem. Figure 3 shows the value of the project

![Diagram](image)

**Figure 3:** Value of project with asymmetric information $W$ and the first-best $W^*$ for a range of $R^r$.

for a range of values of $R^r$ in the presence of asymmetric information with the green line, compared to the first-best with the blue line. The key thresholds highlighted represent those discussed above, and are $R^r = A = \left(1 - \lambda \left(1 - \frac{1}{p^r}\right)\right) R^s$, $B = R^s + (R^s - 1) \frac{\lambda}{p^r \frac{1}{1-\lambda}}$, and $C = R^s \frac{1}{p^r}$.

**Proposition 1.** If $\max\{\lambda, 1 - \lambda\} < \gamma < 1$, the incentive compatible contract is equivalent to first-best if and only if $R^s \leq p^r R^r \leq p^r R^s + (R^s - 1) \frac{\lambda}{1-\lambda}$.

**Proof.** Suppose $R^s = p^r R^r + \epsilon$, and recall that $x^r \geq x^s$ from which we state $\gamma < 1 \Rightarrow x^s < 1$. Using the optimality conditions derived above, the first-best contract would set
\[ x^{**} = 1 \] and \[ x^{r*} = \frac{\gamma - \lambda}{1 - \lambda} \], and the incentive compatible contract would set \( x^r = 1 \) and \( x^s = \frac{\gamma - 1 + \lambda}{\lambda} \); using the definition of \( W \equiv U + V \), this leads to \( W = W^* - (1 - \lambda) \epsilon \).

Now suppose \( p^r R^r = p^r R^s + (R^s - 1) \frac{\lambda}{1 - \lambda} + \epsilon \). The optimality conditions indicate \( x^{r*} = 1 \) and \( x^{s*} = \frac{2 - 1 + \lambda}{\lambda} \) and \( x^r = 1 \) and \( x^s = 0 \). These imply that \( W^* = W + (\gamma - 1 + \lambda) (R^s - 1) \).

Finally, suppose \( p^r R^r = p^r R^s + (R^s - 1) \frac{\lambda}{1 - \lambda} \); the optimality conditions imply \( x^{r*} = x^r \) and \( x^{s*} = x^s \), so \( W = W^* \).

Proposition 1 refers to the region between \( C \) and \( B \) in figure 3. The value of the first-best contract is kinked at \( C \); to the left of \( C \), the lender lends to all the available higher return safe projects with remaining funds to lower return risky projects, the reverse occurs to the right of \( C \) where risky projects have higher expected return. The value of the incentive compatible contract is strictly less than the first-best contract in the region to the left of \( C \) due to misallocation on the intensive margin; capital is being prioritised for lower return risky projects. To the right of \( B \), the misallocation is on the extensive margin as the lender restricts capital to safe borrowers. This occurs because the expected return from risky projects is sufficiently high that the information surplus received by risky borrowers is greater than the surplus generated by safe projects; it is therefore optimal for the lender to stop funding safe projects and set the repayment on risky loans \( \tau^r = R^r \).

Pooling occurs to the left of threshold \( A \) as the contract with asymmetric information offers exactly the same terms to safe and risky projects; a probability of being financed \( x^s = x^r = \gamma \) and repayment \( \tau^s = \tau^r = R^s \). With the information problem, it is not possible for the lender to go better by increasing finance to safe projects as the risky contract would no longer be incentive compatible. This pooling can occur even if \( p^r R^r < 1 \) provided the surplus from safe projects is sufficiently high to subsidise losses from the risky projects, otherwise the lender will cease all lending even if \( p^s R^s > 1 \). To the right of \( A \) is a separating equilibrium with all risky projects receiving finance with the use of remaining funds dependant on \( R^r \).

**Proposition 2.** If \( \gamma = 1 \) and \( p^r R^r \geq 1 \), misallocation is only on the extensive margin.

**Proof.** Under these conditions, and if \( R^s \geq 1 \), the optimality conditions imply that \( x^s = x^r = 1 \) and so there can be no capital misallocation on the intensive margin. If \( R^s < 1 \),
the lender will set \( x^g = 0 \) as in the first-best contract, so there is no misallocation. □

With \( \gamma = 1 \), the kink at \( C \) and the step at \( A \) in figure 3 disappear. The restriction of credit at \( R^r > C \) remain. \( \lambda \) and \( p^r \) have been given exogenously, but we can also consider the effect that these have on figure 3.

**Proposition 3.** If \( p^r \) decreases, \( A, B \) and \( C \) all shift right, and \( C \) and \( A \) diverge.

*Proof.* Using the partial derivatives of \( A, B \) and \( C \) with respect to \( p^r \), for a right-shift in \( C \) of size \( \Delta \) generated by a fall in \( p^r \), \( \mathbf{9} \) threshold \( A \) would move by \( \lambda \Delta \) and \( B \) by \( (1 - \frac{1}{R^r}) \frac{\lambda}{1 - \lambda} \Delta \).

With a lower value of \( p^r \) corresponding to riskier risky projects, all the thresholds will shift right due to a decreased expected value of risky projects. The shift in \( A \) is certainly less than the shift in \( C \) indicating that pooling equilibrium would become relatively less likely for a given distribution of \( p^r \).

**Proposition 4.** For decreases in \( p^r \), \( B \) and \( C \) will diverge if \( \lambda > 1/(2 - \frac{1}{R^r}) \), will exactly co-move if the condition holds with equality, and converge otherwise.

*Proof.* The condition is found by solving \( \frac{\partial B}{\partial p^r} > \frac{\partial C}{\partial p^r} \).

The shift in \( B \) depends on the share of safe and risky projects, and the return on safe projects. \( B \) will move away from \( C \) if the return and share of safe projects are sufficiently high. Consider that if 90% of projects were safe, it would require an expected net return of 12.5% for this to occur, otherwise \( B \) would converge on \( C \) as the risk increased. We discuss this role of risk below, but the results indicate that, under reasonable parametrisations, that as the risk of risky projects increases, the efficient region shrinks, the likelihood of pooling reduces and the restriction of credit to safe projects becomes more likely.

### 2.2.2 Equal Expected Value and the Role of Risk

To think about the role of risk, we are interested in the case where \( \bar{R} \) remains constant but \( p^r \) changes; an increase in risk would be associated with a fall in \( p^r \). The first order

\[ \mathbf{9} \text{It is straightforward to show that } \frac{\partial A}{\partial p^r}, \frac{\partial A}{\partial p^r}, \frac{\partial A}{\partial p^r} < 0 \]

\[ \mathbf{10} \text{Incidentally, the change in } C \text{ of size } \Delta \text{ would correspond to a shift in } p^r \text{ of } -\sqrt{\frac{R^r}{\bar{R}}} \]
conditions become

\[
\begin{align*}
\lambda (\bar{R} - 1) - (1 - \lambda) (\bar{R} - p^R R^s) - \rho \lambda + \varphi^s - \psi &= 0 \quad (2.16) \\
(1 - \lambda) (\bar{R} - 1) - q_t (1 - \lambda) - \varphi^r + \psi &= 0 \quad (2.17)
\end{align*}
\]

Again using the Kuhn-Tucker conditions, we can examine the possible cases which become somewhat simpler. First, we find that provided \( \bar{R} > 1 \), the risky projects get finance with probability one and so the pooling equilibrium disappears. This had occurred when the surplus from higher value safe projects subsidised for low value, or even negative value, risky projects. If it is assumed that \( \bar{R} > 1 \) we find all risky projects will get funded with probability one, otherwise no projects receive funding. Whether or not the remaining funds are used to finance safe projects or are un-utilised depends on the same condition as before. If \( p^r < p^s \left( 1 - \frac{\lambda}{1 - \lambda} \left( 1 - \frac{1}{R} \right) \right) \) then the lender will restrict credit to safe projects entirely, otherwise they will use the remaining \( (\gamma - 1 + \lambda) \frac{1}{\lambda} \) of funds to finance the safe projects. From this condition, it is clear that the probability of this event increases in the share of risky projects \( (1 - \lambda) \). With expected value equal across projects, the higher the risk of the risky projects, the higher the information rents implied by the contract. At the threshold given, it is optimal for the lender to restrict financing safe in order to receive this surplus.

If safe projects are perfectly safe, so \( p^s = 1 \), using the solution to the optimal contract, the \textit{ex ante} value of equity is given by

\[
V = (1 - \lambda) x^s \bar{R} (1 - p^r) \quad (2.18)
\]

and the \textit{ex ante} value of the whole project

\[
W = (x^s \lambda + x^r (1 - \lambda)) (\bar{R} - 1) \quad (2.19)
\]

Whilst the rate of return is unchanged, the overall expected value of the project will fall if the probability of finance falls. The information surplus to the risky borrower increases in risk, leading to an \textit{increase} in the value of equity. As mentioned, when this surplus is sufficiently large, the lender finds it optimal to restrict lending to safe borrowers entirely in order to receive the surplus from risky projects. It follows from the condition above when \( p^s = 1 \), that the probability of financing safe projects is determined by

\[
\lambda (\bar{R} - 1) - (1 - \lambda) \bar{R} (1 - p^r) \quad (2.20)
\]
where if positive, the safe projects will receive funding, but not if negative. Figure 4 shows the value of equity and the entire project for a range of risk values. The surplus to the lender is represented by the green area above $V$ and below $W$. This surplus falls linearly in $p^r$ until the threshold given by

$$p^{r*} = 1 - \frac{\lambda}{1 - \lambda} \left( 1 - \frac{1}{R} \right)$$

at which point, credit to safe borrowers is restricted entirely. The surplus to the borrowers reaches a peak of $(\gamma - 1 + \lambda) W^*$, and drops to zero as the lenders receive all surplus. The level of $W$ drops to a fraction $1 - \lambda$ of the first-best level when safe borrowers are excluded and is the only inefficient region when the expected value of projects are equal.
3 Adverse Selection in General Equilibrium

To analyse the theoretical macroeconomic implications of adverse selection in credit markets, the contract problem is embedded in a general equilibrium framework. To rationalise $\gamma < 1$, we want to think about occupational selection and the value of acting as an entrepreneur seeking funds relative to other occupations. To achieve this in a simple, tractable way, we model the economy as comprised of a continuum of households each with a large number of household members.\footnote{See Christiano, Trabandt & Walentin (2010a) as an example of this assumption, in this case to model the allocation of family members to employment.} This also allows the agency problem to be framed appropriately, by assuming that the entrepreneur must seek external funds from other households, and is equivalent to the islands assumption of Gertler & Kiyotaki (2010). It is assumed that a family head chooses occupational allocation to maximise a utilitarian welfare function that equally weights the utility of all members. Every period, household members can be assigned to three possible roles: entrepreneurs that draw projects and seek funds; workers that receive a wage for providing a unit of work; and unemployed that do neither. Whilst the inclusion of unemployment is not strictly necessary, by including this mechanism, the model implies labour supply conditions analogous to the real business cycle model which is used as a benchmark case. As in Christiano, Trabandt & Walentin (2010a), the utilitarian welfare function implies that all family members receive the same level of consumption.\footnote{Note that this differs from Christiano, Trabandt & Walentin (2010b) and Christiano, Trabandt & Walentin (2011) which focus on involuntary unemployment with search and matching frictions.} We follow the standard timings by assuming that investment decisions are made at the end of the period and employment decisions once shocks are drawn. As in the partial equilibrium model, we assume that each project requires the same quantity of capital input; this is a necessary assumption to capture the assumption $\gamma > \max\{\lambda, 1 - \lambda\}$, key to the results in partial equilibrium. We could alternatively model decreasing returns to scale in projects but choose the simpler method able to capture the salient features. Furthermore, we abstract from the financial structure problem and assume every project requires $n$ units of internal finance, and $\kappa$ units of outside finance.
3.1 Households

With population size normalized to 1, the household head chooses the number of entrepreneurs $e_t$ at the end of the period, and the number of workers $1 - e_{t-1} - u_t$, and unemployed $u_t$ at the start of the new period. The workers provide one unit of labour and, if funded, entrepreneurial activity also uses up a unit of labour. As in Christiano, Trabandt & Walentin (2010a), we assume that household members differ in their disutility of labour, and so household $j \in [0, 1]$ receives the utility

$$\log C^j_t - \frac{\chi}{1 - j}, \quad \chi > 0 \quad (3.1)$$

if assigned as a worker or entrepreneur, and

$$\log C^j_t \quad (3.2)$$

if unemployed. The total number of workers and funded entrepreneurs is given by $1 - u_t$, and so it follows that those members with $0 \leq j \leq 1 - u_t$ will be assigned as workers or entrepreneurs, and those with $j > 1 - u_t$ as unemployed. Recall that entrepreneurs can choose contracts that include lotteries $x^s_t$ and $x^r_t$ as the probability of receiving finance, and a proportion $\lambda$ of projects are safe; of the assigned entrepreneurs, only a proportion $\lambda x^s_t + (1 - \lambda) x^r_t$ will receive external debt finance for their project and so the total household leisure is given by $L_t \equiv \left( [1 - (\lambda x^s_{t-1} + (1 - \lambda) x^r_{t-1})] e_{t-1} + u_t \right)$. Using this definition and equations (3.1) and (3.2), we can write the total household utility

$$U_t = \int_0^1 \log C^j_t \, dj + \int_0^{1-L_t} \frac{\chi}{1 - j} \, dj \quad (3.3)$$

where $\int_0^1 \log C^j_t \, dj = \log C_t$ and $\int_0^{1-L_t} \frac{\chi}{1 - j} \, dj = \chi \log (L_t)$.$^{13}$

Each project requires $n$ units of internal finance and so if the total capital owned by the household is $K_t$, the amount available to invest in external projects is given by $K_t - ne_t$.

At the end of the period, each household simultaneously chooses consumption and next

$^{13}$Because at the time entrepreneurs are allocated, it is not known which project will get funded, this must either satisfy the condition that the number of unfunded entrepreneurs is weakly less than the labour supply, in which case those with lowest dis-utility of labour will be assigned as entrepreneurs and unfunded entrepreneurs will be assigned as workers; or implies an implicit assumption that funding can be switched from entrepreneurs with the highest dis-utility of labour to those with the lowest. The number of workers is always greater than the number of entrepreneurs in the simulations, satisfying this condition. This follows from calibrations to target the number of workers per firm.
period capital stock, posts a contract offer for entrepreneurs from other households, and chooses the number of household members to assign to the entrepreneur role for the following period. The contract will be discussed below, but the household head chooses $e_t$ and $u_t$, and consumption and saving to maximise

$$
\max_{C_{t+s},K_{t+s},e_{t+s},u_{t+s}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} U(C_{t+s}, e_{t+s}, u_{t+s})
$$

(3.4)

Subject to the budget constraint

$$
C_t + K_t = R_t (K_{t-1} - ne_{t-1}) + W_t (1 - e_{t-1} - u_t) + D_t e_{t-1}
$$

where $D_t$ are average dividends payments per project, $W_t$ the wages paid in a perfect labour market, and $R_t$ the ex post return across lending to outside profits. The utilitarian preferences convex in consumption imply all household members receive equal consumption, so $C^j_t = C_t \forall j$. The household consumption savings decision is characterized by

$$
1 = \mathbb{E}_t [\Lambda_{t,t+1} R_{t+1}]
$$

(3.5)

where $\Lambda_{t,t+1} = \beta \frac{C_t}{C_{t+1}}$, and labour supply by

$$
\chi \frac{C_t}{L_t} = W_t.
$$

(3.6)

The occupation allocation decision leads to a further condition

$$
\mathbb{E}_t [\Lambda_{t,t+1} (D_{t+1} - W_{t+1} (\lambda x^s_t + (1 - \lambda) x^r_t)) - n] = 0,
$$

(3.7)

which equates the expected discounted profit from an additional entrepreneur with the opportunity cost which sums the expected welfare gain from the additional leisure and the expected discounted value of an additional $n$ units of funds.

3.2 Entrepreneurs

A measure $e_t$ of entrepreneurs draw investment projects of type $i = s, r$ yielding efficiency units of productive capital $X_i^t \in \{\omega^i, 0\}$. $p_i^s \in [0, 1]$ is the probability of efficiency units $X_i^t = \omega^i$. The two types $i = s, r$ represent safe and risky projects respectively. A proportion $\lambda$ of the projects are safe, the remaining $1 - \lambda$ are risky, and as before
\( \omega^* < \omega^r \) and \( p^* > p^r \). Each project requires \( \kappa \) units of outside investment which must be financed by another island and \( n \) units of internal finance. The number \( \kappa + n \) will be a normalisation device for the units of capital to introduce the idea of one indivisible unit of capital used in each project, and we will calibrate the ratio \( \kappa / (\kappa + n) \) to match the empirical first moment of the leverage ratio.

Entrepreneur \( i \) with a successful, funded project hires \( h^i_t \) units of labour and produces output using

\[
y^i_t = z_t \left( \omega^i_t (\kappa + n) \right)^\alpha (h^i_t)^{1-\alpha} \tag{3.8}
\]

where \( z_t \) is a stationary stochastic process. The entrepreneur will hire workers so that the marginal product of labour equals the real wage,

\[
W_t = (1 - \alpha) \frac{y^i_t}{h^i_t} \tag{3.9}
\]

Capital depreciates at \( \delta \), so the gross surplus equals

\[
\alpha y^i_t + (1 - \delta) (\kappa + n) \omega^i_t \tag{3.10}
\]

of which the entrepreneur must repay the lender \( \tau^i \). There are two types of entrepreneur; risky and safe, but a single labour market. Equation (3.9) implies that the efficiency capital-labour ratio must be equal across firms, and so the risky firms will hire more labour than safe firms.

Let \( R^s_t \) be the gross rate of return on capital for project \( i \). To simplify the problem, we assume that the safe projects always convert one unit of capital input to a single unit of productive capital, so \( \omega^s = 1 \), and so imply a rate of return \( R^s_t \) with probability \( p^s_t = 1 \), whilst risky projects convert one unit of capital into \( \omega^r_t \) leading to a rate of return \( R^r_t \) with probability \( p^r_t \). Recall that the efficiency capital-labour ratio is equal for safe and risky projects; denoting this \( K_t \), we can write

\[
\alpha y^i_t = \alpha \omega^i_t (\kappa + n) z_t K^a_t^{-1} \tag{3.11}
\]

Therefore, if the safe projects yield gross rate of return

\[
R^s_t = \alpha z_t K^a_t^{-1} + (1 - \delta) \tag{3.12}
\]
the risky project will yield
\[ R_t^r = \omega_t R_t^s \] (3.13)
when successful. At the end of each period, remaining surplus is paid to the family as a dividend.

### 3.2.1 Optimal Contract

We have shown that the contract optimal to the lender can be characterised as a static, one period debt contract. We assume that debt repayments are indexed to the aggregate state of the economy, and are only repaid providing the project is successful. We further assume that the success of the project is public information. Modified from above, the individual rationality and incentive compatibility constraints are given by
\[ E_t \left[ \Lambda_{t,t+1} \right] \leq E_t \left[ \Lambda_{t,t+1} R_{t+1}^s \right] \] (3.14)
\[ E_t \left[ \Lambda_{t,t+1} \left( x_{t+1}^r R_{t+1}^r \right) \right] \geq x_{t+1}^r R_{t+1}^r \] (3.15)
for \( i, j \in \{ s, r \} \). As the contract problem is analogous to the partial equilibrium problem described above, we jump straight to the first order conditions
\[ E_t \left[ \Lambda_{t,t+1} \right] = \varrho_t - \psi_t \frac{1}{1-\lambda} + \varphi_t^r \frac{1}{1-\lambda} \] (3.16)
\[ E_t \left[ \Lambda_{t,t+1} \left( \left( \lambda + (1-\lambda) p_{t+1}^r R_{t+1}^s \right) - 1 \right) \right] = \varrho_t + \varphi_t^s - \varphi_t^r \] (3.17)
that imply the same four possible outcomes as in partial equilibrium. The key difference to the partial equilibrium framework relates to outcomes on the defined threshold as the constraints bind. The partial equilibrium model implies a step function; for instance, \((\lambda p^s + (1-\lambda) p^r) R^s > \lambda + (1-\lambda) p^r R^r\) but if a change to \( p^r \) leads to \((\lambda p^s + (1-\lambda) p^r) R^s < \lambda + (1-\lambda) p^r R^r\), then the contract will imply a sudden stop in funding safe projects. What happens when \((\lambda p^s + (1-\lambda) p^r) R^s = \lambda + (1-\lambda) p^r R^r\) is undetermined; at this point, the lender is indifferent about \( x^s \in [0, (\gamma - 1 + \lambda) \frac{1}{\lambda}] \), and the safe borrowers are indifferent anyway as they receive no surplus. In general equilibrium, the level of investment and number of entrepreneurs can be adjusted depending on finance probabilities and expected profits so rather than jumping, the model is likely to remain at the threshold as other variables adjust. Taking the same example, suppose
\[ E_t \left[ \Lambda_{t,t+1} \left( \lambda + (1-\lambda) p_{t+1}^r R_{t+1}^s \right) \right] = E_t \left[ \Lambda_{t,t+1} \left( \lambda + (1-\lambda) p_{t+1}^r R_{t+1}^s \right) \right] \] (3.18)
and that the risky projects have a strictly positive net present value. In this case by examining the first order conditions we can find

\[ \lambda \varphi^r_t + (1 - \lambda) \varphi^s_t > \psi_t \]  
(3.19)

\[ q_t (1 - \lambda) + \varphi^r_t > \psi_t \]  
(3.20)

\[ \lambda q_t + \psi_t = \varphi^s_t \]  
(3.21)

Using the Kuhn-Tucker conditions, it then follows that \( \varphi^r_t > 0 \) and \( \varphi^s_t = q_t = \psi_t = 0 \). The first condition indicates that \( x^r_t = 1 \), but the second that the contract does not specify a value for \( x^s_t \). This will be determined by the general equilibrium conditions and discussed in further detail below.

Summing the return on inside finance \( n \) and the surplus received by entrepreneurs set by the debt contract yields the total profits from entrepreneurial activity

\[ D_t = (1 - \lambda) p^s_t x^s_{t-1} (R^s_t - R^r_t) \kappa + (\lambda p^s_t x^s_{t-1} R^s_t + (1 - \lambda) p^r_t x^r_{t-1} R^r_t) n \]  
(3.22)

If a household does not lend funds, then the capital will not depreciate and the gross return will equal 1. \( \gamma_t \) is the proportion of entrepreneurs that can receive finance and so is naturally bound above by 1. Letting \( \gamma^*_t \) be the value of unbounded \( \gamma_t \), that is, the amount of capital allocated to debt finance relative to equity finance, we can write the total rate of return on capital used as debt finance as

\[ R_t = 1 + \lambda x^r_{t-1} (\tau^s_t - 1) + (1 - \lambda) x^s_{t-1} (p^r_t \tau^r_t - 1) . \]  
(3.23)

### 3.3 Aggregations

Each project requires \( \kappa \) units of debt finance and \( n \) units of internal finance. Furthermore, every entrepreneur has the internal finance irrespective of whether they receive debt finance, in which case capital is not fully utilised. Capital is either allocated to external debt finance or internal equity finance and if there are \( e_t \) entrepreneurs per island, \( K_t \) units of capital in aggregate, and an efficient matching process, using \( \gamma^*_t \equiv (K_t - e_t n) \frac{1}{e_t n} \), we have

\[ \gamma_t = \begin{cases} 
\gamma^*_t & \text{if } K_t < e_t (\kappa + n) \\
1 & \text{otherwise.}
\end{cases} \]  
(3.24)
which is the maximum proportion of entrepreneurs that can get debt finance. Labour market clearing implies that total labour supplied will equal that demanded, so

\[(\lambda x_{t-1}^e h_t^e + (1 - \lambda) x_{t-1}^r p_t^r h_t^r) e_{t-1} = 1 - (\lambda x_{t-1}^e + (1 - \lambda) x_{t-1}^r) e_{t-1} - u_t \] (3.25)

At each firm, the first order condition for labour demand is a linear function of local capital productivity

\[h_t^i = \omega_t^i (\kappa + n) \left( z_t \frac{1 - \alpha}{W_t} \right)^{1/\alpha} \] (3.26)

and using this and letting total labour and efficiency capital inputs be

\[H_t \equiv 1 - (\lambda x_{t-1}^e + (1 - \lambda) x_{t-1}^r) e_{t-1} - u_t \] (3.27)
\[\hat{k}_t \equiv (\lambda x_{t-1}^e + (1 - \lambda) x_{t-1}^r p_t^r \omega_t^i) (\kappa + n) e_{t-1}, \] (3.28)

the labour market clearing condition can be written

\[W_t = (1 - \alpha) z_t \left( \frac{\hat{k}_t}{H_t} \right)^\alpha \] (3.29)

Likewise, total output

\[Y_t = z_t k_t^\alpha H_t^{1-\alpha} \] (3.30)

Using total capital stock \(K_t = (\gamma_t^* \kappa + n) e_t\), we can rewrite this

\[Y_t = A_t K_t^\alpha H_t^{1-\alpha} \] (3.31)

where we can give total factor productivity as

\[A_t \equiv z_t \left( (\lambda x_{t-1}^e + (1 - \lambda) x_{t-1}^r p_{t-1}^r \omega_t^i) \frac{\kappa + n}{\gamma_{t-1}^* \kappa + n} \right)^\alpha \] (3.32)

Finally, we close the model with an aggregate resource constraint

\[Y_t = C_t + I_t \] (3.33)

where investment is the difference between the new capital stock \(K_t\), and the sum of the depreciated returned capital, and the un-depreciated, unused capital

\[I_t = K_t + \delta (\lambda x_{t-1}^e + (1 - \lambda) x_{t-1}^r) (n + \kappa) e_{t-1} - K_{t-1} \] (3.34)
3.4 First-Best Economy

As a point of comparison we use the same model with the information asymmetry removed. As in the partial equilibrium model, the first-best contract sets $\tau^r_t = \tau^s_t = R^*_t$ and again will use all available funds if the expected discounted return from both projects is positive. The project with the highest net present value will then be funded with probability 1, with all funds used to finance the lower value projects. Under the first-best contract, the expected return on equity and debt are equal, and given that both are state contingent, the realised return is also equal. Given the general equilibrium set-up, debt is always preferred to equity as the latter requires the contribution of an entrepreneur so the opportunity cost is greater. One could solve this by making the debt-equity finance decision endogenous with an agency problem, taxes, or some other feature. To keep things simple, we assume in the first-best economy, the households are forced to provide $n$ units of equity finance for every $\kappa$ units of debt finance. This implies that $\gamma_t = 1\forall t$ and so $x^*_t = x^r_t = 1$ providing the expected project value is positive, and so the resulting model corresponds to a standard real business cycle model.

3.5 Efficiency and Investment Wedges

The credit friction emerges both as a wedge between the marginal inter-temporal rate of substitution and the average return to capital, and as an inefficient allocation of capital. To measure the former, referred to as the investment wedge, we take the spread between the expected saving rate $E_t [R_{t+1}]$ and the average return to capital used in production, given by

$$
\Delta_t \equiv \mathbb{E}_t \left[ \frac{\lambda x^*_t + (1 - \lambda) x^r_t p^r_{t+1} \omega^*_t + 1}{\lambda x^*_t + (1 - \lambda) x^r_t} R^*_t - R_{t+1} \right].
$$

(3.35)

Holding the expected project value constant, the investment wedge increases in the risk of risky projects due to higher information rents implied by the incentive compatibility constraint. Whilst this is true for both long-run and short-run increases in risk, short-run increases in risk can also lead to sharp spikes in the spread as the lenders restrict lending to safe borrowers rather opting to store capital instead. This does not occur for long-run increases; lowering the steady state value of $p^r$ will lower the steady state capital without causing a restriction of credit. Although in numerical simulations, a short-term increase in the expected value of a risky project will provide additional information.
surplus to the risky borrower and so a short-term rise in the investment wedge, in steady state, increasing the long-run value of risky projects relative to safe projects reduces the investment wedge. This stems from lower steady state capital holdings and also lowers the steady state value of $\rho$, the marginal value of lending. If the relative value of risky projects is high enough, the investment wedge increases sharply as there is a credit restriction to safe borrowers.

The efficiency wedge emerges due to the misallocation of the capital and defined as the inverse total factor productivity $A_t$ given by equation (3.32). We find that the long-run efficiency wedge increases in the steady-state risk of risky borrowers. This arises from misallocation of capital along the extensive margin; higher risk lowers the value of debt relative to equity and in general equilibrium this leads to a larger number of entrepreneurs than available loans. Finally, misallocation on the intensive margin implies a rise in the efficiency wedge if the value of risky projects fall.$^{14}$

### 3.6 Shocks

To evaluate the model, we consider three possible exogenous transitory shocks to the economy: a total factor productivity shock, a risk shock, and a risky project productivity shock. The first needs little introduction and allows comparison with the real business cycle literature with total factor productivity given by

$$\log z_t = \rho_z \log z_{t-1} + \varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim \mathcal{N}(0, \sigma_z). \quad (3.36)$$

A positive risk shock decreases $\bar{p}_R$ whilst keeping $p_R^t R_t^R$ constant and a risky project productivity shock is the opposite; changing the return on a risky project whilst keeping the risk constant. The former relates the analysis to a literature looking at second moment shocks (e.g Christiano, Motto & Rostagno 2013) with $p_R^t$ given by

$$\logit p_R^t - \logit \bar{p}_R = \rho_p(\logit p_R^t - \logit \bar{p}_R) + \varepsilon_{p,t} \quad (3.37)$$

with $\varepsilon_{p,t} \sim \mathcal{N}(0, \sigma_p)$, and where the logit function, $\logit p = \log \left( \frac{p}{1-p} \right)$, imposes bounds $p_R^t \in (0, 1)$. The risky project productivity shock can generate misallocation of capital as risky projects are always at least as likely as safe projects to get funded. To introduce

$^{14}$Plots demonstrating these observations are given in online appendix D.
this shock, we let \( \nu_t \equiv \omega^r_t p^r_t \) denote the relative value of risky projects subject to a stochastic process given by

\[
\log \nu_t = \rho \log \nu_{t-1} + \varepsilon_{\nu,t}, \quad \varepsilon_{\nu,t} \sim \mathcal{N}(0, \sigma_{\nu}).
\] (3.38)

This leads to changes in risky project productivity for given levels of risk \( p^r_t \).

### 3.7 Parametrisation and Calibration

We choose the steady state expected return on risky projects to equal the return on safe projects so \( p^r \omega^r = 1, \ p^r R^r = R^s \). In addition to the parameters common to the real business cycle literature, we are left with several parameters specific to the adverse selection economy. We calibrate \( \lambda \) and \( p^r \) to target a mean loan default rate and a lending rate spread. For the former, we target a value of 2.8%, taken from the average delinquency rate on commercial and industrial loans over the period 1987Q1 – 2016Q2.\(^\text{15}\) An ergodic mean of the spread between the lending rates \( \tau^r - \tau^s \) of 0.9897% is targeted; this is the average spread between the yields of Moody’s BAA and AAA rated corporate bonds\(^\text{16}\) over the period 1986Q1 – 2016Q3. In the U.S. non-financial firm sector, there is significant difference in financial structure across sectors, and across firm size and age. For a simplified representative framework we look at the aggregate book debt to capital ratio which in the U.S., at the start of 2016, was estimated across 7480 firms at 62.63%\(^\text{17}\). We therefore choose the required debt finance per project, \( \kappa \), to match \( \hat{\kappa} \equiv \kappa \kappa^a + \kappa^b = 0.6263 \). To pin down the average size of the firm, we use data from the Statistics of U.S. Businesses (SUSB) which indicates that there were 7.5 million establishments with a total employment of 118 million in 2013, implying there were 15.8 workers per establishment;\(^\text{18}\); we choose the required internal finance, \( n \), to

\(^{15}\)Board of Governors of the Federal Reserve System (US), Delinquency Rate on Commercial and Industrial Loans, All Commercial Banks [DRBLACBS], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/DRBLACBS, September 4, 2016.

\(^{16}\)Board of Governors of the Federal Reserve System (US), Moody’s Seasoned Baa Corporate Bond Yield©[BAA] and Moody’s Seasoned Baa Corporate Bond Yield©[AAA], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/BAA, September 5, 2016.


\(^{18}\)It is naturally impossible to fit the diverse distribution of firm type and size into a representative framework. There are a large number of smaller business such as sole traders ignored in these numbers.
target the ergodic mean \( \hat{w} = \frac{1-e^{-u}}{e} = 15.8 \). These calibrations are listed in Table 1. For parameters common to the RBC literature, we choose values typically used to ensure a useful comparison and shown in Table 2. We calibrate the shock variance to match second and third moments in aggregate output; this is discussed further below in Section 5.3.

### Table 1: Calibrations of adverse selection model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>Share of safe projects</td>
<td>0.8091</td>
<td>( \frac{(1-\lambda)(1-p^r)}{\lambda} = 0.0280 )</td>
</tr>
<tr>
<td>( p^r )</td>
<td>Risky project success probability</td>
<td>0.8603</td>
<td>( \tau^r - \tau^s = 0.00990 )</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Required debt finance for project</td>
<td>102.46</td>
<td>( \frac{\kappa}{\kappa+n} = 0.6263 )</td>
</tr>
<tr>
<td>( n )</td>
<td>Required equity finance for project</td>
<td>61.14</td>
<td>( \hat{w} = \frac{1-e^{-u}}{e} = 15.8 )</td>
</tr>
</tbody>
</table>

### Table 2: Parametrisation of common real business cycle parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>Capital share of production</td>
<td>0.3</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Household discount factor</td>
<td>0.99</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Capital depreciation rate</td>
<td>0.023</td>
</tr>
<tr>
<td>( \chi )</td>
<td>Utility share of labour</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Some robustness checks of the parametrisation were carried out on both the implied deterministic steady state and the model dynamics. The latter is discussed in the numerical analysis below. The choice of time discounting, and capital share and depreciation are standard and we focus on the novel parametrisation, beginning with their impact on the steady state equilibrium. The number of workers per firm, \( \hat{w} \) is an endogenous variable, but this and the debt-equity ratio are chosen to calibrate \( n \) and \( \kappa \) where \( n + \kappa \) is the but we catch the majority of U.S. workers.
capital per firm. Adjusting \( \hat{w} \) up (down) from 15.8 implies a lower (higher) population share of entrepreneurs and so less (more) firms. There is not a large effect on the steady state equilibrium conditions, except, with \( \gamma \) fixed, reducing (increasing) \( \hat{w} \) reduces (increases) the likelihood\(^{19}\) that all funds are lent, when \( \varrho > 0 \) and \( \varphi^r > 0 \); and increases (reduces) the likelihood that safe projects have restricted access to credit markets, that is with \( \varrho = 0 \) and \( \varphi^r > 0 \). There is a similar effect of adjusting the debt-equity ratio; increasing (reducing) the share of credit in the economy by reducing \( \hat{\kappa} \), increases (reduces) the likelihood that safe borrowers face credit restrictions. The effect disappears for a given \( p^r \), and so whilst having both debt and equity play an important role to ensure an interior solution to the occupational allocation, the exact composition does not have a large impact on steady state values.

As would be expected given their role in the optimal contract, the values of \( p^r \) and \( \lambda \) do have a significant impact on steady state conditions as well as model dynamics. These two parameters are calibrated to target the average loan delinquency rate and lending spread. Setting \( \lambda \) higher implies that risky projects occupy a smaller share of lending opportunities and it shrinks the proportion of the surplus that is received by risky entrepreneurs as information rents. This implies a higher average value of \( \varrho \), so that the lender is less likely to restrict credit to safe borrowers; and a lower average value of \( \varphi^r \) so the lender is more likely to choose a pooling equilibrium. For a given value of \( \gamma \), a higher value of \( p^r \), which indicates safer risky projects implies the opposite: a lower value of \( \varrho \), so that the lender is more likely to restrict credit to safe borrowers; and a higher value of \( \varphi^r \) so the lender is less likely to choose a pooling equilibrium.

4 Numerical Strategy

There are five inequality constraints that introduce significant non-linearity into the model policy function; positivity constraints on the four Lagrange multipliers, \( \varrho_t, \varphi^r_t, \varphi^I_t \) and \( \psi_t \) and an upper bound on the proportion of entrepreneurs that can receive funding, \( \gamma_t \leq 1 \). To evaluate the model dynamics, we compute a third order pruned perturbation approximation to the model transition and decision functions, using the algorithm proposed in Holden (2016\(^a\)) to implement the inequality constraints. This solves a sequence

\(^{19}\)Assessing the steady state across a range of parametrisations.
of news shocks required push the constrained variable back to the bound when it would otherwise be violated; we refer to this as the bounds problem. The local approximation increases the speed of numerical simulation relative to a global solution method. Computing third order approximations improves accuracy relative to lower orders and introduces time-varying precautionary effects; at second order the risk premium is constant whilst at third order is linear in the state. The method outlined in Holden (2016a) proposes an efficient perfect foresight solver that finds a global solution to impose the inequality constraints on a pruned perturbation approximation to a non-linear model. The expected value of the news shocks must be positive and partly predictable from the state, and so to capture the effects of uncertainty arising from the bound, we integrate over future uncertainty up to a finite horizon, following the stochastic extended-path method proposed in Adjemian & Juillard (2013). More specifically, we simulate the model, every period integrating over a fixed number $S$ of future periods to evaluate expected size of news shocks up to this horizon.

We outline some specific details relating to the numerical methods used in the exercises here, but describe the solution algorithm in further detail in online appendix C. Different integration methods are employed for each numerical exercise where, in each case, the choice is made to achieve solutions in reasonable time at as high an accuracy as possible. To compute the simulated time-series, we use a degree 3 monomial rule with $2\hat{S} + 1$ nodes and equal positive weights to integrate over future uncertainty up to a horizon of $S = 16$ periods. For the impulse response functions we use a Gaussian cubature rule with $O\left(\hat{S}^K\right)$ points and maximum monomial degree of $7 = 2K + 1$ again up to a horizon of $S = 16$ periods. A Monte-Carlo simulation is then used to compute impulse response functions as without it, we would miss uncertainty stemming from the bound during the initial period.

The solution method reveals that are some states of the world with either multiple solutions or no solution to the bounds problem. We found that there is always a solution in the vicinity of the steady state, and when there are multiple solutions, the one chosen minimises the size of news shocks. Because the impulse response functions shown in

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20This is adapted from the extended-path method of Fair & Taylor (1983) who present an algorithm to simulate a non-linear model in the absence of uncertainty.

21$\hat{S} \leq S$ is found by setting to zero the smallest eigenvalues of the covariance matrix of the distribution from which we integrate. This is discussed in greater detail in online appendix C.
the section below follow large shocks, the model moves into a region relatively far from the steady state. We found that computing the impulse response functions with the calibrated shock standard deviations would cause the computation to fail; this stems from integrating over an area of the state space where there is no solution to the bounds problem. To deal with this, at some loss of accuracy, the shock standard deviations are scaled down.\textsuperscript{22}

5 Discussion of Numerical Results

We begin the discussion of the numerical results with analysis of the impulse response functions to the three modelled shocks. Following the positive transitory shock to aggregate productivity, $z_t$, in both models there is an increase in output and investment, a rise in employment, $H_t$, as well as an increase in the number of entrepreneurs seeking funds, $e_t$. Plots of impulse response functions to a positive shock to $z_t$ of 1.5% are shown in figure 5. The productivity shock increases the value of both debt and equity finance. In the adverse selection economy, which has unallocated capital in steady state due to a proportion of entrepreneurs being unfunded, for shocks of a typical size, the higher productivity leads to a greater share of capital being allocated to productive projects. This is represented by an increase in the finance rate of safe projects $x^s_t$. Incentive compatibility implies a fall in the relative borrowing rate paid by risky borrowers, increasing the information rents and reducing the spread $\tau^r - \tau^s$. This causes a rise in the investment wedge, $\Delta_t$, which lies behind higher volatility in investment.

We can be more specific about what is meant by ‘of a typical size’. If the shock is sufficiently large, and under the model calibrations, this is of the order of a transitory 12% increase or 8% decrease to total factor productivity, thresholds in the contract will be breached leading to significant non-linear effects.\textsuperscript{23} Following the large negative shock, $\varrho_t$ will fall to zero causing the feasibility constraint on lending to slacken. When the marginal value of lending, $\varrho_t$, is at the zero lower bound, $\gamma$ increases without an equivalent increase in $x^s_t$. This represents the misallocation of capital on the extensive margin; lenders restrict credit as the cost in information rents to risky borrowers would

\textsuperscript{22}We choose $\sigma^a = 0.001$, $\sigma_p = 0.01$ and $\sigma^v = 0.001$, all smaller than the calibrated values in table 3.

\textsuperscript{23}The 8% negative shock would be a 5 standard deviation shock under the estimation of Smets & Wouters (2007).
exceed the gain in additional surplus earned by higher lending to safe entrepreneurs. This causes a sharp increase in the investment wedge, $\Delta_t$, a deeper decline in TFP, and a sharp decline in investment. Plots of such a large negative shock are shown in figure 17 in online appendix E.
5.1 Risk Shock

An increase in the risk of risky projects, caused by a decline in $p_r^t$ with $\omega_r^t$ kept at $1/p_r^t$, generates economic fluctuations in the adverse selection economy whilst leaving the first-best economy unaffected. In the first-best model, the only important factor regarding the financing of projects is the discounted expected value, which is unchanged in the face of a risk shock. With adverse selection, the increased risk leads to higher information rents to the entrepreneur and so a higher value of equity and lower value of debt on average. As shown in figure 6, this causes the marginal value of lending opportunities,

![Diagram](image)

**Figure 6:** Impulse response functions to a transitory risk shock caused by a reduction in the probability of risky project success $p_r^t$ of approximately 1 percentage points. $x^s$ and $x^r$ are the finance probabilities of safe and risky projects, $\varrho$ the marginal value of lending, $p^r$ the probability of risky project success, and $\Delta$ and $A$ the investment wedge and TFP. Plots show level deviation around zero except for $x^s$ and $\varrho$ which are around the ergodic mean.
and the safe project finance probability \( x_t^s \) to both decline, the latter caused by a drop in the availability of loans \( \gamma_t \). The reduction in lending produces a decline in output and investment, and the number of entrepreneurs and level of employment. Quantitatively, under the model calibrations, a 1 percentage point drop in the rate of risky project success leads to a 3% decline in investment, a 0.15 percentage point drop in employment, and a 14 basis point increase in the investment wedge and a 4 basis point increase in the efficiency wedge. The former is an increase in the spread between the gross expected savings rate and the expected return on capital in production, the latter is negative TFP, \(-A_t\). The adverse selection introduces significant non-linearities; for instance, a 2 percentage point drop in \( p_t^r \) causes \( \varrho \) to hit the zero lower bound. This threshold indicates the point at which it is optimal for lenders to restrict credit to safe projects in order to reduce information rents paid to risky borrowers. Following a 3 percentage point drop in \( p_t^r \), the efficiency wedge rises sharply indicating a fall in TFP of about 2.5%. The impact on other variables also becomes more severe: there is approximately a 25% decline in investment on impact and a 3% fall in output; half a percentage point fall in employment; and a 50 basis point increase in the spread. Plots for a 3 percentage point fall in \( p_t^r \) are shown in figure 7.\(^{24}\)

This result is sensitive to the parametrisation of \( \lambda \). Increasing \( \lambda \) to 0.85 from 0.8, so reducing the proportion of risky projects to 15% from 20%, does not have a large impact on the marginal effect on the real economy of changes to \( p_t^r \) close to the steady state, but the steady state value of \( \varrho_t \) shifts to a higher a level. With the alternative parametrisation, a drop in \( p_t^r \) of about 4% is required for \( \varrho \) to become constrained at zero to aggravate a financial crisis. The impulse responses are shown in figures 21 and 22 in online appendix E. The key result to highlight is that by reducing the number of risky projects by 5 percentage points, the drop in \( p_t^r \) needs to be 2 percentage points greater to lead to a financial crisis.

5.2 Risky Project Value Shock

The risky project value shock is more precisely a productivity shock to risky projects. One would expect an increase in the productivity of the risky projects to be positive news,

\(^{24}\)The impulse responses of a larger number of variables to the 1, 2 and 3 percentage point drops in \( p_t^r \) are shown in figures 18 – 20 in online appendix E.
Figure 7: Impulse response functions to a transitory risk shock caused by a reduction in the probability of risky project success \( p_r \) of approximately 3 percentage points. \( x^s \) and \( x^r \) are the finance probabilities of safe and risky projects, \( \varphi \) the marginal value of lending, \( p^r \) the probability of risky project success, and \( \Delta \) and \( A \) the investment wedge and TFP. Plots show level deviation around zero except for \( x^s \) and \( \varphi \) which are around the ergodic mean.

and in the first-best economy, this is certainly true. With adverse selection however, the additional surplus created by risky projects increases the information rents and so the lender fails to profit from the higher productivity. If the shock is sufficiently large, it is optimal for the lender to restrict credit to safe projects in order to access additional risky project surplus. In this case the ‘good news’ can lead to a financial crisis. Following a small negative shock from steady state, there will be misallocation of capital along the intensive margin as a greater proportion of less productive risky project are funded than safe projects. If the shock is sufficiently large, and under model calibrations this is a 5% decrease in the productivity of risky projects, the lender will only offer a single
contract resulting in a pooling equilibrium. However, this did not occur during numerical simulations.\textsuperscript{25}

It turns out that the impact of small shocks to risky project productivity on output is greater in the first-best economy. For small negative shocks for instance, the change in the value of equity is greater than the change in the value of debt in the adverse selection economy, and so there is a relative increase in capital allocated to debt leading to a rise in the proportion of projects funded. This acts as a buffer to the adverse effects of the shock not present in the first-best economy. The reverse occurs following a small positive shock; the value of equity increases more than the value of debt and so the number of entrepreneurs looking for funds increases more than the number of loans available, leading to misallocation of capital on the extensive margin. We find that this effect dominates the misallocation capital on the intensive margin, occurring following an negative shock when less-productive projects receive a higher proportion of funds than safe projects.

As with other shocks, the response of the adverse selection economy to a risky project productivity shock is asymmetric an non-monotonic. The response to shocks are symmetric in the first-best economy,\textsuperscript{26} and the same is true for the adverse selection economy for positive shocks smaller than 2\%, but for larger shocks, the lender begins to restrict credit to safe projects, as shown in figure 8. The shadow value of lending, $\varrho_t$ is constrained at zero and the lender begins to reduce debt finance to safe projects in order to receive a greater share of the additional risky project surplus. This increases the value of debt relative to equity, and so there is a fall in the number of entrepreneurs and an increase in the amount of debt finance. Although $\gamma_t$ rises, there is no equivalent increase in $x_t$ indicating a misallocation of capital on the extensive margin. For a 3\% increase in the productivity of risky projects under the model calibrations, there is a 2\% increase in the efficiency wedge, a 50 basis point increase in the investment wedge, and a 2.5\% fall in output. As with the risk shock, these movements are sensitive to the choice of $\lambda$. A higher $\lambda$ corresponds with a higher steady state shadow value of lending, $\varrho_t$ and so a larger shock is needed to cause a financial crisis. If $\lambda = 0.85$ instead of 0.8, a shock of

\textsuperscript{25}Impulse response functions for negative shocks are shown in figures 23 and 24 in online appendix E.

\textsuperscript{26}This is certainly true for small shocks, but there will be some asymmetries introduced at approximation orders greater than 1.
Figure 8: Impulse response functions to a transitory 3% positive shock to the productivity of risky projects, $\omega_{r_t}$. $x_s$ is the finance probabilities of safe projects, $\varrho$ the marginal value of lending, $\gamma$ the ratio of available loans to the number of projects, and $\Delta$ and $A$ the investment wedge and TFP. Plots show level deviation around the ergodic mean for $\varrho_t$, $\gamma_t$ and $x^*_t$, and around zero for the other variables.

approximately 3% is required for $\varrho_t$ to reach the zero lower bound.\(^{27}\)

5.3 Simulated Moments

The model is simulated over 1000 periods using the extended-path type approach discussed above, and moments computed. Table 3 shows calibrations with the associated moments and output correlations under two environments. Firstly, and by way of com-

\(^{27}\)Impulse response functions to positive shocks of 1.5% and 4% with alternative parametrisation are shown in figures 27 and 28 respectively in online appendix E.
<table>
<thead>
<tr>
<th></th>
<th>( \sigma^2 )</th>
<th>( \sigma_p )</th>
<th>( \sigma^\nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Calibration</strong></td>
<td>0.0030</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>US Data</strong></td>
<td>A.S. F.B. A.S. F.B.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Y )</td>
<td>1.015</td>
<td>1.049</td>
<td>1.014</td>
</tr>
<tr>
<td>( I )</td>
<td>4.24</td>
<td>4.36</td>
<td>3.16</td>
</tr>
<tr>
<td>( C )</td>
<td>0.927</td>
<td>0.815</td>
<td>0.642</td>
</tr>
<tr>
<td>( \tau_r - \tau_s )</td>
<td>0.0957</td>
<td>4x10^{-5}</td>
<td>4x10^{-5}</td>
</tr>
<tr>
<td><strong>Skewness</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Y )</td>
<td>-0.141</td>
<td>0.101</td>
<td>0.100</td>
</tr>
<tr>
<td>( I )</td>
<td>-0.676</td>
<td>0.263</td>
<td>0.239</td>
</tr>
<tr>
<td>( C )</td>
<td>-0.320</td>
<td>0.065</td>
<td>-0.068</td>
</tr>
<tr>
<td>( \tau_r - \tau_s )</td>
<td>3.098</td>
<td>-0.01</td>
<td>0.066</td>
</tr>
<tr>
<td><strong>Correlation w/Y</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I )</td>
<td>0.868</td>
<td>0.792</td>
<td>0.902</td>
</tr>
<tr>
<td>( C )</td>
<td>0.871</td>
<td>0.907</td>
<td>0.829</td>
</tr>
<tr>
<td>( \tau_r - \tau_s )</td>
<td>-0.336</td>
<td>-0.959</td>
<td>0.417</td>
</tr>
<tr>
<td><strong>% periods restricted credit</strong></td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 3:** Calibrations and simulation moments. Two model environments calibrated: (i) first-best economy with only productivity shocks; (ii) adverse selection economy with productivity, and risky project risk and productivity shocks. Data is HP-filtered U.S. time series 1986Q2 – 2015Q4; spread \( \tau_r - \tau_s \) as Moody’s AAA-BAA rated corporate bond yield spread. Further details are given in online appendix A. Standard deviations are in percent for \( Y, I \) and \( C \) and percentage points for \( \tau_r - \tau_s \).
parison, we include just the productivity shock calibrated to target the standard deviation and auto-correlation of output in the first-best economy. As technology shocks within a typical range are not sufficient to produce financial crises in the adverse selection economy, caused by the marginal value of lending, \( \varrho_t \), hitting the zero lower bound, the results of the two economies are relatively similar. As was discussed above, the adverse selection economy generates higher volatility in investment which also causes higher volatility in consumption. The adverse selection model does improve on the cyclicality of the loan rate spread, achieving the correct sign even though implying a far higher correlation with output than is observed in the data. Across the simulated time series with just the productivity shock, the negative skewness in output, investment and consumption and strong positive skewness in the spread is missed entirely, and the volatility in the interest rate spread is widely under-predicted, capturing only the time varying risk premium.

To rectify these issues, the second environments also calibrates the risky project risk and productivity shocks to target output and investment skewness. This exercise shows how introducing adverse selection in credit markets can improve the empirical fit of a typical RBC model by capturing the negative skewness in the time series of output and investment. The risky project risk and productivity shocks also generate volatility in the lending rate spread although miss the very high positive skewness. The mean and volatility of the spread are both much higher in the first-best economy as this perfectly co-moves with the relative expected value of risky projects; in the adverse selection model, the spread is much smaller due to the information surplus earned by risky borrowers. Both higher risk and risky project productivity shock variance lead to lower interest spread skewness and reduce the correlation with output; the adverse selection model finds a very close match for the correlation with output.

6 Conclusion

We have shown that asymmetric information in debt markets for investment projects can lead to adverse selection that manifests as both an investment wedge and an efficiency wedge. Although a stylized model, this has allowed us to match the observed negative skewness in the empirical output time-series and due to non-linearities in fi-
nancial contracts, generate occasional financial crises arising from sharp spikes in the efficiency wedge. These results are an important contribution to the financial frictions literature which typically assumes that the economy is always financially constrained and subject to an investment wedge that depends on the balance sheet of the borrower. Under the standard approach, there are usually insufficient non-linearities to produce the type of financial crisis episodes observed in the data; by focusing on potentially important non-linearities present in financial contracts, we have been able to propose a model that pushes the economy into a financial crisis when thresholds in optimal contracts are breached. The second important result is that by abstracting from the balance sheet channel and focusing on the misallocation of productive resources on both the extensive and intensive margins, we have been able to respond to the issue highlighted in Chari et al. (2007a), that the investment wedge has a less important driver of the business cycle than the efficiency wedge. Large endogenous movements in TFP in the proposed model provide a more convincing explanation of the data than existing models; the drop in investment observed in advanced economies was not deep enough to alone explain the decline in output, leaving many models requiring either large shocks to TFP or a large-scale destruction of physical capital.

Under the model calibration, we have found that a two percentage point increase in the risk of risky projects from the ergodic mean, defined as a two percentage point decline in the probability of project success whilst keeping the expected value constant, is sufficient to generate a financial crisis as lenders begin to restrict credit to safe projects so to reduce the information surplus due risky borrowers. This result is sensitive to the population share of risky borrowers, for example, with 25% less risky projects, over twice as large a risk shock is required to cause a financial crisis. A financial crisis episode can also occur following a shock that increases the productivity of risky projects by over 2% from the ergodic mean; at this point, a similar phenomena occurs as the lenders find it optimal to restrict funding safe projects in order to reduce the information surplus.

To summarise, we have presented a novel contribution to our understanding of the channels by which financial disturbances might have real effects. Particularly interesting given the recent financial crisis, are the effects of increased risk on the market for debt finance. The proposed framework offers opportunities for further research, particularly the analysis of the transmission mechanisms of monetary policies.
References


Crawford, G. S., Pavanini, N. & Schivardi, F. (2015), Asymmetric information and imperfect competition in lending markets.


Online Appendices for “Adverse Selection and Financial Crises”
Jonathan Swarbrick
University of Surrey
September 29, 2017

A Data

The time-series data used for the comparison of moments and to produce plots is described here. Where specific metrics were used as part of the parametrisation procedures, for example, the business loan delinquency rate, the source is given in a footnote. The data listed here was subject to calculations made by the author and so specific detail is given.

Table 4 details the sources of non-banking aggregate time-series. Using 1986Q2 – 2015Q4
Table 4: Non-banking aggregate time-series sources.

data, we make the following calculations for the time-series used in the chapters:

\[
y = \log \left( \frac{GDP}{GDPDEF} \right) \times 100
\]

\[
i = \log \left( \frac{FPI}{GDPDEF} \right) \times 100
\]

\[
c = \log \left( \frac{PCEC}{GDPDEF} \right) \times 100
\]

\[
\Delta = \frac{BAA - AAA}{4}
\]

The resulting time-series for output \(y\), investment \(i\) and consumption \(c\) are decomposed onto a trend component and a business cycle component using proposed in Hodrick & Prescott (1997). The smoothing parameter required by the Hodrick-Prescott filter was set as 1,600, as recommended by the authors to extract business cycle fluctuations from quarterly time-series. The resulting cyclical component is denominated in percentage deviations from trend.
B  Long-term Contracts

The best the lender can expect is for static one-period contracts; a long-term contract is derived here and shown to offer higher expected social value but at the expense of lender value. The one-period contract implied a static trade-off across states, whilst with a long-term contract it is possible to specify a dynamic trade-off between the current pay-off and the future value of equity. Rather than having to restrict the proportion of safe projects invested in today, the restriction can be made in the future, once the risk type has been reset. Denoting borrower type $\theta \in \{s, r\}$, the contract $C$ implies an expected discounted value of future cash flows $V(\theta_t, C)$ and is solved subject to an initial participation constraint, a sequence of incentive compatibility constraints, and a sequence of limited liability constraints that replace the individual rationality constraints. Note that this is a stricter assumption than being sub-game perfect as, by preventing the borrower from becoming insolvent at any point in time, the contract must always promise positive cash-flows. If the reservation value is zero, then the contract will always be subgame perfect, or sequentially rational. Letting $\hat{\theta}^* = \{\hat{\theta}_t\}_{t=1}^\infty$ and $\theta^* = \{\theta_t\}_{t=1}^\infty$ be the reporting strategy and sequence of types respectively. The contract will be incentive compatible if $V(\theta^*, C) \geq V(\hat{\theta}^*, C) \forall \hat{\theta}$ and feasible provided

$$\tau_t(\theta) \leq p_t(\theta) R_t(\theta), \quad \forall t.$$  \hspace{1cm} (B.1)

Given that $V$ is the present discounted value of future profits to the entrepreneur, we refer to this as the value of equity. Every period we allow a project to be discontinued and define $V$ as the value of equity subject to project continuation, given by

$$V_t = \lambda p^s_t (R^s_t - \tau^s_t) + (1 - \lambda) p^r_t (R^r_t - \tau^r_t) + \lambda \beta \hat{V}^s_{t+1} + (1 - \lambda) \beta \hat{V}^r_{t+1},$$  \hspace{1cm} (B.2)

where $\hat{V}^s_{t+1}$ and $\hat{V}^r_{t+1}$ are the future value of equity in reporting safe and risky types respectively. These values must be positive, otherwise a negative future cash-flow is implied that violates the limited liability constraints. As before, the only incentive compatibility constraint required is that which leads to risky borrower truth-telling, and the relevant limited liability constraint is that of the safe borrower. These are written

$$\tau^s_t \leq R^s_t$$  \hspace{1cm} (B.3)

and

$$\tau^r_t \leq R^s_t + \beta [V^r_{t+1} - V^s_{t+1}] \frac{1}{p^r_t},$$  \hspace{1cm} (B.4)
Rather than specify a probability of finance for the current period, the contract now sets a future probability of project liquidation. By the time that liquidation occurs, it is too late for the borrower to match to a new lender that period. We can denote the liquidation value with $Q$ and the value to the lender with $S$. These values are exogenous to the contract, and whilst we assume that it is too late for the borrower to seek new funding, it turns out that whether it is too late for the lender to re-allocate funds does not effect the results. The period following liquidation, borrowers can secure new funding with probability $\eta_t$. If $e_t$ are the proportion of borrowers currently engaged on projects, then a proportion $[\lambda x_t^s + (1 - \lambda) x_t^r]$ of all borrowers will continue. Therefore, the feasibility constraint for funds becomes

$$\gamma \geq [\lambda x_t^s + (1 - \lambda) x_t^r] e_{t-1} + \eta_t (1 - e_{t-1})$$  \hfill (B.5)$$

The proportion of active borrowers evolves according to

$$e_t = [\lambda x_t^s + (1 - \lambda) x_t^r] e_{t-1} + \eta_t (1 - e_{t-1}).$$ \hfill (B.6)$$

As this is external to the contract, we leave this and revisit once the optimal contract has been solved.

As was shown in Atkeson & Lucas (1992), there is a recursive representation for this type of contracting problem with private information. Indeed, as the project requires only a fixed unit funding, so there is no evolution of wealth, apart from possible stochastic variations in returns that would lead to variations in pay-offs, the case is simpler still. The optimal contract problem the lender seeks to solve can then be written

$$U(V) = \max_{\tau_s, \tau_r, \hat{V}^s, \hat{V}^r} \left\{ \lambda (p^s \tau^s - 1) + (1 - \lambda) (p^r \tau^r - 1) \pm \beta \left[ \lambda \hat{U}^s + (1 - \lambda) \hat{U}^r \right] \right\}$$  \hfill (B.7)$$

s.t.

$$V = \lambda p^s (R^s - \tau^s) + (1 - \lambda) p^r (R^r - \tau^r)$$

$$\tau^s \leq R^s$$ \hfill (B.9)$$

$$\tau^r \leq R^s + \beta \left[ \hat{V}^r - \hat{V}^s \right] \frac{1}{p^r}$$ \hfill (B.10)$$

$$\lambda p^s \tau^s + (1 - \lambda) p^r \tau^r \geq 1$$ \hfill (B.11)$$

Where $U$ is the value subject to continuation and $\hat{W}$ prior to liquidation. The value of equity $V$ is the only relevant state variable, with equation (B.2) a constraint to the
To prevent negative lender cash-flows, the last condition is a lender limited liability constraint. The lender then sets the probability of liquidation by solving

\[
\hat{U}(\hat{V}) = \lambda \max_{V'x^s} \hat{U}(\hat{V}^s) + (1 - \lambda) \max_{V'x^r} \hat{U}(\hat{V}^r) \tag{B.12}
\]

subject to

\[
\hat{U}(\hat{V}(\theta)) = x(\theta)U(V) + (1 - x(\theta))S \tag{B.13}
\]

\[
\hat{V}(\theta) = x(\theta)V' + (1 - x(\theta))Q' \tag{B.14}
\]

\[
x^s, x^r \leq 1 \tag{B.15}
\]

\[
\gamma \geq [\lambda x^s + (1 - \lambda) x^r] e + \eta (1 - e) \tag{B.16}
\]

The optimal contract implies that the incentive compatibility and limited liability constraints bind determining repayment schedule \(\tau(\theta)\), and if we focus on the stationary equilibrium, the conditions determining liquidation probability \(x^r\) and \(x^s\) are

\[
U(V') - S' + (1 + \mu) (V' - Q') = \varphi^s + \varphi^r \tag{B.17}
\]

\[
0 = \psi - (1 - \lambda) \varphi^s - \varphi^r \tag{B.18}
\]

where \(\mu\) is the Lagrange multiplier on the new limited liability constraint. By evaluating these conditions subject to the Kuhn-Tucker conditions as before, concentrating initially on the case in which the lender yields weakly positive returns so \(\mu = 0\), we find that providing \(U(V') + V' > S' + Q'\), so that continuation is more valuable than liquidation, the only important constraint is the feasibility constraint (B.16) as \(\varphi > 0\). This condition will only be violated in the presence of negative expected returns, or with particularly high outside options, in which case \(\varphi^s, \psi > 0\) and all projects will be liquidated with probability 1. If we focus on states of the world where the condition is satisfied, then this will lead to a pooling equilibrium as the proportion of borrowers funded in the first period \(e = \gamma\), then \(x^r = x^s = 1\) and no projects will be liquidated, and \(\tau^s = \tau^r = R^s\).

To choose the probability of funding new projects, \(\eta\), the lender solves

\[
\bar{U}(e) = \max_{e', \eta} \left\{ eU + \beta \bar{U}(e') \right\} \tag{B.19}
\]

subject to the evolution

\[
e' = [\lambda x^s + (1 - \lambda) x^r] e + \eta (1 - e) \tag{B.20}
\]

and condition (B.16). Again, provided \(\beta U > 0\), so projects are profitable, the feasibility constraint will bind. It follows that if \(\gamma\) were to increase, the lender would agree...
contracts with new borrowers to ensure all funds are used. If γ falls, then the lender is indifferent about which projects to liquidate; if safe projects were liquidated, then the lender would lose surplus on safe loans, but could receive offsetting increased returns from risky projects in exchange for lower liquidation probabilities, likewise, if the lender liquidated risky projects, then they would need to subsidise losses incurred by reducing repayments from these projects using safe project surplus.

Whilst pooling will occur for a large range of $p^r$, $p^s$ or $\lambda$, a separating equilibrium emerges when risk or the share of risky projects is too high. Specifically, if $p^r$, $p^s$ or $\lambda$ fall low enough to violate the condition

$$\left(\lambda p^s + (1-\lambda) p^r\right) \geq \frac{1}{R^s}$$

then the lender would receive negative profits. The limited liability constraint will bind and we find that safe projects face a positive probability of liquidation so to increase repayments by the risky borrowers above $R^s$. In such states of the world, the liquidation probability of safe projects is given by

$$x^s = 1 - \frac{1 - (\lambda p^s + (1-\lambda) p^r) R^s}{(1-\lambda) \beta (V' - Q')}.$$  \hspace{1cm} (B.22)

By substituting into the expression for $V$ the returns specified by the optimal contract, treating the outside option $Q = \beta V$, and considering the stationary equilibrium in which $V' = V$, the value of the borrower cash-flows is given by

$$V = \frac{(1-\lambda) p^r (R^r - R^s)}{1 - \beta (x^s + (1-x^s) \beta)}.$$  \hspace{1cm} (B.23)

Using this and (B.22), we can give the relationship between the liquidation, risk, return, and project shares as

$$x^s = 1 - \frac{1 - (\lambda p^s + (1-\lambda) p^r) R^s}{\beta (1-\lambda) (1-x^s) p^r (R^r - R^s) - 1 + (\lambda p^s + (1-\lambda) p^r) R^s}.$$  \hspace{1cm} (B.24)

The solution to the problem in (B.19) implies that the lender will maximise funding new projects if profitable, and from the evolution of $e$, we can find a further threshold; once $\lambda (1 - x^s)$ rises above $\frac{1}{\gamma} - 1$, the number of projects liquidated is greater than the number of un-funded borrowers. In this circumstance, the total proportion of funded projects falls from the maximum of $\gamma$ to

$$e = \frac{1}{1 - \lambda (1 - x^s)}.$$  \hspace{1cm} (B.25)

The total surplus falls until $x^s = 0$ so that all safe projects are liquidated with probability one, and the lender receives all information rents from the risky borrowers.
B.1 Equal Expected Value Projects

To compare to the static contracts we can analyse graphical representation of the lender, borrower and social values and, as before, set $p^* = 1$ so $R^* = p^* R^r = \bar{R}$. When the expected returns are equal, unless risky projects are very risky, the social value is equal to that under the first-best solution as all funds are allocated, and there can be no misallocation between projects. However, in this region, the risk does have distributional effects. Figure 9 shows the distribution of surplus for a range of risky project riskiness.

![Figure 9](image_url)

**Figure 9:** Overall value $W$ and value promised to the borrower $V$ by the optimal long-term contract for range of risk. The upper boundary represents total social value, the blue area represents the surplus to the borrowers, and the green area the surplus to the lender. The green and blue lines represent the social value and borrower value with static contracts.

The plot shows the static contract results from figure 3. Whilst a high risk led to an extreme separating equilibrium and a large drop in efficiency, the long-run contract is socially optimal up to point $B$. At point $p^r(A) = (\frac{1}{R} - \lambda) / (1 - \lambda)$, to avoid negative cash-flows to the lender, safe projects begin to get liquidated to reduce the risky borrower’s information rents. The probability of liquidation increases up to threshold $B$ when $x^s = 1 - \frac{1 - \gamma}{\gamma} \frac{1}{\lambda}$ when the proportion of projects getting credit falls from the
maximum of $\gamma$ as the number of liquidations exceeds the possible new projects. This continues to fall to point $C$ when all safe projects are liquidated. This occurs at

$$p^*(C) = \frac{(1 + \beta) (1 - \lambda R^s)}{\beta (1 - \lambda) (1 - \lambda) R^* + (1 + \beta \lambda) (1 - \lambda) R^s}$$  \hspace{1cm} (B.26)$$

Finally, we can recall from the static case that $p^* = 1 - \frac{\lambda}{1 - \lambda} (1 - \frac{1}{R})$, then it follows that $p^* = A + 1 - \frac{1}{R}$ which implies $p^* > A \forall p^r$. In the static contract, the expected social value of the contract for $p^r < p^*$ is given by $W_S = (1 - \lambda) (R - 1)$, whilst in the long-term contract, for $p^r < p^*(C)$, it is given by $W_L = \frac{1}{1 - \lambda} (R - 1)$ so $W_L > W_S$ and the social welfare is always higher under long-term contracts than one-period contracts for any value of $\lambda$ and $p^r$. It is not possible to say that the static contract is preferred by the lender for all distributions of $p^r$ but we can make the following observations: for $p^r > p^*$, we can measure the marginal value of risk in the two contracts for per-period borrower cash-flows as

$$\frac{\partial V_S}{\partial (-p^r)} = \frac{1}{\lambda} (\gamma - 1 + \lambda) R, \quad \forall p^r \leq p^*$$  \hspace{1cm} (B.27)

$$\frac{\partial V_L}{\partial (-p^r)} = \gamma (1 - \lambda) R, \quad \forall p^r \leq p^* - 1 + \frac{1}{R}$$  \hspace{1cm} (B.28)

noting we have transformed our definition of risk to $-p^r$ for convenience. This implies that the marginal value of risk to the borrowers in the long-term contract is always greater than that in the static contract providing $\gamma < 1$, else they are equal\footnote{This compares static and long-term cases by using the ex ante expected per-period value of the contract rather than the value conditional on being under a contract.}. There will be a threshold between points $B$ and $C$ above which the long-term contract is strictly preferred to the static contract by the lender providing our assumption that $\gamma > 1 - \lambda$ holds. This occurs when the long-term contract specifies

$$x_s < \frac{(R - 1) (\gamma - 1 + \lambda)}{(1 - \lambda) (1 - p^r)}$$  \hspace{1cm} (B.29)

Evaluating equation (B.24) at this threshold will yield a quadratic expression that determines the value of $p^r$ at which this occurs.

\footnote{The value here is a per-period value for comparison.}
B.2 Misallocation of Funds in the Long-term Contract

If we relax our assumption that the projects have equal expected value, then we can repeat the exercise above and compare the value yielded by the contract for a range of $R^r$. The condition (B.21) that determines whether there is a pooling equilibrium is independent of $R^r$, and so finding that pooling occurs under most parametrisation, figure 10 adds the per-period value of the long-term contract to figure 3. As the contract implies an equal funding of both types of project, $W_L$ represented by the red dashed line is linear in $R^r$ with no kink. At point C, all projects have equal expected value, and with all funds used, the value of all contracts is equal. Whilst under the static contract, with $R^r > C$ the more profitable risky projects are all funded, under the long-term contract, pooling implies a misallocation of funds as less profitable safe projects have equal chance of funding. Of course with $A < R^r < C$, the long-term contract implies less misallocation of funds than the static contract which funds all less-profitable risky projects. The long-term contract does prevent funds not being used that occurs for $R^r > B$ under the static contract. Whether the long-term or static contract is socially optimal depends on the distribution of $R^r$.

**Figure 10:** Per-period value of project with asymmetric information implied by static contract $W_S$, long-term contract $W_L$, and the first-best $W^*$ for a range of $R^r$. 
C Introducing Inequality Constraints into Local Approximation Solutions

A description of the method for handling inequality constraints in a perturbation approximation is given here.

Once a pruned perturbation approximation to the model transition and decision functions are computed, the inequality constraints are introduced using the algorithm proposed in Holden (2016a). This approach treats a constraint as an endogenous source of news and ensures that where disturbances would cause bounds to be violated, anticipated news shocks return the bounded variable to the constraint. We outline the computational strategy in this section but refer the reader to Holden (2016b) for a description of the necessary and sufficient conditions for the existence and uniqueness of fundamental solutions at the bound. The method builds on an efficient perfect foresight solver that finds a global solution to the bounds problem using a pruned perturbation approximation to a non-linear model. To capture the effects of uncertainty about the bound, we integrate over future uncertainty up to a finite horizon, following the Extended-Path method proposed in Fair & Taylor (1983) and adapted in Adjemian & Juillard (2013).

C.1 Perfect Foresight Solver

The method is described using an example of a linear model with a single constraint, but it generalises to higher order pruned perturbation and multiple bounds. Consider the basic problem in computing the impulse response function under a perfect-foresight simulation, we can write the model as

\[
\begin{align*}
(\hat{A} + \hat{B} + \hat{C}) \hat{\mu} &= \hat{A} \hat{x}_{t-1} + \hat{B} \hat{x}_t + \hat{C} \mathbb{E}_t \hat{x}_{t+1} + D \varepsilon_t \\
\end{align*}
\]  

(C.1)

where \( \mathbb{E}_{t-1} \varepsilon_t = 0 \) and \( \varepsilon_t = 0 \) for \( t > 1 \). \( \hat{x}_t \) is a vector of model variables, \( \varepsilon_t \) a vector of shocks, and \( \hat{\mu} \) a vector of constants where the \( i \)th element of \( \hat{\mu} \) is the steady state value of the \( i \)th element of \( \hat{x}_t \). \( \hat{x}_0 \) is given as an initial condition and we assume a terminal
condition \( \dot{x} \rightarrow \dot{\mu} \) as \( t \rightarrow 0 \) holds. If we define

\[
x_t \equiv \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}, \quad \mu_t \equiv \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}, \quad A \equiv \begin{bmatrix} \hat{A} & \hat{D} \\ 0 & 0 \end{bmatrix},
\]

\( B \equiv \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}, \quad C \equiv \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}
\]

then the model in (C.1) can be written

\[
(A + B + C) \mu = Ax_{t-1} + Bx_t + Cx_{t+1}
\]

where the expectation operator disappears because agents know \( \varepsilon_t = 0 \) for \( t > 1 \). The problem without a bound is to find a path of \( x_t \in \mathbb{R}^n \) that satisfies (C.6). Providing that the generalisation of the Blanchard-Kahn conditions in (Sims 2002) hold, there is a solution of the form \( x_t = (I - F) \mu + Fx_{t-1} \), where \( F = - (B + CF)^{-1} A \) and if \( \det (A + B + C) \neq 0 \), the eigenvalues of \( F \) are strictly inside the unit circle.

Suppose there is a zero lower bound on variable \( x_{1,t} \), where the subscript indicates it is the first element of vector \( x_t \). We extend this notation and write \( I_{1,\cdot}, A_{1,\cdot}, B_{1,\cdot}, C_{1,\cdot} \) for the first row of \( I, A, B, C \), and then \( I_{-1,\cdot}, A_{-1,\cdot}, B_{-1,\cdot}, C_{-1,\cdot} \) for the remaining rows. Using \( \cdot, 1 \) a subscript denotes the first column of the matrix. Using this notation, we can write the bounded equation

\[
x_{1,t} = \max \left\{ 0, I_{1,\cdot} \mu + A_{1,\cdot} (x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot}) (x_t - \mu) + C_{1,\cdot} (x_{t+1} - \mu) \right\}
\]

The problem we wish to solve is to find a path for \( x_t \) to satisfy equations (C.3) and (C.4) where \( x \rightarrow \mu \) as \( t \rightarrow 0 \). As the model returns to steady state asymptotically, there is some horizon \( T \) within which time the constraint will no longer be violated. We will use news shocks to impose the bound and so can write equation (C.4) as

\[
x_{1,t} = I_{1,\cdot} \mu + A_{1,\cdot} (x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot}) (x_t - \mu) + C_{1,\cdot} (x_{t+1} - \mu) + y_{1,t-1}
\]

where \( y_{t,0} \) is a news shock known at period 0, that hits at period \( t \). It follows that for periods \( t \leq T \), \( y_{1,t-1} = y_{t,0} \) whilst for \( t > T \), \( y_{1,t-1} = 0 \). The problem the algorithm must solve then is to find path for \( x_t \in \mathbb{R}^n \) and \( y_t \in \mathbb{R}^T \) that satisfies

\[
(A + B + C) \mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{1,\cdot}y_{1,t-1}
\]

\(^4\)This rules out switching to an alternative steady state.
which modifies the first row of (C.3) to that in (C.5), and conditions on the news shocks

\[ y_{i,t} = y_{i+1,t-1}, \quad \forall i \in \{1, ..., T - 1\} \quad (C.7) \]

\[ y_{T,t} = 0. \quad (C.8) \]

given initial conditions \( x_0 \). The model is linear and so we find that the impulse response to two shocks is equal to the sum of the impulse response to each individual shock. We consider first the path of \( x_{1,t} \) given a vector of news shocks \( y_0 \in \mathbb{R}^T \). Let \( m_k \in \mathbb{R}^T \) be a column vector with the impulse response of \( x_{1,t} \) to a news shock of size 1 at period \( k \) with \( x_0 = \mu \), and let

\[ M \equiv \begin{bmatrix} m_1 & m_2 & \cdots & m_T \end{bmatrix} \quad (C.9) \]

horizontally stack these relative impulse response functions. It follows that the path of \( x_{1,t} \) given \( x_0 = \mu \) and an arbitrary vector of new shocks \( y_0 \) is given by \( My_0 \). Let \( q \in \mathbb{R}^T \) be the path of \( x_{1,t} \) up to period \( T \) that satisfies equation (C.3), that is the model without the constraint, given any \( x_0 \). We can then give the path of \( x_{1,t} \) for any \( x_0 \) and \( y_0 \in \mathbb{R}^T \) that satisfies equation (C.6) as

\[ q + My_0 \quad (C.10) \]

The problem we wish to solve is to find a vector \( y_0 \) and path for \( x_t \) for a given \( x_0 \) that satisfies equation (C.6) and the zero lower bound on \( x_{1,t} \). The path for \( x_{1,t} \) will be given by equation (C.10). The news shocks are only used to impose the bound, so when \( x_{1,t} > 0, y_t = 0 \); this implies firstly that the solution must satisfy

\[ y_0 \circ (q + My_0) = 0 \quad (C.11) \]

and secondly that the news shocks can only act to push the variable up to the bound, that is, \( y_0 \geq 0 \). Finally, the solution must impose the bound, so \( q + My_0 \geq 0 \).

The news shock problem is then characterised as a linear complementarity problem \( LCP(q, M) \) (see Cottle 2009): for a given \( q \) and \( M \), the \( LCP(q, M) \) finds \( y \in \mathbb{R}^T \) to satisfy

\[ q + My \geq 0 \]

\[ y \geq 0 \]

\[ y \circ (q + My) = 0 \]

(C.12)
The structure of matrix $M$ will contain information on whether a unique solution exists; the necessary and sufficient conditions for the existence and stability of a solution are outlined in detail in Holden (2016b).

To extend the bounds problem to $n$ constraints, the vectors $q \in \mathbb{R}^{nT}$ and $y \in \mathbb{R}^{nT}$ stack the impulse responses ignoring the bound for each bounded variable and the impulse response to the news shocks respectively. Matrix $M \in \mathbb{R}^{nT \times nT}$ is a block matrix where block $M_{i,j}$ is the response of variable $x_i$ to the new shock on variable $x_j$. The problem as outlined characterises the inequality conditions as having a zero lower bound that is not binding in steady state, but it is trivial to convert any constraint into this form. Furthermore, the algorithm extends to higher order pruned perturbation approximations when the news shocks are of the form $y^p$, since pruned perturbation approximations of order $p$ are linear in shocks to the power of $p$.

C.2 Integrating Over Uncertainty

Given that the news shocks satisfy $y_t \geq 0$ for all $t$, it follows that they will not be mean zero, and the expected value of $y_t$ will depend on the state. To avoid bias in the formation of expectations, it is necessary to integrate over future uncertainty to capture the effect of positive expected news shocks. As shown in Holden (2016a), it is possible to derive a closed-form formula for the covariance of the expected future path of the bounded variables in the absence of the bound. Using this, we can take a Gaussian approximation to the future distribution of the bounded variables in the absence of the bound, and then integrate over this distribution using Gaussian cubature techniques up to a chosen finite horizon $S$. This implies the perfect foresight problem only needs solving a number of times that is polynomial in the periods of uncertainty, independent of the number of shocks, in contrast to using the stochastic extended-path method of Adjemian & Juillard (2013) under which the order of integration increases exponentially in the number of periods and shocks.

Rather than assuming the news shock variance goes to zero beyond the horizon $S$, a cosine windowing function is used to scale the shock variance. Specifically, if the covariance matrix is $\Sigma$, the covariance matrix used when considering uncertainty at
horizon $k$ is given by

$$\hat{\Sigma}_k = \frac{1}{2} \left[ 1 + \cos \left( \pi \frac{\min\{k - 1, S\}}{S} \right) \right] \Sigma \quad \text{(C.13)}$$

Letting $\Omega$ denote the covariance matrix of the expected future path of the bounded variables derived using $\hat{\Sigma}_k$, and let $w_{t,t+i}$ be the value the bounded variable would take at time $t+i$ if the bound no longer applied after time $t$, we assume that $\left[ w_{t,t+1} \cdots w_{t,t+S} \right]'$ is normally distributed, which it will be at first order, and is a close approximate at higher orders. We take a Schur decomposition of $\Omega$ to given $\Omega = U D U'$ where $D \geq 0$ is a diagonal matrix. By setting any very low values of $D$ to zero, we can reduce the cost of integration which only scales in $\hat{S} < S$ where $\hat{S}$ is the number of remaining non-zero elements of $D$. Using this step and the normal assumption, we arrive at an approximation of the distribution of $\left[ w_{t,t+1} \cdots w_{t,t+S} \right]'$ of $\mathbb{E}_t \left[ w_{t,t+1} \cdots w_{t,t+S} \right]' + \Lambda \zeta$ where $\Lambda \equiv U_1 \sqrt{D_{11}}$, with $D_{11} \in \mathbb{R}^{\hat{S}}$ the matrix block in $D$ that includes the $\hat{S}$ non-zero elements and $U_1$ the corresponding elements of $U$, and where $\zeta \sim \mathcal{N}(0, I_{\hat{S}})$. This simplifies the integration problem to one of integrating over $\hat{S}$ standard normals.

Holden (2016a) suggests three numerical integration methods of which two are uses in this paper. The first is a degree 3 monomial rule with $2\hat{S} + 1$ nodes and equal positive weights to integrate over future uncertainty up to a horizon of $S$ periods. The positive weights provide robustness, so that although computational efficient, gives delivers reasonable accuracy\(^5\). As the integral is evaluated far from the steady state, it is likely to introduce some upward bias. The second is a Gaussian cubature rule with $O(\hat{S} K)$ points and maximum monomial degree of $7 = 2K + 1$. Two techniques are employed to increase computational speed; firstly, an adaptive cubature degree is used, implying a lower degree is used when far from the bound. Secondly, as discussed above, the eigenvalues of the covariance matrix of the distribution from which we integrate smaller than 1% of the largest eigenvalue are set to zero. $\hat{S} \leq S$ is the number of remaining non-zero eigenvalues. This method uses negative weights that are likely to remove the bias introduced in the other method. We compute average impulse response functions around the ergodic mean and whilst a standard perturbation without bounds requires Monte-Carlo simulation to compute the average, the pruning technique allows analytical moments to be calculated. This implies a Monte-Carlo simulation is not required in the absence of

\(^5\)Tests comparing the numerical accuracy of possible methods are discussed in Holden (2016a). Using a degree 3 monomial rule is shown to achieve competitive accuracy but with fast computational speeds.
the bounds, but with bounds, although we integrate out the effects of uncertainty of future news shocks, we miss current uncertainty. To remedy this, we use a Monte Carlo simulation to capture these effects.

D Static Comparatives

Plots of steady-state equilibrium values for a range of parametrisations. Figures 11 – 13 show the investment wedge, efficiency wedge and a number of key variables for a range of the steady-state value of $p^r$ with $\omega^r p^r = 1$ held constant. Figures 14 – 16 show the same but for a range of the steady-state value of $\omega^r p^r$ with $\gamma = 0.952$ held constant.

Figure 11: Steady-state investment wedge for a range of $p^r$ with $\omega^r p^r = 1$ held constant.
Figure 12: Steady-state efficiency wedge for a range of $p^r$ with $\omega^r p^r = 1$ held constant.
Figure 13: Steady-state variables for a range of $p^r$ with $\omega^r p^r = 1$ held constant.
Figure 14: Steady-state investment wedge for a range of risky project value, $\omega^r p^r$, with $\gamma = 0.952$ held constant.

Figure 15: Steady-state efficiency wedge for a range of risky project value, $\omega^r p^r$, with $\gamma = 0.952$ held constant.
Figure 16: Steady-state variables for a range of risky project value, $\omega^r p^r$, with $\gamma = 0.952$ held constant.
Figure 17: Impulse response functions to a very large (+10%) negative transitory shock to total factor productivity $z_t$. 

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Figure 18: Impulse response functions to a transitory risk shock caused by a reduction in the probability of risky project success $p^r_t$ of approximately 1 percentage point.
Figure 19: Impulse response functions to a transitory risk shock caused by a reduction in the probability of risky project success $p_r$ of approximately 2 percentage points.
Figure 20: Impulse response functions to a large transitory risk shock caused by a reduction in the probability of risky project success $p_r^*$ of approximately 3 percentage points.
Figure 21: Impulse response functions to a transitory risk shock caused by a reduction in the probability of risky project success \( p_r \) of approximately 2 percentage points. Alternative parametrisation with \( \lambda = 0.85 \).
Figure 22: Impulse response functions to a transitory risk shock caused by a reduction in the probability of risky project success $p_r^t$ of approximately 4 percentage points. Alternative parametrisation with $\lambda = 0.85$. 

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Figure 23: Impulse response functions to a transitory 1.5% negative shock to the productivity of risky projects, $\omega_r^r$. 
Figure 24: Impulse response functions to a transitory 5% negative shock to the productivity of risky projects, $\omega_r^\tau$. 
Figure 25: Impulse response functions to a transitory 1.5% positive shock to the productivity of risky projects, $\omega^r_t$. 
Figure 26: Impulse response functions to a transitory 3% positive shock to the productivity of risky projects, $\omega^r_t$. 
Figure 27: Impulse response functions to a transitory 1.5% positive shock to the productivity of risky projects, $\omega_r^t$. Alternative parametrisation with \( \lambda = 0.85 \).
Figure 28: Impulse response functions to a transitory 4% positive shock to the productivity of risky projects, $\omega^r_t$. Alternative parametrisation with $\lambda = 0.85.$