Competition over Redistribution*

Galina Zudenkova†

Department of Economics, University of Mannheim

September 2014

Abstract

This paper analyzes a game of redistributive politics in which targeted spending arises as a result of the competition over resources among jurisdictions. I build a political agency model in which a vote-maximizing politician is subject to the oversight of distinct constituencies. Voters’ demand for local public goods in these constituencies is driven down by the competition among them. I characterize the unique equilibrium of this game. In the equilibrium, in order to make his constituency an attractive choice for federal spending, each voter demands somewhat less spending than what he actually receives. As a result, the voters tend to be satisfied with the redistributive policies of the incumbent. The incumbent is then quite likely to win more than half of the votes.

JEL classification: D72.

Keywords: Redistributive politics; Targeted redistribution; Political agency.

---

*The author is grateful to Enriqueta Aragonès, Pierre Boyer, Micael Castanheira, Torun Dewan, Daniel Diermeier, Hülya Eraslan, Jean Guillaume Forand, Eckhard Janeba, Michel Le Breton, Gilat Levy, Volker Nocke, Dan Pemstein, Carlo Prato, Jan-Peter Siedlarek, Michèle Tertilt, Thomas Tröger, seminar and conference participants at several institutions, and especially Jan Zapal for useful comments and suggestions. The grant from Karin-Islinger-Stiftung foundation is gratefully acknowledged. A previous draft of this paper circulated under the title "Persistence of Large Majorities in a Parliamentary Democracy." The usual disclaimer applies.

†Department of Economics, University of Mannheim, D-68131 Mannheim, Germany. E-mail address: galina.zudenkova@gmail.com.
1. Introduction

Redistribution is one of the fundamental tasks which voters delegate to politicians in modern democracies. It deals with allocation of resources among national districts or social groups and normally takes form of local public goods and district- or group-specific public projects. Though motivated by efficiency and equity considerations, redistribution is often used by politicians in order to enhance their election prospects. The growing empirical literature (e.g., Milesi-Ferretti et al. 2002, Johansson 2003, Castells and Solé-Ollé 2005, Aidt and Shvets 2012) provides some evidence on targeted spending and electoral incentives in allocation of intergovernmental grants. These findings are in line with the theoretical literature on redistributive politics which predicts targeted redistribution as the equilibrium outcome of an electoral game. The formal models include Cox and McCubbins (1986), Lindbeck and Weibull (1987), Dixit and Londregan (1996), Snyder (1989), Myerson (1993), Lizzeri and Persico (2001), Roberson (2008), Eguia and Nicolò (2014), and others. These studies analyze redistributive (or zero-sum) games in which political candidates compete over votes by promising targeted spending to national districts or social groups. However, the competition over transfers among constituencies or groups has not been emphasized in the redistributive politics literature, to the best of my knowledge. The present paper aims to contribute to this question and studies a redistributive game in which constituencies compete over resources by setting expectations for redistribution.

I consider a society which delegates redistribution of resources to a vote-maximizing politician. This is a purely redistributive task of dividing spending among national districts. The spending is used for district-specific public policies or local public goods, and I assume stochastic production technology here. A representative voter is each district cares about implementation of a policy which benefits his home district. Only the final policy outcome is observed by the voter, not its corresponding spending share. The voter realizes that the incumbent politician wants to win votes and therefore can be held accountable for the policy outcome at the moment of election. I assume that each voter adopts a cutoff rule and reelects the incumbent only when the corresponding policy outcome in his district exceeds a critical threshold.

My results suggest that in case the voters are quite lenient, the incumbent divides spending among all constituencies. Moreover, more demanding voters receive relatively higher spending shares in their corresponding constituencies. This implies then that the voters tend to be as demanding as possible conditional of being included in the spending allocation. However, if the voters get too demanding then the incumbent prefers to drop the most demanding districts
and rather concentrates spending in more lenient districts. This leads to the competition over being included in the spending allocation which drives reelection thresholds in all districts down to the level at which the incumbent is indifferent between allocating spending among all constituencies and excluding one randomly picked constituency. I show that this is the demand level all voters use in the unique equilibrium. To guarantee a place in the spending bill, each voter demands somewhat less spending than what he actually receives. As a result, the final policy outcomes are quite likely to pass tolerant reelection thresholds. The incumbent is then quite likely to win more than half of the votes.

These results rest on the assumption that political process is modeled as political agency. In this approach, elections are used as a disciplining device. Politicians want to be reelected for another term, and are held accountable for their past performance at the time of election. They therefore have incentives to satisfy the voters’ wishes. The literature on political agency started with the seminal work of Barro (1973) to be followed by Ferejohn (1986), Persson et al. (1997), Austen-Smith and Banks (1989), Banks and Sundaram (1993, 1996), and others. The present paper is mostly related to the political agency model with multiple principals as in Ferejohn (1986, the case of a nonhomogeneous electorate). Ashworth (2012) provided an excellent review of recent strands in this literature. It is worth emphasizing that political agency approach has also received considerable empirical support (see Peltzman 1992, Besley and Case 1995a, 1995b, 2003, Besley 2006) and so may be appropriate for modeling interactions between politicians and voters.

My results suggest that in case the voters are quite demanding, the incumbent prefers to exclude some districts from the allocation of spending. Then the competition over being included in the spending bill drives the voters’ demands down. Similar Bertrand competition among voters arises also in Ferejohn (1986). In this respect, the present paper is also related to the vote buying literature (in particular, to Dal Bo 2007, Felgenhauer and Grüner 2008, Dahm and Glazer 2013). In these studies, payment to each voter is assumed to be conditional on that voter being pivotal, which leads to competition among voters and so to low cost of acceptance.

The paper is also related to the broad literature on coalition and government forma-

\footnote{In contrast with Ferejohn (1986), the politician here is assumed to maximize the expected number of votes rather than the probability of winning the majority. Moreover, I consider a symmetric information case in which neither politicians nor voters observe the noise when they take actions. Ferejohn (1986) studied an asymmetric information environment in which the politician observes the realization of noise before she makes her decision. Finally, I study a purely redistributive task while Ferejohn (1986) considered a redistributive task which also requires a costly action from the incumbent.}
tion (Austen-Smith and Banks 1988, Baron, 1991, 1993, Diermeier and Merlo 2000, Bassi 2013, and others). The standard prediction of that literature is emergence of minimum winning coalitions. In my model, the incumbent politician maximizes her expected votes rather than the probability of winning. Thus, majorities (rather than minimum winning coalitions) emerge in equilibrium. This feature somewhat relates the present study to the vote buying models with supermajorities as an equilibrium outcome (Groseclose and Snyder 1996, 2000, Banks 2000, Boyer and Konrad 2014). Supermajority prediction also arises in Zapal (2014) who analyzes the problem of favor allocation in a committee. In these studies, a vote buyer or a committee chair "purchases" a supermajority of votes by allocating spending or favors to a supermajority. In my model, the incumbent either allocates spending uniformly among all constituencies or excludes one randomly picked district from the spending bill. Still, due to the stochastic element, this cannot guarantee her winning all votes with certainty but is likely to secure more than half of the votes.

My contribution to the aforementioned strands of literature is twofold. Firstly, the present paper contributes to the literature on redistributive politics and uncovers a novel mechanism which generates targeted spending. In the existing literature on zero-sum electoral games, targeted redistribution arises as a result of the competition over votes among the politicians (Cox and McCubbins 1986, Lindbeck and Weibull 1987, Dixit and Londregan 1996, Roberson 2008, and others). These models normally generate the winning probabilities of zero, one or one-half in equilibrium. In the present study, targeted redistribution arises as a result of the competition over spending among the constituencies which set demands for redistribution. My model predicts the winning probabilities strictly larger than one-half. Secondly, the paper contributes to the political agency and vote buying studies which predict Bertrand competition among the voters (Ferejohn 1986, Dal Bo 2007, Felgenhauer and Grüner 2008, Dahm and Glazer 2013). In these models, the competition among the voters drives their demands all the way down as the politician just needs to secure a minimum winning coalition and each voter wants to be a part of it. While the present paper also predicts the competition among the voters, their equilibrium demands do not drop below the level at which the incumbent is indifferent between allocating spending among all constituencies and excluding one randomly picked constituency from the spending bill. The reason is that this study follows the redistributive game literature (see Cox and McCubbins 1986, Lindbeck and Weibull 1987, Dixit and Londregan 1996, Roberson 2008, and others) and assumes a vote-maximizing (rather than winning-probability-maximizing) politician. This results in higher levels of accountability in

\[^{2}\text{In certain contexts, these two objectives (i.e., expected votes and probability of winning maximization) might be equivalent as they yield identical best response functions or identical Nash equilibrium strategies (see}\]
equilibrium.

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 proceeds with the formal analysis. Finally, Section 4 concludes the paper.

2. Model

Consider a homogenous society which consists of \( n \) national districts, with one representative voter in each of those. The society has to elect a politician in an election. There are two candidates running in the election – the incumbent and a challenger. The challenger is identical to the incumbent in all respects. The participation constraints of the candidates are always satisfied.

While in office, the politician has to perform a purely distributive task of allocating spending among national districts. This is distributive spending, i.e., it benefits only those constituencies in which it occurs. Think of this as local public goods or district-specific public projects. The size of total spending is normalized to 1. Denote by \( s_i \in [0, 1] \) a share of spending allocated to district \( i = 1, \ldots, n, \sum_{i=1}^{n} s_i = 1 \). Then a local public good in district \( i \), denoted by \( p_i \), is produced from \( s_i \) with stochastic technology:

\[
p_i = s_i + \varepsilon_i,
\]

where \( \varepsilon_i \) is an independent, unobservable noise. The noise term \( \varepsilon_i \) is meant to represent all other factors apart from immediate spending which affect implementation of a district-specific public project. I assume that \( \varepsilon_i \) follows a single-peaked symmetric bell-shaped continuous distribution with zero mean. In what follows, \( F \) denotes \( \varepsilon_i \)'s cumulative distribution function and \( f \) the corresponding density function.

The incumbent chooses spending shares \( s_i \) to maximize her expected votes in the coming election. The voters in turn care about their home interests such as local public goods or district-specific public projects. Consider now a representative voter in district \( i \), who cares about implementation of district-specific public policy \( p_i \). Only the policy outcome \( p_i \) is observed by the voter, not its composition between the corresponding spending and the noise. The voter realizes that the incumbent wants to secure votes and so can be held accountable for \( p_i \)'s implementation at the moment of election. The voter is assumed to adopt a cutoff strategy: he votes for the incumbent if policy outcome \( p_i \) exceeds a certain threshold denoted by \( \gamma_i \in \mathbb{R} \). Under this cutoff strategy, the probability of the incumbent

Patty 2002, 2007). In the present framework, however, these two objective functions might not be equivalent. I focus on the expected votes maximization also for the sake of tractability.
winning constituency \( i \), \( Pr_i \), is given by

\[
Pr_i \equiv P ( \{ s_i + \varepsilon_i \geq \gamma_i \} ) = F ( s_i - \gamma_i ).
\]

Then the number of votes the incumbent gets in the coming election follows the Poisson
binomial distribution with success probabilities \( Pr_i, \; i = 1, \ldots, n \). The expected number of
votes is therefore equal to

\[
\sum_{i=1}^{n} Pr_i.
\]

This is a sequential political agency game between the incumbent politician and \( n \) re-
presentative voters (one in each national district). The timing of events is as follows. First,
the representative voters in all constituencies simultaneously and non-cooperatively decide on
the cutoff rules to be used in the coming election, i.e., they choose thresholds \( \gamma_i \). Then, the
politician allocates spending among the districts, i.e., specifies a share of spending for each
constituency \( s_i \). Finally, nature chooses noises \( \varepsilon_i \), and the policy outcomes \( p_i \) are observed.
The election takes place and the representative voter in each constituency applies the selected
cutoff rule to reward or punish the politician.

I search for a subgame perfect equilibrium by analyzing the game backwards. First, given
thresholds \( \gamma_i \), I analyze the politician’s decision regarding the allocation of spending. Second,
I solve for the voters’ decision regarding thresholds \( \gamma_i \).

3. Analysis

I start the analysis with the politician’s choice of spending shares \( s_i, \; i = 1, \ldots, n \). Given
thresholds \( \gamma_i \) applied by the representative voters, the politician chooses spending shares \( s_i \)
to maximize her expected votes in the coming election. In case there are several spending
allocations which maximize her expected votes, the politician randomly chooses one of those.
Consider the politician’s maximization problem:

\[
\max_{s_1, \ldots, s_n} \sum_{i=1}^{n} F ( s_i - \gamma_i ) \\
\text{s.t.} \sum_{i=1}^{n} s_i = 1, \\
\quad s_i \geq 0.
\]

She chooses spending shares \( s_i \) before observing realizations of the noise, and takes the voters’
expectations as given. The politician’s maximization problem is analyzed in Appendix A.
show that given thresholds $\gamma_i$ adopted by the representative voters, two cases can occur in equilibrium.

1. First, all constituencies receive positive spending, i.e., $s_i^\gamma > 0$, $i = 1, \ldots, n$, where $s_i^\gamma$ denotes a solution of the politician’s maximization problem. I find that in this case

$$s_i^\gamma = \gamma_i + \frac{1 - \sum_{j=1}^{n} \gamma_j}{n}, \quad i = 1, \ldots, n,$$

and characterize the conditions on thresholds $\gamma_i$ under which this equilibrium exists (see Appendix A for the conditions).

2. Second, some constituencies receive no spending, i.e., $s_i^\gamma = 0$ for some $i$. I introduce the following notation to characterize spending shares in this case. Denote by $\Omega_k$ the set of all subsets of $k$ integers that can be selected from $\{1, \ldots, n\}$. $A$ will denote an element of $\Omega_k$ and $A^c$ the complement of $A$, i.e., $A^c = \{1, \ldots, n\} \setminus A$. I find that for $k = 1, \ldots, n - 1$

$$s_i^\gamma = \begin{cases} 
\gamma_i + \frac{1 - \sum_{j \in A} \gamma_j}{k} & \text{for } i \in A \subset \Omega_k, \\
0 & \text{for } i \in A^c,
\end{cases}$$

and present the conditions on thresholds $\gamma_i$ under which an equilibrium of this type exists.

The results are formalized in Lemma 2 in Appendix A. According to the results, given that a district receives strictly positive spending, a share of spending allocated to this district is determined by its "relative demand". Indeed, the more demanding the representative voter in constituency $i$ relative to other voters (i.e., the higher $\gamma_i$ relative $\gamma_j$, $i \neq j$) the larger share of spending is sent to this constituency. Accordingly, relatively lenient districts receive less spending. In turn, the districts with equal $\gamma$’s receive equal amount of spending. I must stress however that this applies only to the constituencies which receive strictly positive spending, i.e., to those $\gamma_i$ which satisfy the conditions for $s_i^\gamma > 0$.

I turn now to the representative voters’ decision regarding thresholds $\gamma_i$, $i = 1, \ldots, n$. The representative voters simultaneously and non-cooperatively decide on the cutoff rules to be used in the coming election, i.e., they choose thresholds $\gamma$’s. The voters care about their home interests such as local public goods or district-specific public projects and so maximize expected policy outcome $E_{p_i} = Es_i^\gamma$ in their corresponding districts. They want therefore the politician to allocate as much resources as possible to their home districts. I proceed as follows. First, I characterize a symmetric equilibrium. Then I show that it is unique.
3.1. Symmetric Equilibrium

I characterize a symmetric equilibrium in which all voters apply the same threshold which I denote by $\gamma \in \mathbb{R}$. I assume first that such equilibrium exists and find conditions on $\gamma$ under which different types of equilibria in the politician’s maximization problem arise. Then I find $\gamma$ such that $\gamma_i = \gamma$, $i = 1, ..., n$, is a subgame perfect equilibrium of the entire game.

Suppose that $\gamma_i = \gamma$ is an equilibrium. I use the results of Lemma 2 in Appendix A to characterize the conditions on $\gamma$ under which different types of equilibria of the politician’s maximization problem arise. This lemma suggests that given thresholds $\gamma_i = \gamma$, an equilibrium with all constituencies receiving positive spending, $s_i^\gamma = \frac{1}{n}$, $i = 1, ..., n$, arises when $\gamma$ satisfies

$$\gamma \leq \frac{1}{n},$$

$$n F \left( \frac{1}{n} - \gamma \right) \geq k F \left( \frac{1}{k} - \gamma \right) + (n - k) F (-\gamma) \quad \text{for } k = 1, ..., n - 1.$$  \hspace{1cm} (3.1) (3.2)

Note that in this equilibrium, spending is divided equally among all constituencies. Condition (3.1) guarantees that the second-order condition for local maximum of the politician’s maximization problem holds. Condition (3.2) ensures that an allocation with all districts receiving positive spending is a global maximum. This happens when the politician’s objective function $\sum_{i=1}^{n} F (s_i^\gamma - \gamma)$ evaluated at this allocation is higher than at any other local maximum. According to Lemma 2 in Appendix A, in other local maxima of the politician’s maximization problem, only $k = 1, ..., n - 1$ out of $n$ constituencies receive positive spending while $n - k$ constituencies get no resources at all. Evaluating the politician’s objective function in those allocations yields the right-hand side of condition (3.2).

I turn next to an equilibrium in which given thresholds $\gamma_i = \gamma$, only $k = 1, ..., n - 1$ constituencies receive positive spending: $s_i^\gamma = \frac{1}{k}$ for $i = 1, ..., k$, $s_i^\gamma = 0$ for $i = k + 1, ..., n$. Note that spending is divided equally among $k$ constituencies in this type of equilibrium. There are $\frac{n!}{k!(n-k)!}$ allocations which divide spending uniformly among $k$ out of $n$ districts. The politician expects to win the same number of votes under any of those allocations and so randomly chooses one of them. Each district is included in $\frac{(n-1)!}{(k-1)!(n-k)!}$ allocations out of those and so expects to receive spending with probability $\frac{k}{n}$. Then the expected payoff of the representative voters is equal to $Es_i^\gamma = \frac{k}{n} \cdot \frac{1}{k} = \frac{1}{n}$. Lemma 2 in Appendix A suggests that an
equilibrium of this type arises when $\gamma$ satisfies

$$\begin{align*}
\frac{1}{2k} \leq \gamma \leq & \frac{1}{k} \quad \text{for } k = 2, \ldots, n - 1 \quad \left( \frac{1}{2k} \leq \gamma \text{ for } k = 1 \right), \\
& \text{for } l = 1, \ldots, n, l \neq k.
\end{align*}$$

Condition (3.3) combines the Kuhn-Tucker condition for zero spending shares, $\gamma \geq \frac{1}{2k}$, and the second-order condition for strictly positive spending shares, $\gamma \leq \frac{1}{k}$ (except for the case in which $k = 1$). Condition (3.4) guarantees that an allocation with only $k$ constituencies receiving positive spending is a global maximum. This happens if the politician’s objective function $\sum_{i=1}^{n} F(s_i^3 - \gamma)$ evaluated at this allocation is higher than at the allocations with $l \neq k$ districts receiving positive spending, $l = 1, \ldots, n$.

Conditions (3.1)-(3.4) imply that for $\gamma \in \left(-\infty, \frac{1}{2(n-1)}\right]$, an equilibrium with all $n$ districts receiving positive spending arises in the politician’s maximization problem. For $\gamma \in \left[\frac{1}{2k}, \frac{1}{k+1}\right]$ with $k = 2, \ldots, n-1$, an equilibrium with $k+1$ constituencies as well as an equilibrium with $k$ constituencies receiving positive spending might occur. The former will arise for $\gamma \in \left[\frac{1}{2k}, \frac{1}{k+1}\right]$ such that

$$\begin{align*}
(k + 1) F \left( \frac{1}{k+1} - \gamma \right) + (n - k - 1) F (-\gamma) \geq k F \left( \frac{1}{k} - \gamma \right) + (n - k) F (-\gamma)
\end{align*}$$

holds, while the latter for $\gamma \in \left[\frac{1}{2k}, \frac{1}{k+1}\right]$ such that the opposite inequality holds. Finally, for $\gamma \in \left[\frac{1}{3}, \frac{1}{2}\right]$, there arises an equilibrium with two constituencies receiving positive spending while for $\gamma \in \left[\frac{1}{2}, +\infty\right)$, there arises an equilibrium with just one constituency receiving the entire pie. The following lemma summarizes the findings. (The proof is given in the Appendix.)

**Lemma 1.** Suppose that the representative voters adopt equal thresholds $\gamma \in \mathbb{R}$ in all $n$ constituencies. Denote by $\overline{\gamma}(k) \in \left(\frac{1}{2k}, \frac{1}{k+1}\right)$ a threshold under which the politician is indifferent between allocating spending uniformly among $k$ and $k+1$ districts, $k = 2, \ldots, n-1$. This threshold $\overline{\gamma}(k)$ is defined implicitly by

$$\begin{align*}
(k + 1) F \left( \frac{1}{k+1} - \overline{\gamma}(k) \right) - k F \left( \frac{1}{k} - \overline{\gamma}(k) \right) - \overline{\gamma}(k) F (-\overline{\gamma}(k)) = 0
\end{align*}$$

and strictly decreases with $k$. Then the politician allocates spending

- equally among $n$ constituencies when $\gamma \in (-\infty, \overline{\gamma}(n-1)]$;
- equally among $k = 3, \ldots, n-1$ randomly chosen constituencies when $\gamma \in [\overline{\gamma}(k), \overline{\gamma}(k-1)]$;
Figure 3.1: The case of demanding voters, i.e., high thresholds $\gamma$.

- equally among 2 randomly chosen constituencies when $\gamma \in \left[ \gamma(2), \frac{1}{2} \right]$;
- to one randomly chosen constituency when $\gamma \in \left[ \frac{1}{2}, +\infty \right)$.

According to Lemma 1, the more lenient the representative voters (the lower the thresholds $\gamma$), the more constituencies receive the government spending. This is due to the shape of the noise’s cumulative distribution function $F(\cdot)$ which determines the probability of the politician winning the constituencies’ votes, $F(s_i^\gamma - \gamma)$. Recall that $F(\cdot)$ is convex for negative arguments while concave for positive arguments. In case of very demanding voters (i.e., high $\gamma$), allocating spending equally among many districts would lead to a tiny increase in the politician’s reelection chances in each of those districts. However, sending all spending just to one district would boost her reelection chances there considerably and compensate her in terms of expected number of votes. (To see this, recall the convexity of $F(\cdot)$ for negative arguments and Jensen’s inequality. This case is depicted in Figure 3.1.) However, in case of lenient voters (i.e., low $\gamma$), dividing spending among a few constituencies would somewhat increase the politician’s reelection chances there but those increments would largely correspond to a concave part of $F(\cdot)$ where the marginal probability of being reelected is decreasing in spending. In this case, dividing spending among more districts leads to a higher expected number of votes and so is preferred by the politician. (Recall the concavity of $F(\cdot)$
Figure 3.2: The case of lenient voters, i.e., low thresholds $\gamma$. for positive arguments and Jensen’s inequality. This case in depicted in Figure 3.2.) This suggests therefore that in case of very lenient voters, spending will be allocated among all $n$ constituencies. The more demanding the voters, the fewer constituencies will receive positive spending. Finally, in case of quite demanding voters, entire spending will be sent just to one district.

The next step is to find a symmetric equilibrium of the entire game, i.e., to characterize thresholds $\gamma$ the representative voters will adopt in their corresponding districts. Suppose $\gamma_i = \gamma$ is an equilibrium. I check next whether any voter has incentives to deviate and to apply a different threshold in his constituency. When all voters use the same threshold $\gamma$, their expected payoff is equal to $\frac{1}{n}$. Consider next the incentives of voter $i$. In what follows, it is useful to distinguish between several cases.

The first case corresponds to the situation in which $\gamma$ is such that all constituencies receive positive spending in the equilibrium of the politician’s maximization problem, i.e., $\gamma \in (-\infty, \gamma(n-1))$. Suppose district $i$’s voter slightly increases his threshold, say to $\gamma + \varepsilon$, $\varepsilon > 0$. His payoff in this case becomes $\frac{1}{n} + \frac{n-1}{n} \varepsilon$ which exceeds $\frac{1}{n}$. It follows therefore that voter $i$ has incentives to deviate and therefore $\gamma_i = \gamma \in (-\infty, \gamma(n-1))$ is not an equilibrium.

---

3One can check that there always exists positive $\varepsilon$ such that the politician will still find it optimal to allocate spending among all $n$ constituencies when $n-1$ voters adopt threshold $\gamma$ while one voter adopts...
The next case corresponds to the situation in which $\gamma$ is such that some districts receive no spending in the equilibrium of the politician’s maximization problem, i.e., $\gamma \in (\overline{\gamma} (n - 1), +\infty)$. In this case, the politician allocates spending among several randomly picked constituencies. Suppose district $i$’s voter slightly lessens his threshold, say to $\gamma - \varepsilon$, $\varepsilon > 0$. In Appendix C, I show that in this case, for $\varepsilon$ small enough, constituency $i$ will certainly receive positive spending which exceeds $\frac{1}{n}$. It suggests that voter $i$ has a profitable deviation and thus $\gamma_i^* = \gamma \in (\overline{\gamma} (n - 1), +\infty)$ is not an equilibrium.

Finally, consider the case in which $\gamma = \overline{\gamma} (n - 1)$. The politician either allocates spending uniformly among all $n$ districts or excludes one of the districts from the spending allocation. In this case, district $i$’s voter has no incentives to deviate from $\gamma = \overline{\gamma} (n - 1)$. Indeed, assume first that he increases his threshold to $\gamma + \varepsilon$, $\varepsilon > 0$. Then the politician will exclude at least one district from the spending allocation. Following the similar steps as in the previous case analyzed in Appendix C, one can show that it is district $i$ (which is the most demanding district) which will be excluded from the spending allocation in this case. It follows therefore that if voter $i$ increases his threshold then his constituency will certainly receive no spending. Thus, that is not a profitable deviation. Assume next that voter $i$ lessens his demand, say to $\gamma - \varepsilon$, $\varepsilon > 0$. Then the politician will include all $n$ constituencies in the spending allocation and district $i$’s share will become $\frac{1}{n} - \frac{n-1}{n} \varepsilon$ which is lower than $\frac{1}{n}$. Thus, that is not a profitable deviation either. It suggests that voter $i$ has no profitable deviation and therefore $\gamma_i^* = \overline{\gamma} (n - 1)$ is an equilibrium. The findings are summarized in the following proposition where $\gamma_i^*$ denotes equilibrium thresholds and $s_i^*$ equilibrium spending shares.

**Proposition 1.** There exists a symmetric equilibrium such that the representative voters apply equal thresholds

$$\gamma_i^* = \overline{\gamma} (n - 1)$$

in their corresponding districts, the politician allocates spending uniformly either among all $n$ constituencies or among $n - 1$ constituencies. The voters’ expected payoff is given by

$$E s_i^* = \frac{1}{n}.$$  

The expected number of votes the politician wins in the coming election is

$$n F \left( \frac{1}{n} - \overline{\gamma} (n - 1) \right) = (n - 1) F \left( \frac{1}{n-1} - \overline{\gamma} (n - 1) \right) + F (-\overline{\gamma} (n - 1)).$$

According to Proposition 1, there exists a unique symmetric equilibrium $\gamma_i^* = \overline{\gamma} (n - 1)$, i.e., there are no other symmetric equilibria. In the following Section, I show that it is the unique equilibrium of the game. In other words, there are no asymmetric equilibria.

threshold $\gamma + \varepsilon$, where $\gamma \in (-\infty, \overline{\gamma} (n - 1))$. 

12
3.2. Uniqueness

I prove by contradiction that there are no asymmetric equilibria of the game. I first suppose that there exists an equilibrium in which the voters apply different thresholds \( \gamma_i \) in their corresponding constituencies, \( \gamma_i \neq \gamma_j \) for some or all \( i \neq j \). Lemma 2 in Appendix A spells out the conditions on \( \gamma_i \) under which such an equilibrium exists. According to this lemma, the spending will be allocated either among all or among \( k = 1, \ldots, n - 1 \) constituencies. I show next that there is always a voter who would like to deviate in either case. Intuitively, suppose that in equilibrium all constituencies receive positive spending. But then there is always at least one voter who can slightly increase his threshold and get a higher share of spending. So this is not an equilibrium. In case some constituencies receive no spending, either a constituency which gets some resources has incentives to increase its demand to get a larger share or a constituency which gets no resources has incentives to modify its demand to get included in the allocation of spending. So this is not an equilibrium either. It implies that the unique equilibrium of the game is that characterized in Proposition 1. The formal details are given in Appendix D.

3.3. Discussion

Recall from Lemma 1 that \( \tau(n - 1) \in \left( \frac{1}{2(n-1)}, \frac{1}{n} \right) \), which implies that the equilibrium probability of winning each constituency is strictly larger than one-half:

\[
F \left( \frac{1}{n} - \tau(n - 1) \right) > \frac{1}{2}.
\]

Therefore, the politician is more likely to win each constituency than the challenger. Her expected votes exceed \( \frac{n}{2} \) then. This result is novel for the redistributive game literature which often predicts the winning probabilities of 0, 1 or \( \frac{1}{2} \) (e.g., Cox and McCubbins 1986, Lindbeck and Weibull 1987, Dixit and Londregan 1996, Roberson 2008). In these models, the equilibrium forces make the politicians either target distinct constituencies, which generates the winning probabilities of one and zero, or compete over the same constituencies, which yields the equilibrium winning probabilities of one-half. Intuitively, the politicians compete over votes and so behave symmetrically in equilibrium, which results in symmetric outcomes. In the present paper, the incumbent chooses the allocation of spending depending on how productively this spending tips the voters’ decisions about her reelection. Here too, the equilibrium forces make the voters in distinct constituencies set the same expectations for redistribution. However, their demands are lower than what they actually receive in equilibrium, which leads to the winning probabilities strictly larger than one-half. The reason is that
under the equilibrium reelection thresholds, the incumbent is indifferent between allocating spending uniformly among all constituencies and dropping one randomly picked constituency from the spending allocation. No voter therefore wants to increase his demand because in that case, his constituency would be the one to be excluded from the spending allocation.

While majority outcomes are also predicted by the vote-buying models (Groseclose and Snyder 1996, 2000, Banks 2000, Boyer and Konrad 2014), the mechanism at work is novel here. In the present framework, majority outcome arises as a result of the competition among the voters and not among the politicians as in the vote-buying literature. This mechanism therefore uncovers a novel rationale for majority trends.

4. Conclusion

This paper studies a redistributive electoral game in which constituencies compete over resources by setting expectations for redistribution. I build a political agency model in which an agent (politician) is subject to the oversight of multiple principals (representative voters in national districts). The voters realize that the politician maximizes her votes and therefore can hold her accountable through retrospective voting strategies under which she is rewarded for district-specific spending. My results suggest that when the voters are quite lenient, the politician includes all constituencies in the spending bill. In this case, the spending shares are determined by the relative demand of the corresponding constituencies such that more demanding voters end up with higher spending shares conditional on being included in the spending allocation. However, if the voters get too demanding then the politician prefers to exclude at least one of the districts from the spending bill. I show moreover that in this case, a more demanding constituency ends up with no spending while a more lenient constituency certainly receives spending. As a result, the representative voters compete over spending by lowering their demands in order to get included in the spending allocation. They do so until their demands reach the point at which the politician is willing to allocate spending to all constituencies. Then no voter has any more incentives to lessen his demand as this would lead to lower spending share for his constituency. I characterize the unique equilibrium in which all voters demand the same level of spending for their districts under which the politician is indifferent between allocating spending uniformly among all constituencies and excluding one randomly picked constituency. Moreover, the voters’ demands are lower than the equilibrium spending shares and therefore the politician is more likely than a random challenger to win each constituency.

The results presented here are symmetric due to the symmetry of the problem. The model
can be extended to consider agents with asymmetric preferences or to assume asymmetric relationships between spending and corresponding policy outcomes. Such extended model would obviously yield asymmetric solutions for equilibrium spending shares. However, the equilibrium structure would remain unless the politician was assumed to pursue other goals or favor a particular public policy rather than to maximize her expected votes.

Appendix

A. Politician’s Maximization Problem

Given thresholds $\gamma_i \in \mathbb{R}$, the politician solves the following problem:

$$\max_{s_1, \ldots, s_n} \sum_{i=1}^{n} F(s_i - \gamma_i)$$

s.t. $\sum_{i=1}^{n} s_i = 1, \quad s_i \geq 0.$

The Kuhn-Tucker conditions are

$$\frac{\partial L}{\partial s_i} \leq 0, \quad s_i \geq 0, \quad s_i \frac{\partial L}{\partial s_i} = 0, \quad \frac{\partial L}{\partial \lambda} = 0,$$

where

$$L = \sum_{i=1}^{n} F(s_i - \gamma_i) + \lambda \left(1 - \sum_{i=1}^{n} s_i\right)$$

is the Lagrangian function and $\lambda$ denotes the Lagrange Multiplier. The Kuhn-Tucker conditions simplify to

$$f(s_i - \gamma_i) \leq \lambda, \quad s_i \geq 0, \quad s_i (f(s_i - \gamma_i) - \lambda) = 0, \quad \sum_{i=1}^{n} s_i = 1.$$

To characterize equilibrium spending shares, I consider two different cases and check under what conditions each of them arises. In the first case, all constituencies receive positive spending, i.e., $s_i > 0$. In the second case, some constituencies receive no spending, i.e., $s_i = 0$ for some $i$.

Case 1: $s_i > 0$. The Kuhn-Tucker conditions become

$$f(s_i - \gamma_i) = \lambda, \quad \sum_{i=1}^{n} s_i = 1.$$

15
The bordered Hessian for this problem is

\[
H = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & \frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_i^2} & 0 & \cdots & 0 \\
1 & 0 & \frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_1 \partial s_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & \frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_n \partial s_i}
\end{bmatrix}
\]

where

\[
\frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_j^2} = f'(s_j - \gamma_j), \quad (A.1)
\]

\[
\frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_j \partial s_i} = 0.
\]

Then, given the bordered Hessian, its bordered leading principal minors are equal to

\[
|H_k| \equiv \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & \frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_i^2} & 0 & \cdots & 0 \\
1 & 0 & \frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_1 \partial s_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & \frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_n \partial s_i}
\end{bmatrix}
\]

\[
- \sum_{l=1}^{k} \prod_{j \neq l} \frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_j^2} = - \sum_{l=1}^{k} \prod_{j \neq l} f'(s_j - \gamma_j),
\]

\[
k = 2, \ldots, n.
\]

Due to the symmetry of \( f \), \( f(s_i - \gamma_i) = \lambda \) has two solutions. In what follows, I consider first the symmetric case in which \( s_i - \gamma_i = s_j - \gamma_j \) for all \( i \neq j \). I turn next to the case in which \( s_i - \gamma_i = s_j - \gamma_j \) for some \( i \neq j \) while \( s_i - \gamma_i = -(s_j - \gamma_j) \) for the rest of \( i \neq j \).

- Consider first the symmetric case. The Kuhn-Tucker conditions are given by

\[
s_i - \gamma_i = s_j - \gamma_j, \quad \text{for } i \neq j, \quad \sum_{i=1}^{n} s_i = 1,
\]

which yield

\[
s_i = \gamma_i + \frac{1 - \sum_{j=1}^{n} \gamma_j}{n}. \quad (A.2)
\]

Evaluating (A.1) at the critical points characterized by (A.2) yields

\[
\frac{\partial^2 \sum_{i=1}^{n} F(s_i - \gamma_i)}{\partial s_j^2} = f' \left( \frac{1 - \sum_{i=1}^{n} \gamma_i}{n} \right). \quad (A.3)
\]
Therefore, in the plain Hessian evaluated at the critical points (A.2), all entries on the main diagonal are equal to (A.3) while all entries off the main diagonal are equal to zero and so $|H_k|$ becomes

$$|H_k| = -k \left( f' \left( \frac{1 - \sum_{i=1}^{n} \gamma_i}{n} \right) \right)^{k-1}.$$ 

The second-order condition for local maximum is satisfied when the bordered leading principal minors evaluated at the critical points alternate in sign, i.e., $|H_k| > 0$ for even $k$ and $|H_k| < 0$ for odd $k$. This is the case if $f' \left( \frac{1 - \sum_{i=1}^{n} \gamma_i}{n} \right) < 0$, which implies $\sum_{i=1}^{n} \gamma_i < 1$. In case $\sum_{i=1}^{n} \gamma_i = 1$, the determinant of the bordered Hessian is equal to zero and thus (A.2) can be local maximum. Therefore, (A.2) is local maximum when $\sum_{i=1}^{n} \gamma_i \leq 1$. The expected number of votes the politician wins in this case is equal to $nF \left( \frac{1 - \sum_{j=1}^{n} \gamma_j}{n} \right)$.

- I turn now to the case in which $s_i - \gamma_i = s_j - \gamma_j$ for some $i \neq j$ while $s_i - \gamma_i = -(s_j - \gamma_j)$ for the rest of $i \neq j$. In what follows, I show that in this case, the second-order condition for local maximum fails. Indeed, since it is not the symmetric case, there exist at least one (and at most $n - 1$ equal) negative $s_i - \gamma_i$. For $s_i - \gamma_i < 0$, $f' (s_i - \gamma_i)$, denoted in what follows by $f'$, are positive and all equal. The rest of $s_i - \gamma_i$’s are positive and for them, $f' (s_i - \gamma_i) = -f'$ (due to the symmetry of $f$ (·)). Suppose there are $m = 1, ..., n - 1$ negative $s_i - \gamma_i$’s and $n - m$ positive $s_i - \gamma_i$’s. Without loss of generality, I can focus on the situation in which for $i = 1, ..., m$, $s_i - \gamma_i < 0$, while for $i = m + 1, ..., n$, $s_i - \gamma_i > 0$. I next calculate the bordered leading principal minors $|H_k|$, $k = 2, ..., n$, for this case. For $k \leq m$,

$$|H_k| = -k \left( f' \right)^{k-1},$$ 

which is always negative since $f' > 0$. For $k > m$,

$$|H_k| = (-1)^{k-m} (k - 2m) \left( f' \right)^{k-1}. \quad \text{(A.4)}$$

The second-order condition for local maximum requires that the bordered leading principal minors alternate in sign, i.e., $|H_k| > 0$ for even $k$ and $|H_k| < 0$ for odd $k$. This might happen only for $m = 1$ for which (A.4) becomes

$$|H_k| = (-1)^{k-1} (k - 2) \left( f' \right)^{k-1}.$$
Therefore,

\[ |H_2| = 0, \quad |H_3| = (f')^2 > 0, \quad |H_4| = -2(f')^3 < 0, \quad \ldots \]

and so the second-order condition fails. It follows that \( s_i - \gamma_i = s_j - \gamma_j \) for some \( i \neq j \) while \( s_i - \gamma_i = -(s_j - \gamma_j) \) for the rest of \( i \neq j \) is not a local maximum.

\textbf{Case 2}: \( s_i = 0 \) for some \( i \). Denote by \( \Omega_k \) the set of all subsets of \( k \) integers that can be selected from \{1, \ldots, n\}. \( A \) will denote an element of \( \Omega_k \) and \( A^c \) the complement of \( A \), i.e., \( A^c = \{1, \ldots, n\} \setminus A \). I try next strictly positive values for \( k = 1, \ldots, n - 1 \) choice variables and zero values for the other \( n - k \) choice variables, i.e., \( s_i > 0 \) for \( i \in A \subset \Omega_k \) and \( s_i = 0 \) for \( i \in A^c \). In this case, the Kuhn-Tucker conditions become

\[
\begin{align*}
  f(s_i - \gamma_i) &= \lambda \quad \text{for } i \in A, \\
  f(-\gamma_i) &\leq f\left(\frac{1 - \sum_{j \in A} \gamma_j}{k}\right) \quad \text{for } i \in A^c, \\
  \sum_{i \in A} s_i &= 1.
\end{align*}
\]

I follow the similar steps as in \textit{Case 1} to show that the Kuhn-Tucker conditions yield

\[
\begin{align*}
  s_i &= \gamma_i + \frac{1 - \sum_{j \in A} \gamma_j}{k} \quad \text{for } i \in A, \\
  f(-\gamma_i) &\leq f\left(\frac{1 - \sum_{j \in A} \gamma_j}{k}\right) \quad \text{for } i \in A^c.
\end{align*}
\]

Positive spending shares \( s_i, i \in A \), should maximize \( \sum_{i \in A} F(s_i - \gamma_i) \) (except for the case in which \( A \subset \Omega_1 \)). Otherwise, it cannot be an equilibrium. It implies that the second-order condition should hold. The second-order condition is analogous to that in \textit{Case 1} and is given by \( \sum_{i \in A} \gamma_i \leq 1 \). Given that, the Kuhn-Tucker conditions become

\[
\begin{align*}
  s_i &= \gamma_i + \frac{1 - \sum_{j \in A} \gamma_j}{k} \quad \text{for } i \in A, \\
  |\gamma_i| &\geq \frac{1 - \sum_{j \in A} \gamma_j}{k} \quad \text{for } i \in A^c.
\end{align*}
\]

Therefore, this is local maximum when \( \sum_{i \in A} \gamma_i \leq 1 \) (except for the case in which \( A \subset \Omega_1 \)) and \( |\gamma_i| \geq \frac{1 - \sum_{j \in A} \gamma_j}{k} \) for \( i \in A^c \). The expected number of votes the politician wins in this case is equal to

\[
kF\left(\frac{1 - \sum_{j \in A} \gamma_j}{k}\right) + \sum_{i \in A^c} F(-\gamma_i).
\]

The next step is to characterize a global maximum. A spending allocation solves the politician’s maximization problem if it is local maximum and if the politician’s objective function \( \sum_{i=1}^{n} F(s_i - \gamma_i) \) evaluated at this allocation is higher than evaluated at any other
local maximum. Therefore, I can establish the following lemma in which \( s_i^\gamma \) denotes a solution of the politician’s maximization problem. Recall that \( \Omega_k \) is the set of all subsets of \( k \) integers that can be selected from \( \{1, ..., n\} \). \( A \) denotes an element of \( \Omega_k \) and \( A^c \) the complement of \( A \).

**Lemma 2.** Given thresholds \( \gamma_i \in \mathbb{R} \) applied by the representative voters,

- there exists an equilibrium allocation of spending

\[
s_i^\gamma = \gamma_i + \frac{1 - \sum_{j=1}^{n} \gamma_j}{n}, \quad i = 1, ..., n,
\]

when thresholds \( \gamma_i \) satisfy

\[
0 < \gamma_i + \frac{1 - \sum_{j=1}^{n} \gamma_j}{n} < 1, \quad i = 1, ..., n, \quad \text{(A.5)}
\]

\[
\sum_{i=1}^{n} \gamma_i \leq 1, \quad \text{(A.6)}
\]

\[
nF \left( \frac{1 - \sum_{j=1}^{n} \gamma_j}{n} \right) \geq kF \left( \frac{1 - \sum_{j \in A} \gamma_j}{k} \right) + \sum_{i \in A^c} F (-\gamma_i), \quad \text{(A.7)}
\]

for \( A \subset \Omega_k, \ k = 1, ..., n - 1; \)

- for \( k = 1, ..., n - 1 \), there exists an equilibrium allocation of spending

\[
s_i^\gamma = \begin{cases} 
\gamma_i + \frac{1 - \sum_{j \in A} \gamma_j}{k} & \text{for } i \in A \subset \Omega_k, \\
0 & \text{for } i \in A^c,
\end{cases}
\]

when thresholds \( \gamma_i \) satisfy

\[
0 < \gamma_i + \frac{1 - \sum_{j \in A} \gamma_j}{k} < 1, \quad i \in A \subset \Omega_k, \quad \text{(A.8)}
\]

\[
\sum_{i \in A} \gamma_i \leq 1 \text{ (except for the case in which } A \subset \Omega_1), \quad \text{(A.9)}
\]

\[
|\gamma_i| \geq \frac{1 - \sum_{j \in A} \gamma_j}{k} \text{ for } i \in A^c, \quad \text{(A.10)}
\]

\[
kF \left( \frac{1 - \sum_{j \in A} \gamma_j}{k} \right) + \sum_{i \in A^c} F (-\gamma_i) \geq nF \left( \frac{1 - \sum_{j=1}^{n} \gamma_j}{n} \right), \quad \text{(A.11)}
\]

\[
kF \left( \frac{1 - \sum_{j \in A} \gamma_j}{k} \right) + \sum_{i \in A^c} F (-\gamma_i) \geq lF \left( \frac{1 - \sum_{j \in B} \gamma_j}{l} \right) + \sum_{i \in B^c} F (-\gamma_i) \quad \text{(A.12)}
\]

for \( B \subset \Omega_l, \ l = 1, ..., n - 1, \ B \neq A. \)
B. Proof of Lemma 1

Given \( \gamma_i = \gamma \in \left[ \frac{1}{2k}, \frac{1}{k+1} \right] \) with \( k = 2, ..., n-1 \), the politician divides spending equally among \( k+1 \) constituencies if

\[
\Psi(\gamma, k) \geq 0
\]

and among \( k \) constituencies otherwise, where

\[
\Psi(\gamma, k) \equiv (k + 1) F\left(\frac{1}{k+1} - \gamma\right) - k F\left(\frac{1}{k} - \gamma\right) - F\left(-\gamma\right)
\]

Note that \( \Psi(\gamma, k) \) is a strictly decreasing function of \( \gamma \in \left[ \frac{1}{2k}, \frac{1}{k+1} \right] \). Indeed, for \( \gamma \in \left[ \frac{1}{2k}, \frac{1}{k+1} \right] \),

\[
\frac{\partial \Psi(\gamma, k)}{\partial \gamma} = -(k + 1) f\left(\frac{1}{k+1} - \gamma\right) + kf\left(\frac{1}{k} - \gamma\right) + f(-\gamma) < 0 \quad \text{(B.1)}
\]

since \( f\left(\frac{1}{k} - \gamma\right) < f\left(\frac{1}{k+1} - \gamma\right) \) and \( f(-\gamma) < f\left(\frac{1}{k+1} - \gamma\right) \). I next evaluate \( \Psi(\gamma, k) \) in \( \gamma = \frac{1}{2k} \) and in \( \gamma = \frac{1}{k+1} \).

1. Evaluating \( \Psi(\gamma, k) \) in \( \gamma = \frac{1}{2k} \) yields

\[
\Psi\left(\frac{1}{2k}, k\right) = (k + 1) F\left(\frac{k-1}{2k(k+1)}\right) - k F\left(\frac{1}{2k}\right) - F\left(-\frac{1}{2k}\right) =
\]

\[
F\left(\frac{1}{2k}\right) - F\left(-\frac{1}{2k}\right) - (k + 1) \left(F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right)\right) =
\]

\[
\sum_{i=1}^{k+1} \left( F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{i-1}{k(k+1)}\right) \right) - (k + 1) \left(F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right)\right) .
\]

If \( k \) is odd then \( k + 1 \) is even and \( \Psi\left(\frac{1}{2k}, k\right) \) becomes

\[
\Psi\left(\frac{1}{2k}, k\right) = 2 \sum_{i=1}^{\frac{k+1}{2}} \left( F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{i-1}{k(k+1)}\right) \right) - (k + 1) \left(F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right)\right) .
\]

Due to the convexity of \( F(\cdot) \) for negative arguments and Jensen’s inequality, \( F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) < \frac{1}{2} \left(F\left(-\frac{1}{2k} + \frac{i-1}{k(k+1)}\right) + F\left(-\frac{1}{2k} + \frac{i+1}{k(k+1)}\right)\right) \) for \( i = 1, ..., \frac{k+1}{2} \), which yields

\[
F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{i-1}{k(k+1)}\right) < F\left(-\frac{1}{2k} + \frac{i+1}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) .
\]

It follows therefore that \( F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{i-1}{k(k+1)}\right) \) strictly increases with \( i = 1, ..., \frac{k+1}{2} \). Thus, for \( i = 2, ..., \frac{k+1}{2} \),

\[
F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{i-1}{k(k+1)}\right) >
\]

\[
F\left(-\frac{1}{2k} + \frac{1}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{1-1}{k(k+1)}\right) =
\]

\[
F\left(-\frac{1}{2k} + \frac{k-1}{2k(k+1)}\right) - F\left(-\frac{1}{2k}\right) = F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right) ,
\]
where the last equality comes from the symmetry of the cumulative distribution function $F(\cdot)$. It follows that for odd $k$,

$$
\Psi\left(\frac{1}{2k}, k\right) > 2 \sum_{i=1}^{\frac{k+1}{2}} \left( F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right) \right) - (k+1) \left( F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right) \right) = 0.
$$

If $k$ is even then $k+1$ is odd and $\Psi\left(\frac{1}{2k}, k\right)$ becomes

$$
\Psi\left(\frac{1}{2k}, k\right) = 2 \sum_{i=1}^{\frac{k}{2}} \left( F\left(-\frac{1}{2k} + \frac{i}{k(k+1)}\right) - F\left(-\frac{1}{2k} + \frac{i-1}{k(k+1)}\right) \right) +
2 \left( F(0) - F\left(-\frac{1}{2k(k+1)}\right) \right) - (k+1) \left( F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right) \right) .
$$

Taking into account the convexity of $F(\cdot)$ for negative arguments and Jensen’s inequality, and following the similar steps as in the case of odd $k$, I show that in the case of even $k$,

$$
\Psi\left(\frac{1}{2k}, k\right) > 2 \sum_{i=1}^{\frac{k}{2}} \left( F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right) \right) +
2 \left( F(0) - F\left(-\frac{1}{2k(k+1)}\right) \right) - (k+1) \left( F\left(\frac{1}{2k}\right) - F\left(\frac{k-1}{2k(k+1)}\right) \right) =
2 \left( F(0) - F\left(-\frac{1}{2k(k+1)}\right) \right) - F\left(-\frac{k-1}{2k(k+1)}\right) - F\left(-\frac{1}{2k}\right) +
F\left(-\frac{k-1}{2k(k+1)}\right) - F\left(-\frac{k-1}{2k(k+1)}\right) - F\left(-\frac{1}{2k}\right) = 0.
$$

It follows therefore that $\Psi\left(\frac{1}{2k}, k\right) > 0$ both for odd and even $k$.

2. Evaluating $\Psi(\gamma, k)$ in $\gamma = \frac{1}{k+1}$ yields

$$
\Psi\left(\frac{1}{k+1}, k\right) = (k+1) F(0) - k F\left(\frac{1}{k+1}\right) - F\left(-\frac{1}{k+1}\right) .
$$

Taking into account the convexity (concavity) of $F(\cdot)$ for negative (positive) arguments and Jensen’s inequality, I show that $\Psi\left(\frac{1}{k+1}, k\right) < 0$. The derivation of this results is analogous to that of $\Psi\left(\frac{1}{2k}, k\right)$ above and is available upon request.

It follows that $\Psi(\gamma, k)$ decreases with $\gamma \in \left[\frac{1}{2k}, \frac{1}{k+1}\right]$ and takes a strictly positive value in $\gamma = \frac{1}{2k}$ and a strictly negative value in $\gamma = \frac{1}{k+1}$. Therefore, there exists $\gamma(k) \in \left(\frac{1}{2k}, \frac{1}{k+1}\right)$ such that

$$
\Psi(\gamma(k), k) = 0.
$$
Consider next
\[
\frac{\partial \gamma (k)}{\partial k} = - \frac{\partial F (\gamma (k), k)}{\partial k} = \frac{F \left( k^{-1}, \gamma (k) \right) - k F \left( \frac{1}{k}, \frac{1}{k} - \gamma (k) \right) - F \left( \frac{1}{k} - \gamma (k) \right) - \frac{1}{k} f \left( \frac{1}{k} - \gamma (k) \right)}{\partial k}.
\]

The denominator is strictly negative (see (B.1)). To find the sign of the numerator, consider
\[
\frac{\partial \left( F \left( \frac{1}{k} - \gamma (k) \right) - \frac{1}{k} f \left( \frac{1}{k} - \gamma (k) \right) \right)}{\partial k} = \frac{1}{k^3} f' \left( \frac{1}{k} - \gamma (k) \right),
\]
which is strictly negative for \( \gamma (k) \in \left( \frac{1}{2k}, \frac{1}{k+1} \right) \). It follows that \( F \left( \frac{1}{k} - \gamma (k) \right) - \frac{1}{k} f \left( \frac{1}{k} - \gamma (k) \right) \) is a strictly decreasing function of \( k \) and so the numerator is also negative. Therefore
\[
\frac{\partial \gamma (k)}{\partial k} < 0,
\]
which implies that \( \gamma (k) \) is a strictly decreasing function of \( k = 2, \ldots, n - 1 \).

I conclude therefore that for \( \gamma \in (-\infty, \bar{\gamma}(n-1)] \), the politician allocates spending uniformly among \( n \) constituencies; for \( \gamma \in [\bar{\gamma}(k), \bar{\gamma}(k-1)] \) with \( k = 3, \ldots, n - 1 \), among \( k \) constituencies; for \( \gamma \in [\bar{\gamma}(2), \frac{1}{2}] \), among 2 constituencies; for \( \gamma \in \left[ \frac{1}{2}, +\infty \right) \), all spending is sent to one district. Note that for \( \gamma = \gamma (k) \) with \( k = 2, \ldots, n - 1 \), the politician is indifferent between allocating spending among \( k \) and \( k + 1 \) districts. For \( \gamma = \frac{1}{2} \), she is indifferent between sending all spending to one district and dividing it between two districts.

C. Deviation Incentives of Representative Voters

Consider the case in which \( \gamma \in (\bar{\gamma}(n-1), +\infty) \). The politician allocates spending among \( k \) randomly chosen constituencies where \( k = n - 1 \) for \( \gamma \in (\bar{\gamma}(n-1), \bar{\gamma}(n-2)] \); \( k = 3, \ldots, n - 2 \) for \( \gamma \in [\bar{\gamma}(k), \bar{\gamma}(k-1)] \); \( k = 2 \) for \( \gamma \in [\bar{\gamma}(2), \frac{1}{2}] \); and \( k = 1 \) for \( \gamma \in \left[ \frac{1}{2}, +\infty \right] \). Suppose district \( i \)'s voter slightly lessens his threshold, say to \( \gamma - \varepsilon, \varepsilon > 0 \). Then, given that \( n - 1 \) voters adopt thresholds \( \gamma \) while one voter adopts threshold \( \gamma - \varepsilon \), the politician will allocate spending either to the same number of constituencies as in the case of equal thresholds \( \gamma \), or to strictly more constituencies. One can check that there always exists positive \( \varepsilon \) (possibly very small) such that the politician will find it optimal to exclude at least one district from the spending allocation when \( n - 1 \) voters adopt thresholds \( \gamma \) while one voter adopts threshold \( \gamma - \varepsilon \) with \( \gamma \in (\bar{\gamma}(n-1), +\infty) \). The politician in this case is no more indifferent between the spending allocations which include more lenient district \( i \) and those which exclude it. If district \( i \) receives no spending then the expected number of votes is given by
\[
l F \left( \frac{1}{k} - \gamma \right) + (n - l - 1) F (-\gamma) + F (-\gamma + \varepsilon),
\]

where \( l = 1, \ldots, n - 1 \) denotes the number of districts receiving positive spending, \( l \geq k \). If district \( i \) receives positive spending then the expected number of votes is equal to

\[
lf \left( \frac{1}{l} - \gamma + \frac{\varepsilon}{l} \right) + (n - l) F(-\gamma) .
\]

The politician will send spending to constituency \( i \) if

\[
l \left( F \left( \frac{1}{l} - \gamma + \frac{\varepsilon}{l} \right) - F \left( \frac{1}{l} - \gamma \right) \right) \geq F(\gamma) - F(\gamma - \varepsilon), \tag{C.1}
\]

and will exclude it from the spending allocation otherwise. If

\[
\varepsilon < \frac{1}{l} \quad \text{and} \quad \frac{1}{l} - \gamma + \frac{\varepsilon}{l} < \gamma - \varepsilon \tag{C.2}
\]

then due to the concavity (convexity) of \( F(\cdot) \) for positive (negative) arguments and Jensen’s inequality, (C.1) holds and therefore constituency \( i \) will be certainly included in the spending allocation with spending share \( \frac{1}{l} - \frac{l-1}{l} \varepsilon \). It follows that district \( i \)'s voter has a profitable deviation since he can always find \( \varepsilon \) small enough such that

- the politician excludes at least one district from the spending allocation when \( n - 1 \) voters adopt thresholds \( \gamma \) while one voter adopts threshold \( \gamma - \varepsilon \) where \( \gamma \in (\gamma(n - 1), +\infty) \);
- (C.2) holds, i.e., constituency \( i \) receives positive spending;
- and \( \frac{1}{l} - \frac{l-1}{l} \varepsilon > \frac{1}{n} \), i.e., voter \( i \)'s payoff increases when he deviates to \( \gamma - \varepsilon \).

Therefore, \( \gamma_i = \gamma \in (\gamma(n - 1), +\infty) \) is not an equilibrium.

### D. Equilibrium Uniqueness

Suppose \( \gamma_i \neq \gamma_j \) for some or all \( i \neq j \) is an equilibrium. It follows from Lemma 2 in Appendix A that in this equilibrium, the spending is allocated either among all or among \( k = 1, \ldots, n - 1 \) constituencies.

1. Consider first the case in which all districts receive positive spending. Therefore, thresholds \( \gamma_i \) satisfy (A.5)-(A.7).

1. Suppose that inequalities (A.6)-(A.7) are strict. Therefore, if one district, say district \( i \), slightly increases its threshold to \( \gamma_i + \varepsilon \) (where \( \varepsilon \) is positive and sufficiently small), then inequalities (A.5)-(A.7) will still hold while district \( i \) will get a higher spending share. It follows that district \( i \) has a profitable deviation and therefore this is not an equilibrium.
2. Suppose that inequality (A.6) is strict while (A.7) holds with equality. It means that the politician is indifferent between allocating spending among all districts and dropping one or more districts from the spending allocation. Consider a district which receives a positive share of spending in both scenarios (i.e., when all districts are included and when one or more districts are excluded). Obviously, there exists at least one such constituency. Suppose next that this district, say district \( i \), slightly increases its threshold to \( \gamma_i + \varepsilon \) (where \( \varepsilon \) is positive and sufficiently small). Then the left-hand side of (A.7) becomes

\[
nF\left(\frac{1 - \sum_{j=1}^{n} \gamma_j - \varepsilon}{n}\right)
\]

while its right-hand side becomes

\[
kF\left(\frac{1 - \sum_{j \in A^c} \gamma_j - \varepsilon}{k}\right) + \sum_{i \in A^c} F(-\gamma_i).
\]

Then the sign of

\[
n\left(F\left(\frac{1 - \sum_{j=1}^{n} \gamma_j}{n}\right) - F\left(\frac{1 - \sum_{j=1}^{n} \gamma_j - \varepsilon}{n}\right)\right) - k\left(F\left(\frac{1 - \sum_{j \in A} \gamma_j}{k}\right) - F\left(\frac{1 - \sum_{j \in A} \gamma_j - \varepsilon}{k}\right)\right)
\]

(D.1)
determines whether (A.7) holds or not. If the sign of (D.1) is nonpositive then (A.7) holds and thus all districts will still receive positive spending while district \( i \) will get a higher spending share. Therefore, district \( i \) has a profitable deviation in this case and thus this is not an equilibrium. However, if the sign of (D.1) is positive then (A.7) is violated and therefore the politician will drop one or more districts from the spending allocation but still keep district \( i \) in. Whether it is profitable or not for district \( i \) to deviate in this case then depends on the sign of

\[
\frac{1 - \sum_{j=1}^{n} \gamma_j}{n} - \frac{1 - \sum_{j \in A} \gamma_j}{k}.
\]

(D.2)

If it is negative then district \( i \)'s deviation is profitable and thus this is not an equilibrium. Suppose now that it is nonnegative. Then following the similar steps as in the proof of Lemma 1 (numbered item 1) and taking into account the concavity of \( F(\cdot) \) for positive arguments and Jensen’s inequality, one can show that the sign of (D.1) is negative in this case. But this leads to the contradiction.

3. Suppose finally that (A.6) holds with equality. It follows that there is at least one threshold, say in district \( i \), which is strictly positive. The expected number of votes the politician wins in this case is \( nF(0) \). I calculate next the expected
number of votes if district $i$ received no spending:

$$(n - 1) F \left( \frac{1 - (\sum_{j=1}^{n} \gamma_j - \gamma_i)}{n-1} \right) + F (-\gamma_i) = (n - 1) F \left( \frac{\gamma_i}{n-1} \right) + F (-\gamma_i).$$

Following the similar steps as in the proof of Lemma 1 (numbered item 1) and taking into account the convexity of $F(\cdot)$ for negative arguments and Jensen’s inequality, one can show that

$$(n - 1) F \left( \frac{\gamma_i}{n-1} \right) + F (-\gamma_i) > nF (0)$$

and so (A.7) does not hold. Therefore, this is not an equilibrium.

Therefore, the case in which $\gamma_i \neq \gamma_j$ for some or all $i \neq j$ and all districts receive positive spending does not arise in equilibrium.

2. Consider next the case in which $k = 1, \ldots, n - 1$ constituencies receive positive spending while $n - k$ districts get no spending allocated. Then thresholds $\gamma_i$ satisfy (A.8)-(A.12).

1. Suppose that inequalities (A.9), (A.11) and (A.12) are strict. This case is analogous to that analyzed in 1.1 above and so is not an equilibrium.

2. Suppose next that inequality (A.9) is strict while either (A.11) or (A.12) holds with equality. Then there are several cases to consider.

First, the politician might be indifferent between allocating spending among $k$ constituencies and dropping some of those from the spending allocation. This case is analogous to that analyzed in 1.2 and so is not an equilibrium.

Second, the politician might be indifferent between allocating spending among $k$ and $m = 1, \ldots, n$ constituencies so that there is at least one district excluded from the former (the $k$-allocation) but included in the latter (the $m$-allocation). This district then can slightly modify its threshold such that the politician gets strictly in favor of the $m$-allocation. It follows that this district will deviate and so this is not an equilibrium.

Finally, the politician might be indifferent between allocating spending among $k$ constituencies and including some other districts in the spending allocation. Then one of those districts can slightly modify its threshold such that the politician strictly prefers to include it. It follows that this district will deviate and so this is not an equilibrium.

3. Suppose now that (A.9) holds with equality. This case is analogous to that analyzed in 1.3 above and so is not an equilibrium.
Therefore, the case in which \( \gamma_i \neq \gamma_j \) for some or all \( i \neq j \) and \( k = 1, \ldots, n - 1 \) constituencies receive positive spending does not arise in equilibrium.

I conclude then that \( \gamma_i \neq \gamma_j \) for some or all \( i \neq j \) is not an equilibrium of the game.

References


