Optimal Hedging of Option Portfolios with Transaction Costs

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Abstract

One of the most successful approaches to option hedging with transaction costs is the utility-based approach, pioneered by Hodges and Neuberger (1989). Judging against the best possible tradeoff between the risk and the costs of a hedging strategy, this approach seems to achieve excellent empirical performance. However, this approach has one major drawback that prevents the broad application of this approach in practice: the lack of a closed-form solution. The numerical computations are very cumbersome in implementation. Despite some recent advances in finding an explicit description of the utility-based hedging strategy by using either asymptotic, approximation, or other methods, so far they were concerned primarily with hedging a single plain-vanilla option. Yet in practice one faces the problem of hedging a portfolio of options on the same underlying asset. Since the knowledge of the optimal hedging strategy for a portfolio of options is of great practical significance, in this paper we suggest a simplified parameterized functional form of the utility-based hedging strategy for a portfolio of options and a method for finding the optimal parameters. The method is based on simulations and is simple in implementation. Despite the simplicity of the suggested functional form, the performance of our optimized simplified hedging strategy is close to that of the exact utility-based strategy. Moreover, we provide an empirical testing of our optimized hedging strategies against some alternative strategies and show that our strategies outperform all the others.

Key words: option hedging, transaction costs, simulations.


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1 Introduction

One of the most successful approaches to option hedging with transaction costs is the utility based approach, pioneered by Hodges and Neuberger (1989). Judging against the best possible tradeoff between the risk and the costs of a hedging strategy, the utility based approach seems to achieve excellent empirical performance (see Mohamed (1994), Clewlow and Hodges (1997), Martellini and Priaulet (2002), Zakamouline (2004), and Zakamouline (2006a)). However, this approach has one major drawback that prevents the broad application of this approach in practice: the lack of a closed-form solution. Therefore, the solution must be computed numerically. The numerical algorithm is cumbersome to implement and the calculation of the optimal hedging strategy is time consuming.

In particular, the implementation of the numerical algorithm presents a sort of a challenge. In the case of a general utility function one needs to solve numerically an optimal stochastic control problem in four dimensions. The use of the negative exponential utility function reduces the dimensionality of the problem by one, but in this case one faces the problem of overflow or underflow in the values of the exponential utility (see Clewlow and Hodges (1997)). When it comes to the problem of having a large computational time, considering the exploding development within the computer industry this problem becomes less important when one needs to find the optimal hedging strategy for a non-exotic option. In this case the optimal transaction policy is not path-dependent and, hence, the evolution of the underlying price process can be modelled as a recombining binomial tree. However, when an option payoff is path-dependent, one needs to construct a non-recombining binomial tree for the underlying price process. In this case the computation of the optimal hedging policy for a sufficiently large number of trading dates is out of reach.

Since there are no explicit solutions for the utility based hedging with transaction costs and the numerical methods are computationally hard, for practical applications it is of major importance to use other alternatives. One of such alternatives is to obtain an asymptotic solution. In asymptotic analysis one studies the solution to a problem when some parameters in the problem assume large or small values. The asymptotic analysis of the model of Hodges and Neuberger (1989) for the case of a short European call option was performed by Whalley and Wilmott (1997) and Barles and Soner (1998).

Zakamouline (2004) compared the performances of asymptotic strategies against the performance of the exact strategy and found that under realistic model parameters an asymptotic strategy performs noticeably worse than that obtained from the exact numerical solution. He suggested to use an alternative to the asymptotic analysis, namely, the approximation method. Under approximation it is meant the following: one has a rather slow and
cumbersome way to compute the optimal hedging policy with transaction costs and wants to replace it with simple and efficient approximating function(s). To do this, one first specifies a flexible functional form of the optimal hedging policy. Then, given a functional form, the parameters are chosen to provide the best fit to the exact numerical solution. This second stage is known as “model calibration”. Zakamouline performed the approximation of the utility-based hedging strategy for a short European call option and tested empirically his approximation strategy against both the asymptotic strategies and some other well-known strategies. He evaluated the performance of different hedging strategies within the unified mean-variance and the mean-VaR (Value-at-Risk) frameworks and found that his approximation strategy outperforms all the others.

The approximation methodology presented in Zakamouline (2004) produces a clearly superior result as compared to other alternative methods of finding closed-form expressions for the optimal hedging strategy for a plain-vanilla European option. However, this methodology does retain the main disadvantage one would like to get rid off: to approximate the optimal strategy for a specific option one needs to start with the numerical calculations of the optimal hedging strategy for a large set of the model parameters. Since the numerical algorithm is cumbersome to implement and the calculation of the optimal hedging strategy is time consuming, the approximation methodology is unlikely to be commonly used by the practitioners.

In practice one faces the problem of hedging a large portfolio of options on the same underlying asset. Consequently, the knowledge of the optimal hedging strategy for a portfolio of options is of great practical significance. The first contribution of this paper is to present a detailed description of the nature of the optimal hedging strategy for a portfolio of options. We illustrate that the optimal hedging strategy for a portfolio of option is, in fact, rather simple and has three essential features: (i) existence of the no-transaction region, (ii) optimal form of the no-transaction region, and (iii) the volatility adjustment.

The second contribution of this paper is to suggest a simplified parameterized functional form of the optimal hedging strategy for a portfolio of options and a method for finding the optimal parameters. That is, in the essence our idea is based on a pure “financial engineering” way of thinking: We try to “mimic” the essential properties of the optimal hedging strategy by specifying a simple but flexible functional form of the optimal hedging policy. Then, we optimize the performance of the simplified hedging strategy by changing a few parameters in order to find the best possible risk-return tradeoff. Even though our optimization method takes a longer time than the exact numerical computations in the case of a portfolio of non-exotic options, the implementation of our optimization method is simple as the method is based on Monte-Carlos simulations instead of on a straightforward numerical solution of an optimal stochastic control problem. A great
advantage of our method is that it works equally well also for finding the optimal hedging strategy for a portfolio of exotic options. That is, this method makes feasible the finding of the optimal dynamic hedging strategy in cases when the direct numerical computation is out of reach.

Despite the simplicity of the suggested functional form, an optimized strategy showed to be very effective with the performance close to that of the exact utility-based strategy. We provide an empirical testing of our optimized strategy against the alternative strategies and show that our strategy outperforms all the others. We tested our optimization method by simulations of the hedging such popular combinations of standard options as Bull and Bear spreads, long and short positions in Butterfly spread, Call Ratio spread, Put Ratio spread, Straddle, Strip, Strap, Strangle, and Wrangle.

The paper is organized as follow. In Section 2 we introduce the problem of hedging an option portfolio and review some known methods. In Section 3 we review the utility-based option pricing and hedging approach as well as the results of various asymptotic and approximation methods. The purpose of this section is to present a detailed description of the nature of the optimal hedging strategy in the presence of transaction costs. In Section 4 we introduce the rational under our simplified hedging strategy and describe our optimization methodology. In Section 5 we provide an empirical testing of our optimized strategies. Section 6 concludes the paper.

2 The Hedging Problem and Some Solutions

We consider a continuous time economy with one risk-free and one risky asset, which pays no dividends. We will refer to the risky asset as the stock, and assume that the price of the stock, $S_t$, evolves according to a diffusion process given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu$ and $\sigma$ are, respectively, the mean and volatility of the stock returns per unit of time, and $W_t$ is a standard Brownian motion. The risk-free asset, commonly referred to as the bond or bank account, pays a constant interest rate of $r \geq 0$. We assume that a purchase or sale of $\delta$ shares of the stock incurs transaction costs $\lambda |\delta| S$ proportional to the transaction ($\lambda \geq 0$).

We consider hedging a portfolio of European options on the same underlying stock with the same maturity $T$. We denote the value of the option portfolio at time $t$ as $V(t, S_t)$. The terminal payoff of the option portfolio one wishes to hedge is given by $V(T, S_T)$. The general set-up of the hedging problem is as follows: When a hedger sells/buys a portfolio of options, he receives/pays the price $V(t, S_t)$ and sets up a hedging portfolio by buying $\Delta$ shares of the stock and putting $V(t, S_t) - \Delta (1 + \lambda) S_t$ in the bank account. As time goes, the hedger rebalances the hedging portfolio according to some prescribed rule/strategy.
2.1 The Black-Scholes Model

When a market is friction-free ($\lambda = 0$), Black and Scholes (1973) showed that it is possible to replicate the payoff of an option by constructing a self-financing dynamic trading strategy consisting of the risk-free asset and the stock. As a consequence, the absence of arbitrage dictates that the option price be equal to the cost of setting up the replicating portfolio.

The Black-Scholes model can be easily extended for the case of a portfolio of options. That is, in a friction-free market it is possible to replicate the payoff of a portfolio of options. As for a single option, the value of the hedging portfolio that replicates the payoff of a portfolio of options should satisfy the following PDE

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

with a boundary condition given by $V(T, S_T)$. The general solution of the PDE is given by

$$V(t, S_t) = e^{-r(T-t)} E^Q[V(T, S_T)].$$

That is, the (Black-Scholes) price of the portfolio of options equals the present value of the expected portfolio payoff on maturity in the so-called risk-neutral world where the stock price process is given by

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

The Black-Scholes hedging strategy consists in holding $\Delta$ (delta) shares of the stock and some amount in the bank account, where

$$\Delta(t) = \frac{\partial V(t, S_t)}{\partial S}.$$

It should be emphasized that the Black-Scholes hedging is a dynamic replication policy where the trading in the underlying stock has to be done continuously. In the presence of transaction costs in capital markets the absence of arbitrage argument is no longer valid, since perfect hedging is impossible. Due to the infinite variation of the geometric Brownian motion, the continuous replication policy mandated by the Black-Scholes model incurs an infinite amount of transaction costs over any trading interval no matter how small it might be. How should one hedge a portfolio of options in the market with transaction costs?

2.2 The Black-Scholes Hedging at Fixed Regular Intervals

One of the simplest and most straightforward hedging strategies in the presence of transaction costs is to rehedge in the underlying stock at fixed regular intervals. One would simply implement the delta hedging according to the
Black-Scholes strategy, but in discrete time. More formally, the time interval \([t, T]\) is subdivided into \(n\) fixed regular intervals \(\delta t\), such that \(\delta t = \frac{T-t}{n}\).

The hedging proceeds as follows: at time \(t\) the hedger receives/pays \(V(t, S_t)\) and constructs a replicating portfolio by purchasing \(\Delta(t)\) shares of the stock and putting \(V(t, S_t) - \Delta(t)(1 + \lambda)S_t\) into the bank account. At time \(t + \delta t\), an additional number of shares of the stock is bought or sold in order to have the target hedge ratio

\[
\Delta(t + \delta t) = \frac{\partial V(t + \delta t, S_{t+\delta t})}{\partial S}.
\]

At the same time, the bank account is adjusted by

\[
\left[\Delta(t + \delta t) - \Delta(t) - |\Delta(t + \delta t) - \Delta(t)|\lambda\right]S_{t+\delta t}.
\]

Then the hedging is repeated in the same manner at all subsequent times \(t + i\delta t, i = 2, 3, \ldots, n - 1\).

The choice of the number of hedging intervals, \(n\), is somewhat unclear. Obviously, when \(n\) is small, the volume of transaction costs is also small, but the variance of the replication error is large. An increase in \(n\) reduces the variance of the replication error at the expense of increasing the volume of transaction costs. Moreover, as \(\delta t \rightarrow 0\), the volume of transaction costs approach infinity.

### 2.3 The Method of Hoggard, Whalley and Wilmott

A variety of methods have been suggested to deal with the problem of option pricing and hedging with transaction costs. Leland (1985) was the first to initiate this stream of research. He adopted the rehedging at fixed regular intervals and proposed a modified Black-Scholes strategy that permits the replication of a single option with finite volume of transaction costs no matter how small the rehedging interval is. The hedging strategy is adjusted by using a modified volatility.

In particular, the central idea of Leland was to include the expected transaction costs in the cost of a replicating portfolio. That is, according to Leland, the price of an option must equal the expected costs of the replicating portfolio including the transaction costs. As a result, the market maker, who writes, for example, a European call option and constructs the replicating portfolio, should sell it with a premium (as compared to the Black-Scholes price) which offsets the expected transaction costs. On the contrary, the market maker, who buys a European call option and constructs the replicating portfolio, should buy the option with a discount to offset, again, the expected transaction costs. The Leland’s pricing and hedging method is an adjusted Black-Scholes method where one uses a modified volatility in the Black-Scholes formulas for the option price and delta. In
hedging a short European call option the modified volatility is higher than the original volatility which gives a higher option price. On the contrary, in hedging a long European call option the modified volatility is lower than the original volatility which gives a lower option price. Using the Leland’s method one hedges an option with a delta calculated similarly as the Black-Scholes delta, but with adjusted volatility.

Hoggard, Whalley, and Wilmott (1994) extended the method of Leland for the case of a portfolio of options. They showed that to find the option portfolio price and hedging strategy one needs to solve the following nonlinear PDE

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \left[ 1 - K \text{ sign} \left( \frac{\partial^2 V}{\partial S^2} \right) \right] \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

where

$$K = \frac{\lambda}{\sigma} \sqrt{\frac{8}{\pi \delta t}},$$

and where sign(·) is the sign function. The comparison of the PDEs (1) and (4) allows us to introduce a new parameter

$$\sigma^2_m = \sigma^2 \left[ 1 - K \text{ sign} \left( \frac{\partial^2 V}{\partial S^2} \right) \right],$$

and interpret it as the modified volatility. This volatility adjustment depends on the sign of the second derivative of the option portfolio price with respect to the underlying asset price. Recall that this second derivative is known as gamma (the sensitivity of the option portfolio delta to the underlying asset price)

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}.$$  

It is widely known that the modified volatility increases/decreases an option price to account for the amount of hedging transaction cost. However, only a few know that the Leland’s hedging very often outperforms the Black-Scholes hedging at fixed regular intervals even in the case when the hedger starts with the same initial value of the replicating portfolio (see, for example, Mohamed (1994), Clewlow and Hodges (1997) or Zakamouline (2006b)). The only difference in between the Black-Scholes hedging strategy and the Leland’s hedging strategy is in the value of hedging volatility. This implies that the Leland’s modification of volatility optimizes somehow the Black-Scholes hedging strategy in the presence of transaction costs. Zakamouline (2006b) explains in details how the Leland’s modified hedging volatility works. Now we present shortly the explanation.

To understand how the Leland’s modified hedging volatility improves the risk-return tradeoff of the Black-Scholes hedging strategy, we need first
to make the following two observations: Observe that the (total) replication error of a hedging strategy can be subdivided into a hedging error and transaction costs. Note that both the hedging error and the amount of transaction costs of a discretely adjusted delta-neutral replicating strategy are path-dependent. In particular, the amount of transaction costs depends on the absolute value of the option gamma (see Leland (1985) or Hoggard et al. (1994)). The hedging error depends on the values and the signs of the option theta and gamma (see Boyle and Emanuel (1980)). In short, the Leland’s modified hedging volatility makes the hedging error be “negatively” correlated with transaction costs: the hedging error becomes positive when transaction costs are large, and the hedging error becomes negative when transaction costs are small. Thus, the Leland’s modified volatility “equalizes” the replication error across different stock paths. This reduces the risk of the hedging strategy as measured by the variance of the replication error. To illustrate the aforesaid, Figure 1 presents the simulation results of hedging a plain vanilla European call option according to the Leland’s strategy as compared to hedging according to the Black-Scholes strategy.

![Figure 1](image)

Figure 1: Comparison of the expected replication errors of the Leland’s strategy (dashed line) against the Black-Scholes strategy (solid line) across different stock prices at maturity. The option is a vanilla call with strike $K = 100$ that is rehedged every $\delta t = 1/100$.

Note that the Leland’s volatility adjustment reduces the risk of a replicating strategy. However, the reduction of risk can happen at the expense of reducing the returns of a replicating strategy. When a hedger is short gamma, the option is hedged with an increased hedging volatility. An increased hedging volatility decreases the absolute value of the option gamma in the region where the option gamma is high. This reduces the amount of hedging transaction costs (see Zakamouline (2006b)). On the contrary, when a hedger is long gamma, the option is hedged with a decreased hedging volatility. A decreased hedging volatility increases the absolute value of the option gamma in the region where the option gamma is high. This increases
the amount of hedging transaction costs. This is illustrated in Table 1.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Black-Scholes</th>
<th>Leland</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of replication error</td>
<td>-3.632</td>
<td>-3.276</td>
</tr>
<tr>
<td>Std. deviation of replication error</td>
<td>1.443</td>
<td>0.938</td>
</tr>
</tbody>
</table>

(a) Short vanilla call

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Black-Scholes</th>
<th>Leland</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of replication error</td>
<td>-3.632</td>
<td>-4.361</td>
</tr>
<tr>
<td>Std. deviation of replication error</td>
<td>1.329</td>
<td>0.778</td>
</tr>
</tbody>
</table>

(b) Long vanilla call

Table 1: Comparison of the risk-return tradeoffs of the Leland’s strategy against the Black-Scholes strategy. The option is a vanilla call that is re-hedged every $\delta t = 1/100$.

### 2.4 The Delta Tolerance Strategy

This commonly used strategy prescribes rehedging to the Black-Scholes delta when the hedge ratio\(^1\) moves outside of the prescribed tolerance from the perfect hedge position. More formally, the series of stopping times is recursively given by

$$\tau_1 = t, \quad \tau_{i+1} = \inf \left\{ \tau_i < \tau < T : \left| \Delta - \frac{\partial V}{\partial S} \right| > H \right\}, \quad i = 1, 2, \ldots,$$

where $\frac{\partial V}{\partial S}$ is the Black-Scholes hedge, and $H$ is a given constant tolerance. The intuition behind this strategy is pretty obvious: The parameter $H$ is a proxy for the measure of risk of the replicating portfolio. More risk averse hedgers would choose a low $H$, while more risk tolerant hedgers will accept a higher value for $H$. This strategy for hedging a single option seems to be first suggested by Whalley and Wilmott (1993). It is straightforward to apply this strategy to the hedging an option portfolio.

The hedging proceeds as follows: at time $t$ the hedger constructs a replicating portfolio by purchasing/selling $\Delta(t) = \frac{\partial V}{\partial S}$ shares of the stock. Then the hedger monitors continuously until $T$ the discrepancy between the hedge ratio and the perfect hedge position. When this discrepancy exceeds $H$, the rebalancing occurs so as to bring the hedge ratio to the perfect hedge position.

### 2.5 The Asset Tolerance Strategy

This strategy is based on monitoring the moves in the underlying asset price and was suggested first by Henrotte (1993) for hedging a single option. The

\(^1\)This is defined as the relative quantity of the underlying asset held in the hedging portfolio.
strategy prescribes rehedging to the Black-Scholes delta when the percentage change in the value of the underlying asset exceeds the prescribed tolerance. More formally, the series of stopping times is recursively given by

\[ \tau_1 = t, \quad \tau_{i+1} = \inf \left\{ \tau_i < \tau < T : \frac{|S(\tau) - S(\tau_i)|}{S(\tau_i)} > h \right\}, \quad i = 1, 2, \ldots, \]

where \( h \) is a given constant percentage. The intuition behind this strategy is similar to that of the delta tolerance strategy. It is also straightforward to apply this strategy to the hedging an option portfolio.

The hedging proceeds as follows: at time \( t \) the hedger constructs a replicating portfolio by purchasing/selling \( \Delta(t) = \frac{\partial V}{\partial S} \) shares of the stock. Then the hedger monitors continuously until \( T \) the percentage change in the value of the underlying asset. When this percentage change exceeds \( h \), the rebalancing occurs so as to bring the hedge ratio to the perfect hedge position.

## 3 The Utility Based Hedging

### 3.1 The Method

Hodges and Neuberger (1989) pioneered the utility-based option pricing and hedging approach based that explicitly takes into account the hedger’s risk preferences. The key idea behind the utility based approach is the indifference argument: The so-called “reservation” price of an option is defined as the amount of money that makes the hedger indifferent, in terms of expected utility, between trading in the market with and without the option. In many respects such an option price is determined in a similar manner to a certainty equivalent within the expected utility framework, which is a well grounded pricing principle in economics. The difference in the two trading strategies, with and without the option, is interpreted as “hedging” the option.

Initially, the utility-based approach was applied to the pricing and hedging of single options. However, this method can be easily applied for pricing and hedging a portfolio of options. The starting point for the utility based option pricing and hedging approach is to consider the optimal portfolio selection problem of the hedger who faces transaction costs and maximizes expected utility of his terminal wealth. The hedger has a finite horizon \([t, T]\).

For the simplicity of the exposition we assume that there are no transaction costs at terminal time \( T \). The hedger has the amount \( x_t \) in the bank account, and \( y_t \) shares of the stock at time \( t \). We define the value function of the hedger with no options as

\[ J_0(t, x_t, y_t, S_t) = \max E_t[U(x_T + y_T S_T)], \quad (8) \]
where $U(z)$ is the hedger’s utility function. Similarly, the value function of
the hedger who, for example, sells the option portfolio is defined by

$$J_1(t, x_t, y_t, S_t) = \max_t E_t [U(x_T + y_T S_T - V(T, S_T))].$$

Finally, the price of the option portfolio is defined as the compensation $p$
such that

$$J_1(t, x_t + p, y_t, S_t) = J_0(t, x_t, y_t, S_t).$$

The solutions to problems (8), (9), and (10) provide the unique price and,
above all, the optimal hedging strategy. Unfortunately, there are no closed-
form solutions to all these problems. As a result, the solutions have to be
obtained by numerical methods. The existence and uniqueness of the so-
lutions were rigorously proved by Davis, Panas, and Zariphopoulou (1993).
For implementations of numerical algorithms, the interested reader can con-
sult Davis and Panas (1994) and Clewlow and Hodges (1997).

It is usually assumed that the hedger has the negative exponential utility
function

$$U(z) = -\exp(-\gamma z); \quad \gamma > 0,$$

where $\gamma$ is a measure of the hedger’s (constant) absolute risk aversion. This
choice of the utility function satisfies two very desirable properties: (i) the
hedger’s strategy does not depend on his holdings in the bank account, (ii)
the computational effort needed to solve the problem is much lower than
that in case of a utility function that exhibits a non-constant absolute risk
aversion. This particular choice of utility function might seem restrictive.
However, as it was conjectured by Davis et al. (1993) and showed in An-
dersen and Damgaard (1999), an option price is approximately invariant to
the specific form of the hedger’s utility function, and mainly only the level
of absolute risk aversion plays an important role.

The numerical calculations show that when the hedger’s risk aversion is
rather low, the hedger implements mainly a so-called “static” hedge, which
consists in buying $\Delta$ shares of the stock at time $t$ and holding them until the
option maturity $T$. When the hedger’s risk aversion increases, he starts to
rebalance the hedging portfolio in between $(t, T)$. Recall that in the frame-
work of the utility based hedging approach the hedging strategy is defined
as the difference, $\Delta(\tau) = y_1(\tau) - y_0(\tau), \tau \in [t, T]$, between the hedger’s
optimal trading strategies with and without options. When the hedger’s
risk aversion is moderate, it is impossible to give a concise description of
the optimal hedging strategy. However, when the hedger’s risk aversion is
rather high, then we can assume that $y_0(\tau) \equiv 0$ (that is, the hedger does not
invest in the underlying stock in the absence of option contracts) and the
optimal hedging strategy can be conveniently described as $\Delta(\tau) = y_1(\tau)$.
In the latter case the numerical calculations show that the optimal hedge
ratio $\Delta$ is constrained to evolve between two boundaries, $\Delta_l$ and $\Delta_u$, such
that $\Delta_l < \Delta_u$. As long as the hedge lies within these two boundaries, $\Delta_l \leq \Delta \leq \Delta_u$, no rebalancing of the hedging portfolio takes place. That is why the region between the two boundaries is commonly denoted as the no transaction region. As soon as the hedge ratio goes out of the no transaction region, a rebalancing occurs in order to bring the hedge to the nearest boundary of the no transaction region. In other words, if $\Delta$ moves below $\Delta_l$, one should immediately transact to bring it back to $\Delta_l$. Similarly, if $\Delta$ moves above $\Delta_u$, a rebalancing trade occurs to bring it back to $\Delta_u$. Figures 2 and 3 illustrate the optimal hedging strategy.

### 3.2 Hedging to a Fixed Bandwidth Around Delta

Despite a sound economical appeal of the utility based option hedging approach, it does have a number of disadvantages: the model is cumbersome to implement and the numerical computations are time consuming. One commonly used simplification of the utility-based hedging strategy (see, for example, Martellini and Priaulet (2002)) is known as hedging to a fixed bandwidth around delta. This strategy prescribes to rehedge when the hedge ratio moves outside of the prescribed tolerance from the corresponding Black-Scholes delta. More formally, the boundaries of the no transaction region are defined by

$$\Delta = \frac{\partial V}{\partial S} \pm H,$$

where $\frac{\partial V}{\partial S}$ is the middle of the hedging bandwidth that equals the Black-Scholes hedge, and $H$ is some constant which gives a constant size of the
hedging bandwidth. This strategy is closely related to the delta tolerance strategy. The value of $H$ is determined by the hedger’s risk tolerance. The essential difference between these two strategies is that in the hedging to a fixed bandwidth around delta a rebalancing brings the hedge ratio to the nearest boundary of the bandwidth, while in the delta tolerance strategy a rebalancing brings the hedge ratio to the perfect hedge position in the absence of transaction costs.

However, this strategy is, in fact, an over-simplified utility-based hedging strategy. As it is clearly seen from Figures 2 and 3, the middle of the hedging bandwidth does not coincide with the Black-Scholes hedge. Moreover, the size of the hedging bandwidth in the utility-based hedging is not constant. We will illustrate this below.

### 3.3 The Asymptotic Analysis of Whalley and Wilmott

Since there are no explicit solutions for the utility based hedging strategy with transaction costs and the numerical methods are computationally hard, for practical applications it is of major importance to use other alternatives. One of such alternatives is to obtain an asymptotic solution. In asymptotic analysis one studies the solution to a problem when some parameters in the problem assume large or small values.

Whalley and Wilmott (1997) were the first to provide an asymptotic analysis of the model of Hodges and Neuberger (1989) assuming that transaction costs are small. Using formal matched asymptotics, they showed that

![Figure 3: Optimal hedging strategy versus the Black-Scholes delta for a long Bull Spread with exercise prices 80 and 120, and the rest of the model parameters as in Figure 2.](image)
the boundaries of the no transaction region are given by
\[
\Delta = \frac{\partial V}{\partial S} \pm H_{ww} = \frac{\partial V}{\partial S} \pm \left( \frac{3 e^{-r(T-t)\lambda S\Gamma}}{2\gamma} \right)^{\frac{1}{3}}, \tag{12}
\]
where, again, \( \frac{\partial V}{\partial S} \) is the Black-Scholes hedge. It is important to note that the asymptotic method of Whalley and Wilmott (1997) is of general applicability. Even though the authors considered the pricing and hedging of a short European call option, the final expressions for the option price and hedging strategy are valid for any option, including an option portfolio as a special case of a complex option.

It is easy to see that the optimal hedging bandwidth is not constant, but depends in a natural way on a number of parameters. As \( \lambda \to 0 \), the optimal hedge approaches the Black-Scholes hedge. As \( \gamma \) increases, the hedging bandwidth decreases in order to decrease the risk of the hedging portfolio. The dependence of the hedging bandwidth on the option gamma is also natural, as we expect to rehedge more often in regions with high gamma. Moreover, it agrees quite well with the results of exact numerical computations, see Figures 4 and 5. The importance of the dependence of the size of the hedging bandwidth on the option gamma was emphasized by Zakamouline (2005) by comparing the empirical performances of the Whalley and Wilmott asymptotic strategy against the hedging to a fixed bandwidth strategy: without this dependence there are lots of “unnecessary” rebalancing when the price of the underlying changes insignificantly. That is why the Whalley and Wilmott asymptotic strategy outperforms the hedging to a fixed bandwidth strategy. Moreover, Zakamouline (2005) showed that when the option gamma is huge and the size of a constant hedging bandwidth is small relative to the range of the option delta, the performance of the hedging to a fixed bandwidth strategy might be worse than that of the Black-Scholes hedging at fixed intervals strategy.

### 3.4 The Asymptotic Analysis of Barles and Soner

Barles and Soner (1998) performed an alternative asymptotic analysis of the same model assuming that both the transaction costs and the hedger’s risk tolerance are small. They found that to find the option value one needs to solve the following nonlinear PDE
\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \left[ 1 - f \left( e^{r(T-t)\lambda S\Gamma} \right) \right] \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{13}
\]
Again, formally the option value is equal to the Black-Scholes value with an adjusted variable volatility given by
\[
\sigma_m^2 = \sigma^2 (1 - K_{bs}) = \sigma^2 \left( 1 - f \left( e^{r(T-t)\lambda^2 S^2\Gamma} \right) \right). \tag{14}
\]
Figure 4: The form of the optimal hedging bandwidth $(\Delta_u - \Delta_l)S$, obtained using the exact numerics, versus the absolute value of the option portfolio gamma times the stock price, $|\Gamma|S$, for a short Butterfly Spread and the following model parameters: $\gamma = 0.1$, $\lambda = 0.01$, $S_0 = 100$, $\sigma = 0.25$, $\mu = r = 0.05$, $T - t = 0.5$, and exercise prices 80, 100, and 120.

Figure 5: The form of the optimal hedging bandwidth $(\Delta_u - \Delta_l)S$, obtained using the exact numerics, versus the absolute value of the option portfolio gamma times the stock price, $|\Gamma|S$, for a long Bull Spread with exercise prices 80 and 120, and the rest of the model parameters as in Figure 4.
Figure 6: The middle of the optimal hedging bandwidth, obtained using the exact numerics, versus the Black-Scholes delta and the Hoggard, Whalley and Wilmott delta with $K = 0.54$ for a short Butterfly Spread and the following model parameters: $\gamma = 3.0, \lambda = 0.01, S_t = 100, \sigma = 0.25, \mu = r = 0.05, T - t = 0.5$, and exercise prices 80, 100, and 120.

In contrast to (6), this volatility adjustment depends not only on the sign, but also on the value of the option gamma. The optimal hedging strategy consists in keeping the hedge ratio inside the no transaction region given by

$$\Delta = \frac{\partial V(\sigma_m)}{\partial S} \pm H_{bs} = \frac{\partial V(\sigma_m)}{\partial S} \pm \frac{1}{\lambda S} g \left( \lambda^2 \gamma S^2 \Gamma \right),$$

where $\frac{\partial V(\sigma_m)}{\partial S}$ is the Black-Scholes hedge with an adjusted volatility.

The comparative statics for the Barles and Soner optimal hedging bandwidth is similar to that of the Whalley and Wilmott one: the hedging bandwidth increases when either the level of the transaction costs, the hedger’s risk tolerance, or the option gamma increases.

Unfortunately, the functions $f(\cdot)$ and $g(\cdot)$ depend on the option payoff and Barles and Soner gave their forms only for the case of a plain vanilla call. Consequently, one cannot use the asymptotic strategy of Barles and Soner for hedging a portfolio of options. However, it is important to emphasize that the Barles and Soner volatility adjustment works similarly\(^2\) to the Leland’s volatility adjustment. For a short call option the modified volatility is higher than the original volatility. On the contrary, for a long call option the modified volatility is lower than the original volatility.

Even though Barles and Soner performed the asymptotic analysis of the utility based pricing and hedging for a particular type of option, the follow-

\(^2\)This is also clearly seen from the comparison of equations (13) and (4).
Figure 7: The middle of the optimal hedging bandwidth, obtained using the exact numerics, versus the Black-Scholes delta and the Hoggard, Whalley and Wilmott delta with $K = 0.54$ for a long Bull Spread with exercise prices 80 and 120, and the rest of the model parameters as in Figure 6.

This conclusion becomes obvious: the middle of the hedging bandwidth in the utility-based hedging does not coincide with the Black-Scholes delta. This agrees quite well with the results of the exact numerical computations, see Figures 6 and 7. This essential feature of the utility-based hedging strategy was already observed by Hodges and Neuberger (1989) and emphasized by Clewlow and Hodges (1997) using the results of Monte Carlo simulations. Indeed, as the hedger’s risk aversion increases, the hedging bandwidth decreases in order to decrease the risk of the hedging portfolio. Without volatility adjustment, the Whalley and Wilmott asymptotic, delta tolerance, asset tolerance, and hedging to a fixed bandwidth around delta strategies converge to the continuous time Black-Scholes strategy. That is, in the limit as we increase the hedger’s risk aversion, the amount of transaction costs tends to infinity. On the contrary, in the correct utility-based strategy, as the hedger’s risk aversion increases, the decrease in the size of the hedging bandwidth is largely compensated by the increase in the volatility adjustment. Figures 6 and 7 also show that the optimal volatility adjustment in the utility-based hedging is very similar to the Hoggard, Whalley, and Wilmott volatility adjustment when we appropriately choose the value of parameter $K$ in the PDE (4). Note that $K$ determines the “degree” of volatility adjustment.
3.5 The Approximation Method of Zakamouline

Recall that in asymptotic analysis one studies the limiting behavior of the optimal hedging policy as one or several parameters of the problem approach zero. Even though asymptotic analysis can reveal the underlying structure of the solution, under realistic parameters this method provide not quite accurate results. Zakamouline (2004) compared the performance of asymptotic strategies against the exact strategy and found out that under realistic model parameters an asymptotic strategy performs noticeably worse than that obtained from the exact numerical solution. The explanation lies in the fact that, when some of the model parameters are neither very small nor very large, an asymptotic solution provides not quite accurate results. In particular, as compared to the exact numerical solution, under realistic parameters the size of the hedging bandwidth and the volatility adjustment obtained from asymptotic analysis are overvalued. What is more important, an asymptotic solution showed to be unable to sustain a correct inter-relationship between the size of the hedging bandwidth and the degree of volatility adjustment. The significance of the correct interrelationship could hardly be overemphasized: The empirical testing of the hedging strategies revealed that either undervaluation or overvaluation of the volatility adjustment (with respect to the size of the hedging bandwidth) results in a drastic deterioration of the performance of a hedging strategy.

Zakamouline (2004) and Zakamouline (2006a) summarized the stylized facts about the nature of the utility-based hedging strategy and suggested a general specification of the optimal hedging policy for a single plain vanilla European option. The careful visual inspection of the numerically calculated optimal hedging policy together with the insights from asymptotic analysis advocate for the following general specification of the optimal hedging strategy

$$\Delta = \frac{\partial V(\sigma_m)}{\partial S} \pm (H_1 + H_0),$$

(16)

where $\sigma_m$ is the adjusted volatility given by

$$\sigma_m^2 = \sigma^2 (1 - K_\sigma).$$

(17)

The term $H_1$ is closely related to $H_{ww}$ in the Whalley and Wilmott hedging strategy (see equation (12)) and to $H_{bs}$ in the Barles and Soner hedging strategy (see equation (15)). The main feature of $H_1$ is that this term depends on the option gamma. Note that the option gamma approaches zero as the option goes farther either out-of-the-money ($S \to 0$) or in-the-money ($S \to \infty$). This means, in particular, that the size of the no transaction region in an asymptotic strategy also approaches zero. On the contrary, the exact numerics show that, when the option gamma goes to zero, the size of the no transaction region times the stock price approaches a constant value, see, for example, Figures 4 and 5. It turns out that this constant value
is actually the size of the no-transaction region in the optimal portfolio selection problem without options. To reflect this feature of the optimal hedging policy Zakamouline introduced the term $H_0$. The reason for the absence of $H_0$ in an asymptotic strategy is the fact that when either $\gamma \to \infty$ or $\lambda \to 0$ then the size of $H_0$ becomes much less than the size of $H_1$ (see, for example, equations (20) and (21) below). However, when the hedger’s risk aversion is not very high, the presence of $H_0$ is important. This fact was emphasized in Zakamouline (2004). Finally, $K_\sigma$ is closely related to $K_{bs}$ in the Barles and Soner volatility adjustment (see equation (14)) and to $K$ in the Hoggard, Whalley, and Wilmott volatility adjustment (see equation (6)).

Zakamouline (2004) suggested to use an alternative to the asymptotic analysis, namely, the approximation method. The general description of the approximation technique he employed can be found in, for example, Judd (1998) Chapter 6. Under approximation it is meant the following: one has a rather slow and cumbersome way to compute the optimal hedging policy with transaction costs and wants to replace it with simple and efficient approximating function(s). To do this, one first specifies a flexible functional form of the optimal hedging policy. Then, given a functional form, the parameters are chosen to provide the best fit to the exact numerical solution. This second stage is known as “model calibration”.

In short, Zakamouline assumed the following functional form of the approximating function for $H_0$, $H_1$, and $K_\sigma$:

$$H_0 = \alpha \sigma^{\beta_1} \lambda^{\beta_2} \gamma^{\beta_3} S^{\beta_4} (T - t)^{\beta_5}, \quad (18)$$

$$H_1 = K_\sigma = \alpha \sigma^{\beta_1} \lambda^{\beta_2} \gamma^{\beta_3} S^{\beta_4} \Gamma^{\beta_5} e^{-\beta_6 \sigma (T-t)} (T - t)^{\beta_7}, \quad (19)$$

where $\alpha, \beta_1, \ldots, \beta_k$ are parameters to be chosen in order to achieve the best fit. The results of his estimations of the best-fit parameters for hedging a short European call option, after some rounding off the values of parameters $\alpha$ and $\beta_i$, give the following approximating functions

$$H_0 = \frac{\lambda}{\gamma S \sigma^2 (T - t)}, \quad (20)$$

$$H_1 = 1.12 \lambda^{0.31} (T - t)^{0.05} \left(\frac{e^{-r(T-t)}}{\sigma}\right)^{0.25} \left(\frac{\left|\Gamma\right|}{\gamma}\right)^{0.5}, \quad (21)$$

$$K_\sigma = -5.76 \frac{\lambda^{0.78}}{(T - t)^{0.02}} \left(\frac{e^{-r(T-t)}}{\sigma}\right)^{0.25} \left(\gamma S^2 \left|\Gamma\right|\right)^{0.15}. \quad (22)$$

Then, Zakamouline tested empirically his approximation strategy against both the asymptotic strategies and some other well-known strategies. He evaluated the performance of different hedging strategies within the unified mean-variance and the mean-VaR frameworks and found that his approximation strategy outperforms all the others.
A Simplified Utility-Based Hedging Strategy and the Method of Finding the Optimal Parameters

4.1 A Simplified Parameterized Description of the Utility-Based Hedging Strategy

The approximation methodology presented in Zakamouline (2004) produces a clearly superior result as applied to the problem of finding a closed-form expressions for the optimal hedging strategy for a specific option or an option portfolio. However, this methodology does retain the main disadvantage we would like to get rid of: to approximate the optimal strategy for a specific option one needs to start with the numerical calculations of the optimal hedging strategy for a large set of the model parameters. Since the numerical algorithm is cumbersome to implement and the calculation of the optimal hedging strategy is time consuming, the approximation methodology is unlikely to be commonly used by the practitioners.

In the preceding section we presented the description of the nature of the optimal hedging strategy for a portfolio of options. This presentation suggests that the optimal hedging strategy is rather simple and has three essential features:

- The presence of the no transaction region such that if the hedge ratio lies outside of the no transaction region a rebalancing occurs in order to bring the hedge to the nearest boundary of the no transaction region.
- The form of the no transaction region is mainly derived from the form of the absolute value of the option portfolio gamma. In addition we know that when the option portfolio gamma tends to zero, the size of the no transaction region tends to some constant.
- The middle of the no transaction region does not coincide with the Black-Scholes delta. This feature of the optimal hedging strategy can be conveniently described as a modified hedging volatility.

The hedging to a fixed bandwidth around delta strategy given by

$$\Delta = \frac{\partial V}{\partial S} \pm H,$$

is, in fact, an over-simplified utility-based hedging strategy. That is, this strategy lacks two essential features of the utility-based hedging strategy: the optimal form of the hedging bandwidth and the volatility adjustment. The absence of these features leads to a bad empirical performance of this hedging strategy when the size of the no transaction region, $H$, is rather small (i.e., when the hedger’s risk aversion is high) and when the option payoff is either discontinuous or a rather complicated function of the underlying, see Zakamouline (2005). However, in hedging an option position that
has more or less smooth payoff and when the hedger’s risk aversion is not very high, the hedging to a fixed bandwidth strategy shows a good empirical performance that is close to that of the exact utility-based hedging strategy, see Martellini and Priaulet (2002) and Zakamouline (2005).

To reflect all the three essential features of the utility-based hedging strategy, we propose to describe it similarly to as in Zakamouline (2004) and Zakamouline (2006a)

\[
\Delta = \frac{\partial V(\sigma_m)}{\partial S} \pm (H_1 + H_0),
\]

where the delta of an option portfolio, \(\frac{\partial V(\sigma_m)}{\partial S}\), is obtained by solving the following PDE with a proper boundary condition

\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2_m S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,
\]

and where \(\sigma_m\) is the adjusted volatility given by

\[
\sigma^2_m = \sigma^2 (1 - K_\sigma).
\]

We know from Zakamouline (2004) that \(H_0\) is the same for all option positions. The challenge now is to find simple functional forms for \(H_1\) and \(K_\sigma\) that are suitable for any option portfolio.

As a motivation for the choice of the functional form for \(H_1\), let us consider the Whalley and Wilmott asymptotic strategy given by (12). Note that some parameters in (12) are constant during the life of an option position, but the others are variable. For practical applications it makes sense to present the Whalley and Wilmott asymptotic strategy in the following form

\[
\Delta = \frac{\partial V}{\partial S} \pm H_{ww} = \frac{\partial V}{\partial S} \pm h \left( e^{-r(T-t)} S T^2 \right)^{\frac{1}{3}},
\]

where \(h = \left( \frac{3}{2} \right)^{\frac{1}{3}} \) is some constant parameter reciprocal to the hedger’s risk aversion. Similarly to the asymptotic result of Whalley and Wilmott, we assume that the form of the hedging bandwidth \(H_1\) is given by

\[
H_1(t, S) = h_1 e^{-\theta r(T-t)} S^\alpha |\Gamma|^\beta,
\]

where \(\theta\), \(\alpha\) and \(\beta\) are some parameters to be estimated. To estimate these parameters we need to compute the bandwidth \(H_1(t, S)\) and estimate the best-fit parameters\(^3\) for \(\theta\), \(\alpha\) and \(\beta\). To do this, we define an option portfolio, fix the set of parameters \(r, \mu, \sigma, \lambda, \gamma\), and calculate numerically the upper \(y^u_0(t, S)\) and the lower \(y^l_0(t, S)\) boundaries of the no transaction region

without the option portfolio, and the upper \( y_u(t, S) \) and the lower \( y_l(t, S) \) boundaries of the no transaction region with the option portfolio. Then

\[
H_0(t, S) = \frac{y_u^0(t, S) - y_l^0(t, S)}{2},
\]

\[
H_1(t, S) = \frac{y_u^1(t, S) - y_l^1(t, S)}{2} - H_0(t, S).
\]

We measure the goodness of fit using the \( L^2 \) norm. This largely amounts to using the techniques of ordinary linear regression after the log-log transformation of (24). That is, we find the parameters \( \theta, \alpha \) and \( \beta \) by solving the problem

\[
\min_{h_1, \theta, \alpha, \beta} \sum_m \left( \log(H_1(t, S)) - \log(h_1) + \theta r(T-t) - \alpha \log(S) - \beta \log(|\Gamma(t, S)|) \right)^2,
\]

where \( m \) is the number of different data points in \( t \) and \( S \).

Our estimations of the best fit parameters for \( \theta \) show that this parameter is almost insignificant. That is, in the most cases we cannot reject the hypothesis that the value of \( \theta \) is significantly different from zero. This means that we can safely assume the following form for the hedging bandwidth \( H_1 \)

\[
H_1(t, S) = h_1 S^\alpha |\Gamma|^{\beta}.
\]

(25)

Unfortunately, our estimations of the best fit parameters for \( \alpha \) and \( \beta \) show that the values of \( \alpha \) and \( \beta \) depend not only on a particular option portfolio, but also on the hedger’s risk aversion. How do we proceed further? We reformulate the question as follows: How important is the variable \( S \) in (24)? It is easy to find out by comparing the goodness of fit, given by \( R^2_{\alpha, \beta} \) (equal to the regression sum of squares divided by the total sum of squares), from the estimation of (25) and the goodness of fit, \( R^2_{\beta} \), from the estimation of

\[
H_1(t, S) = h_1 |\Gamma|^{\beta}.
\]

(26)

The goodness of fits for some popular combinations of standard options is given in Table 2.

By studying the table it becomes clear that the gamma of an option portfolio alone explains at least 84% of the variation of the bandwidth \( H_1 \). Even though \( \alpha \) is significant in the linear regression and the variable \( S \) helps to improve the goodness of fit, we can disregard \( S \) because the marginal improvement of the goodness of fit provided by this variable is small, and often even insignificant.

Now we are left with only \( \beta \) and estimate (26) for different option portfolios and different values of the hedger’s risk aversion. Our study shows that \( \beta \in (0.3, 2.5) \) and depends on the composition of an option portfolio and the hedger’s risk aversion. That is, there is no single value of \( \beta \) that
Table 2: The goodness of fits for some popular combinations of standard options. The chosen model parameters are: $\gamma = 1.0$, $\lambda = 0.01$, $\sigma = 0.25$, $\mu = r = 0.05$, $T - t = 0.5$, and exercise prices 80, 100, and 120. The goodness of fit was measured in the stock price interval $50 \leq S_t \leq 150$. Note that in the estimation of (25) and (26) we use a log-log transformation, but in the estimation of (27) we do not use one. As a result, $R^2_{\alpha,\beta}$ and $R^2_{\beta}$ are not quite directly comparable with $R^2_{\beta=1}$ and $R^2_{\beta \approx 2/3}$.

<table>
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<th>Combination</th>
<th>$R^2_{\alpha,\beta}$, %</th>
<th>$R^2_{\beta}$, %</th>
<th>$R^2_{\beta=1}$, %</th>
<th>$R^2_{\beta \approx 2/3}$, %</th>
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<tr>
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<td>92</td>
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<td>96</td>
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<tr>
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</table>

suits all possible option portfolios and the hedger’s risk aversions. What we can do is the following: we can either define a single value for $\beta$ that belongs to the interval $(0.3, 2.5)$, for example $\beta = 1$, or we can left it as a parameter to be determined. To illustrate how much we explain in the variation of

$$H_1(t, S) = h_1 |\Gamma|^{\beta=\text{value}}$$

if we hold $\beta$ fixed at some value\(^4\), Table 2 reports also the goodness of fits for $\beta = 1$ and $\beta = 2/3$ (recall that the latter value is obtained in the asymptotic analysis of Whalley and Wilmott). We see that even if $\beta$ is fixed, the goodness of fit remains quite high and when $\beta = 1$ in the worst case we explain at least 54% of the variation of the bandwidth $H_1$. Consequently, for the sake of simplicity we propose to use $\beta = 1$.

\(^4\)That is, in this case we estimate only the value of $h_1$. 

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Now we turn on to the determination of the functional form for $K_\sigma$. As in Zakamouline (2004), we assume that the form of the volatility adjustment is given by

$$K_\sigma = k_\sigma \text{sign}(\Gamma) S^{\alpha|\Gamma|^\beta}$$

such that the sign of the volatility adjustment depends on the sign of the option portfolio gamma and the volatility is adjusted more in regions where the absolute value of gamma is high. $k_\sigma$ in (28) is some parameter that determines the degree of the volatility adjustment. This parameter is constant during the life of the option position and is proportional to the hedger's risk aversion and the level of transaction costs.

When it comes to the estimation of $\alpha$ and $\beta$ in (28), our study shows that the values of $\alpha$ and $\beta$ also depend not only on a particular option portfolio, but also on the hedger’s risk aversion. Again, the option gamma is the major factor that explains the variation of the form of $K_\sigma$. When we try to estimate the best-fit parameter $\beta$ for different option positions and different values of the hedger’s risk aversion, we find that $\beta \in (0, 0.3)$. Again, the optimal value of $\beta$ is not unique. We suggest the choice $\beta = 0$ which might seem a bit surprising, but is completely equivalent to the Hoggard, Whalley, and Wilmott volatility adjustment and could probably be justified by the following: the presence of the volatility adjustment is much more important than its slight dependence on the option gamma. As an illustration of the fact that the Hoggard, Whalley, and Wilmott delta with appropriately chosen volatility adjustment is very close to the middle of the hedging bandwidth in the utility-based hedging strategy, see Figures 6 and 7.

Consequently, after all the experiments, we propose the following simplified description of the utility-based hedging strategy for any option portfolio

$$\Delta = \frac{\partial V(\sigma_m)}{\partial S} \pm \left( h_1|\Gamma| + \frac{h_0}{S(T-t)} \right),$$

where the delta of an option portfolio is obtained by solving the following PDE with a proper boundary condition

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \left[ 1 - k_\sigma \text{sign} \left( \frac{\partial^2 V}{\partial S^2} \right) \right] \frac{\partial^2 V}{\partial S^2} - rV = 0.$$  

In contrast to the hedging to the fixed bandwidth strategy given by (11), our simplified utility-based hedging strategy is not over-simplified, but preserves all the essential features of the utility-based hedging strategy: (i) existence of the no-transaction region, (ii) optimal form of the no-transaction region, and (iii) the volatility adjustment. However, despite a seemingly very drastic

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\(^5\)This is clearly seen for a single option, see, for example, Zakamouline (2004) or Zakamouline (2006a).
simplification, our experiments have shown (see the next subsection) that the empirical performance of the simplified utility-hedging strategy is very close to that of the exact utility-based hedging strategy. In addition, it is important to note that simplification is the only way to be able to determine the optimal parameters using a simple numerical method.

4.2 The Method of Finding the Optimal Parameters of the Hedging Strategy

In the previous subsection we have proposed a simplified parameterized description of the utility-based hedging strategy for a portfolio of options. That is, the simplified utility-based hedging strategy is specified by a triple of parameters \((h_1, h_0, k_\sigma)\). The purpose of this subsection is to present a method of finding the optimal triple of parameters of the simplified utility-based hedging strategy.

To motivate for our method, consider the following: The question of major theoretical and practical importance is how to compare alternative hedging strategies and find out which strategy is better. The widely accepted method of comparison is the following (see, for example, Mohamed (1994), Clewlow and Hodges (1997), Martellini and Priaulet (2002), and Zakanouline (2006a)): first of all we need to decide upon a unified risk-return framework for comparison. Then we perform a simulation analysis of every hedging strategy. Take, as an example, the asymptotic hedging strategy of Whalley and Wilmott given by (23). We fix the value of \(h\), perform simulations of the hedging strategy, and compute the risk and return of the hedging strategy for this particular \(h\). Then, we vary the parameter \(h\) and span all the possible combinations of risk and return. The set of all possible combination of risk and return of a particular hedging strategy comprises the efficient frontier of the hedging strategy. Similarly, we can perform simulations of the hedging to a fixed bandwidth around delta strategy by varying the parameter \(H\) and find its efficient frontier in the same risk-return space. Each hedging strategy produces different efficient frontier, that is, each hedging strategy offers a different risk-return tradeoff. Observe that a rational hedger will always prefer a strategy that minimizes the risk for a given level of return. This is in a clear analogy with the modern portfolio theory where the investor seeks to minimize the volatility of the portfolio (i.e., the risk) for a given level of returns. Consequently, some hedging strategy is better than the others if for a given level of returns it offers the lowest risk. Alternatively, some hedging strategy is better than the others if for a given level of risk it offers the highest returns. A hedging strategy may be superior with respect to the others if for any level of returns it offers the lowest risk.

Consequently, if we choose a suitable risk-return framework, there are some optimal combinations of the triple of parameters \((h_1, h_0, k_\sigma)\) of the sim-
plified utility-based hedging strategy that constitute the efficient frontier of the strategy. The essential feature of a point belonging to the efficient frontier is that the risk-return tradeoff given by this point cannot be improved. That is, any other combinations of \((h_1, h_0, k_\sigma)\) which produce a lower risk have a lower return, or, similarly, any other combinations of \((h_1, h_0, k_\sigma)\) which produce a higher return have a higher risk. Thus, the main idea behind the our method of finding the optimal parameters of the simplified hedging strategy is to use the simulation analysis in order to find the efficient frontier of the hedging strategy in some risk-return space.

Before proceeding to the formal presentation of our method, we need to agree on a suitable risk-return framework. The expected replication error is the only sensible candidate for the return measure of a hedging strategy. As to the risk measure, there are many metrics of risk. In the context of option hedging, the most popular risk metric is the variance of the replication error of a hedging strategy. Therefore, we propose to find the optimal parameters of the simplified utility-base hedging strategy in the following risk-return space: mean replication error - variance of replication error. As a justification for the chosen risk-return space consider the following: We suppose that a hedger has a relatively high risk aversion so that he does not invest in the risky asset if he does not have to. The hedger faces the option hedging problem where he uses the amount of \(p = x_t + y_t S_t\) to construct a self-financing replicating portfolio to hedge the option position. It is widely known that in case of the negative exponential utility the hedger’s choice in \(y_{\tau}, t \leq \tau < T\), is independent of the hedger’s wealth. Consequently, we can assume that the hedger has zero initial wealth. Recall that the hedger’s problem is

\[
\max E[-\exp(-\gamma(x_T + y_T S_T - V(T, S_T)))] = \max E[-\exp(-\gamma RE)],
\]

(31)

where \(x_T + y_T S_T - V(T, S_T)\) is now the replication error, \(RE\), of a hedging strategy. Using a Taylor series expansion it is easy to show that the problem (31) roughly amounts to

\[
\max E[RE] - \frac{1}{2} \gamma \text{Var}[RE].
\]

(32)

That is, the hedger’s problem can be interpreted as finding the optimal risk-return tradeoff between the risk and return of a hedging strategy where the return is given by the mean replication error and the risk is measured by the variance of the replication error.

Now we turn on to the formal presentation of our optimization method in the chosen risk-return space where we search for the combinations of the triple of parameters \((h_1, h_0, k_\sigma)\) which belong to the efficient frontier. We choose some triple of parameters of the hedging strategy, \((h_1, h_0, k_\sigma)\), perform path simulations, and estimate the return, \(\eta = \eta(h_1, h_0, k_\sigma)\), and
the benchmark strategy, $\rho = \rho(h_1, h_0, k_\sigma)$, associated with this particular hedging strategy. Observe that for any risk averse hedger the hedging strategy with parameters $(h'_1, h'_0, k'_\sigma)$ is considered to be better than the strategy with parameters $(h, h_0, k_\sigma)$ if

$$ \eta(h'_1, h'_0, k'_\sigma) \geq \eta(h, h_0, k_\sigma) \quad \text{and} \quad \rho(h'_1, h'_0, k'_\sigma) \leq \rho(h, h_0, k_\sigma). \quad (33) $$

That is, if the strategy with $(h'_1, h'_0, k'_\sigma)$ provides either higher return with less or equal risk, or less risk with higher or equal return than the strategy with $(h, h_0, k_\sigma)$.

The optimization method we propose is based on a sequential improvement of the risk-return tradeoff of a hedging strategy. That is, starting with some $(h_1, h_0, k_\sigma)$ we search for a new $(h'_1, h'_0, k'_\sigma) = (h_1 + \Delta h_1, h_0 + \Delta h_0, k_\sigma + \Delta k_\sigma)$ such that the risk-return tradeoff of the strategy with $(h'_1, h'_0, k'_\sigma)$ is better than that of the strategy with $(h_1, h_0, k_\sigma)$. The most general procedure for finding the risk-return improvement step could be interpreted as finding a unit vector $u = (a, b, c)$ such that

$$ D_u \eta(h_1, h_0, k_\sigma) \geq 0, \quad (34) $$

and

$$ D_u \rho(h_1, h_0, k_\sigma) \leq 0, \quad (35) $$

where $D_u \eta(h_1, h_0, k_\sigma)$ and $D_u \rho(h_1, h_0, k_\sigma)$ are the directional derivatives of $\eta(h_1, h_0, k_\sigma)$ and $\rho(h_1, h_0, k_\sigma)$, respectively, in the direction of the unit vector $u$. The reader is reminded that the directional derivative of some function $f(x, y, z)$ in the direction of the unit vector $u = (a, b, c)$ is defined as

$$ D_u f(x, y, z) = \lim_{h \to 0} \frac{f(x + ha, y + hb, z + hc) - f(x, y, z)}{h} = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \nabla f(x, y, z)u. $$

Consequently, to implement an improvement step from the point with $(h_1, h_0, k_\sigma)$ we need first to find the partial derivatives of $\eta$ and $\rho$ at this point and then to find a vector $u$ such that both (34) and (35) are satisfied. Note that the vector $u$ is not unique. However, since our goal is to find the points on the efficient frontier, the actual path from some $(h_1, h_0, k_\sigma)$ to $(h'_1, h'_0, k'_\sigma)$ belonging to the efficient frontier is not important.

A simple practical realization of the optimization method is based on the implementation of an improvement step with respect to $(h_1, h_0)$ and a consequent improvement step with respect to $(h_1, k_\sigma)$. Note that the starting point for our optimization method is the risk-return tradeoff given by the benchmark strategy $(h_1, 0, 0)$. The introduction of $h_0 > 0$ and $k_\sigma > 0$ helps to improve the risk-return tradeoff of the benchmark strategy. Suppose that some $(h_1, h_0, k_\sigma)$ is an improvement of the benchmark strategy and $\eta(h_1, h_0, k_\sigma)$ and $\rho(h_1, h_0, k_\sigma)$ are the corresponding return and the risk
of this improvement. An improvement step with respect to \((h_1, h_0)\) is carried out in the following manner. We define the step sizes \(\Delta h_1\) and \(\Delta h_0\), implement the simulations and find the risk-returns tradeoffs in the points with \((h_1 + \Delta h_1, h_0, k_\sigma)\) and \((h_1, h_0 + \Delta h_0, k_\sigma)\). Then we estimate the derivatives of \(\eta\) with respect to \(h_1\) and \(h_0\)

\[
\frac{\partial \eta}{\partial h_1} = \frac{\eta(h_1 + \Delta h_1, h_0, k_\sigma) - \eta(h_1, h_0, k_\sigma)}{\Delta h_1},
\]

\[
\frac{\partial \eta}{\partial h_0} = \frac{\eta(h_1, h_0 + \Delta h_0, k_\sigma) - \eta(h_1, h_0, k_\sigma)}{\Delta h_0}.
\]

Similarly, we estimate the derivatives of \(\rho\) with respect to \(h_1\) and \(h_0\). Now given a step size \(\delta h_0\) we want to find a step size \(\delta h_1\) such that

\[
\delta \eta = \eta(h_1 + \delta h_1, h_0 + \delta h_0, k_\sigma) - \eta(h_1, h_0, k_\sigma) > 0,
\]

\[
\delta \rho = \rho(h_1 + \delta h_1, h_0 + \delta h_0, k_\sigma) - \rho(h_1, h_0, k_\sigma) < 0.
\]

The step size \(\delta h_1\) is not unique. That is, there is some degree of freedom in determining \(\delta h_1\). Therefore, we need to impose an additional condition in order to determine the step size \(\delta h_1\). For example, this condition could be

\[
\delta \eta = -\delta \rho. \quad (36)
\]

That is, we want to move in the north-west direction in the chosen risk-return space. It is easy to check that this condition gives us

\[
\delta h_1 = -\frac{\frac{\partial \eta}{\partial h_1} + \frac{\partial \rho}{\partial h_1}}{\frac{\partial \eta}{\partial h_0} + \frac{\partial \rho}{\partial h_0}} \delta h_0.
\]

In addition we need to check that

\[
\delta \rho = \frac{\partial \rho}{\partial h_1} \delta h_1 + \frac{\partial \rho}{\partial h_0} \delta h_0 < 0. \quad (37)
\]

Alternatively, we can check that \(\delta \eta > 0\). If we are able to find the step size \(\delta h_1\) such that the condition (37) is satisfied, we conclude that an improvement step with respect to \((h_1, h_0)\) is feasible and the risk-return tradeoff of the point with \((h_1 + \delta h_1, h_0 + \delta h_0, k_\sigma)\) becomes a new improvement of the benchmark strategy.

Similarly we can implement an improvement step with respect to \((h_1, k_\sigma)\). In this case given the step size \(\delta k_\sigma\) we also search for \(\delta h_1\) such that

\[
\delta \eta = \eta(h_1 + \delta h_1, h_0, k_\sigma + \delta k_\sigma) - \eta(h_1, h_0, k_\sigma) > 0,
\]

\[
\delta \rho = \rho(h_1 + \delta h_1, h_0, k_\sigma + \delta k_\sigma) - \rho(h_1, h_0, k_\sigma) < 0.
\]
Despite the simplicity of the stepwise risk-return improvement algorithm described above, in a practical realization the algorithm is very time consuming since in order to estimate the risk and return of a hedging strategy with a sufficiently high precision we need to simulate a great number of stock paths. For the purpose of comparison, a numerical computation of the “exact” hedging strategy for a portfolio of non-exotic options in a 250-periods model takes 2-10 minutes\(^6\) on a computer with a 2.6 GHz processor, given that the program is implemented on C++. To implement the above described algorithm and get robust optimization results, in order to estimate the risk and return of a hedging strategy we need to simulate approximately 500000 paths, where each path consists of 250 equally spaced trading dates over the life of an option portfolio. In this case given that the algorithm is also implemented on C++, the computational time for finding the optimal parameters \((h_1, h_0, k_\sigma)\) of the risk-return optimized hedging strategy by starting from some benchmark \((h_1, 0, 0)\) amounts, on average, to more than an hour.

To summarize, the optimization algorithm described above is very simple, but computationally time intensive. We want to compute an answer more quickly. After having carried out many experiments, we arrived to a faster modification of the algorithm to determine the optimal triple of parameters \((h_1, h_0, k_\sigma)\). The algorithm is given in the Appendix in order not to overload the paper. Using this algorithm, the computational time to find the optimal parameters of the simplified utility-based hedging strategy is close to that of the exact numerical calculations of the utility-based hedging strategy for an option position with a non path-dependent payoff. In addition, we want to emphasize one more time that this algorithm also works similarly for an option position with a path-dependent payoff, while the exact numerical computations in this case are not feasible.

5 The Simulation Analysis

In this section we provide an empirical testing of our optimized strategy against the Black and Scholes, Hoggard, Whalley and Wilmott modified volatility, delta tolerance, asset tolerance, hedging to a fixed bandwidth around delta, and the Whalley and Wilmott asymptotic strategy. We provide the simulation results for a long Bull Spread (one long call with strike \(K_1\) and one short call with strike \(K_2 > K_1\), both call options with asset settlements) and a short Butterfly Spread (one long call with strike \(K_1\), one long call with strike \(K_3\), and 2 short calls with strike \(K_2\) such that \(K_1 < K_2 < K_3\), all call options with asset settlements). The model parameters are \(T - t = 1\) year to maturity, \(S_t = 100\), the volatility \(\sigma = 0.25\), and the drift \(\mu = r = 0.05\). The proportional transaction costs were \(\lambda = 0.01\).

\(^6\)The computational time depends on the specified precision of computation.
The exercise prices for the long Bull Spread are $K_1 = 80$ and $K_2 = 120$. The exercise prices for the short Butterfly Spread are $K_1 = 80$, $K_2 = 100$, and $K_3 = 120$.

The simulation proceeds as follows: At the beginning, the writer of a complex option position receives the Black-Scholes value of the complex option and sets up a replicating portfolio. The underlying path of the stock is simulated according to

$$S(t + \delta t) = S(t) \exp \left( (\mu - 0.5\sigma^2)\delta t + \sigma \sqrt{\delta t} \varepsilon \right),$$

where $\varepsilon$ is a normally distributed variable with mean 0 and variance 1. At each $\delta t$ a check is made to see if the option needs to be rehedged. If so, the rebalancing trade is performed and transaction costs are drawn from the bank account. Finally, at expiration, we compute the replication error, that is, the cash value of the replicating portfolio minus the due exercise payment.

For the Black and Scholes and Hoggard, Whalley and Wilmott strategies, we vary the parameter $\delta t$ (rehedging interval). For the other strategies, each path consists of 250 equally spaced trading dates over the life of the option. In the delta tolerance, asset tolerance, and hedging to a fixed bandwidth strategies, $H$ and $h$ take values in $[0.01, 0.35]$. For each value of the parameter of these hedging strategies we generate 100,000 paths and compute the mean and variance of the replication error. By varying the value of the parameter of a hedging strategy, we span all the possible combinations of risk and return.

For the benchmark strategy we varied the parameter $h_1$ in $[0.025, 2.5]$. To find the optimal strategy we used fixed steps for $\Delta h_0$ and $\Delta k_\sigma$ and adaptive steps for $\Delta h_1$. To implement the optimization and compute the mean and variance of the replication error with higher precision we generated 200,000 paths for each triple of parameters $(h_1, h_0, k_\sigma)$.

Figures 8 and 9 summarize the results of simulations. For both the complex options our optimized strategies outperform all the others. In addition, we have carried on similar simulations for different values of the model parameters $r$, $\mu$, $\sigma$, $T - t$, and $\lambda$, and different types of complex options. Due to the space limitations, the results of these simulations are not presented. Qualitatively, the relative performance of different hedging strategies remains the same as that in Figures 8 and 9.

**PROVIDE SIMULATIONS RESULTS OF HEDGING A PORTFOLIO OF ASIAN OPTIONS. NOTE THAT IN THIS CASE THE DIRECT NUMERICAL COMPUTATIONS OF THE OPTIMAL HEDGING STRATEGY IS OUT OF REACH.**
Figure 8: Comparison of hedging strategies for a short Butterfly Spread.

Figure 9: Comparison of hedging strategies for a long Bull Spread.
6 Conclusion

This section concludes the paper.

Appendix

A Faster Algorithm to Find the Optimal Parameters of the Simplified Hedging Strategy

The optimization algorithm described in Section 4.2 is very simple, but computationally time intensive. We want to compute an answer more quickly. One possibility is to decrease the number of steps in the stock path. However, this might yield non-optimal computed results. In addition, note that when we simulate a hedging strategy we always estimate the risk and return with some error. In order to compute the risk and return without any error, we can use an $n$-period binary tree for the stock path. Unfortunately, since the transaction policy is path dependent, a binary tree for the stock path becomes non-recombining. Since in this case the number of tree nodes increases exponentially as the number of trading dates increases, this also leads to very time intensive computations. As a matter of fact, using a non-recombining binary tree it is practically feasible to compute the risk and return of a hedging strategy with no more than approximately 32-35 trading dates over the life of an option position. After having carried out many experiments, we found out that we can use a non-recombining binary tree with a few periods of trading for implementing a risk-return improvement step with respect to $(h_1, h_0)$, but for implementing a risk-return improvement step with respect to $(h_1, k_\sigma)$ we need many trading dates over the life on an option position.

Consequently, we arrived to the following fast two-step procedure to determine the optimal triple of parameters $(h_1, h_0, k_\sigma)$: First using a non-recombining binary tree with a few numbers of periods we implement a full improvement of the risk-return tradeoff of a hedging strategy with respect to $(h_1, h_0)$ by starting from some $(h_1, 0)$ (that is, at this step $k_\sigma = 0$). Then, for a given pair of $(h_1, h_0)$ using the simulations of a hedging strategy with many periods we search for $k_\sigma \geq 0$ which minimizes the risk of a hedging strategy. That is, instead of searching for a small improvement step with respect to $(h_1, h_0)$, and then searching for a small improvement step with respect to $(h_1, k_\sigma)$, we implement a full risk-return improvement by introducing the hedging bandwidth $H_0$ while keeping $K_\sigma = 0$, and then we implement a full risk-return improvement by introducing the volatility adjustment $K_\sigma$. This separation of the risk-return improvements by means of $H_0$ and $K_\sigma$ is possible because the bandwidth $H_0$ is important when the hedger has
relatively small risk aversion, while for a highly risk averse hedger $H_0 \approx 0$. To get the idea, see, for example, formulas (20) and (21) and observe that as $\gamma$ increases, $H_0$ decreases much faster than $H_w$. On the other hand, the volatility adjustment is more important for a highly risk averse hedger than for a more risk tolerant hedger. It is clearly seen from, for example, formula (22). Note that as $\gamma$ increases, the volatility adjustment also increases.

Now we turn on to the detailed presentation of our fast risk-return optimization algorithm and the illustration of the improvements by means of Figure 10. In the beginning, we define an initial benchmark set of the triples of parameters $(h_1, 0, 0)$. For example, in our experiments we varied $h_1$ in $(0.005, 50)$. The empirical performance of this benchmark strategy for hedging a short Butterfly Spread is illustrated in Figure 10.

In the first stage, we use a 20-period non-recombining binary tree for the stock path in order to find the hedging strategy and compute its risk and return. Using this tree we search for a full risk-return improvement of a hedging strategy by introducing a hedging bandwidth $H_0$. In particular, in accordance with the algorithm described in Section 4.2, starting with some $(h_1, 0, 0)$ we perform a number of improvements steps with respect to $(h_1, h_0)$ until the improvement is possible. The outcome of this stage is a set of new triples of parameters $(h_1^*, h_0^*, 0)$. This set of parameters become a new benchmark for the second improvement stage. The empirical performance of the new benchmark is illustrated in Figure 10.

In the second stage we use simulations of a hedging strategy with 250 trading dates. In particular, using the algorithm proposed in Zakamouline (2005), we search for $k_\sigma$ that minimizes the risk of a hedging strategy. In short, Zakamouline (2005) proposed to use the risk reduction properties of the modified hedging volatility to improve the performances of some well-known hedging strategies in the presence of transaction costs. As applied to this particular case, the algorithm starts with the benchmark triple of parameters $(h_1^*, h_0^*, k_\sigma = 0)$ obtained in the first stage. We simulate the hedging strategy $N$ times and estimate the risk. Then we change the value of $k_\sigma$ by some step $\Delta k_\sigma$, simulate the new strategy and estimate the risk again. We proceed in such a manner and find the optimal $k_\sigma^* = i\Delta k_\sigma$, $i \in \{0, 1, 2, \ldots\}$, which gives the strategy with the lowest possible risk. For the purpose of illustration, Figure 10 shows the performances of two hedging strategies with the optimal volatility adjustment. One strategy has no hedging bandwidth $H_0$ (that is, the starting point for this strategy is the risk-return tradeoff given by $(h_1, 0, 0)$), and the other starts with the optimal triple $(h_1^*, h_0^*, 0)$ obtained in the first stage.

The computational time to find the optimal combination of $(h_1, h_0)$ depends on the initial value of $h_1$ and the step size $\Delta h_0$. The higher the value of $h_1$ the longer the computational time (that is, the smaller the hedger’s risk aversion, the larger the hedging bandwidths $H_1$ and $H_0$). If $h_1 = 50$
Figure 10: This figure illustrates the performances of the simplified utility-based hedging strategy for the different stages of optimization. For the purpose of comparison, the figure shows the performance of the exact utility-based strategy and the performance of the hedging to a fixed bandwidth strategy.

then the computational time amounts\(^7\) to approximately 5-15 minutes depending on the step size \(\Delta h_0\). The computational time to find the optimal volatility adjustment depends on the number of simulations \(N\), the step size \(\Delta k_\sigma\), and the initial value of \(h_1\). Note that in this case the smaller the value of \(h_1\) the longer the computational time (that is, the higher the hedgers’ risk aversion, the smaller the bandwidth \(H_1\) and the higher the optimal volatility adjustment \(K_\sigma\)). If \(N = 200000\) and \(h_1 = 0.005\), the computational time amounts to approximately 10-20 minutes depending on the step size \(\Delta k_\sigma\). Consequently, the computational time to find the optimal parameters of the simplified hedging strategy is close to that of the exact numerical calculations of the utility-based hedging strategy for a non-path-dependent option position.

For the purpose of comparison, Figure 10 shows also the performance of the exact (numerically computed) utility-based hedging strategy and the performance of the hedging to a fixed bandwidth strategy. Observe that the

\(^7\)As before, all the times are given under conditions that the program is written on C++ and is run on a computer with 2.6 GHz processor.
performance of the optimized simplified hedging strategy is very close to that of the exact hedging strategy when the hedger’s risk aversion is rather low. However, as the hedger’s risk aversion increases, the performance of the optimized simplified hedging strategy becomes worse than that of the exact strategy. We guess that the exact utility based hedging strategy has probably some other feature besides the volatility adjustment that improves the performance when the hedger’s risk aversion is very large. However, the performance of the optimized simplified hedging strategy is much better as compared to that of the hedging to a fixed bandwidth strategy. For this particular option position, the optimized simplified hedging strategy is either 50% less expensive at the same level of risk exposure, or 50% less risky at the same level of the mean hedging error than the hedging to a fixed bandwidth strategy.
References


